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Integral Operators on a Subspace of Holomorphic Functions on the Disc

by

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**Abstract.** Let  $H(D)$  be an algebra of all holomorphic functions on the open unit disc  $D$  and  $X$  a subspace of  $H(D)$ . When  $g$  is a function in  $H(D)$ , put

$$J_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta \text{ and } I_g(f)(z) = \int_0^z f'(\zeta)g(\zeta)d\zeta \quad (z \in D)$$

for  $f$  in  $X$ . In this paper, we study  $J[X] = \{g \in H(D) ; J_g(f) \in X \text{ for all } f \text{ in } X\}$  and  $I[X] = \{g \in H(D) ; I_g(f) \in X \text{ for all } f \text{ in } X\}$ . We apply the results to concrete spaces. For example, we study  $J[X]$  and  $I[X]$  when  $X$  is a weighted Bloch space, a Hardy space or a Privalov space.

## §1. Introduction

Let  $D$  denote the open unit disc in the complex plane  $\mathcal{C}$  and  $H = H(D)$  the set of all holomorphic functions on  $D$ . For a given  $g$  in  $H$ , define three operators :

$$\begin{aligned} (M_g f)(z) &= g(z)f(z) & (f \in H, z \in D) \\ (J_g f)(z) &= \int_0^z f(\zeta)g'(\zeta)d\zeta & (f \in H, z \in D) \end{aligned}$$

and

$$(I_g f)(z) = \int_0^z f'(\zeta)g(\zeta)d\zeta \quad (f \in H, z \in D).$$

Then  $(J_g f)(z) + (I_g f)(z) = (M_g f)(z) - g(0)f(0)$ . If  $g(z) = z$  then  $J_g$  is the Volterra integral operator and if  $g(z) = \log 1/(1-z)$  then  $J_g$  is the Cesàro operator.

In this paper we assume that  $X$  is a subspace of  $H$  which contains constants.  $X_1$  denotes the set  $\{f \in H ; f' \in X\}$ . For each subspace  $X$  put

$$\begin{aligned} M[X] &= \{g \in H ; M_g(X) \subseteq X\}, \\ J[X] &= \{g \in H ; J_g(X) \subseteq X\} \end{aligned}$$

and

$$I[X] = \{g \in H ; I_g(X) \subseteq X\}.$$

We define that  $J^{n+1}[X] = J[J^n[X]]$  and  $I^{n+1}[X] = I[I^n[X]]$  for  $n \geq 1$  where  $J^1[X] = J[X]$  and  $I^1[X] = I[X]$ . For  $X$  and  $Y$  which are subspaces of  $H$ ,  $XY$  denotes a subspace of  $H$  which is generated by a product of a function in  $X$  and one in  $Y$ . Let  $Y^n$  be a subspace of  $H$  which is generated by finite  $n$  products of functions in a subspace  $Y$  of  $H$ . For a subspace  $X$  of  $H$ ,  $B(X)$  denotes the set of all bounded linear operators on  $X$ .

Now we give a lot of examples of  $X$ . For  $0 < p \leq \infty$ ,  $H^p$  is the usual Hardy space on  $D$ ,  $N$  is the Nevalinna class and  $N_+$  is the Smirnov class on  $D$ . These are F-spaces, and  $N$  and  $N_+$  are algebras. It is known that  $J[H^p] = \text{BMOA}$  (see [2], [1]),  $z \notin J[N]$  [5] and  $z \notin J[N_+]$  [7]. The Bloch space  $\mathcal{B}$  is defined to be a Banach space in  $H$  with the norm

$$\|f\| = \sup_{z \in D} (1 - |z|^2)|f'(z)| + |f(0)|.$$

Then  $\mathcal{B}$  contains  $H^\infty$  properly. Recently, R.Yoneda [8] described  $J[\mathcal{B}]$  and he [9] also proved that  $I[\mathcal{B}] = H^\infty$ . It is well known that  $M[H^p] = H^\infty$ .

In Section 2, we assume only that  $X$  is a subspace of  $H$ . Theorem 1 implies that  $J[X]^n \subset X$  for any  $n \geq 1$ . In Section 3, we study  $J[X]$  and  $I[X]$  when  $X$  is an invariant subspace of  $H$  or a subalgebra of  $H$ . Theorem 2 implies that if  $H^\infty X \subset X$  and  $J[X]$  contains  $z$  then  $J[X] \supseteq H_1^\infty$ . In Section 4, assuming that  $X$  is a F-space we show that  $J[X]$  is contained in some weighted Bloch space and  $I[X] \subset H^\infty$ . In Section 5, we define a weighted Bloch space  $\mathcal{B}_\omega$  and we describe  $J[\mathcal{B}_\omega]$ . In Section 6, we study  $J\left[\bigcap_{t < p} H^t\right]$  and  $I\left[\bigcap_{t < p} H^t\right]$ . In Section 7, we show that  $J[N^p]$  is a subalgebra of  $N^p$  which contains  $N_1^p$ , where  $N^p$  is a Privalov space.

