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<tr>
<td>Author(s)</td>
<td>Nakazi, Takahiko</td>
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<tr>
<td>Citation</td>
<td>Hokkaido University Preprint Series in Mathematics, 789, 1-12</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2006</td>
</tr>
<tr>
<td>DOI</td>
<td>10.14943/83939</td>
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<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/69597">http://hdl.handle.net/2115/69597</a></td>
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<td>File Information</td>
<td>pre789.pdf</td>
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Integral Operators on a Subspace of Holomorphic Functions on the Disc

by

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2000 Mathematics Subject Classification : Primary 30 D 45, 30 D 55 ; Secondary 47 B 38

Key words and phrases : Integration operator, Nevanlinna type space, Bloch space, open unit disc

* This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education
Abstract. Let $H(D)$ be an algebra of all holomorphic functions on the open unit disc $D$ and $X$ a subspace of $H(D)$. When $g$ is a function in $H(D)$, put

$$J_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta$$

and

$$I_g(f)(z) = \int_0^z f'(\zeta)g(\zeta)d\zeta \quad (z \in D)$$

for $f$ in $X$. In this paper, we study $J[X] = \{g \in H(D) : J_g(f) \in X \text{ for all } f \in X\}$ and $I[X] = \{g \in H(D) : I_g(f) \in X \text{ for all } f \in X\}$. We apply the results to concrete spaces. For example, we study $J[X]$ and $I[X]$ when $X$ is a weighted Bloch space, a Hardy space or a Privalov space.

§1. Introduction

Let $D$ denote the open unit disc in the complex plane $\mathbb{C}$ and $H = H(D)$ the set of all holomorphic functions on $D$. For a given $g$ in $H$, define three operators :

$$(M_g f)(z) = g(z)f(z) \quad (f \in H, \ z \in D)$$

$$(J_g f)(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta \quad (f \in H, \ z \in D)$$

and

$$(I_g f)(z) = \int_0^z f'(\zeta)g(\zeta)d\zeta \quad (f \in H, \ z \in D).$$

Then $(J_g f)(z) + (I_g f)(z) = (M_g f)(z) - g(0)f(0)$. If $g(z) = z$ then $J_g$ is the Voltera integral operator and if $g(z) = \log 1/(1-z)$ then $J_g$ is the Cesàro operator.

In this paper we assume that $X$ is a subspace of $H$ which contains constants. $X_1$ denotes the set $\{f \in H ; f' \in X\}$. For each subspace $X$ put

$$M[X] = \{g \in H ; M_g(X) \subseteq X\},$$

$$J[X] = \{g \in H ; J_g(X) \subseteq X\}$$

and

$$I[X] = \{g \in H ; I_g(X) \subseteq X\}.$$  

We define that $J^{n+1}[X] = J[J^n[X]]$ and $I^{n+1}[X] = I[I^n[X]]$ for $n \geq 1$ where $J^1[X] = J[X]$ and $I^1[X] = I[X]$. For $X$ and $Y$ which are subspaces of $H$, $XY$ denotes a subspace of $H$ which is generated by a product of a function in $X$ and one in $Y$. Let $Y^n$ be a subspace of $H$ which is generated by finite $n$ products of functions in a subspace $Y$ of $H$. For a subspace $X$ of $H$, $B(X)$ denotes the set of all bounded linear operators on $X$.

Now we give a lot of examples of $X$. For $0 < p \leq \infty$, $H^p$ is the usual Hardy space on $D$, $N$ is the Nevalinna class and $N_+$ is the Smirnov class on $D$. These are $\mathcal{F}$-spaces, and $N$ and $N_+$ are algebras. It is known that $J[H^p] = \text{BMOA}$ (see [2], [1]), $z \notin J[N]$ [5] and $z \notin J[N_+]$ [7]. The Bloch space $\mathcal{B}$ is defined to be a Banach space in $H$ with the norm

$$\|f\| = \sup_{z \in D}(1-|z|^2)|f'(z)| + |f(0)|.$$

In Section 2, we assume only that $X$ is a subspace of $H$. Theorem 1 implies that $J[X]^n \subset X$ for any $n \geq 1$. In Section 3, we study $J[X]$ and $I[X]$ when $X$ is an invariant subspace of $H$ or a subalgebra of $H$. Theorem 2 implies that if $H^\infty X \subset X$ and $J[X]$ contains $z$ then $J[X] \supset H^\infty_1$. In Section 4, assuming that $X$ is a F-space we show that $J[X]$ is contained in some weighted Bloch space and $I[X] \subset H^\infty$. In Section 5, we define a weighted Bloch space $B_\omega$ and we describe $J[B_\omega]$.

