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# An Elliptic Analogue of the Generalized Dedekind-Rademacher Sums

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## Abstract

In this paper we introduce an elliptic analogue of the generalized Dedekind-Rademacher sums which satisfy reciprocity laws. In these sums, Kronecker's double series play a role of *elliptic Bernoulli functions*. This paper gives an answer to the problem of S. Fukuhara and N. Yui concerning the elliptic Apostol-Dedekind sums. We also mention a relation between the generating function of Kronecker's double series and that of the (Debye) elliptic polylogarithms studied by A. Levin.

## 1 Introduction

After pioneering works by Dedekind [De], much attention has been directed to generalizations of Dedekind sums and their reciprocity laws (see [Be] for a good picture of previously defined generalizations of the Dedekind sum).

Let  $a, b$  and  $c$  be positive integers with no common factor, and  $x, y$  and  $z$  real numbers. U. Halbritter [Ha] and R.R. Hall *et al.* [HWZ] have defined the *generalized Dedekind-Rademacher sums*  $S_{m,n} \begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix}$  by

$$S_{m,n} \begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix} := \sum_{j \pmod{c}} \tilde{B}_m \left( a \frac{j+z}{c} - x \right) \tilde{B}_n \left( b \frac{j+z}{c} - y \right), \quad (1)$$

and have proved their reciprocity laws. Here  $\tilde{B}_m(x)$  is the  $m$ -th Bernoulli function defined as follows: We denote by  $B_m(x)$  the Bernoulli polynomials.

$$\frac{e^{x\xi}}{e^\xi - 1} = \sum_{m=0}^{\infty} \frac{B_m(x)}{m!} \xi^{m-1}.$$

$B_m := B_m(0)$  is the  $m$ -th Bernoulli number. Let  $\{x\}$  be the fractional part of  $x$ . The Bernoulli functions  $\tilde{B}_m(x)$  are defined by

$$\tilde{B}_m(x) = \begin{cases} 0, & (m = 1, x \in \mathbb{Z}), \\ B_m(\{x\}), & (\text{otherwise}). \end{cases}$$

As special cases, these sums involve the generalizations of the Dedekind sums introduced by T.M. Apostol [Apo], L. Carlitz [Ca], M. Mikolás [Mi], U. Dieter [Di], C. Meyer [Me], H. Rademacher [Rad], and L. Takács [Ta]. Note that the so-called *Apostol-Dedekind sum* [Apo] is expressed by  $S_{1,n} \begin{pmatrix} 1 & b & c \\ 0 & 0 & 0 \end{pmatrix}$ .

Moreover, various elliptic analogues of the classical Dedekind sum have been studied by several authors ([Ba1], [Ba2], [Ega], [FY], [Ito1], [Ito2], [Scz]). R. Sczech introduced the *elliptic Dedekind sums* as an elliptic analogue of the classical Dedekind sums. A. Bayad [Ba1, Ba2] and S. Egami [Ega] studied the *multiple elliptic Dedekind sums* as an elliptic analogue of Zagier's multiple Dedekind sums [Za1].

Recently, S. Fukuhara and N. Yui [FY] have introduced the *elliptic Apostol-Dedekind sums* as an elliptic analogue of the classical Apostol-Dedekind sums. However their elliptic analogue did not involve an elliptic analogue of the classical Bernoulli functions. So they have raised the problem to find true analogues in the elliptic world of the classical Apostol-Dedekind sums which involve the Bernoulli functions.

In this paper we answer to the above problem. We define an elliptic analogue of the generalized Dedekind-Rademacher sums (*elliptic Dedekind-Rademacher sums*) which involve an elliptic analogue of the classical Bernoulli functions. For the definition of this elliptic analogue, we make use of Kronecker's double series [We]. E.V. Ivashkevich *et al.* [III Sect.3.1] noted that these series can be considered as an elliptic generalization of the classical Bernoulli functions, and K. Katayama [Ka] also indicated that these series are an analogue of those functions.

The principal aim of this paper is to establish the reciprocity laws for the elliptic Dedekind-Rademacher sums (Theorem 3.1). As a corollary, we reproduce the reciprocity laws [HWZ] for the classical generalized Dedekind-Rademacher sums (Theorem 3.2). We also mention a relation between the generating function of Kronecker's double series and that of the (Debye) elliptic polylogarithms studied by A. Levin [Le] in order to enforce the validity of our elliptic generalization (see Section 2).