## §2. Subspace

In this section, we study  $M[X]$ ,  $J[X]$  and  $I[X]$  assuming only that  $X$  is a subspace of  $H$ .

**Lemma 1.** *Let  $X$  be a subspace of  $H$  and  $f, g$  in  $H$ .*

(1)  $I_g I_f = I_{gf} = I_f I_g$  on  $X$

(2)  $I_g J_f = J_f M_g$  on  $X$

Proof. (1) For  $k \in X$ ,

$$((I_g I_f)k)(z) = \int_0^z (I_f k)'(\zeta) g(\zeta) d\zeta = \int_0^z k'(\zeta) f(\zeta) g(\zeta) d\zeta = (I_{fg} k)(z)$$

(2) For  $k \in X$ ,

$$\begin{aligned} ((I_g J_f)k)(z) &= \int_0^z (J_f k)'(\zeta) g(\zeta) d\zeta = \int_0^z k(\zeta) f'(\zeta) g(\zeta) d\zeta \\ &= (J_f(gk))(z) = ((J_f M_g)k)(z). \end{aligned}$$

**Theorem 1.** *Let  $X$  be a subspace of  $H$  with constants. Then  $J[X]$  is a subspace of  $X$  with constants and  $J[X]^n \subset X$ .*

Proof. If  $g \in J[X]$  then  $J_g(1) = g - g(0) \in X$  and so  $g \in X$  because  $1 \in X$ . Hence  $J[X]$  is a subspace of  $X$  with constants.

Assuming  $J[X]^n \subset X$ , we will show that  $J[X]^{n+1} \subset X$ . Suppose that  $g \in J[X]$  and  $\{g_j\}_{j=1}^n \subset J[X]$ . In order to prove that  $g \prod_{j=1}^n g_j$  belongs to  $X$ , we will use the following

equalities.

$$\begin{aligned} & \int_0^z g(\zeta) \left( \prod_{j=1}^n g_j \right)'(\zeta) d\zeta \\ &= g(z) \left( \prod_{j=1}^n g_j \right)(z) - g(0) \left( \prod_{j=1}^n g_j \right)(0) - \int_0^z g'(\zeta) \left( \prod_{j=1}^n g_j \right)(\zeta) d\zeta \end{aligned}$$

and

$$\int_0^z g(\zeta) \left( \prod_{j=1}^n g_j \right)'(\zeta) d\zeta = \sum_{\ell=1}^n \int_0^z (g(\zeta) \prod_{j \neq \ell} g_j(\zeta)) g'_\ell(\zeta) d\zeta.$$

By hypothesis on induction,  $\prod_{j=1}^n g_j \in X$  and so  $\int_0^z g'(\zeta) \left( \prod_{j=1}^n g_j \right)(\zeta) d\zeta \in X$  because  $g \in J[X]$ . By hypothesis on induction, for  $\ell = 1, \dots, n$   $g \prod_{j \neq \ell} g_j \in X$  and so  $\int_0^z (g(\zeta) \prod_{j \neq \ell} g_j(\zeta)) g'_\ell(\zeta) d\zeta \in X$  because  $g_\ell \in J[X]$ . By the above two equalities,  $g \prod_{j=1}^n g_j$  belongs to  $X$ . This implies that  $J[X]^{n+1} \subset X$ .

**Proposition 1.** *Let  $X$  be a subspace of  $H$  with constants. Then  $I[X]$  is a subalgebra of  $H$ .*

Proof. If  $k \in I[X]$  and  $g \in I[X]$  then it is easy to see that  $I_k I_g = I_{kg}$  (see Proposition 3). Hence  $I_k I_g(X) = I_k(I_g(X)) \subseteq I_k(X) \subseteq X$  and so  $kg$  belongs to  $I[X]$ . It is clear that  $I[X]$  is a subspace of  $H$ .

**Proposition 2.** *Suppose  $X$  is a subspace of  $H$  with constants.*

- (1)  $M[X]$  is an algebra in  $X$ .
- (2)  $J[X] \cap M[X] = I[X] \cap M[X]$ .
- (3)  $J[X] \cap I[X] \subseteq M[X]$ .
- (4)  $J[X] \subset M[X]$  if and only if  $J[X] \subset I[X]$ . Similarly  $I[X] \subset M[X]$  if and only if  $I[X] \subset J[X]$ .