In Section 6, we study $J[\bigcap_{t<p} H^t]$ and $I[\bigcap_{t<p} H^t]$. In Section 7, we show that $J[N^p]$ is a subalgebra of $N^p$ which contains $N^p_1$, where $N^p_1$ is a Privalov space.

§2. Subspace

In this section, we study $M[X]$, $J[X]$ and $I[X]$ assuming only that $X$ is a subspace of $H$.

**Lemma 1.** Let $X$ be a subspace of $H$ and $f, g$ in $H$.

1. $I_g I_f = I_{gf} = I_f I_g$ on $X$
2. $I_g J_f = J_f M_g$ on $X$

**Proof.** (1) For $k \in X$,

$$((I_g I_f)k)(z) = \int_0^z (I_f k)'(\zeta)g(\zeta)d\zeta = \int_0^z k'(\zeta)f(\zeta)g(\zeta)d\zeta = (I_{gf}k)(z)$$

(2) For $k \in X$,

$$((I_g J_f)k)(z) = \int_0^z (J_f k)'(\zeta)g(\zeta)d\zeta = \int_0^z k(\zeta)f'(\zeta)g(\zeta)d\zeta = (J_f M_g)k)(z) = (J_{fg}k)(z).$$

**Theorem 1.** Let $X$ be a subspace of $H$ with constants. Then $J[X]$ is a subspace of $X$ with constants and $J[X]^n \subset X$.

**Proof.** If $g \in J[X]$ then $J_g(1) = g - g(0) \in X$ and so $g \in X$ because $1 \in X$. Hence $J[X]$ is a subspace of $X$ with constants.

Assuming $J[X]^n \subset X$, we will show that $J[X]^{n+1} \subset X$. Suppose that $g \in J[X]$ and $\{g_j\}_{j=1}^n \subset J[X]$. In order to prove that $g \prod_{j=1}^n g_j$ belongs to $X$, we will use the following
equalities.

\[
\int_0^z g(\zeta) \left( \prod_{j=1}^n g_j \right) (\zeta)d\zeta = g(z) \left( \prod_{j=1}^n g_j \right) (z) - g(0) \left( \prod_{j=1}^n g_j \right) (0) - \int_0^z g'(\zeta) \left( \prod_{j=1}^n g_j \right) (\zeta)d\zeta
\]

and

\[
\int_0^z g(\zeta) \left( \prod_{j=1}^n g_j \right) (\zeta)d\zeta = \sum_{\ell=1}^n \int_0^z (g(\zeta) \prod_{j \neq \ell} g_j(\zeta)) g'_\ell(\zeta)d\zeta.
\]

By hypothesis on induction, \( \prod_{j=1}^n g_j \in X \) and so \( \int_0^z g'(\zeta) \left( \prod_{j=1}^n g_j \right) (\zeta)d\zeta \in X \) because \( g \in J[X] \). By hypothesis on induction, for \( \ell = 1, \ldots, n \) \( g\prod_{j \neq \ell} g_j \in X \) and so \( \int_0^z (g(\zeta) \prod_{j \neq \ell} g_j(\zeta)) g'_\ell(\zeta)d\zeta \in X \) because \( g_\ell \in J[X] \). By the above two equalities, \( g\prod_{j=1}^n g_j \) belongs to \( X \). This implies that \( J[X]^{n+1} \subseteq X \).

**Proposition 1.** Let \( X \) be a subspace of \( H \) with constants. Then \( I[X] \) is a subalgebra of \( H \).

Proof. If \( k \in I[X] \) and \( g \in I[X] \) then it is easy to see that \( I_k I_g = I_{kg} \) (see Proposition 3). Hence \( I_k I_g(X) = I_k(I_g(X)) \subseteq I_k(X) \subseteq X \) and so \( kg \) belongs to \( I[X] \). It is clear that \( I[X] \) is a subspace of \( H \).

**Proposition 2.** Suppose \( X \) is a subspace of \( H \) with constants.

(1) \( M[X] \) is an algebra in \( X \).

