Integral Operators on a Subspace of Holomorphic Functions on the Disc

by

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Abstract. Let $H(D)$ be an algebra of all holomorphic functions on the open unit disc $D$ and $X$ a subspace of $H(D)$. When $g$ is a function in $H(D)$, put

$$J_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta \text{ and } I_g(f)(z) = \int_0^z f'(\zeta)g(\zeta)d\zeta \quad (z \in D)$$

for $f$ in $X$. In this paper, we study $J[X] = \{g \in H(D) : J_g(f) \in X \text{ for all } f \in X\}$ and $I[X] = \{g \in H(D) : I_g(f) \in X \text{ for all } f \in X\}$. We apply the results to concrete spaces. For example, we study $J[X]$ and $I[X]$ when $X$ is a weighted Bloch space, a Hardy space or a Privalov space.

§1. Introduction

Let $D$ denote the open unit disc in the complex plane $\mathbb{C}$ and $H = H(D)$ the set of all holomorphic functions on $D$. For a given $g$ in $H$, define three operators :

$$(M_gf)(z) = g(z)f(z) \quad (f \in H, \; z \in D)$$
$$I_gf(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta \quad (f \in H, \; z \in D)$$

and

$$(I_gf)(z) = \int_0^z f'(\zeta)g(\zeta)d\zeta \quad (f \in H, \; z \in D).$$

Then $(J_gf)(z) + (I_gf)(z) = (M_gf)(z) - g(0)f(0)$. If $g(z) = z$ then $J_g$ is the Voltera integral operator and if $g(z) = \log 1/(1 - z)$ then $J_g$ is the Cesàro operator.

In this paper we assume that $X$ is a subspace of $H$ which contains constants. $X_1$ denotes the set of all holomorphic functions on $X_1$ and $I[X_1]$ denotes the set of all bounded linear operators on $X$. For each subspace $X$ put

$$M[X] = \{g \in H ; \; M_g(X) \subseteq X\},$$
$$J[X] = \{g \in H ; \; J_g(X) \subseteq X\}$$

and

$$I[X] = \{g \in H ; \; I_g(X) \subseteq X\}.$$ We define that $J^{n+1}[X] = J[J^n[X]]$ and $I^{n+1}[X] = I[I^n[X]]$ for $n \geq 1$ where $J^1[X] = J[X]$ and $I^1[X] = I[X]$. For $X$ and $Y$ which are subspaces of $H$, $XY$ denotes a subspace of $H$ which is generated by a product of a function in $X$ and one in $Y$. Let $Y^n$ be a subspace of $H$ which is generated by finite $n$ products of functions in a subspace $Y$ of $H$. For a subspace $X$ of $H$, $B(X)$ denotes the set of all bounded linear operators on $X$.

Now we give a lot of examples of $X$. For $0 < p \leq \infty$, $H^p$ is the usual Hardy space on $D$. $N$ is the Nevalinna class and $N_+$ is the Smirnov class on $D$. These are F-spaces, and $N$ and $N_+$ are algebras. It is known that $J[H^p] = \text{BMOA}$ (see [2], [1]), $z \notin J[N]$ [5] and $z \notin J[N_+]$ [7]. The Bloch space $B$ is defined to be a Banach space in $H$ with the norm

$$\|f\| = \sup_{z \in D} (1 - |z|^2)|f'(z)| + |f(0)|.$$
Then $\mathcal{B}$ contains $H^\infty$ properly. Recently, R. Yoneda [8] described $J[\mathcal{B}]$ and he [9] also proved that $I[\mathcal{B}] = H^\infty$. It is well known that $M[H^p] = H^\infty$.

In Section 2, we assume only that $X$ is a subspace of $H$. Theorem 1 implies that $J[X]^n \subset X$ for any $n \geq 1$. In Section 3, we study $J[X]$ and $I[X]$ when $X$ is an invariant subspace of $H$ or a subalgebra of $H$. Theorem 2 implies that if $H^\infty X \subset X$ and $J[X]$ contains $z$ then $J[X] \supseteq H^\infty$. In Section 4, assuming that $X$ is a F-space we show that $J[X]$ is contained in some weighted Bloch space and $I[X] \subset H^\infty$. In Section 5, we define a weighted Bloch space $B_\omega$ and we describe $J[B_\omega]$. In Section 6, we study $J \left[ \bigcap_{t<p} H^t \right]$ and $I \left[ \bigcap_{t<p} H^t \right]$. In Section 7, we show that $J[N^p]$ is a subalgebra of $N^p$ which contains $N^p_1$, where $N^p$ is a Privalov space.

