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Hyponormal Toeplitz Operators And Zeros Of Polynomials

By

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Abstract. The problem of hyponormality for Toeplitz operators with (trigonometric) polynomial symbols is studied. We give a necessary and sufficient condition using the zeros of the analytic polynomial induced by the Fourier coefficients of the symbol.

Let $L^p$ be the Lebesgue space on the unit circle $T$ and let $H^p$ be the corresponding Hardy space for $1 \leq p \leq \infty$. The Toeplitz operator $T_\phi$ with symbol $\phi$ in $L^\infty$ is the operator on $H^2$ defined by $T_\phi f = P(\phi f)$ for $f$ in $H^2$, where $P$ is the orthogonal projection from $L^2$ onto $H^2$. In this paper, we are interested in when $T_\phi$ is hyponormal.

Two characterizations of the hyponormality of $T_\phi$ are known as the following:

(I) Suppose $\phi_1$ and $\phi_2$ are functions in $H^2$ with $\phi = \phi_1 + \overline{\phi}_2$ in $L^\infty$. Then $T_\phi$ is hyponormal if and only if there exists a constant $c$ and a function $k$ in $H^\infty$ with $\|k\|_\infty \leq 1$ such that $\phi_2 = c + T_k \phi_1$.

(II) $T_\phi$ is hyponormal if and only if there exist two functions $k$ and $g$ in $H^\infty$ such that $\phi = k \overline{\phi} + g$ and $\|k\|_\infty \leq 1$.

The characterization (I) is due to Cowen [1]. Cowen [1] and Zhu [6] used this characterization. (II) is due to Nakazi - Takahashi [4, Lemma 1]. It is easy to prove (II) if we use (I). Nakazi-Takahashi [4] and Hwang-Lee [3] used this one. Hwang-Lee [3] established an explicit and useful criterion using (II) when the symbol $\phi$ is a trigonometric polynomial. Their criterion involves the zeros of an analytic polynomial induced by the Fourier coefficients of $\phi$. On the other hand, Zhu [6] gave a characterization which is related to the coefficients of the analytic polynomial induced by the Fourier coefficients of $\phi$, using (I) and a theorem of Schur [5]. In this paper, we give a necessary and sufficient condition which is related to the zeros of an analytic polynomial induced by the Fourier coefficient of $\phi$, using (II) and the Caratheodory-Shur interpolation theorem (cf. [2]).

**Theorem 1.** Suppose $\phi$ is a trigonometric polynomial such that $\phi = z^t \prod_{j=1}^{t} (z - \alpha_j) \prod_{j=1}^{s} (z - \beta_j)$ where $\ell \geq 1$, $|\alpha_j| < 1$ and $|\beta_j| \geq 1$. When $t = 0$ or $s = 0$, we assume that $\prod_{j=1}^{t} (z - \alpha_j) = 1$ or $\prod_{j=1}^{s} (z - \beta_j) = 1$. Let $f = \sum_{j=1}^{t} \frac{z - \alpha_j}{1 - \alpha_j z}$ and $h = \sum_{j=1}^{s} \frac{1 - \beta_j z}{z - \beta_j}$. Then, $T_\phi$ is hyponormal if and only if $2\ell \leq t + s$ and there exists a solution $a_0, \cdots, a_{\ell-1}$ of the linear system of equations

$$f^{(i)}(0) = \sum_{j=0}^{i} (i-1)(i-2) \cdots (i-j+1)a_j h^{(i-j)}(0) \quad (0 \leq i \leq \ell - 1)$$
for which the associated lower triangular Toeplitz matrix

\[
T(a_0, \cdots, a_{\ell-1}) = \begin{bmatrix}
a_0 & \cdots & 0 \\
a_1 & a_0 & \cdots & 0 \\
& \vdots & \ddots & \vdots \\
a_{\ell-1} & a_{\ell-2} & \cdots & a_0
\end{bmatrix}
\]

has \(\|T(a_0, \cdots, a_{\ell-1})\| \leq 1\).

Proof. By the characterization (II), \(T_{\phi}\) is hyponormal if and only if there exists a function \(K\) in \(H^\infty\) with \(\|K\|_\infty \leq 1\) and a function \(g\) in \(H^\infty\) such that \(\phi = K\bar{\phi} + g\). Hence \(T_{\phi}\) is hyponormal if and only if \(2\ell \leq t + s\) by (1) of Corollary 5 in [4] and there exists a function \(K\) in \(H^\infty\) with \(\|K\|_\infty \leq 1\) and a function \(g\) in \(H^\infty\) such that

\[
z^t \prod_{j=1}^{t} (z - \alpha_j) \prod_{j=1}^{s} (z - \beta_j) = Kz^t \prod_{j=1}^{t} (\bar{z} - \bar{\alpha}_j) \prod_{j=1}^{s} (\bar{z} - \bar{\beta}_j) + g.
\]

The above equality can be written as follows:

\[
f = Kz^{2\ell - (t+s)}h + z^tG
\]

where \(G = g/\prod_{j=1}^{t} (1 - \bar{\alpha}_jz) \prod_{j=1}^{s} (1 - \bar{\beta}_jz)\). Since \(z^{(t+s)-2\ell}(f - z^tG) = Kh\),

\[
z^{(t+s)-2\ell}(f - z^tG) \prod_{j=1}^{s} (z - \beta_j) = K \prod_{j=1}^{s} (1 - \bar{\beta}_jz).
\]