(2) \( J[X] \cap M[X] = I[X] \cap M[X] \).

(3) \( J[X] \cap I[X] \subseteq M[X] \).

(4) \( J[X] \subseteq M[X] \) if and only if \( J[X] \subseteq I[X] \). Similarly \( I[X] \subseteq M[X] \) if and only if \( I[X] \subseteq J[X] \).

Proof. (1) is clear. (2) and (3) follow from the equality : \( J_g f + I_g f = M_g f - g(0) f(0) \). (4) If \( J[X] \subseteq M[X] \) then by (2) \( J[X] \subseteq I[X] \). Conversely if \( J[X] \subseteq I[X] \) then by (3) \( J[X] \subseteq M[X] \).

§3. Invariant subspace and subalgebra

In this section, we study \( J[X] \) and \( I[X] \) when \( X \) is an invariant subspace or a subalgebra of \( H \).
Theorem 2. Suppose that $X$ is a subspace of $H$ with constants and $kX \subset X$ for any $k$ in $H^\infty$.

(1) If $g_0$ is an arbitrary function in $J[X]$, then $J[X]$ contains $\{g \in H ; |g'(z)| \leq |g_0'(z)| (z \in D)\}$.

(2) If $J[X]$ contains $z$ then it contains $H^\infty_1$.

(3) Suppose $J[X]$ contains $z$. If $\{g_n\}$ is in $J[X]$ and $g'_n \to g'$ uniformly on $D$ then $g$ belongs to $J[X]$.

(4) $zJ[X] \subset J[X]$ if and only if $J_z(J[X]X) \subset X$.

(5) $J[X] \cap H^\infty \subset I[X]$ and hence $I[X]$ contains $H^\infty_1$ if $z \in J[X]$.

Proof. (1) If $g \in H$ and $|g'(\zeta)| \leq |g_0'(\zeta)| (\zeta \in D)$, then $g'(g_0^{-1}) \in H^\infty$ and so $fg'(g_0^{-1}) \in X$ for any $f \in X$. Hence for any $f \in X$ 

$$\int_0^z f(\zeta)g'(\zeta)d\zeta = \int_0^z f(\zeta)g'(\zeta)g_0'(\zeta)^{-1}g_0'(\zeta)d\zeta$$

belongs to $X$ because $f(g'(g_0^{-1}) \in X$ and $g_0 \in J[X]$. This implies that $g$ belongs to $J[X]$.

(2) Since $z \in J[X]$, by (1) and the definition of $H^\infty_1$, $H^\infty_1$ is contained in $J[X]$.

(3) If $g'_n \to g'$ uniformly on $D$, then $(g - g_n)' \in H^\infty$. Hence $f(g - g_n)' \in X$ for any $f \in X$. Therefore $g$ belongs to $J[X]$ because $z \in J[X]$ and 

$$\int_0^z f(\zeta)g'(\zeta)d\zeta = \int_0^z f(\zeta)(g(\zeta) - g_n(\zeta))d\zeta + \int_0^z f(\zeta)g_n(\zeta)d\zeta.$$ 

(4) follows trivially from the following equality:

$$\int_0^z f(\zeta)(\zeta g(\zeta))'d\zeta = \int_0^z f(\zeta)g(\zeta)d\zeta + \int_0^z f(\zeta)\zeta g'(\zeta)d\zeta$$

for $f \in X$ and $g \in J[X]$.

(5) By the equality : $I_g(f) = fg - (fg)(0) - J_g(f)$, if $g \in J[X] \cap H^\infty$ and $f \in X$ then $I_g(f)$ belongs to $X$ because $gX \subset X$.

Proposition 3. If $X$ is a subalgebra of $H$ which contains constants then $M[X] = X$, $J[X]$ is also a subalgebra of $X$ and $J[X] = I[X] \cap X$.

Proof. $M[X] = X$ is clear. If both $g$ and $h$ are in $J[X]$, then by Theorem 1 both $fh$ and $fg$ belongs to $X$ for any $f \in X$ because $X$ is an algebra. Hence $gh$ belongs to $J[X]$ by the following equality : $J_{gh}(f) = J_g(fh) + J_h(fg)$ for any $f \in X$. This implies that $J[X]$ is a subalgebra of $X$ by Theorem 1. From (2) of Proposition 2 $J[X] = I[X] \cap X$ follows.

