Integral Operators on a Subspace of Holomorphic Functions on the Disc

by

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Abstract. Let $H(D)$ be an algebra of all holomorphic functions on the open unit disc $D$ and $X$ a subspace of $H(D)$. When $g$ is a function in $H(D)$, put

$$J_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta \quad \text{and} \quad I_g(f)(z) = \int_0^z f'(\zeta)g(\zeta)d\zeta \quad (z \in D)$$

for $f$ in $X$. In this paper, we study $J[X] = \{g \in H(D) : J_g(f) \in X \text{ for all } f \in X\}$ and $I[X] = \{g \in H(D) : I_g(f) \in X \text{ for all } f \in X\}$. We apply the results to concrete spaces. For example, we study $J[X]$ and $I[X]$ when $X$ is a weighted Bloch space, a Hardy space or a Privalov space.

§1. Introduction

Let $D$ denote the open unit disc in the complex plane $\mathbb{C}$ and $H = H(D)$ the set of all holomorphic functions on $D$. For a given $g$ in $H$, define three operators:

$$(M_g f)(z) = g(z)f(z) \quad (f \in H, \ z \in D)$$

$$(J_g f)(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta \quad (f \in H, \ z \in D)$$

and

$$(I_g f)(z) = \int_0^z f'(\zeta)g(\zeta)d\zeta \quad (f \in H, \ z \in D).$$

Then $(J_g f)(z) + (I_g f)(z) = (M_g f)(z) - g(0)f(0)$. If $g(z) = z$ then $J_g$ is the Voltera integral operator and if $g(z) = \log 1/(1 - z)$ then $J_g$ is the Cesàro operator.

In this paper we assume that $X$ is a subspace of $H$ which contains constants. $X_1$ denotes the set $\{f \in H : f' \in X\}$. For each subspace $X$ put

$$M[X] = \{g \in H : M_g(X) \subseteq X\},$$

$$J[X] = \{g \in H : J_g(X) \subseteq X\}$$

and

$$I[X] = \{g \in H : I_g(X) \subseteq X\}.$$}

We define that $J^{n+1}[X] = J[J^n[X]]$ and $I^{n+1}[X] = I[I^n[X]]$ for $n \geq 1$ where $J^1[X] = J[X]$ and $I^1[X] = I[X]$. For $X$ and $Y$ which are subspaces of $H$, $XY$ denotes a subspace of $H$ which is generated by a product of a function in $X$ and one in $Y$. Let $Y^n$ be a subspace of $H$ which is generated by finite $n$ products of functions in a subspace $Y$ of $H$. For a subspace $X$ of $H$, $B(X)$ denotes the set of all bounded linear operators on $X$.

Now we give a lot of examples of $X$. For $0 < p \leq \infty$, $H^p$ is the usual Hardy space on $D$, $N$ is the Nevalinna class and $N_+$ is the Smirnov class on $D$. These are $F$-spaces, and $N$ and $N_+$ are algebras. It is known that $J[H^p] = BMOA$ (see [2], [1]), $z \notin J[N]$ [5] and $z \notin J[N_+]$ [7]. The Bloch space $\mathcal{B}$ is defined to be a Banach space in $H$ with the norm

$$\|f\| = \sup_{z \in D}(1 - |z|^2)|f'(z)| + |f(0)|.$$
Then $\mathcal{B}$ contains $H^\infty$ properly. Recently, R. Yoneda [8] described $J[\mathcal{B}]$ and he [9] also proved that $I[\mathcal{B}] = H^\infty$. It is well known that $M[H^p] = H^\infty$.

In Section 2, we assume only that $X$ is a subspace of $H$. Theorem 1 implies that $J[X]^n \subset X$ for any $n \geq 1$. In Section 3, we study $J[X]$ and $I[X]$ when $X$ is an invariant subspace of $H$ or a subalgebra of $H$. Theorem 2 implies that if $H^\infty X \subset X$ and $J[X]$ contains $z$ then $J[X] \supseteq H^\infty$. In Section 4, assuming that $X$ is a $F$-space we show that $J[X]$ is contained in some weighted Bloch space and $I[X] \subset H^\infty$. In Section 5, we define a weighted Bloch space $\mathcal{B}_\omega$ and we describe $J[\mathcal{B}_\omega]$. In Section 6, we study $J\left[\bigcap_{t<p} H^t\right]$ and $I\left[\bigcap_{t<p} H^t\right]$. In Section 7, we show that $J[N^p]$ is a subalgebra of $N^p$ which contains $N^p_1$, where $N^p$ is a Privalov space.