The paper is organized as follows: In Section 2 we review Kronecker's double series and their basic properties. A relation between the generating function of Kronecker's double series and that of the (Debye) elliptic polylogarithms is also discussed. In Section 3 we define elliptic Dedekind-Rademacher sums, and obtain their reciprocity laws (Theorem 3.1). The reciprocity laws [HWZ] for the classical generalized Dedekind-Rademacher sums are reproduced by the degeneration of the elliptic Dedekind-Rademacher sums (Theorem 3.2).

## 2 Kronecker's double series

In this section we review Kronecker's double series and some of their basic properties. we also mention a relation between the generating function of Kronecker's double series and that of the (Debye) elliptic polylogarithms studied by A. Levin [Le].

Let  $\tau$  be in the upper half-plane. We shall use the following notations:  $e(x) := e^{2\pi ix}$ ,  $q := e(\tau)$ , and Jacobi's theta function

$$\theta(x; \tau) := \sum_{m \in \mathbb{Z}} e\left(\frac{1}{2}\left(m + \frac{1}{2}\right)^2 \tau + \left(m + \frac{1}{2}\right)\left(x + \frac{1}{2}\right)\right).$$

We define the function  $F(x, \xi; \tau)$  as follows:

$$F(x, \xi; \tau) = \frac{\theta'(0; \tau)\theta(x + \xi; \tau)}{\theta(x; \tau)\theta(\xi; \tau)}, \quad (x, \xi \in \mathbb{C} \setminus \mathbb{Z} + \tau\mathbb{Z}),$$

where  $\theta'(x; \tau) = \frac{\partial}{\partial x} \theta(x; \tau)$ . For fixed  $\xi \in \mathbb{C} \setminus \mathbb{Z} + \tau\mathbb{Z}$ , the function  $F(x, \xi; \tau)$  with respect to  $x$  is meromorphic with only simple poles on the lattice  $\mathbb{Z} + \tau\mathbb{Z}$ . In addition, it satisfies the following properties (see [Za2]):

$$F(x+1, \xi; \tau) = F(x, \xi; \tau), \quad F(x+\tau, \xi; \tau) = e(-\xi)F(x, \xi; \tau), \quad (2)$$

$$\operatorname{Res}_{x=n\tau+n'} F(x, \xi; \tau) = e(-n\xi), \quad (n, n' \in \mathbb{Z}). \quad (3)$$

Let us introduce Kronecker's double series. Set  $\underline{F}(x', x; \xi; \tau) := e(x\xi)F(-x' + x\tau, \xi; \tau)$ , and define functions  $B_m(x', x; \tau)$  by

$$\underline{F}(x', x; \xi; \tau) = \sum_{m=0}^{\infty} \frac{B_m(x', x; \tau)}{m!} (2\pi i)^m \xi^{m-1}. \quad (4)$$

Namely  $\underline{F}(x', x; \xi; \tau)$  gives the generating function of  $B_m(x', x; \tau)$ 's. It is easily seen from (2) that the functions  $B_m(x', x; \tau)$  have the following periodicity:

$$B_m(x'+1, x; \tau) = B_m(x', x+1; \tau) = B_m(x', x; \tau). \quad (5)$$

In view of the expansion (7) below,  $B_m(x', x; \tau)$ 's are called Kronecker's double series.

We will give explicit forms of  $B_m(x', x; \tau)$ 's. The function  $F(x, \xi; \tau)$  has another expression ([We], [Jor] p.446, [Za2] Sect.3):

$$F(x, \xi; \tau) = 2\pi i \left[ \sum_{j=1}^{\infty} \frac{q^j}{e(x) - q^j} e(-j\xi) - \sum_{j=1}^{\infty} \frac{q^j}{e(-x) - q^j} e(j\xi) + \frac{1}{e(x) - 1} + \frac{1}{e(\xi) - 1} + 1 \right], \quad (|\operatorname{Im} x|, |\operatorname{Im} \xi| < \operatorname{Im} \tau).$$

So one gets

$$B_m(x', x; \tau) = m \left( \sum_{j=1}^{\infty} (x-j)^{m-1} \frac{e(-x\tau)q^j}{e(-x') - e(-x\tau)q^j} - \sum_{j=1}^{\infty} (x+j)^{m-1} \frac{e(x\tau)q^j}{e(x') - e(x\tau)q^j} + x^{m-1} \frac{e(-x' + x\tau)}{e(-x' + x\tau) - 1} \right) + B_m(x). \quad (6)$$

We note that  $B_1(0, 0; \tau)$  can not be defined since the function  $\frac{e(-x' + x\tau)}{e(-x' + x\tau) - 1}$  becomes infinity if  $x = 0$  and  $x' \in \mathbb{Z}$ .

