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The stability of the family of B_2-type arrangements

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Abstract

We introduce a B_2-type arrangements as a generalization of the classical Coxeter arrangement of type B_2, and consider the stability and the freeness of it. We show their (semi)stability is determined by the combinatorics. Moreover, we give a partial answer to the 4-shift problem, which is the conjecture on the combinatorics and geometry induced from B_2-type arrangements.

0 Introduction

A *hyperplane arrangement* (or simply an *arrangement*) $\mathcal{A}$ is a finite collection of affine hyperplanes in a fixed vector space V. There are a lot of studies of arrangements and recently the associated reflexive sheaf $\widetilde{D_0(\mathcal{A})}$ on $\mathbf{P}(V)$ is intensively studied. Roughly speaking, $\widetilde{D_0(\mathcal{A})}$ has, as local sections, vector fields tangent to hyperplanes in the arrangement $\mathcal{A}$. It is known that when $\dim V = 3$, the reflexive sheaf $\widetilde{D_0(\mathcal{A})}$ is a rank two vector bundle on $\mathbf{P}^2_{\mathbb{K}}$, and we consider the algebraic and geometric structure of this vector bundle in this article. Especially, we are interested in the stability and splitting of $\widetilde{D_0(\mathcal{A})}$, where the stability means the slope stability (see Definition 1.10) and splitting means the bundle is a direct sum of line bundles. Both concepts are important in algebraic geometry, hence to consider these properties of $\widetilde{D_0(\mathcal{A})}$ is useful for the study of arrangement problems by using algebraic geometry.

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Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and V a three-dimensional vector space over $\mathbb{K}$. Moreover, let us fix a basis $\{X, Y, Z\}$ for the dual vector space V^*. In this article, we are interested in the following family of arrangements.

Definition 0.1
A family of arrangements $\{\mathcal{A}(k)\}_{k\in\mathbb{Z}_{>0}}$ in V is called a family of B_2-type arrangements if $\mathcal{A}(k)$ is defined as follows:

$$\begin{aligned}
X &= (-k+1)Z, \ldots, (k+c-1)Z \ (c \geq 0), \\
Y &= (-k+1)Z, \ldots, (k+f-1)Z \ (f \geq 0), \\
Y+X &= (-k+a)Z, \ldots, (k+a+b-1)Z \ (b \geq -1), \\
Y-X &= (-k+d)Z, \ldots, (k+d+e-1)Z \ (e \geq -1), \\
Z &= 0,
\end{aligned}$$

where a, b, c, d, e, f are all integers. Moreover, we call each arrangement $\mathcal{A}(k)$ in a family of B_2-type arrangements $\{\mathcal{A}(k)\}$ a B_2-type arrangement.

For example, the pictures of $\{\mathcal{A}(k)\}|_{Z=1}$ $(k = 1, 2, 3)$ when $(a, b, c, d, e, f) = (1, -1, 0, 1, -1, 0)$ are as follows:

