Hyponormal Toeplitz Operators And Zeros Of Polynomials

By

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* This research was partially supported by Grant-in-Aid for Scientific Research, Japan Society for the Promotion of Science

2000 Mathematics Subject Classification : Primary 47 B 20, 47 B 35.

Key words and phrases : Trigonometric Toeplitz operators, hyponormal operator, Pick problem

Abstract. The problem of hyponormality for Toeplitz operators with (trigonometric) polynomial symbols is studied. We give a necessary and sufficient condition using the zeros of the analytic polynomial induced by the Fourier coefficients of the symbol.

Let L^p be the Lebesgue space on the unit circle T and let H^p be the corresponding Hardy space for $1 \leq p \leq \infty$. The Toeplitz operator T_ϕ with symbol ϕ in L^∞ is the operator on H^2 defined by $T_\phi f = P(\phi f)$ for f in H^2, where P is the orthogonal projection from L^2 onto H^2. In this paper, we are interested in when T_ϕ is hyponormal.

Two characterizations of the hyponormality of T_ϕ are known as the following :

(I) Suppose ϕ_1 and ϕ_2 are functions in H^2 with $\phi = \phi_1 + \bar{\phi}_2$ in L^∞. Then T_ϕ is hyponormal if and only if there exists a constant c and a function k in H^∞ with $\|k\|_\infty \leq 1$ such that $\phi_2 = c + T_{\bar{k}}\phi_1$.

(II) T_ϕ is hyponormal if and only if there exist two functions k and g in H^∞ such that $\phi = k\bar{\phi} + g$ and $\|k\|_\infty \leq 1$.

The characterization (I) is due to Cowen [1]. Cowen [1] and Zhu [6] used this characterization. (II) is due to Nakazi - Takahashi [4, Lemma 1]. It is easy to prove (II) if we use (I). Nakazi-Takahashi [4] and Hwang-Lee [3] used this one. Hwang-Lee [3] established an explicit and useful criterion using (II) when the symbol ϕ is a trigonometric polynomial. Their criterion involves the zeros of an analytic polynomial induced by the Fourier coefficients of ϕ. On the other hand, Zhu [6] gave a characterization which is related to the coefficients of the analytic polynomial induced by the Fourier coefficients of ϕ, using (I) and a theorem of Schur [5]. In this paper, we give a necessary and sufficient condition which is related to the zeros of an analytic polynomial induced by the Fourier coefficient of ϕ, using (II) and the Caratheodory-Shur interpolation theorem (cf. [2]).

Theorem 1. *Suppose ϕ is a trigonometric polynomial such that* $\phi = \bar{z}^\ell \prod_{j=1}^{t}(z-\alpha_j)\prod_{j=1}^{s}(z-\beta_j)$ *where* $\ell \geq 1$, $|\alpha_j| < 1$ *and* $|\beta_j| \geq 1$. *When* $t = 0$ *or* $s = 0$, *we assume that* $\prod_{j=1}^{t}(z-\alpha_j) = 1$ *or* $\prod_{j=1}^{s}(z-\beta_j) = 1$. *Let* $f = \prod_{j=1}^{t}\dfrac{z-\alpha_j}{1-\bar{\alpha}_j z}$ *and* $h = \prod_{j=1}^{s}\dfrac{1-\bar{\beta}_j z}{z-\beta_j}$. *Then,* T_ϕ *is hyponormal if and only if* $2\ell \leq t+s$ *and there exists a solution* $a_0, \cdots, a_{\ell-1}$ *of the linear system of equations*

$$f^{(i)}(0) = \sum_{j=0}^{i}(i-1)(i-2)\cdots(i-j+1)a_j h^{(i-j)}(0) \quad (0 \leq i \leq \ell-1)$$

for which the associated lower triangular Toeplitz matrix

$$T(a_0, \cdots, a_{\ell-1}) = \begin{bmatrix} a_0 & & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{\ell-1} & a_{\ell-2} & \cdots & a_0 \end{bmatrix}$$

has $\|T(a_0, \cdots, a_{\ell-1})\| \leq 1$.

Proof. By the characterization (II), T_ϕ is hyponormal if and only if there exists a function K in H^∞ with $\|K\|_\infty \leq 1$ and a function g in H^∞ such that $\phi = K\bar{\phi} + g$. Hence T_ϕ is hyponormal if and only if $2\ell \leq t + s$ by (1) of Corollary 5 in [4] and there exists a function K in H^∞ with $\|K\|_\infty \leq 1$ and a function g in H^∞ such that

$$\bar{z}^\ell \prod_{j=1}^{t}(z - \alpha_j) \prod_{j=1}^{s}(z - \beta_j) = K z^\ell \prod_{j=1}^{t}(\bar{z} - \bar{\alpha}_j) \prod_{j=1}^{s}(\bar{z} - \bar{\beta}_j) + g.$$

The above equality can be written as follows :

$$f = K z^{2\ell-(t+s)} h + z^\ell G$$

where $G = g / \prod_{j=1}^{t}(1 - \bar{\alpha}_j z) \prod_{j=1}^{s}(1 - \bar{\beta}_j z)$. Since $z^{(t+s)-2\ell}(f - z^\ell G) = Kh$,

$$z^{(t+s)-2\ell}(f - z^\ell G) \prod_{j=1}^{s}(z - \beta_j) = K \prod_{j=1}^{s}(1 - \bar{\beta}_j z).$$