Proof. (1) is clear. (2) and (3) follow from the equality :  $J_g f + I_g f = M_g f - g(0)f(0)$ . (4) If  $J[X] \subset M[X]$  then by (2)  $J[X] \subset I[X]$ . Conversely if  $J[X] \subset I[X]$  then by (3)  $J[X] \subset M[X]$ .

### §3. Invariant subspace and subalgebra

In this section, we study  $J[X]$  and  $I[X]$  when  $X$  is an invariant subspace or a subalgebra of  $H$ .

**Theorem 2.** Suppose that  $X$  is a subspace of  $H$  with constants and  $kX \subset X$  for any  $k$  in  $H^\infty$ .

(1) If  $g_0$  is an arbitrary function in  $J[X]$ , then  $J[X]$  contains  $\{g \in H ; |g'(z)| \leq |g'_0(z)|(z \in D)\}$ .

(2) If  $J[X]$  contains  $z$  then it contains  $H_1^\infty$ .

(3) Suppose  $J[X]$  contains  $z$ . If  $\{g_n\}$  is in  $J[X]$  and  $g'_n \rightarrow g'$  uniformly on  $D$  then  $g$  belongs to  $J[X]$ .

(4)  $zJ[X] \subset J[X]$  if and only if  $J_z(J[X]X) \subset X$ .

(5)  $J[X] \cap H^\infty \subset I[X]$  and hence  $I[X]$  contains  $H_1^\infty$  if  $z \in J[X]$ .

Proof. (1) If  $g \in H$  and  $|g'(\zeta)| \leq |g'_0(\zeta)|(\zeta \in D)$ , then  $g'(g'_0)^{-1} \in H^\infty$  and so  $fg'(g'_0)^{-1} \in X$  for any  $f \in X$ . Hence for any  $f \in X$

$$\int_0^z f(\zeta)g'(\zeta)d\zeta = \int_0^z f(\zeta)g'(\zeta)g'_0(\zeta)^{-1}g'_0(\zeta)d\zeta$$

belongs to  $X$  because  $fg'(g'_0)^{-1} \in X$  and  $g_0 \in J[X]$ . This implies that  $g$  belongs to  $J[X]$ .

(2) Since  $z \in J[X]$ , by (1) and the definition of  $H_1^\infty$ ,  $H_1^\infty$  is contained in  $J[X]$ .

(3) If  $g'_n \rightarrow g'$  uniformly on  $D$ , then  $(g - g_n)' \in H^\infty$ . Hence  $f(g - g_n)' \in X$  for any  $f \in X$ . Therefore  $g$  belongs to  $J[X]$  because  $z \in J[X]$  and

$$\int_0^z f(\zeta)g'(\zeta)d\zeta = \int_0^z f(\zeta)(g(\zeta) - g_n(\zeta))'d\zeta + \int_0^z f(\zeta)g'_n(\zeta)d\zeta.$$

(4) follows trivially from the following equality :

$$\int_0^z f(\zeta)(\zeta g(\zeta))'d\zeta = \int_0^z f(\zeta)g(\zeta)d\zeta + \int_0^z f(\zeta)\zeta g'(\zeta)d\zeta$$

for  $f \in X$  and  $g \in J[X]$ .

(5) By the equality :  $I_g(f) = fg - (fg)(0) - J_g(f)$ , if  $g \in J[X] \cap H^\infty$  and  $f \in X$  then  $I_g(f)$  belongs to  $X$  because  $gX \subset X$ .

**Proposition 3.** If  $X$  is a subalgebra of  $H$  which contains constants then  $M[X] = X$ ,  $J[X]$  is also a subalgebra of  $X$  and  $J[X] = I[X] \cap X$ .

Proof.  $M[X] = X$  is clear. If both  $g$  and  $h$  are in  $J[X]$ , then by Theorem 1 both  $fh$  and  $fg$  belongs to  $X$  for any  $f \in X$  because  $X$  is an algebra. Hence  $gh$  belongs to  $J[X]$  by the following equality :  $J_{gh}(f) = J_g(fh) + J_h(fg)$  for any  $f \in X$ . This implies that  $J[X]$  is a subalgebra of  $X$  by Theorem 1. From (2) of Proposition 2  $J[X] = I[X] \cap X$  follows.