§2. Subspace

In this section, we study $M[X]$, $J[X]$ and $I[X]$ assuming only that $X$ is a subspace of $H$.

**Lemma 1.** Let $X$ be a subspace of $H$ and $f, g$ in $H$.

1. $I_g I_f I_g f = I_f I_g$ on $X$
2. $I_g J_f = J_f M_g$ on $X$

Proof. (1) For $k \in X$,

$$((I_g I_f)k)(z) = \int_0^z (I_f k)'(\zeta)g(\zeta)d\zeta = \int_0^z k'(\zeta)f(\zeta)g(\zeta) d\zeta = (I_f g k)(z)$$

(2) For $k \in X$,

$$((I_g J_f)k)(z) = \int_0^z (J_f k)'(\zeta)g(\zeta)d\zeta = \int_0^z k(\zeta)f'(\zeta)g(\zeta) d\zeta = (J_f M_g k)(z).$$

**Theorem 1.** Let $X$ be a subspace of $H$ with constants. Then $J[X]$ is a subspace of $X$ with constants and $J[X]^n \subset X$.

Proof. If $g \in J[X]$ then $J_g(1) = g - g(0) \in X$ and so $g \in X$ because $1 \in X$. Hence $J[X]$ is a subspace of $X$ with constants.

Assuming $J[X]^n \subset X$, we will show that $J[X]^{n+1} \subset X$. Suppose that $g \in J[X]$ and $\{g_j\}_{j=1}^n \subset J[X]$. In order to prove that $g \prod_{j=1}^n g_j$ belongs to $X$, we will use the following
equalities.

\[
\int_{0}^{z} g(\zeta) \left( \prod_{j=1}^{n} g_{j} \right) (\zeta) d\zeta
= g(z) \left( \prod_{j=1}^{n} g_{j} \right) (z) - g(0) \left( \prod_{j=1}^{n} g_{j} \right) (0) - \int_{0}^{z} g'(\zeta) \left( \prod_{j=1}^{n} g_{j} \right) (\zeta) d\zeta
\]

and

\[
\int_{0}^{z} g(\zeta) \left( \prod_{j=1}^{n} g_{j} \right) ' (\zeta) d\zeta = \sum_{\ell=1}^{n} \int_{0}^{z} (g(\zeta) \prod_{j=1}^{n} g_{j}(\zeta)) g_{\ell}(\zeta) d\zeta.
\]

By hypothesis on induction, \( \prod_{j=1}^{n} g_{j} \in X \) and so \( \int_{0}^{z} g'(\zeta) \left( \prod_{j=1}^{n} g_{j} \right) (\zeta) d\zeta \in X \) because \( g \in J[X] \). By hypothesis on induction, for \( \ell = 1, \ldots, n \) \( \prod_{j\neq\ell} g_{j} \in X \) and so \( \int_{0}^{z} (g(\zeta) \prod_{j=1}^{n} g_{j}(\zeta)) g_{\ell}(\zeta) d\zeta \in X \) because \( g_{\ell} \in J[X] \). By the above two equalities, \( \prod_{j=1}^{n} g_{j} \) belongs to \( X \). This implies that \( J[X]^{n+1} \subset X \).

**Proposition 1.** Let \( X \) be a subspace of \( H \) with constants. Then \( I[X] \) is a subalgebra of \( H \).

Proof. If \( k \in I[X] \) and \( g \in I[X] \) then it is easy to see that \( I_{k}I_{g} = I_{kg} \) (see Proposition 3). Hence \( I_{k}I_{g}(X) = I_{k}(I_{g}(X)) \subset I_{k}(X) \subset X \) and so \( kg \) belongs to \( I[X] \). It is clear that \( I[X] \) is a subspace of \( H \).

**Proposition 2.** Suppose \( X \) is a subspace of \( H \) with constants.

1. \( M[X] \) is an algebra in \( X \).
2. \( J[X] \cap M[X] = I[X] \cap M[X] \).
3. \( J[X] \cap I[X] \subset M[X] \).
4. \( J[X] \subset M[X] \) if and only if \( J[X] \subset I[X] \). Similarly \( I[X] \subset M[X] \) if and only if \( I[X] \subset J[X] \).