§4. F-space

Let $X$ be an F-space in $H$ with an invariant metric $d$. For each a in $D$, put for $f$ in $X$

$$E_a f = f(a) \text{ and } D_a f = f'(a).$$
In this section we assume that both $\mathcal{E}_a$ and $\mathcal{D}_a$ are bounded on $X$. Put

$$S(a) = \sup\{|\mathcal{E}_a(f)| ; \ f \in X, \ d(f,0) \leq 1\}$$

and

$$s(a) = \sup\{|\mathcal{D}_a(f)| ; \ f \in X, \ d(f,0) \leq 1\},$$

then $S(a) < \infty$ and $s(a) < \infty$ if $a \in D$. Suppose $v$ is a nonnegative function on $D$. For a function $f$ in $H$ put

$$\|f\|_\omega = \sup_{z \in D} \omega(z)|f'(z)| + |f(0)|$$

and

$$\mathcal{B}_\omega = \{f \in H ; \ \|f\|_\omega < \infty\}.$$  

If $\omega$ is bounded, $\mathcal{B}_\omega$ contains all holomorphic functions on the closed unit disc $\bar{D}$.

**Proposition 4.** If $X$ is an $F$-space such that $S(a) < \infty$ and $s(a) < \infty$ for each $a \in D$, then $M[X], J[X]$ and $I[X]$ belongs to $B[X]$.

Proof. We will prove only that $J[X] \subset B[X]$ because the other statements are similar. By the closed graph theorem, it is enough to prove that for $a \in \mathcal{B}_\omega \cap X$ and $I[f_n] \to f$ in $X$ and $J[a] \to F$. Since $S(a) < \infty$, $f_n(a) \to f(a)$ ($a \in D$). Since $s(a) < \infty$, $f_n(a) \to F'(a)$ ($a \in D$). Thus $f(a) \to F'(a)$ and so $J[a] = F$ because $F(0) = 0$.

**Theorem 3.** Let $X$ be an $F$-space in $H$ with an invariant metric $d$. Suppose that

$$\sup_{|a| \leq 1-\epsilon} S(a) < \infty$$

for any $\epsilon > 0$. Then $J[X] \subset \mathcal{B}_\omega \cap X$ and $I[X] \subset H^\omega$, where $\omega_0 = 1/sS$.

Proof. If $g \in J[X]$ then by Proposition 4, for any $f \in X$ $d(J_g f,0) \leq \|J_g\|d(f,0)$. Since $J_g f \in X$, by definition of $\mathcal{D}_z$ $|D_z(J_g f)| \leq s(z)d(J_g f,0)$ ($z \in D$). Hence

$$s(z)^{-1}|f(z)||g'(z)| \leq \|J_g\|d(f,0) \quad (z \in D)$$

and so

$$s^{-1}(z)S^{-1}(z)|g'(z)| \leq \|J_g\| \quad (z \in D).$$

By Theorem 1 $g$ belongs to $\mathcal{B}_\omega \cap X$ where $\omega_0 = 1/sS$. If $g \in I[X]$ then by Proposition 4, for any $f \in X$ $d(I_g f,0) \leq \|I_g\|d(f,0)$. Since $I_g f \in X$, by definition of $\mathcal{D}_z$ $|D_z(I_g f)| \leq s(z)d(I_g f,0)$ ($z \in D$). Hence

$$s(z)^{-1}|f'(z)||g(z)| \leq \|I_g\|d(f,0) \quad (z \in D)$$

and so

$$|g(z)| \leq \|I_g\| \quad (z \in D).$$

**Proposition 5.** Let $X$ be a subspace of $H$ with constants which is of finite dimension. Then $J[X] = I[X] = M[X] = \emptyset$. 

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Proof. Suppose \( \{f_j\}_{j=1}^n \) is a basis in \( X \) with \( f_1 \equiv 1 \). We will show that \( J[X] = \mathcal{C} \).