#### §4. F-space

Let  $X$  be an F-space in  $H$  with an invariant metric  $d$ . For each  $a$  in  $D$ , put for  $f$  in  $X$

$$\mathcal{E}_a f = f(a) \quad \text{and} \quad \mathcal{D}_a f = f'(a).$$

In this section we assume that both  $\mathcal{E}_a$  and  $\mathcal{D}_a$  are bounded on  $X$ . Put

$$S(a) = \sup\{|\mathcal{E}_a(f)| ; f \in X, d(f, 0) \leq 1\}$$

and

$$s(a) = \sup\{|\mathcal{D}_a(f)| ; f \in X, d(f, 0) \leq 1\},$$

then  $S(a) < \infty$  and  $s(a) < \infty$  if  $a \in D$ . Suppose  $v$  is a nonnegative function on  $D$ . For a function  $f$  in  $H$  put

$$\|f\|_\omega = \sup_{z \in D} \omega(z)|f'(z)| + |f(0)|$$

and

$$\mathcal{B}_\omega = \{f \in H ; \|f\|_\omega < \infty\}.$$

If  $\omega$  is bounded,  $\mathcal{B}_\omega$  contains all holomorphic functions on the closed unit disc  $\bar{D}$ .

**Proposition 4.** *If  $X$  is an  $F$ -space such that  $S(a) < \infty$  and  $s(a) < \infty$  for each  $a \in D$ , then  $M[X]$ ,  $J[X]$  and  $I[X]$  belongs to  $B[X]$ .*

*Proof.* We will prove only that  $J[X] \subset B[X]$  because the other statements are similar. By the closed graph theorem, it is enough to prove that for  $\phi \in J[X]$  if  $f_n \rightarrow f$  in  $X$  and  $J_\phi(f_n) \rightarrow F$  then  $J_\phi(f) = F$ . Since  $S(a) < \infty$ ,  $f_n(a) \rightarrow f(a)$  ( $a \in D$ ). Since  $s(a) < \infty$ ,  $f_n(a)\phi'(a) \rightarrow F'(a)$  ( $a \in D$ ). Thus  $f(a)\phi'(a) = F'(a)$  and so  $J_\phi(f) = F$  because  $F(0) = 0$ .

**Theorem 3.** *Let  $X$  be an  $F$ -space in  $H$  with an invariant metric  $d$ . Suppose that  $\sup_{|a| \leq 1-\varepsilon} S(a) < \infty$  for any  $\varepsilon > 0$ . Then  $J[X] \subset \mathcal{B}_{\omega_0} \cap X$  and  $I[X] \subset H^\infty$ , where  $\omega_0 = 1/sS$ .*

*Proof.* If  $g \in J[X]$  then by Proposition 4, for any  $f \in X$   $d(J_g f, 0) \leq \|J_g\|d(f, 0)$ . Since  $J_g f \in X$ , by definition of  $\mathcal{D}_z$   $|\mathcal{D}_z(J_g f)| \leq s(z)d(J_g f, 0)$  ( $z \in D$ ). Hence

$$s(z)^{-1}|f(z)||g'(z)| \leq \|J_g\|d(f, 0) \quad (z \in D)$$

and so

$$s^{-1}(z)S^{-1}(z)|g'(z)| \leq \|J_g\| \quad (z \in D).$$

By Theorem 1  $g$  belongs to  $\mathcal{B}_{\omega_0} \cap X$  where  $\omega_0 = 1/sS$ . If  $g \in I[X]$  then by Proposition 4, for any  $f \in X$   $d(I_g f, 0) \leq \|I_g\|d(f, 0)$ . Since  $I_g f \in X$ , by definition of  $\mathcal{D}_z$   $|\mathcal{D}_z(I_g f)| \leq s(z)d(I_g f, 0)$  ( $z \in D$ ). Hence

$$s(z)^{-1}|f'(z)||g(z)| \leq \|I_g\|d(f, 0) \quad (z \in D)$$

and so

$$|g(z)| \leq \|I_g\| \quad (z \in D).$$

**Proposition 5.** *Let  $X$  be a subspace of  $H$  with constants which is of finite dimension. Then  $J[X] = I[X] = M[X] = \mathcal{C}$ .*