Proof. (1) is clear. (2) and (3) follow from the equality: \( J_{g}f + I_{g}f = M_{g}f - g(0)f(0) \). (4) If \( J[X] \subset M[X] \) then by (2) \( J[X] \subset I[X] \). Conversely if \( J[X] \subset I[X] \) then by (3) \( J[X] \subset M[X] \).

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§3. Invariant subspace and subalgebra

In this section, we study \( J[X] \) and \( I[X] \) when \( X \) is an invariant subspace or a subalgebra of \( H \).
Theorem 2. Suppose that $X$ is a subspace of $H$ with constants and $kX \subset X$ for any $k$ in $H^\infty$.

(1) If $g_0$ is an arbitrary function in $J[X]$, then $J[X]$ contains $\{g \in H : |g'(z)| \leq |g'_0(z)|(z \in D)\}$.

(2) If $J[X]$ contains $z$ then it contains $H^\infty$.

(3) Suppose $J[X]$ contains $z$. If $\{g_n\}$ is in $J[X]$ and $g_n' \to g'$ uniformly on $D$ then $g$ belongs to $J[X]$.

(4) $zJ[X] \subset J[X]$ if and only if $J_z(J[X]X) \subset X$.

(5) $J[X] \cap H^\infty \subset I[X]$ and hence $I[X]$ contains $H^\infty$ if $z \in J[X]$.

Proof. (1) If $g \in H$ and $|g'(\zeta)| \leq |g'_0(\zeta)|(\zeta \in D)$, then $g'(g'_0)^{-1} \in H^\infty$ and so $fg'(g'_0)^{-1} \in X$ for any $f \in X$. Hence for any $f \in X$

$$\int_0^z f(\zeta)g'(\zeta)d\zeta = \int_0^z f(\zeta)g'(\zeta)g'_0(\zeta)^{-1}g'_0(\zeta)d\zeta$$

belongs to $X$ because $fg'(g'_0)^{-1} \in X$ and $g_0 \in J[X]$. This implies that $g$ belongs to $J[X]$. (2) Since $z \in J[X]$, by (1) and the definition of $H^\infty$, $H^\infty$ is contained in $J[X]$.

(3) If $g_n' \to g'$ uniformly on $D$, then $(g - g_n)' \in H^\infty$. Hence $f(g - g_n)' \in X$ for any $f \in X$. Therefore $g$ belongs to $J[X]$ because $z \in J[X]$ and

$$\int_0^z f(\zeta)g'(\zeta)d\zeta = \int_0^z f(\zeta)(g(\zeta) - g_n(\zeta))'d\zeta + \int_0^z f(\zeta)g'_n(\zeta)d\zeta.$$

(4) follows trivially from the following equality:

$$\int_0^z f(\zeta)(g(\zeta))'d\zeta = \int_0^z f(\zeta)g(\zeta)d\zeta + \int_0^z f(\zeta)g'_n(\zeta)d\zeta$$

for $f \in X$ and $g \in J[X]$.

(5) By the equality : $I_g(f) = fg - (fg)(0) - J_g(f)$, if $g \in J[X] \cap H^\infty$ and $f \in X$ then $I_g(f)$ belongs to $X$ because $gX \subset X$.

Proposition 3. If $X$ is a subalgebra of $H$ which contains constants then $M[X] = X$, $J[X]$ is also a subalgebra of $X$ and $J[X] = I[X] \cap X$.

Proof. $M[X] = X$ is clear. If both $g$ and $h$ are in $J[X]$, then by Theorem 1 both $fh$ and $fg$ belongs to $X$ for any $f \in X$ because $X$ is an algebra. Hence $gh$ belongs to $J[X]$ by the following equality : $J_{gh}(f) = J_g(fh) + J_h(fg)$ for any $f \in X$. This implies that $J[X]$ is a subalgebra of $X$ by Theorem 1. From (2) of Proposition 2 $J[X] = I[X] \cap X$ follows.

\[4.] F-space

Let $X$ be an F-space in $H$ with an invariant metric $d$. For each $a$ in $D$, put for $f$ in $X$

$$\mathcal{E}_af = f(a) \quad \text{and} \quad \mathcal{D}_af = f'(a).$$
In this section we assume that both \( \mathcal{E}_a \) and \( \mathcal{D}_a \) are bounded on \( X \). Put
\[
S(a) = \sup\{|\mathcal{E}_a(f)| ; f \in X, \ d(f,0) \leq 1\}
\]
and
\[
s(a) = \sup\{|\mathcal{D}_a(f)| ; f \in X, \ d(f,0) \leq 1\},
\]
then \( S(a) < \infty \) and \( s(a) < \infty \) if \( a \in D \). Suppose \( \nu \) is a nonnegative function on \( D \). For a function \( f \) in \( H \) put
\[
\|f\|_{\nu} = \sup_{z \in D} \nu(z)|f'(z)| + |f(0)|
\]
and
\[
\mathcal{B}_\nu = \{f \in H ; \|f\|_{\nu} < \infty\}.
\]
If \( \nu \) is bounded, \( \mathcal{B}_\nu \) contains all holomorphic functions on the closed unit disc \( \bar{D} \).