If \( g \in J[X] \) then by Theorem 1 \( g^\ell \in X \) for any \( \ell \geq 0 \) and so there exist \( \{\alpha^\ell_j\}^n_{j=1} \subset \mathcal{C} \) such that \( g^\ell = \sum_{j=1}^n \alpha^\ell_j f_j \). Hence there exist \( \{b_j\}^n_{j=0} \subset \mathcal{C} \) such that \( \sum_{\ell=0}^n b_\ell g^\ell = 0 \). This implies that \( g \) is just constant because \( g \) is analytic. Therefore \( J[X] = \mathcal{C} \). We will show that \( I[X] = \mathcal{C} \). Put \( X_1 = \{f' : f \in X\} \). If \( g \in I[X] \) then by Proposition 1 \( g^\ell X_1 \subset X_1 \) for any \( \ell \geq 1 \) and so there exist \( \{\alpha^\ell_j\}^n_{j=1} \subset \mathcal{C} \) such that \( g^\ell f_2 = \sum_{j=2}^n \alpha^\ell_j f_j \).

By the same argument above \( g f_2 \) is constant. Similary it follows that \( \{g f_2\}^2_1 \) are constants and so \( g \) is constant because \( \{f_2\}^n_1 \) is a basis in \( X_1 \). Therefore \( I[X] = \mathcal{C} \).

§5. Weighted Bloch space

Let \( \omega \) be a positive bounded function on \( D \). For a function \( f \) in \( H \) put

\[
\|f\|_\omega = \sup_{z \in D} \omega(z)|f'(z)| + |f(0)|
\]

and

\[
\mathcal{B}_\omega = \{ f \in H : \|f\|_\omega < \infty \}.
\]

Since \( \omega \) is bounded, \( \mathcal{B}_\omega \) contains all holomorphic functions on the closed unit disc \( \overline{D} \). \( \mathcal{B}_\omega \) is called a weighted Bloch space. A weight \( \omega \) is called measurable when \( \omega(at) \) is measurable on \( [0,1] \) for each \( a \) in \( D \). Put \( \varepsilon(r) = \inf\{\omega(z) : |z| \leq r\} \) and \( r < 1 \).

**Lemma 2.** If \( \varepsilon(r) > 0 \) for \( 0 \leq r < 1 \) then \( \mathcal{B}_\omega \) is a Banach space with norm \( \| \cdot \|_\omega \).

Proof. Suppose that \( \{f_n\} \) is a Cauchy sequence in \( \mathcal{B}_\omega \). For any \( \varepsilon > 0 \), there exist a positive integer \( n_0 \) such that \( \|f_n - f_m\|_\omega < \varepsilon \) if \( n, m \geq n_0 \). Hence if \( r < 1 \) and \( z \in D_r = \{z : |z| < r\} \) then

\[
|f_n'(z) - f_m'(z)| \leq \frac{\varepsilon}{\omega(z)} \leq \frac{\varepsilon}{\varepsilon(r)}.
\]

By the normal family argument, there exists a function \( f' \in H(D_r) \) such that \( f_n' \to f' \) uniformly on \( D_r \). Hence as \( n \to \infty \),

\[
|f'(z) - f_m'(z)| \leq \frac{\varepsilon}{\omega(z)} \leq \frac{\varepsilon}{\varepsilon(r)} \quad (z \in D_r).
\]

Since \( r \) is arbitrary, \( f \) belongs to \( H(D) \) and

\[
\omega(z)|f'(z) - f_m'(z)| \leq \varepsilon \quad (z \in D)
\]

if \( m \geq n_0 \). Since \( f_m(0) \to f(0), \|f - f_m\|_\omega \to 0 \).
Theorem 4. Let $\omega$ be a measurable, $\varepsilon(r) > 0$ for $0 \leq r < 1$ and $X = B_\omega$. Then

$$B_{\omega S} = J[\mathcal{B}_\omega] \text{ and } I[\mathcal{B}_\omega] \subset H^\infty$$

where $S(z) = \sup \{|f(z)| \mid f \in B_\omega, \|f\|_\omega \leq 1\}$. Moreover $\|J_g\| = \|g\|_\omega S$ for each $g$ in $J[\mathcal{B}_\omega]$ with $g(0) = 0$.