Proof. Suppose  $\{f_j\}_{j=1}^n$  is a basis in  $X$  with  $f_1 \equiv 1$ . We will show that  $J[X] = \mathcal{C}$ . If  $g \in J[X]$  then by Theorem 1  $g^\ell \in X$  for any  $\ell \geq 0$  and so there exist  $\{\alpha_j^\ell\}_1^n \subset \mathcal{C}$  such that  $g^\ell = \sum_{j=1}^n \alpha_j^\ell f_j$ . Hence there exist  $\{b_\ell\}_{\ell=0}^n \subset \mathcal{C}$  such that  $\sum_{\ell=0}^n b_\ell g^\ell = 0$ . This implies that  $g$  is just constant because  $g$  is analytic. Therefore  $J[X] = \mathcal{C}$ . We will show that  $I[X] = \mathcal{C}$ . Put  $X_1 = \{f' ; f \in X\}$ . If  $g \in I[X]$  then by Proposition 1  $g^\ell X_1 \subset X_1$  for any  $\ell \geq 1$  and so there exist  $\{\alpha_j^\ell\}_1^n \subset \mathcal{C}$  such that  $g^\ell f'_2 = \sum_{j=2}^n \alpha_j^\ell f'_j$ . By the same argument above  $g f'_2$  is constant. Similary it follows that  $\{g f'_j\}_2^n$  are constants and so  $g$  is constant because  $\{f'_j\}_\ell^n$  is a basis in  $X_1$ . Therefore  $I[X] = \mathcal{C}$ .

## §5. Weighted Bloch space

Let  $\omega$  be a positive bounded function on  $D$ . For a function  $f$  in  $H$  put

$$\|f\|_\omega = \sup_{z \in D} \omega(z) |f'(z)| + |f(0)|$$

and

$$\mathcal{B}_\omega = \{f \in H ; \|f\|_\omega < \infty\}.$$

Since  $\omega$  is bounded,  $\mathcal{B}_\omega$  contains all holomorphic functions on the closed unit disc  $\bar{D}$ .  $\mathcal{B}_\omega$  is called a weighted Bloch space. A weight  $\omega$  is called measurable when  $\omega(at)$  is measurable on  $[0,1]$  for each  $a$  in  $D$ . Put  $\varepsilon(r) = \inf\{\omega(z) ; |z| \leq r\}$  and  $r < 1$ .

**Lemma 2.** *If  $\varepsilon(r) > 0$  for  $0 \leq r < 1$  then  $\mathcal{B}_\omega$  is a Banach space with norm  $\|\cdot\|_\omega$ .*

Proof. Suppose that  $\{f_n\}$  is a Cauchy sequence in  $\mathcal{B}_\omega$ . For any  $\varepsilon > 0$ , there exist a positive integer  $n_0$  such that  $\|f_n - f_m\|_\omega < \varepsilon$  if  $n, m \geq n_0$ . Hence if  $r < 1$  and  $z \in D_r = \{z ; |z| < r\}$  then

$$|f'_n(z) - f'_m(z)| \leq \frac{\varepsilon}{\omega(z)} \leq \frac{\varepsilon}{\varepsilon(r)}.$$

By the normal family argument, there exists a function  $f' \in H(D_r)$  such that  $f'_n \rightarrow f'$  uniformly on  $D_r$ . Hence as  $n \rightarrow \infty$ ,

$$|f'(z) - f'_m(z)| \leq \frac{\varepsilon}{\omega(z)} \leq \frac{\varepsilon}{\varepsilon(r)} \quad (z \in D_r).$$

Since  $r$  is arbitrary,  $f$  belongs to  $H(D)$  and

$$\omega(z) |f'(z) - f'_m(z)| \leq \varepsilon \quad (z \in D)$$

if  $m \geq n_0$ . Since  $f_m(0) \rightarrow f(0)$ ,  $\|f - f_m\|_\omega \rightarrow 0$ .



**Theorem 4.** Let  $\omega$  be a measurable,  $\varepsilon(r) > 0$  for  $0 \leq r < 1$  and  $X = \mathcal{B}_\omega$ . Then

$$\mathcal{B}_{\omega S} = J[\mathcal{B}_\omega] \text{ and } I[\mathcal{B}_\omega] \subset H^\infty$$

where  $S(z) = \sup\{|f(z)|; f \in \mathcal{B}_\omega, \|f\|_\omega \leq 1\}$ . Moreover  $\|J_g\| = \|g\|_{\omega S}$  for each  $g$  in  $J[\mathcal{B}_\omega]$  with  $g(0) = 0$ .

Proof. By Theorem 1,  $J[\mathcal{B}_\omega] \subseteq \mathcal{B}_\omega$ . If  $g \in J[\mathcal{B}_\omega]$  then  $\|J_g f\|_\omega \leq \|J_g\| \|f\|_\omega$  ( $f \in \mathcal{B}_\omega$ ) and so  $\omega(z) |f(z)| \cdot |g'(z)| \leq \|J_g\| \cdot \|f\|_\omega$ . Hence

$$\omega(z) S(z) |g'(z)| \cdot \frac{|f(z)|}{S(z)} \leq \|J_g\| \cdot \|f\|_\omega$$