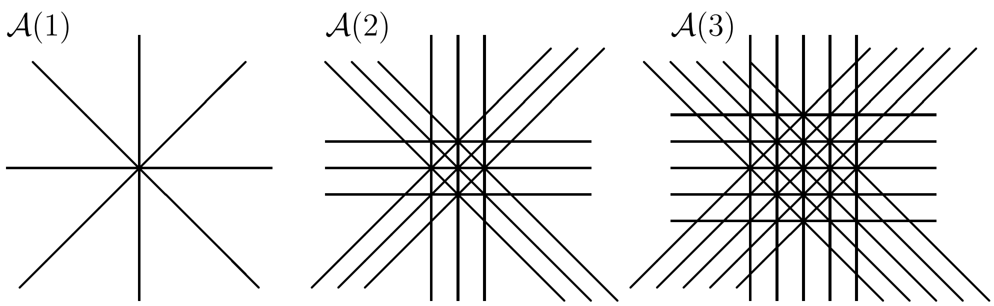

Figure 1: Example of B_2-type arrangements

Since $\mathcal{A}(1)$ in Figure 1 is the classical Coxeter arrangement of type B_2, we call the family in Definition 0.1 that of B_2-type arrangements (by the same manner, a family of A_2-type arrangements is defined in [A]). Note the B_2-type arrangement is a special case of the deformation of the Coxeter arrangement of type B_2 defined in [PS]. In this article we classify families of B_2-type arrangements from the view point of stability and freeness. Here we say the family of arrangements $\{\mathcal{A}(k)\}_{k\in\mathbb{Z}_{>0}}$ is *free* (resp:*stable, semistable*) if there exists an integer k_0 such that for all $k \geq k_0$ the rank two vector bundle $\widetilde{D_0(\mathcal{A}(k))}$ is a direct sum of line bundles (resp:stable, semistable). There are some motivations to study the stability of these arrangements. The first motivation is the relation between the stability and combinatorics of arrangements. The stability of normal crossing arrangements is considered in [DK], and that of A_2-type in [A], in which the stability is related to the combinatorics. These results pose a problem whether the stability of arrangements is determined by the combinatorial structure of arrangements. This is similar to the Terao conjecture, which asserts the freeness of arrangements depends only on the combinatorics of arrangements. Since splitting vector bundles are never stable and vice versa, this problem is worth being considered to study the Terao conjecture. The author gave a complete classification of the family of A_2-type arrangements from the viewpoint of the stability and freeness in [A]. Hence it is natural to consider B_2-type arrangements as a further problem. The second motivation is the shift problem, posed by Yoshinaga. Roughly speaking, it asserts the interesting combinatorial behavior of a family of B_2-type arrangements $\{\mathcal{A}(k)\}$ is governed by the geometric property of the sheaf of logarithmic vector fields $\{D_0(\widetilde{\mathcal{A}(k)})\}$, i.e., it asserts the existences of isomorphisms

$$D_0(\widetilde{\mathcal{A}(k+1)}) \simeq D_0(\widetilde{\mathcal{A}(k)}) \otimes \mathcal{O}(-4) \ (k \gg 0).$$

Since the shift and the Coxeter number of the root system of type B_2 are both 4, we call this conjecture the 4-*shift problem*. By the same way, we have formulated the 3-shift problem for A_2-type in [A] and gave a partial answer to it. Since the stability implies the simplicity and sometimes determines the structure of vector bundles (through Beilinson's spectral sequence), it is important to consider the stability of arrangements of this type. For details of 4-shift problem, see Remark 2.3.

Our main theorem is the following classification of the stability, semistability and the freeness of B_2-type arrangements.

Theorem 0.2
Let $\{\mathcal{A}(k)\}$ be the family of B_2-type arrangements such that $\mathcal{A}(1)$ is defined by

$$\begin{aligned} X &= 0, Z, \ldots, cZ \ (c \geq 0), \\ Y &= 0, Z, \ldots, fZ \ (f \geq 0), \\ Y+X &= (a-1)Z, \ldots, (a+b)Z \ (b \geq -1), \\ Y-X &= (d-1)Z, \ldots, (d+e)Z \ (e \geq -1), \\ Z &= 0. \end{aligned}$$

Let us put

$$B_1 := 2(a + \frac{1}{2}b - \frac{1}{2}c - \frac{1}{2}f - \frac{1}{2}),$$

$$B_2 := 2(d + \frac{1}{2}e + \frac{1}{2}c - \frac{1}{2}f - \frac{1}{2}),$$

$$B_3 := 2(\frac{1}{2}c - \frac{1}{2}f),$$

and put

$$M := B_1^2 + B_2^2 + B_3^2.$$

(a) *The family $\{\mathcal{A}(k)\}$ is free if and only if*

$$M = 0, 1, 2$$

or

$$a + d \text{ is even and } M = 3, \text{ or } 4.$$

Moreover, let $(1, d_1^k, d_2^k)$ be the exponents of the arrangement $\mathcal{A}(k)$ and $d^k := |d_1^k - d_2^k|$. Then d^k is two if and only if $M = 0$, $a + d$ is even, b and e are both odd numbers and $\sum_{i=1}^{3} B_i$ is even. Otherwise d^k is zero or one.

(b) *The normalization of the vector bundle $\widetilde{D_0(\mathcal{A}(k))}$ is isomorphic to $T_{\mathbf{P}^2}(-2)$ if and only if*

$$M = 3 \text{ and } a + d \text{ is odd}$$

or

$$M = 5.$$

(c') *The family $\{\mathcal{A}(k)\}$ is not free but semistable if and only if*

$$M = 6,$$

or

$$M = 4 \text{ and } a + d \text{ is odd.}$$

or

$$M = 8, \ 12 \text{ and } a + d \text{ is even.}$$

(c) *The family $\{\mathcal{A}(k)\}$ is stable if and only if*

$$M = 9, \ 10, \ 11,$$

or

$$M = 8, \ 12 \text{ and } a + d \text{ is odd.}$$

or

$$M \geq 13.$$

We can see that the conditions (c') and (c) have different combinatorics, so this theorem also implies the stability and semistability of B_2-type arrangements are determined by the combinatorics. The statement (a) in the main theorem is proved through the addition-deletion theorem, so they are inductively free. Since Terao conjecture is true for inductively free arrangements, so is for B_2-type arrangements. As an application of the main theorem, we give a partial answer to the 4-shift problem.