Originally, Kronecker's double series have been introduced in the following way. Let  $x'$  and  $x$  be real numbers with  $-1 < x < 0$ . Kronecker proved the following equation [We].

$$\underline{F}(x', x; \xi; \tau) = - \sum^e \frac{e(n'x' + nx)}{-\xi + n'\tau + n},$$

where  $\sum^e$  denotes Eisenstein summation [We], i.e.,

$$\sum^e = \lim_{N' \rightarrow \infty} \lim_{N \rightarrow \infty} \sum_{n'=-N'}^{N'} \sum_{n=-N}^N.$$

So we have

$$B_m(x', x; \tau) = \begin{cases} 1 & (m = 0), \\ -\frac{m!}{(2\pi i)^m} \sum \frac{e^{n'x' + nx}}{(n'\tau + n)^m} & (m \geq 1). \end{cases} \quad (7)$$

The series in the right-hand-side was studied by Kronecker [We]. As noted in [IIH], Kronecker's double series degenerate into the classical Bernoulli functions. We reconstruct this statement and give its proof for reader's convenience.

**PROPOSITION 2.1.** *Let  $x'$  and  $x$  be real numbers with  $x' \notin \mathbb{Z}$ . Then*

$$\lim_{\tau \rightarrow i\infty} B_m(x', x; \tau) = \begin{cases} \frac{1}{2} \frac{1 + e(x')}{1 - e(x')} & (m = 1, x \in \mathbb{Z}), \\ \tilde{B}_m(x) & (\text{otherwise}). \end{cases} \quad (8)$$

*Proof.* The proposition is trivial if  $m = 0$ . Suppose that  $m \geq 1$ . One obtains from (5) that

$$B_m(x', \{x\}; \tau) = B_m(x', x; \tau).$$

So it is sufficient to prove the proposition when  $0 \leq x < 1$ . Then we have

$$\lim_{\tau \rightarrow i\infty} e(x\tau)q^j = \lim_{\tau \rightarrow i\infty} e(-x\tau)q^j = 0 \quad (j \in \mathbb{Z}_{\geq 1}), \quad (9)$$

and

$$\lim_{\tau \rightarrow i\infty} x^{m-1} \frac{e(-x' + x\tau)}{e(-x' + x\tau) - 1} = \begin{cases} \frac{1}{1 - e(x')} & (m = 1, x = 0), \\ 0 & (\text{otherwise}). \end{cases} \quad (10)$$

Eqs.(6), (9) and (10) induce Eq.(8) since  $B_1 = -1/2$ .  $\square$

The generating function  $\underline{F}(x', x; \xi; \tau)$  of Kronecker's double series is related to that of the (Debye) elliptic polylogarithms studied by A. Levin [Le] in the following sense. Set  $\Lambda(\xi; -2\pi ix) := 2\pi i \int_{\xi}^{i\infty} \frac{e(-xt)}{e(-t) - 1} dt$ . The Debye polylogarithms  $\Lambda_m(\xi)$  are defined by [Le]:

$$2\pi i \int_{\xi}^{i\infty} \frac{e(-xt)}{e(-t) - 1} dt = \sum_{m=0}^{\infty} \Lambda_{m+1}(\xi) (2\pi i)^{m+1} (-x)^m. \quad (11)$$

Namely  $\Lambda(\xi; -2\pi ix)$  gives the generating function of  $\Lambda_m(\xi)$ 's. Since

$$\frac{\partial}{\partial \xi} \Lambda(-\xi; -2\pi ix) = 2\pi i \frac{e(x\xi)}{e(\xi) - 1} = \sum_{m=0}^{\infty} \frac{B_m(x)}{m!} (2\pi i)^m \xi^{m-1}, \quad (12)$$

