Hyponormal Toeplitz Operators And Zeros Of Polynomials

By

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Abstract. The problem of hyponormality for Toeplitz operators with (trigonometric) polynomial symbols is studied. We give a necessary and sufficient condition using the zeros of the analytic polynomial induced by the Fourier coefficients of the symbol.

Let $L^p$ be the Lebesgue space on the unit circle $T$ and let $H^p$ be the corresponding Hardy space for $1 \leq p \leq \infty$. The Toeplitz operator $T_\phi$ with symbol $\phi$ in $L^\infty$ is the operator on $H^2$ defined by $T_\phi f = P(\phi f)$ for $f$ in $H^2$, where $P$ is the orthogonal projection from $L^2$ onto $H^2$. In this paper, we are interested in when $T_\phi$ is hyponormal.

Two characterizations of the hyponormality of $T_\phi$ are known as the following:

(I) Suppose $\phi_1$ and $\phi_2$ are functions in $H^2$ with $\phi = \phi_1 + \bar{\phi}_2$ in $L^\infty$. Then $T_\phi$ is hyponormal if and only if there exists a constant $c$ and a function $k$ in $H^\infty$ with $\|k\|_\infty \leq 1$ such that $\phi_2 = c + T\bar{k}\phi_1$.

(II) $T_\phi$ is hyponormal if and only if there exist two functions $k$ and $g$ in $H^\infty$ such that $\phi = k\bar{\phi} + g$ and $\|k\|_\infty \leq 1$.

The characterization (I) is due to Cowen [1]. Cowen [1] and Zhu [6] used this characterization. (II) is due to Nakazi - Takahashi [4, Lemma 1]. It is easy to prove (II) if we use (I). Nakazi-Takahashi [4] and Hwang-Lee [3] used this one. Hwang-Lee [3] established an explicit and useful criterion using (II) when the symbol $\phi$ is a trigonometric polynomial. Their criterion involves the zeros of an analytic polynomial induced by the Fourier coefficients of $\phi$. On the other hand, Zhu [6] gave a characterization which is related to the coefficients of the analytic polynomial induced by the Fourier coefficients of $\phi$, using (I) and a theorem of Schur [5]. In this paper, we give a necessary and sufficient condition which is related to the zeros of an analytic polynomial induced by the Fourier coefficient of $\phi$, using (II) and the Caratheodory-Shur interpolation theorem (cf. [2]).

Theorem 1. Suppose $\phi$ is a trigonometric polynomial such that $\phi = z^\ell \prod_{j=1}^t (z - \alpha_j) \prod_{j=1}^s (z - \beta_j)$ where $\ell \geq 1$, $|\alpha_j| < 1$ and $|\beta_j| \geq 1$. When $t = 0$ or $s = 0$, we assume that $\prod_{j=1}^t (z - \alpha_j) = 1$ or $\prod_{j=1}^s (z - \beta_j) = 1$. Let $f = \prod_{j=1}^t \frac{z - \alpha_j}{1 - \bar{\alpha}_j z}$ and $h = \prod_{j=1}^s \frac{1 - \bar{\beta}_j z}{z - \beta_j}$. Then, $T_\phi$ is hyponormal if and only if $2\ell \leq t + s$ and there exists a solution $a_0, \ldots, a_{\ell-1}$ of the linear system of equations

$$f^{(i)}(0) = \sum_{j=0}^i (i - 1)(i - 2) \cdots (i - j + 1) a_j h^{(i-j)}(0) \quad (0 \leq i \leq \ell - 1)$$
for which the associated lower triangular Toeplitz matrix

\[
T(a_0, \ldots, a_{\ell-1}) = \begin{bmatrix}
  a_0 & \cdots & 0 \\
  a_1 & a_0 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \vdots \\
  a_{\ell-1} & a_{\ell-2} & \cdots & a_0
\end{bmatrix}
\]

has \(\|T(a_0, \ldots, a_{\ell-1})\| \leq 1\).

Proof. By the characterization (II), \(T_\phi\) is hyponormal if and only if there exists a function \(K\) in \(H^\infty\) with \(\|K\|_\infty \leq 1\) and a function \(g\) in \(H^\infty\) such that \(\phi = K\bar{\phi} + g\). Hence \(T_\phi\) is hyponormal if and only if \(2\ell \leq t + s\) by (1) of Corollary 5 in [4] and there exists a function \(K\) in \(H^\infty\) with \(\|K\|_\infty \leq 1\) and a function \(g\) in \(H^\infty\) such that

\[
z^t \prod_{j=1}^{t} (z - \alpha_j) \prod_{j=1}^{s} (z - \beta_j) = K z^t \prod_{j=1}^{t} (\bar{z} - \bar{\alpha}_j) \prod_{j=1}^{s} (\bar{z} - \bar{\beta}_j) + g.
\]

The above equality can be written as follows :

\[
f = K z^{2\ell-(t+s)} h + z^t G
\]

where \(G = g/\prod_{j=1}^{t} (1 - \bar{\alpha}_j z) \prod_{j=1}^{s} (1 - \bar{\beta}_j z)\). Since \(z^{(t+s)-2\ell} (f - z^t G) = Kh\),

\[
z^{(t+s)-2\ell} (f - z^t G) \prod_{j=1}^{s} (z - \beta_j) = K \prod_{j=1}^{s} (1 - \bar{\beta}_j z).
\]