§2. Subspace

In this section, we study $M[X]$, $J[X]$ and $I[X]$ assuming only that $X$ is a subspace of $H$.

**Lemma 1.** Let $X$ be a subspace of $H$ and $f, g$ in $H$.
(1) $I_g I_f = I_f I_g$ on $X$
(2) $I_g J_f = J_f M_g$ on $X$

Proof. (1) For $k \in X$,

$$((I_g I_f)k)(z) = \int_0^z (I_f)\eta(\zeta)g(\zeta)d\zeta = \int_0^z k'(\zeta)f(\zeta)g(\zeta)d\zeta = (I_g k)(z)$$

(2) For $k \in X$,

$$((I_g J_f)k)(z) = \int_0^z (J_f)\eta(\zeta)g(\zeta)d\zeta = \int_0^z k(\zeta)f'(\zeta)g(\zeta)d\zeta = (J_f M_g)(z) = (J_f (gk))(z).$$

**Theorem 1.** Let $X$ be a subspace of $H$ with constants. Then $J[X]$ is a subspace of $X$ with constants and $J[X]^n \subset X$.

Proof. If $g \in J[X]$ then $J_g(1) = g - g(0) \in X$ and so $g \in X$ because $1 \in X$. Hence $J[X]$ is a subspace of $X$ with constants.

Assuming $J[X]^n \subset X$, we will show that $J[X]^{n+1} \subset X$. Suppose that $g \in J[X]$ and $\{g_j\}_{j=1}^n \subset J[X]$. In order to prove that $g\prod_{j=1}^n g_j$ belongs to $X$, we will use the following
equalities.

\[
\int_0^z g(\zeta) \left( \prod_{j=1}^n g_j \right)'(\zeta) d\zeta \\
= g(z) \left( \prod_{j=1}^n g_j \right)(z) - g(0) \left( \prod_{j=1}^n g_j \right)(0) - \int_0^z g'(\zeta) \left( \prod_{j=1}^n g_j \right)(\zeta) d\zeta
\]

and

\[
\int_0^z g(\zeta) \left( \prod_{j=1}^n g_j \right)'(\zeta) d\zeta = \sum_{\ell=1}^n \int_0^z (g(\zeta) \prod_{j \neq \ell} g_j(\zeta)) g'_\ell(\zeta) d\zeta.
\]

By hypothesis on induction, \( \prod_{j=1}^n g_j \in X \) and so \( \int_0^z g'(\zeta) \left( \prod_{j=1}^n g_j \right)(\zeta) d\zeta \in X \) because \( g \in J[X] \). By hypothesis on induction, for \( \ell = 1, \ldots, n \), \( g \prod_{j \neq \ell} g_j \in X \) and so \( \int_0^z (g(\zeta) \prod_{j \neq \ell} g_j(\zeta)) g'_\ell(\zeta) d\zeta \in X \) because \( g_\ell \in J[X] \). By the above two equalities, \( g \prod_{j=1}^n g_j \) belongs to \( X \). This implies that \( J[X]^{n+1} \subset X \).

**Proposition 1.** Let \( X \) be a subspace of \( H \) with constants. Then \( I[X] \) is a subalgebra of \( H \).

**Proof.** If \( k \in I[X] \) and \( g \in I[X] \) then it is easy to see that \( I_k I_g = I_{kg} \) (see Proposition 3). Hence \( I_k I_g(X) = I_k(I_g(X)) \subset I_k(X) \subset X \) and so \( kg \) belongs to \( I[X] \). It is clear that \( I[X] \) is a subspace of \( H \).

**Proposition 2.** Suppose \( X \) is a subspace of \( H \) with constants.

1. \( M[X] \) is an algebra in \( X \).
2. \( J[X] \cap M[X] = I[X] \cap M[X] \).
3. \( J[X] \cap I[X] \subset M[X] \).
4. \( J[X] \subset M[X] \) if and only if \( J[X] \subset I[X] \). Similarly \( I[X] \subset M[X] \) if and only if \( I[X] \subset J[X] \).