$\frac{\partial}{\partial \xi} \Lambda(-\xi; -2\pi ix)$  becomes the generating function of the classical Bernoulli functions (polynomials). On the other hand, A. Levin has shown that the modified generating function  $\underline{\Lambda}(\xi, \tau; x', x)$  of the (Debye) elliptic polylogarithms satisfies the following equation (see [Le] Proposition 3.1):

$$\frac{\partial}{\partial \xi} \underline{\Lambda}(\xi, \tau; -2\pi ix', -2\pi ix) = \underline{F}(x', x; \xi; \tau). \quad (13)$$

Comparing Eqs.(12) and (13), we can consider that  $\underline{F}(x', x; \xi; \tau)$  is an elliptic analogue of the generating function of the classical Bernoulli functions (polynomials). In this point of view  $B_m(x', x; \tau)$  become an elliptic analogue of the classical Bernoulli functions since  $\underline{F}(x', x; \xi; \tau)$  is the generating function of  $B_m(x', x; \tau)$ 's.

**REMARK 2.1.** Let  $Li_m(z)$  denote the Euler polylogarithms, i.e.,

$$Li_m(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^m}.$$

One of the basic property of the Euler polylogarithms is the iterated integral representation. We can formally consider that Eq.(11) is based on it in the following trick:

$$\begin{aligned} 2\pi i \int \frac{e(x\xi)}{e(\xi) - 1} d\xi &= 2\pi i \int e(x\xi) Li_0(e(-\xi)) d\xi \\ &= -e(x\xi) Li_1(e(-\xi)) + 2\pi i x \int e(x\xi) Li_1(e(-\xi)) d\xi \\ &= \dots\dots \\ &= -e(x\xi) \left( \sum_{m=0}^{\infty} Li_{m+1}(e(-\xi)) x^m \right) \\ &= - \sum_{m=0}^{\infty} \left( \sum_{k=0}^m Li_{k+1}(e(-\xi)) \frac{(2\pi i)^{-k} \xi^{m-k}}{(m-k)!} \right) (2\pi i x)^m \\ &= \sum_{m=0}^{\infty} \Lambda_{m+1}(-\xi) (2\pi i)^{m+1} (-x)^m. \end{aligned}$$

Here we assume that the series is formal, and use Proposition 1.1(a) in [Le] for the last equation. We wish that this formal calculation gives us a key to an elliptic analogue of the classical Euler polylogarithms and their iterated integral representation.

### 3 Elliptic Dedekind-Rademacher sums

In this section we define *elliptic Dedekind-Rademacher sums*, and obtain their reciprocity laws. We also reproduce the reciprocity laws [HWZ] for the classical generalized Dedekind-Rademacher sums by the degeneration of the *elliptic Dedekind-Rademacher sums*.

We begin with the following proposition before defining the elliptic Dedekind-Rademacher sums.

**PROPOSITION 3.1.** *Let  $a, b$  be positive integers, and  $x, y$  real numbers. Let  $\langle a, b \rangle$  be the greatest common divisor of  $a$  and  $b$ . Then the following statements are equivalent.*

- (i)  $ay - bx \notin \langle a, b \rangle \mathbb{Z}$ .
- (ii)  $a \frac{j+y}{b} - x \notin \mathbb{Z} \quad (j = 0, \dots, b-1)$ .
- (iii)  $\left(\frac{x}{a} + \frac{1}{a}\mathbb{Z}\right) \cap \left(\frac{y}{b} + \frac{1}{b}\mathbb{Z}\right) = \emptyset$ .

This is proved by simple calculation, so we omit the proof. Let  $a, a', b, b', c, c'$  are positive integers, and  $x, x', y, y', z, z'$  real numbers. Suppose that

$$a'z' - c'x' \notin \langle a', c' \rangle \mathbb{Z}, \quad b'z' - c'y' \notin \langle b', c' \rangle \mathbb{Z}. \quad (14)$$

Set

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) := ((a', a), (b', b), (c', c)), \quad (\mathbf{x}, \mathbf{y}, \mathbf{z}) := ((x', x), (y', y), (z', z)).$$

We define the *elliptic Dedekind-Rademacher sum* as follows:

$$S_{m,n}^\tau \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \end{pmatrix} := \frac{1}{c'} \sum_{\substack{j \pmod{c} \\ j' \pmod{c'}}} B_m \left( a' \frac{j' + z'}{c'} - x', a \frac{j + z}{c} - x; \frac{a'}{a} \tau \right) \\ \times B_n \left( b' \frac{j' + z'}{c'} - y', b \frac{j + z}{c} - y; \frac{b'}{b} \tau \right). \quad (15)$$

It follows from (14) and Proposition 3.1(ii) that  $B_1(0, 0; \tau)$  does not appear in (15). So this sum is well-defined.