Proof. By Theorem 1, $J[\mathcal{B}_\omega] \subset B_\omega$. If $g \in J[\mathcal{B}_\omega]$ then $\|J_g f\|_\omega \leq \|J_g\| \|f\|_\omega$ ($f \in B_\omega$) and so $\omega(z) |f(z)| \cdot |g'(z)| \leq \|J_g\| \cdot \|f\|_\omega$. Hence

$$\omega(z)S(z) |g'(z)| \cdot \frac{|f(z)|}{S(z)} \leq \|J_g\| \cdot \|f\|_\omega$$

and so

$$\omega(z)S(z) |g'(z)| \leq \|J_g\|.$$

Therefore $g$ belongs to $B_{\omega S}$ and $\|g\|_{\omega S} \leq \|J_g\| + |g(0)|$. Thus $J[\mathcal{B}_\omega] \subset B_{\omega S}$. Note that $B_{\omega S} \subset B_\omega$ because $S(z) \geq 1$ ($z \in D$). Conversely if $g \in B_{\omega S}$ then

$$\omega(z)|J_g f'(z)| = \omega(z)S(z) |g'(z)| \cdot \frac{|f(z)|}{S(z)} \leq \|g\|_{\omega S} \|f\|_\omega$$

and so $g$ belongs to $J[\mathcal{B}_\omega]$. Thus

$$\|g\|_{\omega S} \leq \|J_g\| + |g(0)| \leq \|g\|_{\omega S} + |g(0)|.$$

In Theorem 4, if $\omega$ is an absolute value of some analytic function and a radial function, R.Yoneda ([8],[9]) showed those under some special technical conditions on $\omega$.

§6. Hardy space

For $0 < p \leq \infty$, $H^p$ denotes $\bigcap_{1 < p} H^1$ and $H^{\infty}$ is written as $H^\omega$. For $0 < p < \infty$, when $W = |h|^p$ for an outer function $h$ in $H^p$, $H^p(W)$ denotes a weighted Hardy space that is, the closure of $H^\omega$ in $L^p(Wd\theta/2\pi)$.

Lemma 3 is well known (cf. [3, Theorem 5.12]). In Proposition 6 it is known ([1],[2]) that $J[H^p] = \text{BMOA}$. Hence our result is weaker than that. However if $J[H^p] = \text{BMOA}$ then our result shows that $I[H^p] = H^\infty$.

Lemma 3. (1) For $0 < p < 1$, if $f$ is a function in $H^p$ then $\int_0^z f(\zeta)d\zeta$ belongs to $H^{1-p}$. (2) If $f$ is a function in $H^1$ then $\int_0^z f(\zeta)d\zeta$ belongs to $H^{\infty}$.

Proposition 6. For $0 < p < \infty$, $H_1^{\infty} \subset J[H^p] \subset H^\omega$ and $zJ[H^p] \subset J[H^p]$. Moreover $M[H^p] = H^\omega$ and $I[H^p] = J[H^p] \cap H^\omega$. 

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Proof. By Lemma 3, $z \in J[H^p]$ and so by (2) of Theorem 2 $H_1^\infty \subset J[H^p]$. Theorem 1 implies that $J[H^p] \subset H^\omega$. By (5) of Theorem 2, $J[H^p] \cap H^\infty \subset I[H^p]$. Theorem 3 implies that $I[H^p] \subset H^\infty$. Hence $I[H^p] \subset H^\infty$ and so (4) of Proposition 2 $I[H^p] \subset J[H^p]$. It is well known that $M[H^p] = H^\omega$. By (2) of Proposition 2 $I[H^p] = J[H^p] \cap H^\infty$. By (4) of Theorem 2, to prove that $zJ[H^p] \subset J[H^p]$ it is sufficient to show that $J_z(J[H^p]H^p) \subset H^p$. Since $J[H^p]H^p \subset H^p^-$, by Lemma 2, $J_z(J[H^p]H^p) \subset H^p$.

**Theorem 5.** For $0 < p < \infty$, $\bigcap_{t<1} H^t \subset J[H^p^-] \subset H^\omega$ and so log$(1-z)^{-1}$ belongs to $J[H^p^-]$. Moreover $zJ[H^p^-] \subset J[H^p^-]$, $M[H^p^-] = H^\omega$ and so $J[H^p^-] \cap H^\infty = I[H^p^-] \cap H^\infty$. When $p = \infty$, $J[H^\omega] = J[H^\omega] \cap H^\omega$ and $J[H^\omega]$ is a subalgebra of $H^\omega$ which contains $H_1^\infty$.