**Proposition 4.** If \( X \) is an \( F \)-space such that \( S(a) < \infty \) and \( s(a) < \infty \) for each \( a \in D \), then \( M[X], J[X] \) and \( I[X] \) belongs to \( B[X] \).

**Proof.** We will prove only that \( J[X] \subset B[X] \) because the other statements are similar. By the closed graph theorem, it is enough to prove that for \( \phi \in \mathcal{E}_X \) if \( f_n \to f \) in \( X \) and \( J_\phi(f_n) \to F \) then \( J_\phi(f) = F \). Since \( S(a) < \infty \), \( f_n(a) \to f(a) \) \( (a \in D) \). Since \( s(a) < \infty \), \( f_n(a)\phi'(a) \to F'(a) \) \( (a \in D) \). Thus \( f(a)\phi'(a) = F'(a) \) and so \( J_\phi(f) = F \) because \( F(0) = 0 \).

**Theorem 3.** Let \( X \) be an \( F \)-space in \( H \) with an invariant metric \( d \). Suppose that \( \sup |s(a)| \leq 1/\epsilon \) for any \( \epsilon > 0 \). Then \( J[X] \subset \mathcal{B}_{\omega_0} \cap X \) and \( I[X] \subset H^\infty \), where \( \omega_0 = 1/sS \).

**Proof.** If \( g \in J[X] \) then by Proposition 4, for any \( f \in X \)
\[
d(J_g f,0) \leq \|J_g\|d(f,0).
\]
Since \( J_g f \in X \), by definition of \( \mathcal{D}_z \) \( |\mathcal{D}_z(J_g f)| \leq s(z)d(J_g f,0)(z \in D) \). Hence
\[
s(z)^{-1} |f(z)| |g'(z)| \leq \|J_g\|d(f,0) \quad (z \in D)
\]
and so
\[
s^{-1}(z)S^{-1}(z)|g'(z)| \leq \|J_g\| \quad (z \in D).
\]
By Theorem 1 \( g \) belongs to \( \mathcal{B}_{\omega_0} \cap X \) where \( \omega_0 = 1/sS \). If \( g \in I[X] \) then by Proposition 4, for any \( f \in X \)
\[
d(I_g f,0) \leq \|I_g\|d(f,0).
\]
Since \( I_g f \in X \), by definition of \( \mathcal{D}_z \) \( |\mathcal{D}_z(I_g f)| \leq s(z)d(I_g f,0) \) \( (z \in D) \). Hence
\[
s(z)^{-1} |f'(z)||g(z)| \leq \|I_g\|d(f,0) \quad (z \in D)
\]
and so
\[
|g(z)| \leq \|I_g\| \quad (z \in D).
\]

**Proposition 5.** Let \( X \) be a subspace of \( H \) with constants which is of finite dimension. Then \( J[X] = I[X] = M[X] = \emptyset \).
Proof. Suppose \( \{f_j\}_{j=1}^n \) is a basis in \( X \) with \( f_1 \equiv 1 \). We will show that \( J[X] = \mathcal{C} \).

If \( g \in J[X] \) then by Theorem 1 \( g^\ell \in X \) for any \( \ell \geq 0 \) and so there exist \( \{\alpha_j^\ell\}_{j=1}^n \subset \mathcal{C} \) such that \( g^\ell = \sum_{j=1}^n \alpha_j^\ell f_j \). Hence there exist \( \{b_j\}_{\ell=0}^n \subset \mathcal{C} \) such that \( \sum_{\ell=0}^n b_\ell g^\ell = 0 \). This implies that \( g \) is just constant because \( g \) is analytic. Therefore \( J[X] = \mathcal{C} \). We will show that \( I[X] = \mathcal{C} \). Put \( X_1 = \{f' \; ; \; f \in X\} \). If \( g \in I[X] \) then by Proposition 1 \( g^\ell X_1 \subset X_1 \) for any \( \ell \geq 1 \) and so there exist \( \{\alpha_j^\ell\}_{j=1}^n \subset \mathcal{C} \) such that \( g^\ell f'_j = \sum_{j=2}^n \alpha_j^\ell f'_j \). By the same argument above \( g f'_j \) is constant. Similarly it follows that \( \{g f'_j\}_{j=2}^n \) are constants and so \( g \) is constant because \( \{f'_j\}_{j=2}^n \) is a basis in \( X_1 \). Therefore \( I[X] = \mathcal{C} \).