**Proof.** (1) is clear. (2) and (3) follow from the equality : \( J_g f + I_g f = M_g f - g(0) f(0) \). (4) If \( J[X] \subset M[X] \) then by (2) \( J[X] \subset I[X] \). Conversely if \( J[X] \subset I[X] \) then by (3) \( J[X] \subset M[X] \).

§3. Invariant subspace and subalgebra

In this section, we study \( J[X] \) and \( I[X] \) when \( X \) is an invariant subspace or a subalgebra of \( H \).
Theorem 2. Suppose that $X$ is a subspace of $H$ with constants and $kX \subset X$ for any $k$ in $H^\infty$.

1. If $g_0$ is an arbitrary function in $J[X]$, then $J[X]$ contains $\{g \in H \mid |g'(z)| \leq |g_0'(z)|(z \in D)\}$.
2. If $J[X]$ contains $z$ then it contains $H_1^\infty$.
3. Suppose $J[X]$ contains $z$. If $\{g_n\}$ is in $J[X]$ and $g'_n \to g'$ uniformly on $D$ then $g$ belongs to $J[X]$.
4. If $J[X] \subset J[X]$ if and only if $J_z(J[X]X) \subset X$.
5. $J[X] \cap H^\infty \subset I[X]$ and hence $I[X]$ contains $H_1^\infty$ if $z \in J[X]$.

Proof. (1) If $g \in H$ and $|g'(\zeta)| \leq |g_0'(\zeta)|(\zeta \in D)$, then $g'(g_0')^{-1} \in H^\infty$ and so $fg'(g_0')^{-1} \in X$ for any $f \in X$. Hence for any $f \in X$

$$\int_0^zf(\zeta)g'(\zeta)d\zeta = \int_0^zf(\zeta)\frac{g'(\zeta)g_0'(\zeta)}{g_0'(\zeta)}d\zeta$$

belongs to $X$ because $fg'(g_0')^{-1} \in X$ and $g_0 \in J[X]$. This implies that $g$ belongs to $J[X]$.

(2) Since $z \in J[X]$, by (1) and the definition of $H_1^\infty$, $H_1^\infty$ is contained in $J[X]$.

(3) If $g'_n \to g'$ uniformly on $D$, then $(g - g_n)' \in H^\infty$. Hence $f(g - g_n)' \in X$ for any $f \in X$. Therefore $g$ belongs to $J[X]$ because $z \in J[X]$ and

$$\int_0^zf(\zeta)g'(\zeta)d\zeta = \int_0^zf(\zeta)(g(\zeta) - g_n(\zeta))d\zeta + \int_0^zf(\zeta)g'_n(\zeta)d\zeta.$$ 

(4) follows trivially from the following equality:

$$\int_0^zf(\zeta)(\zeta g(\zeta))'d\zeta = \int_0^zf(\zeta)g(\zeta)d\zeta + \int_0^zf(\zeta)\zeta g'(\zeta)d\zeta$$

for $f \in X$ and $g \in J[X]$.

(5) By the equality: $I_g(f) = fg - (fg)(0) - J_g(f)$, if $g \in J[X] \cap H^\infty$ and $f \in X$ then $I_g(f)$ belongs to $X$ because $gX \subset X$.

Proposition 3. If $X$ is a subalgebra of $H$ which contains constants then $M[X] = X$, $J[X]$ is also a subalgebra of $X$ and $J[X] = I[X] \cap X$.

Proof. $M[X] = X$ is clear. If both $g$ and $h$ are in $J[X]$, then by Theorem 1 both $fh$ and $fg$ belongs to $X$ for any $f \in X$ because $X$ is an algebra. Hence $gh$ belongs to $J[X]$ by the following equality: $J_{gh}(f) = J_g(fh) + J_h(fg)$ for any $f \in X$. This implies that $J[X]$ is a subalgebra of $X$ by Theorem 1. From (2) of Proposition 2 $J[X] = I[X] \cap X$ follows.