If  $m \neq 1$  and  $n \neq 1$ , or if  $az - cx \notin \langle a, c \rangle \mathbb{Z}$  and  $bz - cy \notin \langle b, c \rangle \mathbb{Z}$ , then it follows from (8) that

$$\lim_{\tau \rightarrow i\infty} S_{m,n}^\tau \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \end{pmatrix} = S_{m,n} \begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix}. \quad (16)$$

This fact justifies our elliptic generalization of the classical generalized Dedekind-Rademacher sums. The assumption can not be removed. The reason is that  $\lim_{\tau \rightarrow i\infty} B_1(x', 0; \tau)$  is affected by the variable  $x'$ , i.e.,  $\lim_{\tau \rightarrow i\infty} B_1(x', 0; \tau)$  is indeterminate.

For any positive integer  $k$ , we have

$$S_{m,n}^\tau \begin{pmatrix} \mathbf{a}_k & \mathbf{b}_k & \mathbf{c}_k \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \end{pmatrix} = k S_{m,n}^\tau \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \end{pmatrix}, \\ S_{m,n}^\tau \begin{pmatrix} \mathbf{a}'_k & \mathbf{b}'_k & \mathbf{c}'_k \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \end{pmatrix} = S_{m,n}^\tau \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \end{pmatrix},$$

where

$$(\mathbf{a}_k, \mathbf{b}_k, \mathbf{c}_k) = ((a', ka), (b', kb), (c', kc)), \\ (\mathbf{a}'_k, \mathbf{b}'_k, \mathbf{c}'_k) = ((ka', a), (kb', b), (kc', c)).$$

So we may assume that the integers  $a, b$  and  $c$  (resp.  $a', b'$  and  $c'$ ) have no common factor.

Next we give the reciprocity laws for the elliptic Dedekind-Rademacher sums. As R.R. Hall *et al.* have noted in [HWZ], these laws mix various pairs of indices  $(m, n)$ , so they are conveniently stated in terms of the generating function

$$\sigma^\tau \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \\ X & Y & Z \end{pmatrix} = \sum_{m,n=0}^{\infty} \frac{(2\pi i)^{m+n}}{m!n!} S_{m,n}^\tau \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \end{pmatrix} \left(\frac{X}{a}\right)^{m-1} \left(\frac{Y}{b}\right)^{n-1}.$$

Here  $X$  and  $Y$  are non-zero variables and the variable  $Z$ , which does not appear explicitly on the right-hand side of the definition, is defined by  $-X - Y$ . If

$az - cx \notin \langle a, c \rangle \mathbb{Z}$  and  $bz - cy \notin \langle b, c \rangle \mathbb{Z}$ , then we see from (16) that  $\lim_{\tau \rightarrow i\infty} \sigma^\tau \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \\ X & Y & Z \end{pmatrix}$  gives the generating function  $\sigma \begin{pmatrix} a & b & c \\ x & y & z \\ X & Y & Z \end{pmatrix}$  of the classical generalized Dedekind-Rademacher sums. Because of (4) and (15), this generating function has another expression:

$$\sigma^\tau \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \\ X & Y & Z \end{pmatrix} = \frac{1}{c'} \sum_{\substack{j \pmod{c} \\ j' \pmod{c'}}} \underline{F} \left( a' \frac{j' + z'}{c'} - x', a \frac{j + z}{c} - x; \frac{X}{a}; \frac{a'}{a} \tau \right) \\ \times \underline{F} \left( b' \frac{j' + z'}{c'} - y', b \frac{j + z}{c} - y; \frac{Y}{b}; \frac{b'}{b} \tau \right). \quad (17)$$

The reciprocity laws for the elliptic Dedekind-Rademacher sums are the following:

**THEOREM 3.1.** *Let  $X, Y, Z$  be variables with  $X+Y+Z=0$ , and  $a, a', b, b', c, c'$  positive integers, and  $x, y, z$  real numbers. Let  $x', y'$  and  $z'$  be real numbers such that*

$$a'y' - b'x' \notin \langle a', b' \rangle \mathbb{Z}, \quad a'z' - c'x' \notin \langle a', c' \rangle \mathbb{Z}, \quad b'z' - c'y' \notin \langle b', c' \rangle \mathbb{Z}, \quad (18)$$

and let

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) := ((a', a), (b', b), (c', c)), \quad (\mathbf{x}, \mathbf{y}, \mathbf{z}) := ((x', x), (y', y), (z', z)).$$