The organization of this article is as follows. In Section 1, we review some definitions and results on the arrangement theory and stability of vector bundles. In Section 2, we prove the main theorem through several steps. In that process, we give a useful criterion to construct a stable arrangements (Theorem 2.5) from inductively free arrangements. One of the key roles in the proof is played by freeness criterions Proposition 2.3 and 2.4. Since the proofs of these propositions by using the addition-deletion theorem need a lot of calculations, we show them in Section 3. Moreover, as a corollary, we give exponents of some multiarrangements of type B_2 in Corollary 3.3.

Notation. $\mathbb{Z}$ denotes the ring of integers and $\mathbb{K}$ denotes an algebraically closed field of characteristic zero. In this article, let the variables a, b, c, d, e, f denote integers. For a vector space V over $\mathbb{K}$, V^* denotes the dual vector space of V. Let $S = \oplus_{d\in\mathbb{Z}} S_d$ be a commutative graded ring with a unit, where S_d is a homogeneous part of S with degree d. We assume that S is noetherian, $S_d = 0$ for all $d < 0$, $S_0 = \mathbb{K}$ and S is generated by S_1 as a $\mathbb{K}$-algebra for every graded ring S. $\mathrm{Der}_{\mathbb{K}}(S)$ is the S-module of $\mathbb{K}$-linear derivations of S. For any integer $d \in \mathbb{Z}$ and a graded S-module M which is finitely generated over S, M_d is a homogeneous part of M with degree d. We assume that $M_d = 0$ for all $d < 0$. $\widetilde{M}$ denotes the sheafification of M, so $\widetilde{M}$ is a coherent sheaf on $\mathbf{Proj}(S)$. For a vector bundle E on the projective space $\mathbf{P}^n_{\mathbb{K}}$, $c_i(E)$ denotes the i-th Chern class of E and we put the Chern polynomial $c_t(E)$ of E as

$$c_t(E) := \sum_{i=0}^{n} c_i(E)t^i.$$

For a finite set A, its cardinality is denoted by $|A|$.

1 Preliminaries

We introduce and review some results and definitions which will be used in the rest of this article. First we recall those of hyperplane arrangements, for which we refer the reader to [OT]. Let us fix an l-dimensional $\mathbb{K}$-vector space

$V \simeq \mathbb{K}^l$. A *hyperplane arrangement* (or a *simple arrangement*) $\mathcal{A}$ is a finite collection of affine hyperplanes in V. We often say an "arrangement" instead of a " hyperplane arrangement", and call an arrangement in an l-dimensional vector space an "l-arrangement". We say an arrangement $\mathcal{A}$ is *central* if each hyperplane in $\mathcal{A}$ is a vector subspace of V. In this article, we assume all arrangements are non-empty and "central" unless otherwise specified. Note we can regard a central l-arrangement as the arrangement in $\mathbf{P}^{l-1} \simeq \mathbf{P}(V)$. Let $\{X_1, \ldots, X_l\}$ be a basis for V^* and put $S := \mathrm{Sym}(V^*) \simeq \mathbb{K}[X_1, \ldots, X_l]$. For each hyperplane $H \in \mathcal{A}$, let us fix a nonzero linear form $\alpha_H \in V^*$ such that its kernel is H, and put

$$Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H.$$

Definition 1.1
For an arrangement $\mathcal{A}$, the S-module $D(\mathcal{A})$ is defined by

$$\begin{aligned} D(\mathcal{A}) : &= \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(\alpha_H) \in S \cdot \alpha_H \ (\forall H \in \mathcal{A})\} \\ &= \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(Q(\mathcal{A})) \in S \cdot Q(\mathcal{A})\}. \end{aligned}$$

We call $D(\mathcal{A})$ the *module of logarithmic vector fields* (with respect to $\mathcal{A}$). We say a nonzero element $\theta = \sum_{i=1}^{l} f_i \dfrac{\partial}{\partial X_i} \in D(\mathcal{A})$ is *homogeneous of degree p* if $f_i \in S_p$ for $1 \leq i \leq l$. An arrangement $\mathcal{A}$ is *free* if $D(\mathcal{A})$ is a free S-module. When $\mathcal{A}$ is free, there exists a homogeneous basis $\{\theta_1, \ldots, \theta_l\}$ for $D(\mathcal{A})$. Then the *exponents* of the free arrangement $\mathcal{A}$ are defined by

$$\exp(\mathcal{A}) := (\deg(\theta_1), \ldots, \deg(\theta_l)).$$

It is known that $\exp(\mathcal{A})$ do not depend on the choice of a basis.

Next, we define a multiarrangement, which was introduced and studied by Ziegler in [Z].

