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HOKKAIDO UNIVERSITY
Hyponormal Toeplitz Operators And Zeros Of Polynomials

By

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The problem of hyponormality for Toeplitz operators with (trigonometric) polynomial symbols is studied. We give a necessary and sufficient condition using the zeros of the analytic polynomial induced by the Fourier coefficients of the symbol.

Let $L^p$ be the Lebesgue space on the unit circle $T$ and let $H^p$ be the corresponding Hardy space for $1 \leq p \leq \infty$. The Toeplitz operator $T_\phi$ with symbol $\phi$ in $L^\infty$ is the operator on $H^2$ defined by $T_\phi f = P(\phi f)$ for $f$ in $H^2$, where $P$ is the orthogonal projection from $L^2$ onto $H^2$. In this paper, we are interested in when $T_\phi$ is hyponormal.

Two characterizations of the hyponormality of $T_\phi$ are known as the following:

(I) Suppose $\phi_1$ and $\phi_2$ are functions in $H^2$ with $\phi = \phi_1 + \bar{\phi}_2$ in $L^\infty$. Then $T_\phi$ is hyponormal if and only if there exists a constant $c$ and a function $k$ in $H^\infty$ with $\|k\|_\infty \leq 1$ such that $\phi_2 = c + T_k \bar{\phi}_1$.

(II) $T_\phi$ is hyponormal if and only if there exist two functions $k$ and $g$ in $H^\infty$ such that $\phi = k \bar{\phi} + g$ and $\|k\|_\infty \leq 1$.

The characterization (I) is due to Cowen [1]. Cowen [1] and Zhu [6] used this characterization. (II) is due to Nakazi - Takahashi [4, Lemma 1]. It is easy to prove (II) if we use (I). Nakazi-Takahashi [4] and Hwang-Lee [3] used this one. Hwang-Lee [3] established an explicit and useful criterion using (II) when the symbol $\phi$ is a trigonometric polynomial. Their criterion involves the zeros of an analytic polynomial induced by the Fourier coefficients of $\phi$. On the other hand, Zhu [6] gave a characterization which is related to the coefficients of the analytic polynomial induced by the Fourier coefficients of $\phi$, using (I) and a theorem of Schur [5]. In this paper, we give a necessary and sufficient condition which is related to the zeros of an analytic polynomial induced by the Fourier coefficient of $\phi$, using (II) and the Caratheodory-Shur interpolation theorem (cf. [2]).

**Theorem 1.** Suppose $\phi$ is a trigonometric polynomial such that $\phi = z^t \prod_{j=1}^t (z - \alpha_j) \prod_{j=1}^s (z - \beta_j)$ where $\ell \geq 1$, $|\alpha_j| < 1$ and $|\beta_j| \geq 1$. When $t = 0$ or $s = 0$, we assume that $\prod_{j=1}^t (z - \alpha_j) = 1$ or $\prod_{j=1}^s (z - \beta_j) = 1$. Let $f = \prod_{j=1}^t \frac{z - \alpha_j}{1 - \alpha_j z}$ and $h = \prod_{j=1}^s \frac{1 - \beta_j z}{z - \beta_j}$. Then, $T_\phi$ is hyponormal if and only if $2\ell \leq t + s$ and there exists a solution $a_0, \ldots, a_{\ell-1}$ of the linear system of equations

$$f^{(i)}(0) = \sum_{j=0}^{\ell-1} (i-1)(i-2) \cdots (i-j+1) a_j h^{(i-j)}(0) \quad (0 \leq i \leq \ell - 1)$$
for which the associated lower triangular Toeplitz matrix

\[ T(a_0, \cdots, a_{\ell-1}) = \begin{bmatrix}
  a_0 & \cdots & 0 \\
  a_1 & a_0 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \vdots \\
  a_{\ell-1} & a_{\ell-2} & \cdots & a_0
\end{bmatrix} \]

has \( \|T(a_0, \cdots, a_{\ell-1})\| \leq 1 \).

Proof. By the characterization (II), \( T_\phi \) is hyponormal if and only if there exists a function \( K \in H^\infty \) with \( \|K\|_\infty \leq 1 \) and a function \( g \in H^\infty \) such that \( \phi = K\bar{\phi} + g \). Hence \( T_\phi \) is hyponormal if and only if \( 2\ell \leq t + s \) by (1) of Corollary 5 in [4] and there exists a function \( K \in H^\infty \) with \( \|K\|_\infty \leq 1 \) and a function \( g \in H^\infty \) such that

\[ z^t \prod_{j=1}^{t} (z - \alpha_j) \prod_{j=1}^{s} (z - \beta_j) = K z^t \prod_{j=1}^{t} (\bar{z} - \bar{\alpha}_j) \prod_{j=1}^{s} (\bar{z} - \bar{\beta}_j) + g. \]

The above equality can be written as follows:

\[ f = K z^{2\ell - (t+s)} h + z^{t} G \]

where \( G = g / \prod_{j=1}^{t} (1 - \bar{\alpha}_j z) \prod_{j=1}^{s} (1 - \bar{\beta}_j z) \). Since \( z^{(t+s)-2\ell}(f - z^{t} G) = Kh, \)

\[ z^{(t+s)-2\ell}(f - z^{t} G) \prod_{j=1}^{s} (z - \beta_j) = K \prod_{j=1}^{s} (1 - \bar{\beta}_j z). \]