§5. Weighted Bloch space

Let \( \omega \) be a positive bounded function on \( D \). For a function \( f \) in \( H \) put

\[
\|f\|_\omega = \sup_{z \in D} \omega(z)|f'(z)| + |f(0)|
\]

and

\[
\mathcal{B}_\omega = \{f \in H \; ; \; \|f\|_\omega < \infty\}.
\]

Since \( \omega \) is bounded, \( \mathcal{B}_\omega \) contains all holomorphic functions on the closed unit disc \( \bar{D} \). \( \mathcal{B}_\omega \) is called a weighted Bloch space. A weight \( \omega \) is called measurable when \( \omega(at) \) is measurable on \([0,1]\) for each \( a \) in \( D \). Put \( \varepsilon(r) = \inf \{\omega(z) \; ; \; |z| \leq r\} \) and \( r < 1 \).

**Lemma 2.** If \( \varepsilon(r) > 0 \) for \( 0 \leq r < 1 \) then \( \mathcal{B}_\omega \) is a Banach space with norm \( \|\cdot\|_\omega \).

Proof. Suppose that \( \{f_n\} \) is a Cauchy sequence in \( \mathcal{B}_\omega \). For any \( \varepsilon > 0 \), there exist a positive integer \( n_0 \) such that \( \|f_n - f_m\|_\omega < \varepsilon \) if \( n, m \geq n_0 \). Hence if \( r < 1 \) and \( z \in D_r = \{z \; ; \; |z| < r\} \) then

\[
|f'_n(z) - f'_m(z)| \leq \frac{\varepsilon}{\omega(z)} \leq \frac{\varepsilon}{\varepsilon(r)}.
\]

By the normal family argument, there exists a function \( f' \in H(D_r) \) such that \( f'_n \to f' \) uniformly on \( D_r \). Hence as \( n \to \infty \),

\[
|f'(z) - f'_m(z)| \leq \frac{\varepsilon}{\omega(z)} \leq \frac{\varepsilon}{\varepsilon(r)} \quad (z \in D_r).
\]

Since \( r \) is arbitrary, \( f \) belongs to \( H(D) \) and

\[
\omega(z)|f'(z) - f'_m(z)| \leq \varepsilon \quad (z \in D)
\]

if \( m \geq n_0 \). Since \( f_m(0) \to f(0) \), \( \|f - f_m\|_\omega \to 0 \).
Theorem 4. Let $\omega$ be a measurable, $\varepsilon(r) > 0$ for $0 \leq r < 1$ and $X = \mathcal{B}_\omega$. Then

$$\mathcal{B}_{\omega S} = J[\mathcal{B}_\omega] \quad \text{and} \quad I[\mathcal{B}_\omega] \subset H^\infty$$

where $S(z) = \sup\{|f(z)| \mid f \in \mathcal{B}_\omega, \|f\|_{\omega} \leq 1\}$. Moreover $\|J_g\| = \|g\|_{\omega S}$ for each $g$ in $J[\mathcal{B}_\omega]$ with $g(0) = 0$.

Proof. By Theorem 1, $J[\mathcal{B}_\omega] \subset \mathcal{B}_\omega$. If $g \in J[\mathcal{B}_\omega]$ then $\|J_g f\|_{\omega} \leq \|J_g\| \|f\|_{\omega}$ ($f \in \mathcal{B}_\omega$) and so $\omega(z) |f(z)| \cdot |g'(z)| \leq \|J_g\| \|f\|_{\omega}$. Hence