Proof. By Theorem 1, $J[H^p^-] \subset H^\omega$. We will show that $\bigcap_{t<1} H^t \subset J[H^p^-]$. If $g \in \bigcap_{t<1} \bigcap_{s<t/(t+1)} H^s$. By Lemma 3, $\int_0^{2\pi} f(e^{it}g)g'(e^{it}) \, d\theta/2\pi$ and so $f'g'$ belongs to $\bigcap_{s<t/(t+1)} H^s$. By Lemma 3, $\int_0^{2\pi} f(z)g'(z) \, d\zeta$ belongs to $H^\infty$. As $s \to t/(t+1)$, $s/(1-s) \to t$ and so $\int_0^{2\pi} f(z)g'(z) \, d\zeta$ belongs to $H^1$. As $s \to p$, $\int_0^{2\pi} f(z)g'(z) \, d\zeta$ belongs to $H^p^-$. Thus $J_z[H^p^-] \subset H^p^-$ and so $\bigcap_{t<1} H^t \subset J[H^p^-]$. By (4) of Theorem 2, if we show that $J_z(J[H^p^-]H^p^-) \subset H^p^-$ then it follows that $zJ[H^p^-] \subset J[H^p^-]$. Since $J[H^p^-]H^p^- \subset H^p^-$, by Lemma 4 $J_z(J[H^p^-]H^p^-) \subset H^p^-$. It is known that $M[H^p^-] = H^\omega$. The last statement is a result of (2) of Proposition 2.

When $p = \infty$, by Proposition 3 $J[H^\omega] = H^\omega \cap H^\omega$ and $J[H^\omega]$ is a subalgebra of $H^\omega$. Theorem 2 implies $J[H^\omega] \subset H_1^\infty$.

**Theorem 6.** Let $1 \leq p < \infty$ and $W = \{ h \mid h \, |^p \, \text{for some outer function } h \in H^p \}$. Then $\{ g \in H; g(z) = \int_0^{2\pi} h(z)k(z) \, d\zeta \}$ and $k \in H^\infty \subset J[H^p(W)] \subset H^\omega(W)$. $M[H^p(W)] = H^\omega$ and $J[H^p(W)] \cap H^\infty = I[H^p(W)]$. There exists a weight $W$ such that $z$ does not belong to $J[H^p(W)]$.

Proof. If $g(z) = \int_0^{2\pi} h(z)k(z) \, d\zeta$ and $k \in H^\infty$, then

$h(z) \{ J_z(h^{-1}f) \}(z) = h(z) \int_0^{2\pi} f(z)k(z) \, d\zeta$

and so $hJ_zh^{-1}f$ belongs to $H^p$ for all $f \in H^p$ by Lemma 3 because $fk \in H^1$. Therefore $\{ g \in H; g(z) = \int_0^{2\pi} h(z)k(z) \, d\zeta \}$ and $k \in H^\infty \subset J[H^p(W)]$. By Theorem 1, $J[H^p(W)] \subset H^\infty$. Therefore $\{ g \in H; g(z) = \int_0^{2\pi} h(z)k(z) \, d\zeta \} \subset J[H^p(W)]$.
∪ \mathcal{H}^p(W)$. In fact, since $g^n h \in \mathcal{H}^p$ for any $n \geq 1$, $gh^{1/n} \in \mathcal{H}^{np}$ and so $g$ belongs to $\mathcal{H}^{np}(W)$. If $\phi \in M(\mathcal{H}^p(W))$ then $\phi(h^{-1}H^p) \subset h^{-1}H^p$ and so $\phi H^p \subset H^p$. Hence $\phi \in M(\mathcal{H}^p) = \mathcal{H}^\infty$. Therefore $M(\mathcal{H}^p(W)) = \mathcal{H}^\infty$ and so by (2) of Proposition 2 $\mathcal{J}[\mathcal{H}^p(W)] \cap H^\infty = \mathcal{I}[\mathcal{H}^p(W)] \cap H^\infty$. For $a \in D$ it is easy to see that

$$\sup\{|f(a)| : f \in \mathcal{H}^p(W) \text{ and } ||f||_{W,p} \leq 1\} = (1 - |a|^2)^{-1/p} |h(a)|^{-p} < \infty$$

and so by Theorem 3 $I[\mathcal{H}^p(W)] \subset H^\infty$. Thus $J[\mathcal{H}^p(W)] \cap H^\infty = I[\mathcal{H}^p(W)]$. If $J_+(\mathcal{H}^p(W)) \subset \mathcal{H}^p(W)$ for any $W$ with $\log W \in L^1(d\theta/2\pi)$ then $J_+ (N_+) \subseteq N_+$. For by a theorem of H. Helson [6] $N_+$ is the union of all $\mathcal{H}^p(W)$ as $W$ ranges over the set of weights with sumable log $W$. Hence there exists a weight $W$ such that $z \notin J[\mathcal{H}^p(W)]$. Because it is known that $J_+(N_+) \not\subset N_+$ [7].