Suppose that the integers  $a, b$  and  $c$  (resp.  $a', b'$  and  $c'$ ) have no common factor. Then we have

$$\sigma^\tau \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \\ X & Y & Z \end{pmatrix} + \sigma^\tau \begin{pmatrix} \mathbf{b} & \mathbf{c} & \mathbf{a} \\ \mathbf{y} & \mathbf{z} & \mathbf{x} \\ Y & Z & X \end{pmatrix} + \sigma^\tau \begin{pmatrix} \mathbf{c} & \mathbf{a} & \mathbf{b} \\ \mathbf{z} & \mathbf{x} & \mathbf{y} \\ Z & X & Y \end{pmatrix} = 0. \quad (19)$$

Here the left is over the three terms obtained by cyclic permutation of the columns of  $\begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \\ X & Y & Z \end{pmatrix}$ .

The strategy of our proof, which is the same method employed by A. Bayad [Ba2] and S. Egami [Ega], is to apply the residue theorem.

*Proof of Theorem 3.1.* Let

$$w_a = -\frac{x'}{a'} + \frac{x}{a}\tau, \quad w_b = -\frac{y'}{b'} + \frac{y}{b}\tau, \quad w_c = -\frac{z'}{c'} + \frac{z}{c}\tau.$$

We define the function  $G(\alpha)$  as follows:

$$G(\alpha) = F \left( a'(\alpha - w_a), \frac{X}{a}; \frac{a'}{a}\tau \right) F \left( b'(\alpha - w_b), \frac{Y}{b}; \frac{b'}{b}\tau \right) F \left( c'(\alpha - w_c), \frac{Z}{c}; \frac{c'}{c}\tau \right).$$

It follows from (2) that  $G(\alpha + 1) = G(\alpha + \tau) = G(\alpha)$ . The poles of  $G(\alpha)$  are on the following lattices:

$$w_a + \frac{1}{a'}\mathbb{Z} + \frac{\tau}{a}\mathbb{Z}, \quad w_b + \frac{1}{b'}\mathbb{Z} + \frac{\tau}{b}\mathbb{Z}, \quad w_c + \frac{1}{c'}\mathbb{Z} + \frac{\tau}{c}\mathbb{Z}.$$



By virtue of (18) and Proposition 3.1(iii), we have

$$(w_i + \frac{1}{i'}\mathbb{Z} + \frac{\tau}{i}\mathbb{Z}) \cap (w_j + \frac{1}{j'}\mathbb{Z} + \frac{\tau}{j}\mathbb{Z}) = \emptyset \quad (i, j = a, b, c \ (i \neq j)).$$

This implies that the order of every pole of  $G(\alpha)$  is equal to one. Because the function  $G(\alpha)$  is doubly periodic, the sum of the residues of  $G(\alpha)$  at its poles in any period parallelogram is equal to zero. So it follows that

$$0 = \sum_{s=a,b,c} \sum_{\substack{j \pmod{s} \\ j' \pmod{s'}}} \operatorname{Res}_{\alpha=w_s - j'/s' + (j/s)\tau} G(\alpha).$$

Consider the case  $s = c$ . By virtue of (3), We have

$$\begin{aligned} & \sum_{\substack{j \pmod{c} \\ j' \pmod{c'}}} \operatorname{Res}_{\alpha=w_c - j'/c' + (j/c)\tau} G(\alpha) \\ &= \frac{1}{c'} \sum_{\substack{j \pmod{c} \\ j' \pmod{c'}}} e(j\frac{-Z}{c}) F\left(-\left(a'\frac{j'+z'}{c'} - x'\right) + \left(a\frac{j+z}{c} - x\right)\frac{a'}{a}\tau; \frac{X}{a}; \frac{a'}{a}\tau\right) \\ & \quad \times F\left(-\left(b'\frac{j'+z'}{c'} - y'\right) + \left(b\frac{j+z}{c} - y\right)\frac{b'}{b}\tau; \frac{Y}{b}; \frac{b'}{b}\tau\right) \\ &= \frac{1}{c'} e\left(\frac{xX}{a} + \frac{yY}{b} + \frac{zZ}{c}\right) \sum_{\substack{j_c \pmod{c} \\ j'_c \pmod{c'}}} \underline{F}\left(a'\frac{j'+z'}{c'} - x', a\frac{j+z}{c} - x; \frac{X}{a}; \frac{a'}{a}\tau\right) \\ & \quad \times \underline{F}\left(b'\frac{j'+z'}{c'} - y', b\frac{j+z}{c} - y; \frac{Y}{b}; \frac{b'}{b}\tau\right). \end{aligned}$$