$$\omega(z) S(z) \cdot |g'(z)| \frac{|f(z)|}{S(z)} \leq \|J_g\| \|f\|_{\omega}$$

and so

$$\omega(z) S(z) \cdot |g'(z)| \leq \|J_g\| \cdot |f\|_{\omega}$$

Therefore $g$ belongs to $\mathcal{B}_{\omega S}$ and $\|g\|_{\omega S} \leq \|J_g\| + |g(0)|$. Thus $J[\mathcal{B}_\omega] \subset \mathcal{B}_{\omega S}$. Note that $\mathcal{B}_{\omega S} \subset \mathcal{B}_\omega$ because $S(z) \geq 1$ ($z \in D$). Conversely if $g \in \mathcal{B}_{\omega S}$ then

$$\omega(z)|J_g (f)'(z)| = \omega(z) S(z) \cdot |g'(z)| \frac{|f(z)|}{S(z)} \leq \|g\|_{\omega S} \|f\|_{\omega}$$

and so $g$ belongs to $J[\mathcal{B}_\omega]$. Thus

$$\|g\|_{\omega S} \leq \|J_g\| + |g(0)| \leq \|g\|_{\omega S} + |g(0)|.$$

In Theorem 4, if $\omega$ is an absolute value of some analytic function and a radial function, R.Yoneda ([8],[9]) showed those under some special technical conditions on $\omega$.

§6. Hardy space

For $0 < p \leq \infty$, $H^p$ denotes $\bigcap_{t<p} H^t$ and $H^\infty$ is written as $H^\omega$. For $0 < p < \infty$, when $W = h^p$ for an outer function $h$ in $H^p$, $H^p(W)$ denotes a weighted Hardy space that is, the closure of $H^\omega$ in $L^p(Wd\theta/2\pi)$.

Lemma 3 is well known (cf. [3, Theorem 5.12]). In Proposition 6 it is known ([1],[2]) that $J[H^p] = \text{BMOA}$. Hence our result is weaker than that. However if $J[H^p] = \text{BMOA}$ then our result shows that $I[H^p] = H^\infty$.

**Lemma 3.** (1) For $0 < p < 1$, if $f$ is a function in $H^p$ then $\int \frac{f(\zeta)d\zeta}{\zeta}$ belongs to $H^{p/(1-p)}$. (2) If $f$ is a function in $H^1$ then $\int \frac{f(\zeta)d\zeta}{\zeta}$ belongs to $H^\infty$.

**Proposition 6.** For $0 < p < \infty$, $H_1^\infty \subset J[H^p] \subset H^\omega$ and $J[H^p] \subset J[H^p]$. Moreover $M[H^p] = H^\infty$ and $I[H^p] = J[H^p] \cap H^\infty$. 

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Proof. By Lemma 3, \( z \in J[H^p] \) and so by (2) of Theorem 2 \( H_1^\infty \subset J[H^p] \).

Theorem 1 implies that \( J[H^p] \subset H^\omega \). By (5) of Theorem 2, \( J[H^p] \cap H^\infty \subset I[H^p] \). Theorem 3 implies that \( I[H^p] \subset H^\infty \). Hence \( I[H^p] \cap H^\infty \subset H^\infty \) and so (4) of Proposition 2 \( I[H^p] \subset J[H^p] \). It is well known that \( M[H^p] = H^\infty \). By (2) of Proposition 2 \( I[H^p] = J[H^p] \cap H^\infty \). By (4) of Theorem 2, to prove that \( zJ[H^p] \subset J[H^p] \) it is sufficient to show that \( J_z(J[H^p]H^p) \subset H^p \). Since \( J[H^p]H^p \subset H^p \), by Lemma 2, \( J_z(J[H^p]H^p) \subset H^p \).

Theorem 5. For \( 0 < p < \infty \), \( \bigcap_{t<1} H^t \subset J[H^p] \subset H^\omega \) and so \( \log(1-z)^{-1} \) belongs to \( J[H^p] \). Moreover \( zJ[H^p] \subset J[H^p] \), \( M[H^p] = H^\infty \) and so \( J[H^p] \cap H^\infty = I[H^p] \cap H^\infty \).

When \( p = \infty \), \( J[H^\omega] = I[H^\omega] \cap H^\omega \) and \( J[H^\omega] \) is a subalgebra of \( H^\omega \) which contains \( H_1^\infty \).