Definition 1.2 ([Z])
We say a pair $(\mathcal{A}, m)$ is a multiarrangement if $\mathcal{A}$ is a simple arrangement and

$$m : \mathcal{A} \to \mathbb{Z}_{>0}$$

is a map from $\mathcal{A}$ to positive integers. The map m is called a multiplicity function.

A simple arrangement $\mathcal{A}$ can be thought of as a multiarrangement with $m \equiv 1$. By the same way as for simple arrangements, we define the module of logarithmic vector fields $D(\mathcal{A}, m)$ for a multiarrangement $(\mathcal{A}, m)$.

Definition 1.3
For a multiarrangement $(\mathcal{A}, m)$, the S-module $D(\mathcal{A}, m)$ is defined by

$$D(\mathcal{A}, m) := \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(\alpha_H) \in S \cdot \alpha_H^{m(H)} \ (\forall H \in \mathcal{A})\}.$$

Let $H_0 \in \mathcal{A}$ be a hyperplane in an arrangement $\mathcal{A}$. The restriction of $\mathcal{A}$ to H_0 is a simple arrangement $\mathcal{A} \cap H_0 := \{H \cap H_0 \mid H \in \mathcal{A} \setminus \{H_0\}\}$. This restriction has a natural structure of the multiarrangement $(\mathcal{A} \cap H_0,\ m)$, i.e., the multiplicity function $m : \mathcal{A} \cap H_0 \to \mathbb{Z}_{>0}$ is defined by

$$m : \mathcal{A} \cap H_0 \ni H' \mapsto |\{H \in \mathcal{A} \mid H \cap H_0 = H'\}| \in \mathbb{Z}.$$

For details, see [Z] or [Y]. It is known that $D(\mathcal{A}, m)$ is a reflexive module (e.g., see Theorem 4.75 in [OT] and Theorem 5 in [Z]). We can define the freeness and exponents of the multiarrangements by the same way as for simple arrangements. The exponents of a free multiarrangement are sometimes called *multi-exponents*. The following theorem due to K. Saito is useful to see the freeness of an arrangement and determine its basis.

Theorem 1.4 (Saito's criterion)
Let $(\mathcal{A}, m)$ be an l-multiarrangement, $D(\mathcal{A}, m)$ be its module of logarithmic vector fields, and $\theta_1, \ldots, \theta_l \in D(\mathcal{A}, m)$ be elements. Then the following two conditions are equivalent:

(1) $\{\theta_1, \ldots, \theta_l\}$ forms a basis for $D(\mathcal{A}, m)$ over S.

(2) $\theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_l = u \prod_{H \in \mathcal{A}} \alpha_H^{m(H)} (\dfrac{\partial}{\partial X_1} \wedge \ldots \wedge \dfrac{\partial}{\partial X_l})$ for some $u \in \mathbb{K} \setminus \{0\}$.

For the proof, see Theorem 4.19 in [OT] and Theorem 8 in [Z]. In this article we often consider the sheafification of $D(\mathcal{A})$. The module $D(\mathcal{A})$ and the global section of $\widetilde{D(\mathcal{A})}$ are related by the following isomorphism, which is due to Silvotti.

Lemma 1.5
For a central l-arrangement $\mathcal{A}$, there is a degree-preserving S-module isomorphism:

$$f : D(\mathcal{A}) \to \oplus_{d \in \mathbb{Z}} H^0(\mathbf{P}_{\mathbb{K}}^{l-1},\ \widetilde{D(\mathcal{A})}(d)).$$

For the proof, see Lemma 4.4 in [AY]. The Chern polynomial of $\widetilde{D(\mathcal{A})}$ can be calculated from the combinatorics of $\mathcal{A}$. To see this, let us introduce some notations. The *characteristic polynomial* of an arrangement $\mathcal{A}$ is defined by

$$\chi(\mathcal{A}, t) := \sum_{X \in L_{\mathcal{A}}} \mu(X) t^{\dim X},$$

where $L_{\mathcal{A}}$ is a lattice which consists of the intersections of elements of $\mathcal{A}$, ordered by reverse inclusion, $\hat{0} := V$ is the unique minimal element of $L_{\mathcal{A}}$ and $\mu : L_{\mathcal{A}} \longrightarrow \mathbb{Z}$ is the Möbius function defined as follows:

$$\begin{aligned} \mu(\hat{0}) &= 1, \\ \mu(X) &= -\sum_{Y<X} \mu(Y), \text{ if } \hat{0} < X. \end{aligned}$$