Since one can calculate the other cases in a similar way, we obtain (19).  $\square$

**REMARK 3.1.** As R.R. Hall *et al.* have remarked in [HWZ] Section 4, the left-hand side of (19) is invariant under all permutations of the columns since  $\sigma^\tau$  is symmetric in its first two columns. We can also see this fact since the function

$$G(\alpha) \text{ is invariant under all permutations of the columns of } \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \\ X & Y & Z \end{pmatrix}.$$

Now we are going to reproduce from Theorem 3.1 the reciprocity laws [HWZ] for the classical generalized Dedekind-Rademacher sums. To do this we prepare two lemmas:

**LEMMA 3.1.** *Let  $a, b, c$  be positive integers, and  $x, y, z$  real numbers. Then the following statements are equivalent:*

- (i)  $(x, y, z) \in (a, b, c)\mathbb{R} + \mathbb{Z}^3$ .
- (ii) *There is an integer  $j_0$  such that  $a\frac{j_0+z}{c} - x, b\frac{j_0+z}{c} - y \in \mathbb{Z}$ .*

*Proof.* Suppose that (i) holds. Then there are integers  $j, m, n$  and a real number  $r$  such that  $(x, y, z) = (a, b, c)r + (-m, -n, -j)$ . So we have

$$r = \frac{m+x}{a} = \frac{n+y}{b} = \frac{j+z}{c}.$$

Therefore one gets

$$a \frac{j+z}{c} - x = m, \quad b \frac{j+z}{c} - y = n.$$

Hence (ii) follows. The reverse is also proved in a similar way.  $\square$

**LEMMA 3.2.** *Let  $z'$  be a variable,  $a, b, c$  positive integers,  $j, j_0$  integers, and  $x, z$  real numbers. Suppose that  $a \frac{j_0+z}{c} - x, b \frac{j_0+z}{c} - y \in \mathbb{Z}$ . Then we have*

$$\begin{aligned} \lim_{\tau \rightarrow i\infty} B_m(z' - \frac{1}{2}, a \frac{j+z}{c} - x; \frac{a'}{a} \tau) \\ = \begin{cases} -\frac{\pi i}{2} z' + O(z'^2) & (m=1, j \equiv j_0 \pmod{c}), \\ \tilde{B}_m(a \frac{j+z}{c} - x) & (\text{otherwise}), \end{cases} \end{aligned} \quad (20)$$

$$\begin{aligned} \lim_{\tau \rightarrow i\infty} B_m(z', b \frac{j+z}{c} - y; \frac{b'}{b} \tau) \\ = \begin{cases} -\frac{1}{2\pi i} z'^{-1} - \frac{\pi i}{6} z' + O(z'^2) & (m=1, j \equiv j_0 \pmod{c}), \\ \tilde{B}_m(b \frac{j+z}{c} - y) & (\text{otherwise}). \end{cases} \end{aligned} \quad (21)$$

Here  $O(z')$  denotes the Landau symbol.

*Proof.* Suppose that  $m=1$  and  $j \equiv j_0 \pmod{c}$ . Then it follows from (8) that

$$\lim_{\tau \rightarrow i\infty} B_m(z' - \frac{1}{2}, a \frac{j+z}{c} - x; \frac{a'}{a} \tau) = \frac{1}{2} \frac{1 - e(z')}{1 + e(z')} = -\frac{\pi i}{2} z' + O(z'^2).$$