This implies that \(K\) is divisible in \(H^\infty\) by \(z^{(t+s)-2\ell}\) because \(|\beta_j| \geq 1\). Hence if \(k = z^{2\ell - (t+s)}K\) then \(k\) belongs to \(H^\infty\) and \(f = kh + z^tG\). Hence \(T_{\phi}\) is hyponormal if and only if \(2\ell \leq t + s\) and there exists a function \(k \in H^\infty\) with \(\|k\|_\infty \leq 1\) such that

\[
f^{(i)}(0) = \sum_{j=0}^{i} \binom{i}{j} C_j k^{(j)}(0) h^{(i-j)}(0) \quad (0 \leq i \leq \ell - 1)
\]

where \(\binom{i}{j} = i!/j!(i-j)!\). Put \(k = \sum_{j=0}^{\infty} a_j z^j\) then \(k^{(j)}(0) = j!a_j\) and so

\[
f^{(i)}(0) = \sum_{j=0}^{i} (i - 1)(i - 2) \cdots (i - j + 1)a_j h^{(i-j)}(0)
\]

for \(0 \leq i \leq \ell - 1\). Now the theorem follows from the Carathéodory-Shur interpolation theorem (cf.[2]).
In the characterization (II) of the hyponormality, put $\mathcal{E}(\phi) = \{ k \in H^\infty \mid \phi = k\overline{\phi} + g, g \in H^\infty, \text{ and } \|k\|_\infty \leq 1\}$. $\mathcal{E}(\phi)$ has been studied and it may contain more than two elements (see [4]). Hence the $k$ in the proof of Theorem 1 may not be unique in general and so $(a_j)_j^{t=0}$ may not be unique.

By a result in the previous paper [4, Corollary 5], if $\{1/\beta_j\}_{j=1}^{t} \subseteq \{\alpha_j\}_{j=1}^{t}$ (see Theorem 1) then $T_\phi$ is hyponormal. Here we give a necessary and sufficient condition for hyponormality of $T_\phi$ in terms of a relation between $\{\alpha_j\}_{j=1}^{t}$ and $\{\beta_j\}_{j=1}^{t}$ when $t = 1$ or 2.

**Corollary 1.** Let $t = 1$ in Theorem 1. Then, $T_\phi$ is hyponormal if and only if

$$
\prod_{j=1}^{t} |\alpha_j| \times \prod_{j=1}^{s} |\beta_j| \leq 1.
$$

When $t = 0$ or $s = 0$, we assume $\prod_{j=1}^{t} |\alpha_j| = 1$ or $\prod_{j=1}^{s} |\beta_j| = 1$.

Proof. By Theorem 1, $T_\phi$ is hyponormal if and only if $f(0) = a_0h(0)$ and $|a_0| \leq 1$.

**Corollary 2.** Let $t = 2$ in Theorem 1. Then, $T_\phi$ is hyponormal if and only if there exist constants $a_0, a_1$ such that $|a_1| \leq 1 - |a_0|^2$ and

$$
\sum_{k=1}^{t} \left\{ (1 - |\alpha_k|^2) \prod_{j \neq k} (-\alpha_j) \right\} = a_0 \sum_{k=1}^{s} (|\beta_k|^2 - 1) \prod_{j \neq k} (-1/\beta_j) + a_1 \prod_{j=1}^{s} (-1/\beta_j).
$$

If $s = 0$ then $\sum_{k=1}^{t} \left\{ (1 - |\alpha_k|^2) \prod_{j \neq k} (-\alpha_j) \right\} = 1$ and if $t = 0$ there exists constants $a_0, a_1$ such that $|a_1| \leq 1 - |a_0|^2$ and $a_0 \sum_{k=1}^{s} (|\beta_k|^2 - 1) \prod_{j \neq k} (-1/\beta_j) + a_1 \prod_{j=1}^{s} (-1/\beta_j) = 0$.

Proof. By Theorem 1, $T_\phi$ is hyponormal if and only if $f(0) = a_0h(0)$, $f'(0) = a_\phi h(0) + a_1 h'(0)$ and $|a_1| \leq 1 - |a_0|^2$.

**Corollary 3.** Let $s = 0$ in Theorem 1. Then, $T_\phi$ is hyponormal if and only if $|f^{(i)}(0)| \leq 1$ ($0 \leq i \leq t - 1$).

Proof. By Theorem 1, $T_\phi$ is hyponormal if and only if $f^{(i)}(0) = a_0$ for $i = 0, 1, 2$ and $\|T(a_0, 0, 0)\| = |a_0| \leq 1$.

**Corollary 4.** Let $t = 0$ in Theorem 1. Then, $T_\phi$ is hyponormal if and only if $1 = a_0h(0)$, $\sum_{j=0}^{i} a_j h^{(j)}(0) = 0$ ($1 \leq i \leq t - 1$) and $\|T(a_0, a_1, \cdots, a_{(t-1)})\| \leq 1$.

Our corollaries are new and different from Examples 6 and 7 in [6]. The author in [6] proved them under some condition, $a_2 \neq 0$ in Example 6. Of course, his result is not for zeros of a polynomial.
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References