It is known that for a central arrangement $\mathcal{A}$, its characteristic polynomial $\chi(\mathcal{A}, t)$ can be divided by $(t-1)$. Moreover, the *reduced characteristic polynomial* $\chi_0(\mathcal{A}, t)$ is defined by

$$\chi_0(\mathcal{A}, t) := \chi(\mathcal{A}, t)/(t-1)$$

and the *Poincaré polynomial* $\pi(\mathcal{A}, t)$ by

$$\pi(\mathcal{A}, t) := \sum_{X \in L_{\mathcal{A}}} \mu(X)(-t)^{\operatorname{codim} X}.$$

The polynomials $\chi(\mathcal{A},\, t)$ and $\pi(\mathcal{A},\, t)$ are related as follows:

$$\chi(\mathcal{A},\, t) = t^l \pi(\mathcal{A},\, -1/t),$$

and these polynomials are important concepts in the theory of hyperplane arrangements. Actually there are a lot of combinatorial or geometric interpretations of the characteristic polynomial. For details, see [OT]. We can use $\pi(\mathcal{A}, t)$ to calculate the Chern polynomial.

Theorem 1.6 ([MS], Theorem 4.1)
For a polynomial $F(t) \in \mathbb{Z}[t]$, let $\overline{F(t)}$ denote the class of $F(t)$ in $\mathbb{Z}[t]/(t^l)$. Let $\mathcal{A}$ be a central l-arrangement and assume $\widetilde{D(\mathcal{A})}$ is a vector bundle on $\mathbf{P}(V)$. Then it holds that

$$c_t(\widetilde{D(\mathcal{A})}) = \overline{\pi(\mathcal{A}, -t)}.$$

In particular, if $l = 3$ and

$$\chi_0(\mathcal{A}, t) = t^2 - c_1 t + c_2,$$

then for any central 3*-arrangement* $\mathcal{A}$ *it holds that*

$$c_t(\widetilde{D(\mathcal{A})}) = (1 - c_1 t + c_2 t^2)(1 - t).$$

To show the freeness of arrangements, we often use the addition-deletion theorem. Let $\mathcal{A} \neq \emptyset$ be an arrangement, $H \in \mathcal{A}$ be a hyperplane, $\mathcal{A}' := \mathcal{A} \setminus H$ and let $\mathcal{A}'' := \mathcal{A}' \cap H$.

Theorem 1.7 ([OT], Theorem 4.51)
Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ *be a triple defined above. Any two of the following statements imply the third:*

$$\begin{array}{rcl} \mathcal{A} \textit{ is free with } \exp(\mathcal{A}) & = & (b_1, \ldots, b_{l-1}, b_l), \\ \mathcal{A}' \textit{ is free with } \exp(\mathcal{A}') & = & (b_1, \ldots, b_{l-1}, b_l - 1), \\ \mathcal{A}'' \textit{ is free with } \exp(\mathcal{A}'') & = & (b_1, \ldots, b_{l-1}). \end{array}$$

Next, let us consider the theory of 3-arrangements. Let $\mathcal{A}$ be an arrangement in a three-dimensional vector space V. Then the sheaf $\widetilde{D(\mathcal{A})}$ is a rank three vector bundle on $\mathbf{P}^2$ since $\widetilde{D(\mathcal{A})}$ is reflexive (e.g., see [H]). Fix a basis $\{X, Y, Z\}$ for V^* in such a way that the hyperplane $\{Z = 0\}$ is an element of $\mathcal{A}$. Regarding $\{Z = 0\}$ as the infinite line in $\mathbf{P}^2$, we define the *deconing* $d\mathcal{A}$ of a 3-arrangement $\mathcal{A}$ with respect to $\{Z = 0\}$ as

$$d\mathcal{A} := \{dH := H|_{Z=1} \mid H \in \mathcal{A} \setminus \{Z = 0\}\}.$$

Let us define $S := \mathrm{Sym}(V^*) \simeq \mathbb{K}[X, Y, Z]$. We define the *module of reduced logarithmic vector fields* $D_0(\mathcal{A})$ as follows:

Definition 1.8
The S*-module* $D_0(\mathcal{A})$ *is defined by*

$$D_0(\mathcal{A}) := \{\theta \in D(\mathcal{A}) \mid \theta(Z) = 0\}.$$

Note that for any (central) arrangement $\mathcal{A}$, there exists an derivation

$$\theta_E := X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z} \in D(\mathcal{A}).$$