and so

$$\omega(z) S(z) |g'(z)| \leq \|J_g\|.$$

Therefore  $g$  belongs to  $\mathcal{B}_{\omega S}$  and  $\|g\|_{\omega S} \leq \|J_g\| + |g(0)|$ . Thus  $J[\mathcal{B}_\omega] \subseteq \mathcal{B}_{\omega S}$ . Note that  $\mathcal{B}_{\omega S} \subseteq \mathcal{B}_\omega$  because  $S(z) \geq 1$  ( $z \in D$ ). Conversely if  $g \in \mathcal{B}_{\omega S}$  then

$$\omega(z) |J_g(f)'(z)| = \omega(z) S(z) |g'(z)| \cdot \frac{|f(z)|}{S(z)} \leq \|g\|_{\omega S} \|f\|_\omega$$

and so  $g$  belongs to  $J[\mathcal{B}_\omega]$ . Thus

$$\|g\|_{\omega S} \leq \|J_g\| + |g(0)| \leq \|g\|_{\omega S} + |g(0)|.$$

In Theorem 4, if  $\omega$  is an absolute value of some analytic function and a radial function, R.Yoneda ([8],[9]) showed those under some special technical conditions on  $\omega$ .

## §6. Hardy space

For  $0 < p \leq \infty$ ,  $H^{p-}$  denotes  $\bigcap_{t < p} H^t$  and  $H^{\infty-}$  is written as  $H^\omega$ . For  $0 < p < \infty$ , when  $W = |h|^p$  for an outer function  $h$  in  $H^p$ ,  $H^p(W)$  denotes a weighted Hardy space that is, the closure of  $H^\infty$  in  $L^p(W d\theta/2\pi)$ .

Lemma 3 is well known (cf. [3, Theorem 5.12]). In Proposition 6 it is known ([1],[2]) that  $J[H^p] = \text{BMOA}$ . Hence our result is weaker than that. However if  $J[H^p] = \text{BMOA}$  then our result shows that  $I[H^p] = H^\infty$ .

**Lemma 3.** (1) For  $0 < p < 1$ , if  $f$  is a function in  $H^p$  then  $\int_0^z f(\zeta) d\zeta$  belongs to  $H^{p/1-p}$ . (2) If  $f$  is a function in  $H^1$  then  $\int_0^z f(\zeta) d\zeta$  belongs to  $H^\infty$ .

**Proposition 6.** For  $0 < p < \infty$ ,  $H_1^\infty \subset J[H^p] \subset H^\omega$  and  $zJ[H^p] \subset J[H^p]$ . Moreover  $M[H^p] = H^\infty$  and  $I[H^p] = J[H^p] \cap H^\infty$ .

Proof. By Lemma 3,  $z \in J[H^p]$  and so by (2) of Theorem 2  $H_1^\infty \subset J[H^p]$ . Theorem 1 implies that  $J[H^p] \subset H^\omega$ . By (5) of Theorem 2,  $J[H^p] \cap H^\infty \subset I[H^p]$ . Theorem 3 implies that  $I[H^p] \subset H^\infty$ . Hence  $I[H^p] \cap H^p \subset H^p$  and so (4) of Proposition 2  $I[H^p] \subset J[H^p]$ . It is well known that  $M[H^p] = H^\infty$ . By (2) of Proposition 2  $I[H^p] = J[H^p] \cap H^\infty$ . By (4) of Theorem 2, to prove that  $zJ[H^p] \subset J[H^p]$  it is sufficient to show that  $J_z(J[H^p]H^p) \subset H^p$ . Since  $J[H^p]H^p \subset H^{p-}$ , by Lemma 2,  $J_z(J[H^p]H^p) \subset H^p$ .

**Theorem 5.** For  $0 < p < \infty$ ,  $\bigcap_{t < 1} H^t \subset J[H^{p-}] \subset H^\omega$  and so  $\log(1-z)^{-1}$  belongs to  $J[H^{p-}]$ . Moreover  $zJ[H^{p-}] \subset J[H^{p-}]$ ,  $M[H^{p-}] = H^\infty$  and so  $J[H^{p-}] \cap H^\infty = I[H^{p-}] \cap H^\infty$ . When  $p = \infty$ ,  $J[H^\omega] = I[H^\omega] \cap H^\omega$  and  $J[H^\omega]$  is a subalgebra of  $H^\omega$  which contains  $H_1^\infty$ .