This implies that \( K \) is divisible in \( H^\infty \) by \( z^{(t+s)-2\ell} \) because \( |\beta_j| \geq 1 \). Hence if \( k = z^{2\ell - (t+s)} K \) then \( k \) belongs to \( H^\infty \) and \( f = kh + z^{t} G \). Hence \( T_\phi \) is hyponormal if and only if \( 2\ell \leq t + s \) and there exists a function \( k \in H^\infty \) with \( \|k\|_\infty \leq 1 \) such that

\[ f^{(i)}(0) = \sum_{j=0}^{i} C_j k^{(j)}(0) h^{(i-j)}(0) \quad (0 \leq i \leq \ell - 1) \]

where \( C_j = i! / j! (i - j)! \). Put \( k = \sum_{j=0}^{\infty} a_j z^j \) then \( k^{(j)}(0) = j! a_j \) and so

\[ f^{(i)}(0) = \sum_{j=0}^{i} (i-1)(i-2) \cdots (i-j+1) a_j h^{(i-j)}(0) \]

for \( 0 \leq i \leq \ell - 1 \). Now the theorem follows from the Carathéodory-Shur interpolation theorem (cf.[2]).
In the characterization (II) of the hyponormality, put $\mathcal{E}(\phi) = \{ k \in H^\infty \mid \phi = k\overline{\phi} + g, g \in H^\infty \text{, and } \| k \|_\infty \leq 1 \}$. $\mathcal{E}(\phi)$ has been studied and it may contain more than two elements (see [4]). Hence the $k$ in the proof of Theorem 1 may not be unique in general and so $(a_\ell)^{k}_{j=0}$ may not be unique.

By a result in the previous paper [4, Corollary 5], if $\{ 1/\overline{\beta_j} \}_{j=1}^s \subseteq \{ \alpha_j \}_{j=1}^t$ (see Theorem 1) then $T_\phi$ is hyponormal. Here we give a necessary and sufficient condition for hyponormality of $T_\phi$ in terms of a relation between $\{ \alpha_j \}_{j=1}^t$ and $\{ \beta_j \}_{j=1}^s$ when $\ell = 1$ or 2.

**Corollary 1.** Let $\ell = 1$ in Theorem 1. Then, $T_\phi$ is hyponormal if and only if

$$\prod_{j=1}^t |\alpha_j| \times \prod_{j=1}^s |\beta_j| \leq 1.$$  

When $t = 0$ or $s = 0$, we assume $\prod_{j=1}^t |\alpha_j| = 1$ or $\prod_{j=1}^s |\beta_j| = 1$.

Proof. By Theorem 1, $T_\phi$ is hyponormal if and only if $f(0) = a_0 h(0)$ and $|a_0| \leq 1$.

**Corollary 2.** Let $\ell = 2$ in Theorem 1. Then, $T_\phi$ is hyponormal if and only if there exist constants $a_0, a_1$ such that $|a_1| \leq 1 - |a_0|^2$ and

$$\sum_{k=1}^{t} \left( 1 - |\alpha_k|^2 \right) \prod_{j \neq k} (-\alpha_j) = a_0 \sum_{k=1}^{s} |\beta_k|^2 - 1 \prod_{j \neq k} (-\frac{1}{\beta_j}) + a_1 \prod_{j=1}^{s} \left( -\frac{1}{\beta_j} \right).$$

If $s = 0$ then $\sum_{k=1}^{t} \left( 1 - |\alpha_k|^2 \right) \prod_{j \neq k} (-\alpha_j) = 1$ and if $t = 0$ then there exists constants $a_0, a_1$ such that $|a_1| \leq 1 - |a_0|^2$ and $a_0 \sum_{k=1}^{s} |\beta_k|^2 - 1 \prod_{j \neq k} (-\frac{1}{\beta_j}) + a_1 \prod_{j=1}^{s} \left( -\frac{1}{\beta_j} \right) = 0$.

Proof. By Theorem 1, $T_\phi$ is hyponormal if and only if $f(0) = a_0 h(0)$, $f'(0) = a_0 h(0) + a_1 h'(0)$ and $|a_1| \leq 1 - |a_0|^2$.

**Corollary 3.** Let $s = 0$ in Theorem 1. Then, $T_\phi$ is hyponormal if and only if $|f^{(i)}(0)| \leq 1 (0 \leq i \leq \ell - 1)$.

Proof. By Theorem 1, $T_\phi$ is hyponormal if and only if $f^{(i)}(0) = a_0$ for $i = 0, 1, 2$ and $\|T(a_0, 0, 0)\| = |a_0| \leq 1$.

**Corollary 4.** Let $t = 0$ in Theorem 1. Then, $T_\phi$ is hyponormal if and only if $1 = a_0 h(0)$, $\sum_{j=0}^{t} a_j h^{(j)}(0) = 0 (1 \leq i \leq \ell - 1)$ and $\|T(a_0, a_1, \cdots, a_{(\ell-1)})\| \leq 1$.

Our corollaries are new and different from Examples 6 and 7 in [6]. The author in [6] proved them under some condition, $a_2 \neq 0$ in Example 6. Of course, his result is not for zeros of a polynomial.
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References