Proof. By Theorem 1, \( J[H^p] \subset H^\omega \). We will show that \( \bigcap_{t<1} H^t \subset J[H^p] \). If \( g \in \bigcap_{t<1} H^t \) then \( g' \) belongs to \( H^{1-} \). If \( f \in H^{1-} \) then \( f \) belongs to \( H^t \) for any \( 0 < t < p \). If \( 0 < s < t/(t+1) \) then \( t/s > 1 \) and \( 1/(t+s) + 1/(t/t - s) = 1 \). By the Hölder inequality,

\[
\int_0^{2\pi} |f(e^{i\theta})g'(e^{i\theta})|^s \, d\theta / 2\pi \leq \left( \int_0^{2\pi} |f(e^{i\theta})|^t \, d\theta / 2\pi \right)^{\frac{s}{t}} \left( \int_0^{2\pi} |g'(e^{i\theta})|^{t/s} \, d\theta / 2\pi \right)^{\frac{s}{t}}
\]

and so \( f g' \) belongs to \( \bigcap_{s<t/(t+1)} H^s \). By Lemma 3, \( \int_0^{2\pi} f(\zeta)g'(\zeta) d\zeta \) belongs to \( H^{1-} \). As \( s \to t/(t+1) \), \( s/(1-s) \to t \) and so \( \int_0^{2\pi} f(\zeta)g'(\zeta) d\zeta \) belongs to \( H^{1-} \). As \( t \to p \), \( \int_0^{2\pi} f(\zeta)g'(\zeta) d\zeta \) belongs to \( H^p \). Thus \( J_z[H^p] \subset H^p \) and so \( \bigcap_{t<1} H^t \subset J[H^p] \). By (4) of Theorem 2, if we show that \( J_z(J[H^p]H^p) \subset H^p \) then it follows that \( zJ[H^p] \subset J[H^p] \). Since \( J[H^p] \cap H^p \subset H^p \), by Lemma 4 \( J_z(J[H^p]H^p) \subset H^p \). It is known that \( M[H^p] = H^\infty \). The last statement is a result of (2) of Proposition 2.

When \( p = \infty \), by Proposition 3 \( J[H^\omega] = I[H^\omega] \cap H^\omega \) and \( J[H^\omega] \) is a subalgebra of \( H^\omega \). Theorem 2 implies \( J[H^\omega] \supset H_1^\infty \).

Theorem 6. Let \( 1 \leq p < \infty \) and \( W = |h|^p \) for some outer function \( h \in H^p \).

Then \( \{g \in H; g(z) = \int_0^z h(\zeta)k(\zeta) d\zeta \text{ and } k \in H^{p-1} \} \subset J[H^p(W)] \subset H^w(W) \). \( M[H^p(W)] = H^\infty \) and \( J[H^p(W)] \cap H^\infty = I[H^p(W)] \). There exists a weight \( W \) such that \( z \) does not belong to \( J[H^p(W)] \).

Proof. If \( g(z) = \int_0^z h(\zeta)k(\zeta) d\zeta \) and \( k \in H^{p-1} \), then

\[
h(z)\{J_g(h^{-1}f)\}(z) = h(z)\int_0^z f(\zeta)k(\zeta) d\zeta
\]

and so \( h J_z h^{-1} f \) belongs to \( H^p \) for all \( f \in H^p \) by Lemma 3 because \( f k \in H^1 \). Therefore \( \{g \in H; g(z) = \int_0^z h(\zeta)k(\zeta) d\zeta \text{ and } k \in H^{p-1} \} \subset J[H^p(W)] \). By Theorem 1, \( J[H^p(W)] \subset 9 \)
\[ \bigcap_{p<\infty} H^p(W). \] In fact, since \( g^n h \in H^p \) for any \( n \geq 1 \), \( gh^{1/n} \in H^{np} \) and so \( g \) belongs to \( H^{np}(W) \). If \( \phi \in M(H^p(W)) \) then \( \phi(h^{-1}H^p) \subset h^{-1}H^p \) and so \( \phi H^p \subset H^p \). Hence \( \phi \in M(H^p) = H^\infty \). Therefore \( M(H^p(W)) = H^\infty \) and so by (2) of Proposition 2 \( J[H^p(W)] \cap H^\infty = I[H^p(W)] \cap H^\infty \). For \( a \in D \) it is easy to see that

\[
\sup \{ |f(a)| : f \in H^p(W) \text{ and } \|f\|_{W, p} \leq 1 \} = (1 - |a|^{2-1/p})^{-1} / |h(a)|^{-p} < \infty
\]

and so by Theorem 3 \( I[H^p(W)] \subset H^\infty \). Thus \( J[H^p(W)] \cap H^\infty = I[H^p(W)] \). If \( J_z(H^p(W)) \subset H^p(W) \) for any \( W \) with \( \log W \in L^1(d\theta/2\pi) \) then \( J_z(N_+) \subset N_+ \). For by a theorem of H.Helson [6] \( N_+ \) is the union of all \( H^p(W) \) as \( W \) ranges over the set of weights with summable \( \log W \). Hence there exists a weight \( W \) such that \( z \notin J[H^p(W)] \). Because it is known that \( J_z(N_+) \not\subset N_+ \) [7].