\section{§7. Privalov space}

We denote by $N^p$, for $1 \leq p < \infty$, the set of all functions $f$ in $H$ which satisfy

$$\sup_{0 < r < 1} \int_0^{2\pi} (\log^+ |f(re^{i\theta})|)^p d\theta < \infty.$$  

When $p = 1$, $N^p$ is just $N$. Then

$$\bigcup_{p > 0} \mathcal{H}^p \subset \bigcap_{p > 1} N^p \text{ and } \bigcup_{p > 1} N^p \subset N_+ \subset N_1 = N.$$  

\textbf{Proposition 7.} Let $X = N_+$ or $N$. Then $J[X]$ is a subalgebra of $X$ and $J[X] = I[X] \cap X$. If $(f)^{-1}$ is in $H^\infty$ then $f$ does not belong to $J[X]$.

Proof. It is known that $N_+$ and $N$ are subalgebras of $H$. Hence the first part of this proposition is a result of Theorem 1 and Proposition 3. By [5] and [7], $z \notin J[X]$ and so the second part follows from (1) of Theorem 2.

In Proposition 7, it is known ([5],[7]) that $z \notin J[X]$. Hence $I[X] \not= z$. We don’t know whether $J[X] = \mathcal{C}$ and $I[X] = \mathcal{C}$.

\textbf{Theorem 7.} If $1 < p < \infty$ then $J[N^p]$ is a subalgebra of $N^p$ which contains $N^p_1$, and $J[N^p] = I[N^p] \cap N^p$.

Proof. Suppose $1 < p < \infty$ and $g \in N^p_1$. If $f \in N^p$ then

$$\left\{ \int_0^{2\pi} (\log^+ |(J_g f)(re^{i\theta})|)^p d\theta / 2\pi \right\}^{1/p} = \left\{ \int_0^{2\pi} \left( \log^+ \left| \int_0^t f(te^{i\theta})g'(te^{i\theta}) \, dt \right|^p \right) d\theta / 2\pi \right\}^{1/p}$$
\[
\leq \left\{ \int_0^{2\pi} \left( \log^+ \int_0^1 |f(te^{i\theta})g'(te^{i\theta})|dt \right)^p \, d\theta / 2\pi \right\}^{1/p}
\]

\[
\leq \left\{ \int_0^{2\pi} \left( \log^+ \sup_{0 \leq t < 1} |f(te^{i\theta})| + \log^+ \sup_{0 \leq t < 1} |g'(te^{i\theta})| \right)^p \, d\theta / 2\pi \right\}^{1/p}
\]

\[
\leq \left\{ \int_0^{2\pi} \left( \log^+ \sup_{0 \leq t < 1} |f(te^{i\theta})| \right)^p \, d\theta / 2\pi \right\}^{1/p} + \left\{ \int_0^{2\pi} \left( \log^+ \sup_{0 \leq t < 1} |g'(te^{i\theta})| \right)^p \, d\theta / 2\pi \right\}^{1/p}.
\]

Put \( u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} P_r(t-\theta) \log^+ |f(e^{it})|dt \), then \( u(r, \theta) \geq \log^+ |f(re^{i\theta})| \). Since \( \log^+ |f(e^{it})| \in L^p \), by a theorem of Hardy and Littlewood (cf. [3, Proposition 1.8]), \( \sup_{0 \leq r < 1} u(r, \theta) \) belongs to \( L^p \) and so \( \log^+ \sup_{0 \leq r < 1} |f(re^{i\theta})| \) belongs to \( L^p \). Similarly we can prove that \( \log^+ \sup_{0 \leq r < 1} |g'(re^{i\theta})| \) belongs to \( L^p \). Thus \( J_g \) belongs to \( N^p \). Hence \( N_1^p \subset J[N^p] \). It is known that \( N^p \) is a subalgebra of \( H \). Hence, by Proposition 3 \( J[N^p] \) is a subalgebra of \( N^p \) and \( J[N^p] = I[N^p] \cap N^p \).
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