§4. F-space

Let $X$ be an F-space in $H$ with an invariant metric $d$. For each $a$ in $D$, put for $f$ in $X$

$$E_a f = f(a) \text{ and } D_a f = f'(a).$$
In this section we assume that both $\mathcal{E}_a$ and $\mathcal{D}_a$ are bounded on $X$. Put

$$S(a) = \sup\{|\mathcal{E}_a(f)| \ ; \ f \in X, \ d(f, 0) \leq 1\}$$

and

$$s(a) = \sup\{|\mathcal{D}_a(f)| \ ; \ f \in X, \ d(f, 0) \leq 1\},$$

then $S(a) < \infty$ and $s(a) < \infty$ if $a \in D$. Suppose $v$ is a nonnegative function on $D$. For a function $f$ in $H$ put

$$\|f\|_v = \sup_{z \in \bar{D}} v(z)|f'(z)| + |f(0)|$$

and

$$\mathcal{B}_v = \{f \in H \ ; \ \|f\|_v < \infty\}.$$  

If $v$ is bounded, $\mathcal{B}_v$ contains all holomorphic functions on the closed unit disc $\bar{D}$.

**Proposition 4.** If $X$ is an $F$-space such that $S(a) < \infty$ and $s(a) < \infty$ for each $a \in D$, then $M[X], J[X]$ and $I[X]$ belongs to $B[X]$.

Proof. We will prove only that $J[X] \subset B[X]$ because the other statements are similar. By the closed graph theorem, it is enough to prove that for $\phi \in X$ and $\omega$ and $\epsilon$ and $\in D$.

Then $S(a) < \infty$, since $f_n(a)\phi'(a) \to F'(a) (a \in D)$. Thus $f(a)\phi'(a) = F'(a)$ and so $J_\phi(f) = F$ because $F(0) = 0$.

**Theorem 3.** Let $X$ be an $F$-space in $H$ with an invariant metric $d$. Suppose that $\sup_{|a| \leq 1-\epsilon} S(a) < \infty$ for any $\epsilon > 0$. Then $J[X] \subset \mathcal{B}_{\omega_0} \cap X$ and $I[X] \subset H^\infty$, where $\omega_0 = 1/sS$.

Proof. If $g \in J[X]$ then by Proposition 4, for any $f \in X$ $d(J_g f, 0) \leq \|J_g\|d(f, 0)$. Since $J_g f \in X$, by definition of $D_z$ $|D_z(J_g f)| \leq s(z)d(J_g f, 0) (z \in D)$.

Hence

$$s(z)^{-1}|f(z)||g'(z)| \leq \|J_g\|d(f, 0) \quad (z \in D)$$

and so

$$s^{-1}(z)S^{-1}(z)|g'(z)| \leq \|J_g\| \quad (z \in D).$$

By Theorem 1 $g$ belongs to $\mathcal{B}_{\omega_0} \cap X$ where $\omega_0 = 1/sS$. If $g \in I[X]$ then by Proposition 4, for any $f \in X$ $d(I_g f, 0) \leq \|I_g\|d(f, 0)$. Since $I_g f \in X$, by definition of $D_z$ $|D_z(I_g f)| \leq s(z)d(I_g f, 0) \quad (z \in D)$. Hence

$$s(z)^{-1}|f'(z)||g(z)| \leq \|I_g\|d(f, 0) \quad (z \in D)$$

and so

$$|g(z)| \leq \|I_g\| \quad (z \in D).$$

**Proposition 5.** Let $X$ be a subspace of $H$ with constants which is of finite dimension. Then $J[X] = I[X] = M[X] = \emptyset$. 

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Proof. Suppose \( \{f_j\}_{j=1}^n \) is a basis in \( X \) with \( f_1 \equiv 1 \). We will show that \( J[X] = \mathcal{C} \).

If \( g \in J[X] \) then by Theorem 1 \( g^\ell \in X \) for any \( \ell \geq 0 \) and so there exist \( \{\alpha_j^\ell\}_{j=1}^n \subset \mathcal{C} \) such that

\[
g^\ell = \sum_{j=1}^n \alpha_j^\ell f_j.
\]

Hence there exist \( \{b_j\}_{j=0}^n \subset \mathcal{C} \) such that \( \sum_{\ell=0}^n b_\ell g^\ell = 0 \). This implies that \( g \) is just constant because \( g \) is analytic. Therefore \( J[X] = \mathcal{C} \). We will show that \( I[X] = \mathcal{C} \). Let \( X_1 = \{f' : f \in X\} \). If \( g \in I[X] \) then by Proposition 1 \( g^\ell X_1 \subset X_1 \) for any \( \ell \geq 1 \) and so there exist \( \{\alpha_j^\ell\}_{j=1}^n \subset \mathcal{C} \) such that \( g^\ell f'_2 = \sum_{j=2}^n \alpha_j^\ell f'_j \). By the same argument above \( gf'_2 \) is constant. Similarly it follows that \( \{gf'_\ell\}_{\ell=2}^\infty \) are constants and so \( g \) is constant because \( \{f'_\ell\}_{\ell=1}^n \) is a basis in \( X_1 \). Therefore \( I[X] = \mathcal{C} \).