This implies that \(K\) is divisible in \(H^\infty\) by \(z^{(t+s)-2\ell}\) because \(|\beta_j| \geq 1\). Hence if \(k = z^{2\ell-(t+s)} K\) then \(k\) belongs to \(H^\infty\) and \(f = kh + z^t G\). Hence \(T_\phi\) is hyponormal if and only if \(2\ell \leq t + s\) and there exists a function \(k\) in \(H^\infty\) with \(\|k\|_\infty \leq 1\) such that

\[
f^{(i)}(0) = \sum_{j=0}^{i} \binom{i}{j} C_j k^{(j)}(0) h^{(i-j)}(0) \quad (0 \leq i \leq \ell - 1)
\]

where \(\binom{i}{j} = i!/(j!(i-j)!).\) Put \(k = \sum_{j=0}^{\infty} a_j z^j\) then \(k^{(j)}(0) = j! a_j\) and so

\[
f^{(i)}(0) = \sum_{j=0}^{i} (i-1)(i-2) \cdots (i-j+1) a_j h^{(i-j)}(0)
\]

for \(0 \leq i \leq \ell - 1\). Now the theorem follows from the Carathéodory-Shur interpolation theorem (cf.[2]).
In the characterization (II) of the hyponormality, put \( \mathcal{E}(\phi) = \{ k \in H^\infty : \phi = k\bar{\phi} + g, g \in H^\infty, \text{ and } \|k\|_\infty \leq 1 \} \). \( \mathcal{E}(\phi) \) has been studied and it may contain more than two elements (see [4]). Hence the \( k \) in the proof of Theorem 1 may not be unique in general and so \( (a_j)_{j=0}^\ell \) may not be unique.

By a result in the previous paper [4, Corollary 5], if \( \{1/\tilde{\beta}_j\}_{j=1}^s \subseteq \{\alpha_j\}_{j=1}^1 \) (see Corollary 1) then \( T_\phi \) is hyponormal. Here we give a necessary and sufficient condition for hyponormality of \( T_\phi \) in terms of a relation between \( \{\alpha_j\}_{j=1}^1 \) and \( \{\beta_j\}_{j=1}^s \) when \( \ell = 1 \) or 2.

**Corollary 1.** Let \( \ell = 1 \) in Theorem 1. Then, \( T_\phi \) is hyponormal if and only if
\[
\prod_{j=1}^t |\alpha_j| \times \prod_{j=1}^s |\beta_j| \leq 1.
\]
When \( t = 0 \) or \( s = 0 \), we assume \( \prod_{j=1}^t |\alpha_j| = 1 \) or \( \prod_{j=1}^s |\beta_j| = 1 \).

**Proof.** By Theorem 1, \( T_\phi \) is hyponormal if and only if \( f(0) = a_0 h(0) \) and \( |a_0| \leq 1 \).

**Corollary 2.** Let \( \ell = 2 \) in Theorem 1. Then, \( T_\phi \) is hyponormal if and only if there exist constants \( a_0, a_1 \) such that \( |a_1| \leq 1 - |a_0|^2 \) and
\[
\sum_{k=1}^t \left( (1 - |\alpha_k|^2) \prod_{j\neq k}^t (-\alpha_j) \right) = a_0 \sum_{k=1}^s (|\beta_k|^2 - 1) \prod_{j\neq k}^s \left( -\frac{1}{\beta_j} \right) + a_1 \prod_{j=1}^s \left( -\frac{1}{\beta_j} \right).
\]

If \( s = 0 \) then \( \sum_{k=1}^t \left( (1 - |\alpha_k|^2) \prod_{j\neq k}^t (-\alpha_j) \right) = 1 \) and if \( t = 0 \) then there exists constants \( a_0, a_1 \) such that \( |a_1| \leq 1 - |a_0|^2 \) and \( a_0 \sum_{k=1}^s (|\beta_k|^2 - 1) \prod_{j\neq k}^s \left( -\frac{1}{\beta_j} \right) + a_1 \prod_{j=1}^s \left( -\frac{1}{\beta_j} \right) = 0 \).

**Proof.** By Theorem 1, \( T_\phi \) is hyponormal if and only if \( f(0) = a_0 h(0) \), \( f'(0) = a_0 h(0) + a_1 h'(0) \) and \( |a_1| \leq 1 - |a_0|^2 \).

**Corollary 3.** Let \( s = 0 \) in Theorem 1. Then, \( T_\phi \) is hyponormal if and only if \( |f^{(i)}(0)| \leq 1 \) \((0 \leq i \leq \ell - 1)\).

**Proof.** By Theorem 1, \( T_\phi \) is hyponormal if and only if \( f^{(i)}(0) = a_0 \) for \( i = 0, 1, 2 \) and \( \|T(a_0, 0, 0)\| = |a_0| \leq 1 \).

**Corollary 4.** Let \( t = 0 \) in Theorem 1. Then, \( T_\phi \) is hyponormal if and only if
\[
1 = a_0 h(0), \sum_{j=0}^i a_j h^{(j)}(0) = 0 \text{ } (1 \leq i \leq \ell - 1) \text{ and } \|T(a_0, a_1, \cdots, a_{(\ell-1)})\| \leq 1.
\]

Our corollaries are new and different from Examples 6 and 7 in [6]. The author in [6] proved them under some condition, \( a_2 \neq 0 \) in Example 6. Of course, his result is not for zeros of a polynomial.
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References