The other cases also follows from (8) in a similar way.  $\square$

**THEOREM 3.2** ([HWZ]). *Let  $X, Y, Z$  be variables with  $X + Y + Z = 0$ , and  $a, b, c$  positive integers with no common factor, and  $x, y, z$  real numbers. Then we have*

$$\begin{aligned} \sigma \begin{pmatrix} a & b & c \\ x & y & z \\ X & Y & Z \end{pmatrix} + \sigma \begin{pmatrix} b & c & a \\ y & z & x \\ Y & Z & X \end{pmatrix} + \sigma \begin{pmatrix} c & a & b \\ z & x & y \\ Z & X & Y \end{pmatrix} \\ = \begin{cases} -\frac{1}{4} (2\pi i)^2 & ((x, y, z) \in (a, b, c)\mathbb{R} + \mathbb{Z}^3), \\ 0 & (\text{otherwise}). \end{cases} \end{aligned} \quad (22)$$

Here

$$\sigma \begin{pmatrix} a & b & c \\ x & y & z \\ X & Y & Z \end{pmatrix} = \sum_{m,n=0}^{\infty} \frac{(2\pi i)^{m+n}}{m!n!} S_{m,n} \begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix} \left(\frac{X}{a}\right)^{m-1} \left(\frac{Y}{b}\right)^{n-1}.$$

*Proof.* We only consider the case  $(x, y, z) \in (a, b, c)\mathbb{R} + \mathbb{Z}^3$ ; the other case can be proved analogously. Let

$$\mathbf{a} = (1, a), \quad \mathbf{b} = (1, b), \quad \mathbf{c} = (1, c), \quad \mathbf{x} = \left(\frac{1}{2}, x\right), \quad \mathbf{y} = (0, y), \quad \mathbf{z} = (z', z).$$

When  $\tau$  tends to  $i\infty$ , Theorem 3.1 with  $a' = b' = c' = 1$ ,  $x' = 1/2$  and  $y' = 0$  implies that

$$\begin{aligned} & \sum_{m,n=0}^{\infty} \frac{(2\pi i)^{m+n}}{m!n!} \left(\frac{X}{a}\right)^{m-1} \left(\frac{Y}{b}\right)^{n-1} \lim_{\tau \rightarrow i\infty} S_{m,n}^{\tau} \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \end{pmatrix} \\ & + \sum_{m,n=0}^{\infty} \frac{(2\pi i)^{m+n}}{m!n!} \left(\frac{Y}{b}\right)^{m-1} \left(\frac{Z}{c}\right)^{n-1} \lim_{\tau \rightarrow i\infty} S_{m,n}^{\tau} \begin{pmatrix} \mathbf{b} & \mathbf{c} & \mathbf{a} \\ \mathbf{y} & \mathbf{z} & \mathbf{x} \end{pmatrix} \\ & + \sum_{m,n=0}^{\infty} \frac{(2\pi i)^{m+n}}{m!n!} \left(\frac{Z}{c}\right)^{m-1} \left(\frac{X}{a}\right)^{n-1} \lim_{\tau \rightarrow i\infty} S_{m,n}^{\tau} \begin{pmatrix} \mathbf{c} & \mathbf{a} & \mathbf{b} \\ \mathbf{z} & \mathbf{x} & \mathbf{y} \end{pmatrix} = 0. \quad (23) \end{aligned}$$

In order to show (22), let us compare the coefficient of  $z'^0 (= 1)$  in (23). We denote by  $\alpha_{m,n}$  the coefficient of  $z'^0$  of  $\lim_{\tau \rightarrow i\infty} S_{m,n}^{\tau} \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \end{pmatrix}$ . Lemma 3.1 and 3.2 together with (15) implies that

$$\alpha_{m,n} = \begin{cases} S_{1,1} \begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix} + \frac{1}{4} & (m = n = 1), \\ S_{m,n} \begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix} & (\text{otherwise}). \end{cases}$$

So the coefficient of  $z'^0$  in the first term of the left-hand-side of (23) is equal to

$$\sigma \begin{pmatrix} a & b & c \\ x & y & z \\ X & Y & Z \end{pmatrix} + \frac{1}{4}(2\pi i)^2. \text{ Similarly we can deduce that the coefficient of } z'^0$$

in the rest terms of the left-hand-side of (23) is equal to  $\sigma \begin{pmatrix} b & c & a \\ y & z & x \\ Y & Z & X \end{pmatrix} +$

$\sigma \begin{pmatrix} c & a & b \\ z & x & y \\ Z & X & Y \end{pmatrix}$ . This completes the proof.  $\square$

**REMARK 3.2.** The theorem in [HWZ] Sect.4 is incorrect, i.e., the right-hand-side of (7) in it should be multiplied by  $-1$ . The cause is that the proposition in [HWZ] Sect.3 has a small error.

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