We call this derivation θ_E the *Euler derivation*. It is obvious that

$$D_0(\mathcal{A}) \simeq D(\mathcal{A})/(S \cdot \theta_E).$$

Hence the structure of $D_0(\mathcal{A})$ does not depend on the choice of the coordinates of V. Moreover, in the notation of Theorem 1.6, it holds that

$$c_t(\widetilde{D_0(\mathcal{A})}) = 1 - c_1 t + c_2 t^2.$$

When $\emptyset \neq \mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, d_1, d_2)$, $|\exp(\mathcal{A})|$ denotes the value $|d_1 - d_2|$. We often write $|\exp|$ instead of $|\exp(\mathcal{A})|$ if the arrangement we are considering is clear. As we saw above, we can restrict a given arrangement $\mathcal{A}$ on the plane $H_0 := \{Z = 0\} \in \mathcal{A}$. Moreover, we can obtain a multiarrangement $(\mathcal{A} \cap H_0,\ m)$ and the restriction homomorphism

$$\varphi : D_0(\mathcal{A}) \to D(\mathcal{A} \cap H_0,\ m),$$

defined as follows:

$$D_0(\mathcal{A}) \ni \theta \mapsto \theta|_{Z=0} \in D(\mathcal{A} \cap H_0,\ m).$$

For the details of this homomorphism, see [Z]. We can compute the codimension (as $\mathbb{K}$-vector spaces) of the image of φ from the characteristic polynomial of $\mathcal{A}$ and the exponents of $D(\mathcal{A} \cap H_0,\ m)$ by the following theorem, which is a variant of Theorem 3.2 in [Y].

Theorem 1.9 (Yoshinaga)
With the above notation, let $\{\theta_1, \theta_2\}$ be a homogeneous basis for a free $S/(S \cdot Z)$-module $D(\mathcal{A} \cap H_0,\ m)$ such that $\deg(\theta_i) = d_i$ $(i = 1, 2)$. Then the dimension of $\operatorname{coker}(\varphi)$ *(as a $\mathbb{K}$-vector space) is finite and is given by*

$$\chi_0(\mathcal{A}, 0) - d_1 d_2.$$

In particular, $\mathcal{A}$ is free if and only if

$$\chi_0(\mathcal{A}, 0) = d_1 d_2.$$

In [Y], Yoshinaga showed the same statement as in Theorem 1.9 for the logarithmic differential module $\Omega(\mathcal{A})$, and we can prove Theorem 1.9 by the same way as in [Y].

Next, we review some definitions and results on the stability of vector bundles on projective spaces. The reference for the stability of vector bundles is Chapter II of [OSS]. First, we define the stability and semistability of torsion free sheaves.

Definition 1.10
A torsion free sheaf E on the projective space $\mathbf{P}^n_{\mathbb{K}}$ is said to be stable if for any coherent subsheaf $F \subset E$ with $0 < \mathrm{rank}(F) < \mathrm{rank}(E)$ we have

$$\frac{c_1(F)}{\mathrm{rank}(F)} < \frac{c_1(E)}{\mathrm{rank}(E)},$$

and semistable if

$$\frac{c_1(F)}{\mathrm{rank}(F)} \leq \frac{c_1(E)}{\mathrm{rank}(E)}.$$

Moreover, we say E is *unstable* if E is not stable. We call a 3-arrangement $\mathcal{A}$ is *stable* (resp:*semistable*) if $\widetilde{D_0(\mathcal{A})}$ is stable (resp:semistable). In this article, we will use the following definitions and results.

Lemma 1.11 ([OSS], Lemma 1.2.4, Ch. II)
A torsion free sheaf E on the projective space $\mathbf{P}^n_{\mathbb{K}}$ is stable if and only if $E \otimes \mathcal{O}_{\mathbf{P}^n}(d)$ is stable for some $d \in \mathbb{Z}$.

Definition 1.12
For a rank two vector bundle E on $\mathbf{P}^n_{\mathbb{K}}$ $(n \geq 2)$, there exists the unique integer $d_E \in \mathbb{Z}$ such that

$$c_1(E \otimes \mathcal{O}_{\mathbf{P}^n}(d_E)) \in \{0, -1\}.$$

$E \otimes \mathcal{O}_{\mathbf{P}^n}(d_E)$ is called the normalization of E or the normalized E and denoted by E_{norm}. The normalized Chern polynomial of E denotes the Chern polynomial of E_{norm}.

Lemma 1.13 ([OSS], Lemma 1.2.5, Ch. II)
Let E be a rank two vector bundle on $\mathbf{P}^n$ $(n \geq 2)$. Then E is stable if and only if

$$H^0(\mathbf{P}^n,\, E_{\mathrm{norm}}) = 0.$$

Moreover, if $c_1(E)$ is even, then E is semistable if and only if $H^0(\mathbf{P}^n,\, E_{\mathrm{norm}}(-1)) = 0$.

Lemma 1.14 ([OSS], Lemma 1.2.7, Ch. II)
For a rank two vector bundle E on $\mathbf{P}^2_{\mathbb{K}}$, put $c_i := c_i(E)$ $(i = 1, 2)$. If

$$c_1^2 - 4c_2 \geq 0,$$

then E is unstable. Moreover, if $c_1^2 - 4c_2 = -4$, then E is unstable.