Proof. By Theorem 1,  $J[H^{p-}] \subset H^\omega$ . We will show that  $\bigcap_{t < 1} H_1^t \subset J[H^{p-}]$ . If  $g \in \bigcap_{t < 1} H_1^t$  then  $g'$  belongs to  $H^{1-}$ . If  $f \in H^{p-}$  then  $f$  belongs to  $H^t$  for any  $0 < t < p$ . If  $0 < s < t/(t+1)$  then  $t/s > 1$  and  $1/(t/s) + 1/(t/t-s) = 1$ . By the Hölder inequality,

$$\begin{aligned} & \int_0^{2\pi} |f(e^{i\theta})g'(e^{i\theta})|^s d\theta/2\pi \\ & \leq \left( \int_0^{2\pi} |f(e^{i\theta})|^t d\theta/2\pi \right)^{\frac{s}{t}} \left( \int_0^{2\pi} |g'(e^{i\theta})|^{\frac{st}{t-s}} d\theta/2\pi \right)^{\frac{s-t}{t}} \end{aligned}$$

and so  $fg'$  belongs to  $\bigcap_{s < t/(t+1)} H^s$ . By Lemma 3,  $\int_0^z f(\zeta)g'(\zeta)d\zeta$  belongs to  $H^{\frac{s}{1-s}}$ . As  $s \rightarrow t/(t+1)$ ,  $s/(1-s) \rightarrow t$  and so  $\int_0^z f(\zeta)g'(\zeta)d\zeta$  belongs to  $H^{t-}$ . As  $t \rightarrow p$ ,  $\int_0^z f(\zeta)g'(\zeta)d\zeta$  belongs to  $H^{p-}$ . Thus  $J_g[H^{p-}] \subseteq H^{p-}$  and so  $\bigcap_{t < 1} H_1^t \subset J[H^{p-}]$ . By (4) of Theorem 2, if we show that  $J_z(J[H^{p-}]H^{p-}) \subset H^{p-}$  then it follows that  $zJ[H^{p-}] \subset J[H^{p-}]$ . Since  $J[H^{p-}]H^{p-} \subset H^{p-}$ , by Lemma 4  $J_z(J[H^{p-}]H^{p-}) \subset H^{p-}$ . It is known that  $M[H^{p-}] = H^\infty$ . The last statement is a result of (2) of Proposition 2.

When  $p = \infty$ , by Proposition 3  $J[H^\omega] = I[H^\omega] \cap H^\omega$  and  $J[H^\omega]$  is a subalgebra of  $H^\omega$ . Theorem 2 implies  $J[H^\omega] \supset H_1^\infty$ .

**Theorem 6.** Let  $1 \leq p < \infty$  and  $W = |h|^p$  for some outer function  $h$  in  $H^p$ . Then  $\{g \in H; g(z) = \int_0^z h(\zeta)k(\zeta)d\zeta \text{ and } k \in H^{\frac{p}{p-1}}\} \subset J[H^p(W)] \subset H^\omega(W)$ .  $M[H^p(W)] = H^\infty$  and  $J[H^p(W)] \cap H^\infty = I[H^p(W)]$ . There exists a weight  $W$  such that  $z$  does not belong to  $J[H^p(W)]$ .

Proof. If  $g(z) = \int_0^z h(\zeta)k(\zeta)d\zeta$  and  $k \in H^{\frac{p}{p-1}}$ , then

$$h(z)\{J_g(h^{-1}f)\}(z) = h(z) \int_0^z f(\zeta)k(\zeta)d\zeta$$

and so  $hJ_g h^{-1}f$  belongs to  $H^p$  for all  $f \in H^p$  by Lemma 3 because  $fk \in H^1$ . Therefore  $\{g \in H; g(z) = \int_0^z h(\zeta)k(\zeta)d\zeta \text{ and } k \in H^{\frac{p}{p-1}}\} \subset J[H^p(W)]$ . By Theorem 1,  $J[H^p(W)] \subset$

$\bigcap_{p < \infty} H^p(W)$ . In fact, since  $g^n h \in H^p$  for any  $n \geq 1$ ,  $gh^{1/n} \in H^{np}$  and so  $g$  belongs to  $H^{np}(W)$ . If  $\phi \in M(H^p(W))$  then  $\phi(h^{-1}H^p) \subset h^{-1}H^p$  and so  $\phi H^p \subset H^p$ . Hence  $\phi \in M(H^p) = H^\infty$ . Therefore  $M(H^p(W)) = H^\infty$  and so by (2) of Proposition 2  $J[H^p(W)] \cap H^\infty = I[H^p(W)] \cap H^\infty$ . For  $a \in D$  it is easy to see that

$$\sup\{|f(a)| ; f \in H^p(W) \text{ and } \|f\|_{W,p} \leq 1\} = (1 - |a|^2)^{-1/p} |h(a)|^{-p} < \infty$$

and so by Theorem 3  $I[H^p(W)] \subset H^\infty$ . Thus  $J[H^p(W)] \cap H^\infty = I[H^p(W)]$ . If  $J_z(H^p(W)) \subseteq H^p(W)$  for any  $W$  with  $\log W \in L^1(d\theta/2\pi)$  then  $J_z(N_+) \subseteq N_+$ . For by a theorem of H.Helson [6]  $N_+$  is the union of all  $H^p(W)$  as  $W$  ranges over the set of weights with sumable  $\log W$ . Hence there exists a weight  $W$  such that  $z \notin J[H^p(W)]$ . Because it is known that  $J_z(N_+) \not\subseteq N_+$  [7].