\section{5. Weighted Bloch space}

Let \( \omega \) be a positive bounded function on \( D \). For a function \( f \) in \( H \) put

\[
\|f\|_\omega = \sup_{z \in D} \omega(z) |f'(z)| + |f(0)|
\]

and

\[
\mathcal{B}_\omega = \{f \in H : \|f\|_\omega < \infty\}.
\]

Since \( \omega \) is bounded, \( \mathcal{B}_\omega \) contains all holomorphic functions on the closed unit disc \( \bar{D} \). \( \mathcal{B}_\omega \) is called a weighted Bloch space. A weight \( \omega \) is called measurable when \( \omega(at) \) is measurable on \( [0,1] \) for each \( a \) in \( D \). Put \( \varepsilon(r) = \inf \{\omega(z) : |z| \leq r\} \) and \( r < 1 \).

**Lemma 2.** If \( \varepsilon(r) > 0 \) for \( 0 \leq r < 1 \) then \( \mathcal{B}_\omega \) is a Banach space with norm \( \|\cdot\|_\omega \).

Proof. Suppose that \( \{f_n\} \) is a Cauchy sequence in \( \mathcal{B}_\omega \). For any \( \varepsilon > 0 \), there exist a positive integer \( n_0 \) such that \( \|f_n - f_m\|_\omega < \varepsilon \) if \( n, m \geq n_0 \). Hence if \( r < 1 \) and \( z \in D_r = \{z : |z| < r\} \) then

\[
|f_n'(z) - f_m'(z)| \leq \frac{\varepsilon}{\omega(z)} \leq \frac{\varepsilon}{\varepsilon(r)}.
\]

By the normal family argument, there exists a function \( f' \in H(D_r) \) such that \( f'_n \to f' \) uniformly on \( D_r \). Hence as \( n \to \infty \),

\[
|f'(z) - f'_m(z)| \leq \frac{\varepsilon}{\omega(z)} \leq \frac{\varepsilon}{\varepsilon(r)} \quad (z \in D_r).
\]

Since \( r \) is arbitrary, \( f \) belongs to \( H(D) \) and

\[
\omega(z)|f'(z) - f'_m(z)| \leq \varepsilon \quad (z \in D)
\]

if \( m \geq n_0 \). Since \( f_m(0) \to f(0) \), \( \|f - f_m\|_\omega \to 0. \)
Theorem 4. Let \( \omega \) be a measurable, \( \varepsilon(r) > 0 \) for \( 0 \leq r < 1 \) and \( X = B_\omega \). Then

\[
B_\omega S = J[B_\omega] \quad \text{and} \quad I[B_\omega] \subset H^\infty
\]

where \( S(z) = \sup \{ |f(z)| : f \in B_\omega, \|f\|_\omega \leq 1 \} \). Moreover \( \|J_g\| = \|g\|_\omega \) for each \( g \in J[B_\omega] \) with \( g(0) = 0 \).

Proof. By Theorem 1, \( J[B_\omega] \subset B_\omega \). If \( g \in J[B_\omega] \) then \( \|J_g f\|_\omega \leq \|J_g\| \|f\|_\omega \) (\( f \in B_\omega \)) and so \( \omega(z) \cdot |f(z)| \cdot |g'(z)| \leq \|J_g\| \cdot \|f\|_\omega \). Hence

\[
\omega(z) S(z) \cdot |g'(z)| \cdot \left| \frac{f(z)}{S(z)} \right| \leq \|J_g\| \cdot \|f\|_\omega
\]

and so

\[
\omega(z) S(z) \cdot |g'(z)| \leq \|J_g\|.
\]