This implies that K is divisible in H^∞ by $z^{(t+s)-2\ell}$ because $|\ \beta_j\ | \geq 1$. Hence if $k = z^{2\ell-(t+s)}K$ then k belongs to H^∞ and $f = kh + z^\ell G$. Hence T_ϕ is hyponormal if and only if $2\ell \leq t + s$ and there exists a function $k \in H^\infty$ with $\|k\|_\infty \leq 1$ such that

$$f^{(i)}(0) = \sum_{j=0}^{i} {}_iC_j k^{(j)}(0) h^{(i-j)}(0) \quad (0 \leq i \leq \ell - 1)$$

where ${}_iC_j = i!/j!(i-j)!$. Put $k = \sum_{j=0}^{\infty} a_j z^j$ then $k^{(j)}(0) = j! a_j$ and so

$$f^{(i)}(0) = \sum_{j=0}^{i} (i-1)(i-2)\cdots(i-j+1) a_j h^{(i-j)}(0)$$

for $0 \leq i \leq \ell - 1$. Now the theorem follows from the Carathéodory-Shur interpolation theorem (cf.[2]).

In the characterization (II) of the hyponormality, put $\mathcal{E}(\phi) = \{k \in H^\infty \ ; \ \phi = k\bar{\phi} + g, g \in H^\infty$, and $\|k\|_\infty \le 1\}$. $\mathcal{E}(\phi)$ has been studied and it may contain more than two elements (see [4]). Hence the k in the proof of Theorem 1 may not be unique in general and so $(a_j)_{j=0}^{\ell}$ may not be unique.

By a result in the previous paper [4, Corollary 5], if $\{1/\bar{\beta}_j\}_{j=1}^{s} \subseteq \{\alpha_j\}_{j=1}^{t}$ (see Theorem 1) then T_ϕ is hyponormal. Here we give a necessary and sufficient condition for hyponormality of T_ϕ in terms of a relation between $\{\alpha_j\}_{j=1}^{t}$ and $\{\beta_j\}_{j=1}^{s}$ when $\ell = 1$ or 2.

Corollary 1. *Let $\ell = 1$ in Theorem 1. Then, T_ϕ is hyponormal if and only if $\prod_{j=1}^{t} |\alpha_j| \times \prod_{j=1}^{s} |\beta_j| \le 1$. When $t = 0$ or $s = 0$, we assume $\prod_{j=1}^{t} |\alpha_j| = 1$ or $\prod_{j=1}^{s} |\beta_j| = 1$.*

Proof. By Theorem 1, T_ϕ is hyponormal if and only if $f(0) = a_0 h(0)$ and $|a_0| \le 1$.

Corollary 2. *Let $\ell = 2$ in Theorem 1. Then, T_ϕ is hyponormal if and only if there exist constants a_0, a_1 such that $|a_1| \le 1 - |a_0|^2$ and*

$$\sum_{k=1}^{t} \left\{ (1 - |\alpha_k|^2) \prod_{j \neq k}^{t} (-\alpha_j) \right\} = a_0 \sum_{k=1}^{s} (|\beta_k|^2 - 1) \prod_{j \neq k}^{s} \left(-\frac{1}{\beta_j} \right) + a_1 \prod_{j=1}^{s} \left(-\frac{1}{\beta_j} \right).$$

If $s = 0$ then $\sum_{k=1}^{t} \left\{ (1 - |\alpha_k|^2) \prod_{j \neq k}^{t} (-\alpha_j) \right\} = 1$ and if $t = 0$ then there exists constants a_0, a_1 such that $|a_1| \le 1 - |a_0|^2$ and $a_0 \sum_{k=1}^{s} (|\beta_k|^2 - 1) \prod_{j \neq k}^{s} \left(-\frac{1}{\beta_j} \right) + a_1 \prod_{j=1}^{s} \left(-\frac{1}{\beta_j} \right) = 0.$

Proof. By Theorem 1, T_ϕ is hyponormal if and only if $f(0) = a_0 h(0)$, $f'(0) = a_0 h(0) + a_1 h'(0)$ and $|a_1| \le 1 - |a_0|^2$.

Corollary 3. *Let $s = 0$ in Theorem 1. Then, T_ϕ is hyponormal if and only if $|f^{(i)}(0)| \le 1$ $(0 \le i \le \ell - 1)$.*

Proof. By Theorem 1, T_ϕ is hyponormal if and only if $f^{(i)}(0) = a_0$ for $i = 0, 1, 2$ and $\|T(a_0, 0, 0)\| = |a_0| \le 1$.

Corollary 4. *Let $t = 0$ in Theorem 1. Then, T_ϕ is hyponormal if and only if $1 = a_0 h(0)$, $\sum_{j=0}^{i} a_j h^{(j)}(0) = 0$ $(1 \le i \le \ell - 1)$ and $\|T(a_0, a_1, \cdots, a_{(\ell-1)}\| \le 1$.*

Our corollaries are new and different from Examples 6 and 7 in [6]. The author in [6] proved them under some condition, $a_2 \neq 0$ in Example 6. Of course, his result is not for zeros of a polynomial.

Acknowlegement. The author would like to thank the referee for many comments improving the original manuscript.

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