§7. Privalov space

We denote by \( N^p \), for \( 1 \leq p < \infty \), the set of all functions \( f \) in \( H \) which satisfy

\[
\sup_{0<r<1} \int_0^{2\pi} (\log^+ |f(r\, e^{i\theta})|^p) d\theta < \infty.
\]

When \( p = 1 \), \( N^p \) is just \( N \). Then

\[
\bigcup_{p>0} H^p \subset \bigcap_{p>1} N^p \text{ and } \bigcup_{p>1} N^p \subset N_+ \subset N^1 = N.
\]

**Proposition 7.** Let \( X = N_+ \) or \( N \). Then \( J[X] \) is a subalgebra of \( X \) and \( J[X] = I[X] \cap X \). If \( (f)^{-1} \) is in \( H^\infty \) then \( f \) does not belong to \( J[X] \).

Proof. It is known that \( N_+ \) and \( N \) are subalgebras of \( H \). Hence the first part of this proposition is a result of Theorem 1 and Proposition 3. By [5] and [7], \( z \notin J[X] \) and so the second part follows from (1) of Theorem 2.

In Proposition 7, it is known ([5],[7]) that \( z \notin J[X] \). Hence \( I[X] \not\subset z \). We don’t know whether \( J[X] = \mathcal{C} \) and \( I[X] = \mathcal{C} \).

**Theorem 7.** If \( 1 < p < \infty \) then \( J[N^p] \) is a subalgebra of \( N^p \) which contains \( N_+^p \), and \( J[N^p] = I[N^p] \cap N^p \).

Proof. Suppose \( 1 < p < \infty \) and \( g \in N_+^p \). If \( f \in N^p \) then

\[
\left\{ \int_0^{2\pi} (\log^+ |(J_g f)(r\, e^{i\theta})|^p) d\theta / 2\pi \right\}^{1/p} = \left\{ \int_0^{2\pi} \left( \log^+ \left| \int_0^t f(t\, e^{i\theta}) g'(t\, e^{i\theta}) \, dt \right|^p \right) d\theta / 2\pi \right\}^{1/p}
\]

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\[ \leq \left\{ \int_0^{2\pi} \left( \log^+ \int_0^1 |f(te^{i\theta})g'(te^{i\theta})| \, dt \right)^p \, d\theta / 2\pi \right\}^{1/p} \]

\[ \leq \left\{ \int_0^{2\pi} \left( \log^+ \sup_{0 \leq t < 1} |f(te^{i\theta})| + \log^+ \sup_{0 \leq t < 1} |g'(te^{i\theta})| \right)^p \, d\theta / 2\pi \right\}^{1/p} \]

\[ \leq \left\{ \int_0^{2\pi} \left( \log^+ \sup_{0 \leq t < 1} |f(te^{i\theta})| \right)^p \, d\theta / 2\pi \right\}^{1/p} + \left\{ \int_0^{2\pi} \left( \log^+ \sup_{0 \leq t < 1} |g'(te^{i\theta})| \right)^p \, d\theta / 2\pi \right\}^{1/p} \]

Put \( u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} P_r(t-\theta) \log^+ |f(e^{it})| \, dt \), then \( u(r, \theta) \geq \log^+ |f(re^{i\theta})| \). Since \( \log^+ |f(e^{it})| \in L^p \), by a theorem of Hardy and Littlewood (cf. [3, Proposition 1.8]), \( \sup_{0 \leq r < 1} u(r, \theta) \) belongs to \( L^p \) and so \( \log^+ \sup_{0 \leq r < 1} |f(re^{i\theta})| \) belongs to \( L^p \). Similarly we can prove that \( \log^+ \sup_{0 \leq r < 1} |g'(re^{i\theta})| \) belongs to \( L^p \). Thus \( Jg \) belongs to \( N^p \). Hence \( N^p_1 \subset J[N^p] \). It is known that \( N^p \) is a subalgebra of \( H \). Hence, by Proposition 3 \( J[N^p] \) is a subalgebra of \( N^p \) and \( J[N^p] = I[N^p] \cap N^p \).
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