Therefore \( g \) belongs to \( B_\omega S \) and \( \|g\|_{\omega S} \leq \|J_g\| + \|g(0)\| \). Thus \( J[B_\omega] \subset B_\omega S \). Note that \( B_\omega S \subset B_\omega \) because \( S(z) \geq 1 \) (\( z \in D \)). Conversely if \( g \in B_\omega S \) then

\[
\omega(z) |J_g f(z)| = \omega(z) S(z) \cdot |g'(z)| \cdot \left| \frac{f(z)}{S(z)} \right| \leq \|g\|_{\omega S} \|f\|_\omega
\]

and so \( g \) belongs to \( J[B_\omega] \). Thus

\[
\|g\|_{\omega S} \leq \|J_g\| + \|g(0)\| \leq \|g\|_{\omega S} + \|g(0)\|.
\]

In Theorem 4, if \( \omega \) is an absolute value of some analytic function and a radial function, R.Yoneda ([8],[9]) showed those under some special technical conditions on \( \omega \).

§6. Hardy space

For \( 0 < p \leq \infty \), \( H^p_\omega \) denotes \( \bigcap_{1 < p} H^1 \) and \( H^{\infty}_\omega \) is written as \( H^\infty \). For \( 0 < p < \infty \), when \( W = \{h^p\} \) for an outer function \( h \) in \( H^p \), \( H^p(W) \) denotes a weighted Hardy space that is, the closure of \( H^\infty \) in \( L^p(Wd\theta/2\pi) \).

Lemma 3 is well known (cf. [3, Theorem 5.12]). In Proposition 6 it is known ([1],[2]) that \( J[H^p] = \text{BMOA} \). Hence our result is weaker than that. However if \( J[H^p] = \text{BMOA} \) then our result shows that \( I[H^p] = H^\infty \).

Lemma 3. (1) For \( 0 < p < 1 \), if \( f \) is a function in \( H^p \) then \( \int_0^z f(\zeta)d\zeta \) belongs to \( H^{p/(1-p)} \). (2) If \( f \) is a function in \( H^1 \) then \( \int_0^z f(\zeta)d\zeta \) belongs to \( H^\infty \).

Proposition 6. For \( 0 < p < \infty \), \( H^\infty_1 \subset J[H^p] \subset H^\omega \) and \( zJ[H^p] \subset J[H^p] \). Moreover \( M[H^p] = H^\infty \) and \( I[H^p] = J[H^p] \cap H^\infty \).
Proof. By Lemma 3, \( z \in J[H^p] \) and so by (2) of Theorem 2 \( H_1^\infty \subset J[H^p] \).

Theorem 1 implies that \( J[H^p] \subset H^\omega \). By (5) of Theorem 2, \( J[H^p] \cap H^\infty \subset I[H^p] \). Theorem 3 implies that \( I[H^p] \subset H^\infty \). Hence \( I[H^p] \cap H^p \subset H^p \) and so (4) of Proposition 2 \( I[H^p] \subset J[H^p] \). It is well known that \( M[H^p] = H^\infty \). By (2) of Proposition 2 \( I[H^p] = J[H^p] \cap H^\infty \). By (4) of Theorem 2, to prove that \( zJ[H^p] \subset J[H^p] \) it is sufficient to show that \( J_z(J[H^p]H^p) \subset H^p \). Since \( J[H^p]H^p \subset H^p \), by Lemma 2, \( J_z(J[H^p]H^p) \subset H^p \).

Theorem 5. For \( 0 < p < \infty \), \( \bigcap_{t < 1} H^t \subset J[H^p] \subset H^\omega \) and so \( \log(1-z)^{-1} \) belongs to \( J[H^p] \). Moreover \( zJ[H^p] \subset J[H^p] \), \( M[H^p] = H^\infty \) and so \( J[H^p] \cap H^\infty = I[H^p] \cap H^\infty \).

When \( p = \infty \), \( J[H^\omega] = I[H_\omega] \cap H^\omega \) and \( J[H^\omega] \) is a subalgebra of \( H^\omega \) which contains \( H_1^\infty \).