## §7. Privalov space

We denote by  $N^p$ , for  $1 \leq p < \infty$ , the set of all functions  $f$  in  $H$  which satisfy

$$\sup_{0 < r < 1} \int_0^{2\pi} (\log^+ |f(re^{i\theta})|)^p d\theta < \infty.$$

When  $p = 1$ ,  $N^p$  is just  $N$ . Then

$$\bigcup_{p > 0} H^p \subset \bigcap_{p > 1} N^p \text{ and } \bigcup_{p > 1} N^p \subset N_+ \subset N^1 = N.$$

**Proposition 7.** *Let  $X = N_+$  or  $N$ . Then  $J[X]$  is a subalgebra of  $X$  and  $J[X] = I[X] \cap X$ . If  $(f')^{-1}$  is in  $H^\infty$  then  $f$  does not belong to  $J[X]$ .*

Proof. It is known that  $N_+$  and  $N$  are subalgebras of  $H$ . Hence the first part of this proposition is a result of Theorem 1 and Proposition 3. By [5] and [7],  $z \notin J[X]$  and so the second part follows from (1) of Theorem 2.

In Proposition 7, it is known ([5],[7]) that  $z \notin J[X]$ . Hence  $I[X] \not\ni z$ . We don't know whether  $J[X] = \mathcal{C}$  and  $I[X] = \mathcal{C}$ .

**Theorem 7.** *If  $1 < p < \infty$  then  $J[N^p]$  is a subalgebra of  $N^p$  which contains  $N_1^p$ , and  $J[N^p] = I[N^p] \cap N^p$ .*

Proof. Suppose  $1 < p < \infty$  and  $g \in N_1^p$ . If  $f \in N^p$  then

$$\begin{aligned} & \left\{ \int_0^{2\pi} (\log^+ |(J_g f)(re^{i\theta})|)^p d\theta / 2\pi \right\}^{1/p} \\ &= \left\{ \int_0^{2\pi} \left( \log^+ \left| \int_0^r f(te^{i\theta}) g'(te^{i\theta}) dt \right| \right)^p d\theta / 2\pi \right\}^{1/p} \end{aligned}$$

$$\begin{aligned}
&\leq \left\{ \int_0^{2\pi} \left( \log^+ \int_0^{1-} |f(te^{i\theta})g'(te^{i\theta})| dt \right)^p d\theta/2\pi \right\}^{1/p} \\
&\leq \left\{ \int_0^{2\pi} \left( \log^+ \sup_{0 \leq t < 1} |f(te^{i\theta})| + \log^+ \sup_{0 \leq t < 1} |g'(te^{i\theta})| \right)^p d\theta/2\pi \right\}^{1/p} \\
&\leq \left\{ \int_0^{2\pi} \left( \log^+ \sup_{0 \leq t < 1} |f(te^{i\theta})| \right)^p d\theta/2\pi \right\}^{1/p} + \left\{ \int_0^{2\pi} \left( \log^+ \sup_{0 \leq t < 1} |g'(te^{i\theta})| \right)^p d\theta/2\pi \right\}^{1/p}.
\end{aligned}$$

Put  $u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} P_r(t-\theta) \log^+ |f(e^{it})| dt$ , then  $u(r, \theta) \geq \log^+ |f(re^{i\theta})|$ . Since  $\log^+ |f(e^{it})| \in L^p$ , by a theorem of Hardy and Littlewood (cf. [3, Proposition 1.8]),  $\sup_{0 \leq r < 1} u(r, \theta)$  belongs to  $L^p$  and so  $\log^+ \sup_{0 \leq r < 1} |f(re^{i\theta})|$  belongs to  $L^p$ . Similarly we can prove that  $\log^+ \sup_{0 \leq r < 1} |g'(re^{i\theta})|$  belongs to  $L^p$ . Thus  $J_g f$  belongs to  $N^p$ . Hence  $N_1^p \subset J[N^p]$ . It is known that  $N^p$  is a subalgebra of  $H$ . Hence, by Proposition 3  $J[N^p]$  is a subalgebra of  $N^p$  and  $J[N^p] = I[N^p] \cap N^p$ .

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