Theorem 1.15 ([Sch], Theorem 4.5)
Let $\mathcal{A}$ be an arrangement of d lines in $\mathbf{P}^2$, H_0 be a line in $\mathcal{A}$, and let us put $\mathcal{A}' := \mathcal{A} \setminus \{H_0\}$. Then the followings hold:

(i) If d is odd, then $\mathcal{A}$ is stable if $\mathcal{A}'$ is stable and $|\mathcal{A} \cap H_0| > (d+1)/2$.

(ii) If d is odd, then $\mathcal{A}$ is semistable if $\mathcal{A}'$ is semistable and $|\mathcal{A} \cap H_0| > (d-1)/2$.

(iii) If d is even, then $\mathcal{A}$ is stable if $\mathcal{A}'$ is semistable and $|\mathcal{A} \cap H_0| > d/2$.

Theorem 1.16
Assume that E is a stable rank two bundle on $\mathbf{P}^2_{\mathbb{K}}$ such that $c_1(E) = -1$ and $c_2(E) = 1$. Then

$$E \simeq \Omega_{\mathbf{P}^2}(1) \simeq T_{\mathbf{P}^2}(-2).$$

For the proof of Theorem 1.16, see Theorem 3.1.3 and Example 1 of section 3.2 in Chapter II of [OSS].

2 Proof of the main theorem

In this section we prove Theorem 0.2. We fix a notation. Let V be a three-dimensional vector space over $\mathbb{K}$, $\{X, Y, Z\}$ be a basis for V^* and $S := \mathrm{Sym}(V^*) \simeq \mathbb{K}[X, Y, Z]$. A family of B_2-type arrangements is the family of arrangements $\{\mathcal{A}(k)\}$ in V defined in Definition 0.1 such that $\mathcal{A}(1)$ is expressed as follows:

$$\begin{aligned} X &= 0, Z, \ldots, cZ \ (c \geq 0), \\ Y &= 0, Z, \ldots, fZ \ (f \geq 0), \\ Y + X &= (a-1)Z, \ldots, (a+b)Z \ (b \geq -1), \\ Y - X &= (d-1)Z, \ldots, (d+e)Z \ (e \geq -1), \\ Z &= 0. \end{aligned} \tag{1}$$

For the rest of this section, we use the notaion (1) for the family of B_2-type arrangements. We begin with the following lemma, which describes the behavior of Chern polynomials of B_2-type arrangements for sufficiently large k.

Lemma 2.1
Let $\{\mathcal{A}(k)\}$ be the family of B_2-type arrangements. Let us put

$$\begin{aligned} A_1 := 8k &+ b+c+e+f-2, \\ A_2 := (a &+ \frac{1}{2}b-\frac{1}{2}c-\frac{1}{2}f-\frac{1}{2})^2+(d+\frac{1}{2}e+\frac{1}{2}c-\frac{1}{2}f-\frac{1}{2})^2 \\ &+ (\frac{1}{2}c-\frac{1}{2}f)^2+(4k+\frac{1}{2}b+\frac{1}{2}c+\frac{1}{2}e+\frac{1}{2}f-1)^2. \end{aligned}$$

(a) *If $a+d$ is even and $k \gg 0$, then it holds that*

$$c_t(\widetilde{D_0(\mathcal{A}(k))}) = 1 - A_1 t + (A_2 - 1)t^2,$$

if b and e are both odd numbers, and

$$c_t(\widetilde{D_0(\mathcal{A}(k))}) = 1 - A_1 t + (A_2 - \frac{1}{2})t^2,$$

if otherwise.

(b) *If $a+d$ is odd and $k \gg 0$, then it holds that*

$$c_t(\widetilde{D_0(\mathcal{A}(k))}) = 1 - A_1 t + A_2 t^2,$$

if b and e are both odd numbers, and

$$c_t(\widetilde{D_0(\mathcal{A}(k))}) = 1 - A_1 t + (A_2 - \frac{1}{2})t^2,$$

if otherwise.