Proof. By Theorem 1, \( J[H^p] \subset H^\omega \). We will show that \( \bigcap_{t < 1} H^t \subset J[H^p] \). If \( g \in \bigcap_{t < 1} H^t \) then \( g' \) belongs to \( H^{1-t} \). If \( f \in H^p \) then \( f \) belongs to \( H^t \) for any \( 0 < t < p \). If \( 0 < s < t/(t+1) \) then \( ts > 1 \) and \( 1/(t+s) + 1/(t-t) = 1 \). By the Hölder inequality,

\[
\int_0^{2\pi} |f(e^{i\theta})g'(e^{i\theta})|^s d\theta / 2\pi \leq \left( \int_0^{2\pi} |f(e^{i\theta})|^t d\theta / 2\pi \right)^{s/t} \left( \int_0^{2\pi} |g'(e^{i\theta})|^t d\theta / 2\pi \right)^{s/t},
\]

and so \( fg' \) belongs to \( \bigcap_{s < t/(t+1)} H^s \). By Lemma 3, \( \int_0^t f(z)g'(z)dz \) belongs to \( H^{1-t} \). As \( s \to t/(t+1) \), \( s/(1-s) \to t \) and so \( \int_0^t f(z)g'(z)dz \) belongs to \( H^{1-t} \). As \( t \to p \), \( \int_0^t f(z)g'(z)dz \) belongs to \( H^p \). Thus \( J_z[H^p] \subset H^p \) and so \( \bigcap_{t < 1} H^t \subset J[H^p] \). By (4) of Theorem 2, if we show that \( J_z(J[H^p]H^p) \subset H^p \) then it follows that \( zJ[H^p] \subset J[H^p] \). Since \( J[H^p]H^p \subset H^p \), by Lemma 4 \( J_z(J[H^p]H^p) \subset H^p \). It is known that \( M[H^p] = H^\infty \). The last statement is a result of (2) of Proposition 2.

When \( p = \infty \), by Proposition 3 \( J[H^\omega] = I[H_\omega] \cap H^\omega \) and \( J[H^\omega] \) is a subalgebra of \( H^\omega \). Theorem 2 implies \( J[H^\omega] \supset H_1^\infty \).

Theorem 6. Let \( 1 \leq p < \infty \) and \( W = |h|^p \) for some outer function \( h \) in \( H^p \).

Then \( \{g \in H; g(z) = \int_0^t h(\zeta)k(\zeta)dz \text{ and } k \in H^{\frac{p}{p-1}} \} \subset J[H^p(W)] \subset H^\omega(W) \). \( M[H^p(W)] = H^\infty \) and \( J[H^p(W)] \cap H^\infty = I[H^p(W)] \). There exists a weight \( W \) such that \( z \) does not belong to \( J[H^p(W)] \).

Proof. If \( g(z) = \int_0^t h(\zeta)k(\zeta)dz \) and \( k \in H^{\frac{p}{p-1}} \), then

\[
h(z)\{J_g(h^{-1}f)\}(z) = h(z)\int_0^t f(\zeta)k(\zeta)d\zeta
\]

and so \( hJ_gh^{-1}f \) belongs to \( H^p \) for all \( f \in H^p \) by Lemma 3 because \( f k \in H^1 \). Therefore \( \{g \in H; g(z) = \int_0^t h(\zeta)k(\zeta)dz \text{ and } k \in H^{\frac{p}{p-1}} \} \subset J[H^p(W)] \). By Theorem 1, \( J[H^p(W)] \subset \)
\[ \bigcup_{p<\infty} H^p(W). \] In fact, since \( g^n h \in H^p \) for any \( n \geq 1, gh^{1/n} \in H^{np} \) and so \( g \) belongs to \( H^{np}(W) \). If \( \phi \in M(H^p(W)) \) then \( \phi(h^{-1}H^p) \subset h^{-1}H^p \) and so \( \phi H^p \subset H^p \). Hence \( \phi \in M(H^p) = H^\infty \). Therefore \( M(H^p(W)) = H^\infty \) and so by (2) of Proposition 2 \( J[H^p(W)] \cap H^\infty = I[H^p(W)] \cap H^\infty \). For \( a \in D \) it is easy to see that

\[
\sup\{ |f(a)| : f \in H^p(W) \text{ and } ||f||_{W,p} \leq 1 \} = (1 - |a|^{-1/p} |h(a)|^{-p} < \infty
\]

and so by Theorem 3 \( I[H^p(W)] \subset H^\infty \). Thus \( J[H^p(W)] \cap H^\infty = I[H^p(W)] \). If \( J_+(H^p(W)) \subset H^p(W) \) for any \( W \) with \( \log W \in L^1(d\theta/2\pi) \) then \( J_+(N_+) \subset N_+ \). For by a theorem of H. Helson [6] \( N_+ \) is the union of all \( H^p(W) \) as \( W \) ranges over the set of weights with sumable \( \log W \). Hence there exists a weight \( W \) such that \( z \notin J[H^p(W)] \). Because it is known that \( J_+(N_+) \not\subset N_+ \) [7].

\section*{§7. Privalov space}

We denote by \( N^p \), for \( 1 \leq p < \infty \), the set of all functions \( f \) in \( H \) which satisfy

\[
\sup_{0<r<1} \int_0^{2\pi} (\log^+ |f(re^{i\theta})|)^p d\theta < \infty.
\]

When \( p = 1 \), \( N^p \) is just \( N \). Then

\[
\bigcup_{p>0} H^p \subset \bigcap_{p>1} N^p \text{ and } \bigcup_{p>1} N^p \subset N_+ \subset N^1 = N.
\]

Proposition 7. Let \( X = N_+ \) or \( N \). Then \( J[X] \) is a subalgebra of \( X \) and \( J[X] = I[X] \cap X \). If \( (f')^{-1} \) is in \( H^\infty \) then \( f \) does not belong to \( J[X] \).

Proof. It is known that \( N_+ \) and \( N \) are subalgebras of \( H \). Hence the first part of this proposition is a result of Theorem 1 and Proposition 3. By [5] and [7], \( z \notin J[X] \) and so the second part follows from (1) of Theorem 2.

In Proposition 7, it is known ([5],[7]) that \( z \notin J[X] \). Hence \( I[X] \not\equiv z \). We don’t know whether \( J[X] = C \) and \( I[X] = C \).

Theorem 7. If \( 1 < p < \infty \) then \( J[N^p] \) is a subalgebra of \( N^p \) which contains \( N_+^p \), and \( J[N^p] = I[N^p] \cap N^p \).

Proof. Suppose \( 1 < p < \infty \) and \( g \in N_+^p \). If \( f \in N^p \) then

\[
\left\{ \int_0^{2\pi} (\log^+ |(J_g f)(re^{i\theta})|)^p d\theta / 2\pi \right\}^{1/p} = \left\{ \int_0^{2\pi} \left( \log^+ \left| \int_0^t f(te^{i\theta}) g'(te^{i\theta}) dt \right|^p d\theta / 2\pi \right) \right\}^{1/p}
\]
\leq \left\{ \int_{0}^{2\pi} \left( \log^+ \int_{0}^{1} |f(te^{i\theta})g'(te^{i\theta})| dt \right)^p \frac{d\theta}{2\pi} \right\}^{1/p}

\leq \left\{ \int_{0}^{2\pi} \left( \log^+ \sup_{0\leq t<1} |f(te^{i\theta})| + \log^+ \sup_{0\leq t<1} |g'(te^{i\theta})| \right)^p \frac{d\theta}{2\pi} \right\}^{1/p}

\leq \left\{ \int_{0}^{2\pi} \left( \log^+ \sup_{0\leq t<1} |f(te^{i\theta})| \right)^p \frac{d\theta}{2\pi} \right\}^{1/p} + \left\{ \int_{0}^{2\pi} \left( \log^+ \sup_{0\leq t<1} |g'(te^{i\theta})| \right)^p \frac{d\theta}{2\pi} \right\}^{1/p}

Put \( u(r, \theta) = \frac{1}{2\pi} \int_{0}^{2\pi} P_r(t-\theta) \log^+ |f(e^{it})| dt \), then \( u(r, \theta) \geq \log^+ |f(re^{i\theta})| \). Since \( \log^+ |f(e^{it})| \in L^p \), by a theorem of Hardy and Littlewood (cf. [3, Proposition 1.8]), \( \sup_{0\leq r<1} u(r, \theta) \) belongs to \( L^p \) and so \( \log^+ \sup_{0\leq r<1} |f(re^{i\theta})| \) belongs to \( L^p \). Similarly we can prove that \( \log^+ \sup_{0\leq r<1} |g'(re^{i\theta})| \) belongs to \( L^p \). Thus \( J_gf \) belongs to \( N^p \). Hence \( N^p_1 \subset J[N^p] \). It is known that \( N^p \) is a subalgebra of \( H \). Hence, by Proposition 3 \( J[N^p] \) is a subalgebra of \( N^p \) and \( J[N^p] = I[N^p] \cap N^p \).
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