Proof. By Theorem 1.6, we can obtain the Chern polynomial through combinatorial calculations of the deconing $d\mathcal{A}(k)$ of the B_2-type arrangement $\mathcal{A}(k)$. First we consider the cardinality of the set of points in $L_2(k) := \{p \in L(d\mathcal{A}(k)) \mid \dim(p) = 0\}$ which have as coordinates half integers, i.e., the points in $L_2(k)$ with coordinates $((2s+1)/2,\ (2t+1)/2) \in \mathbb{K}^2$ $(s,t \in \mathbb{Z})$. Let $HF(\mathcal{A}(k))$ denote the cardinality of this set. We can calculate $HF(\mathcal{A}(k))$ as follows:

(a) When $a+d$ is even and $b,\ e$ are both odd numbers. In this case,

$$\begin{aligned} HF(\mathcal{A}(k)) &= (k+\frac{1}{2}b-\frac{1}{2})(k+\frac{1}{2}e+\frac{1}{2})+(k+\frac{1}{2}b+\frac{1}{2})(k+\frac{1}{2}e-\frac{1}{2}) \\ &= 2(k+\frac{1}{2}b)(k+\frac{1}{2}e)-\frac{1}{2}. \end{aligned}$$

(b) When $a+d$ is odd and $b,\ e$ are both odd numbers. In this case,

$$\begin{aligned} HF(\mathcal{A}(k)) &= (k+\frac{1}{2}b+\frac{1}{2})(k+\frac{1}{2}e+\frac{1}{2})+(k+\frac{1}{2}b-\frac{1}{2})(k+\frac{1}{2}e-\frac{1}{2}) \\ &= 2(k+\frac{1}{2}b)(k+\frac{1}{2}e)+\frac{1}{2}. \end{aligned}$$

(c) When other cases. In this case,

$$\begin{aligned} HF(\mathcal{A}(k)) &= (k+\frac{1}{2}b+\frac{1}{2})(k+\frac{1}{2}e)+(k+\frac{1}{2}b-\frac{1}{2})(k+\frac{1}{2}e) \\ &= 2(k+\frac{1}{2}b)(k+\frac{1}{2}e). \end{aligned}$$

Next we consider the cardinality of the set of points of $L_2(k)$ whose coordinates are integers. There are $(2k+c-1)(2k+f-1)$-intersection points of the lattice L_i which consists of lines $\{X=s,\ Y=t \mid -k+1 \leq s \leq k+c-1,\ -k+1 \leq s \leq k+f-1\}$. To count the cardinality of these points, it suffices to calculate the cardinality $|\{\text{line}\} \cap L_i|$ for lines in $L_+ = \{Y+X=u \mid -k+a \leq u \leq k+a+b-1\}$ and $L_- = \{Y-X = v \mid -k+d \leq v \leq k+d+e-1\}$. First, let us assume $f \geq c$ and consider those with L_+ as follows:

line	$\lvert L_i \cap \{\text{line}\}\rvert$
$Y+X=k+a+b-1$	$3k+a+b-2$
$Y+X=k+a+b-2$	$3k+a+b-3$
$\vdots$	$\vdots$
$Y+X=f$	$2k+f-1$
$\vdots$	$\vdots$
$Y+X=c$	$2k+f-1$
$Y+X=c-1$	$2k+f$
$\vdots$	$\vdots$
$Y+X=-k+a$	$3k+c+f-a-1$

The sum of intersection points above is $(2k+f-1)(2k+b)+\frac{1}{2}(k+a+b-f-1)(k+a+b-f)+\frac{1}{2}(k+c-a)(k+c-a+1)$. Next we consider those with L_-. They are as follows:

line	$\lvert L_i \cap \{\text{line}\}\rvert$
$Y-X=k+d+e-1$	$3k+d+e+c-2$
$Y-X=k+d+e-2$	$3k+d+e+c-3$
$\vdots$	$\vdots$
$Y-X=f-c$	$2k+f-1$
$\vdots$	$\vdots$
$Y-X=0$	$2k+f-1$
$Y-X=-1$	$2k+f$
$\vdots$	$\vdots$
$Y-X=-k+d$	$3k+f-d-1$

The sum of intersection points above is $(2k+f-1)(2k+e)+\frac{1}{2}(k-d)(k-d+1)+\frac{1}{2}(k+d+e+c-f-1)(k+d+e+c-f)$. Summing these quantities, we have the lemma. When $c \geq f$, we can do the same calculation as above and have the same formula. $\square$

Depending on whether $c_1(D_0(\widetilde{\mathcal{A}(k)}))$ is odd or even, the normalization of Chern polynomials calculated in Lemma 2.1 can be obtained as follows:

Lemma 2.2
Let $\{\mathcal{A}(k)\}$ be the family of B_2-type arrangements. Put

$$L := A_2 - (4k+\frac{1}{2}b+\frac{1}{2}c+\frac{1}{2}e+\frac{1}{2}f-1)^2.$$

Then for $k \gg 0$, the normalized Chern polynomial $c_t(D_0(\widetilde{\mathcal{A}(k))}_{norm})$ is as follows:

	$a+d$	b and e	A_1	normalized Chern polynomial
a)	even	both odd	even	$1+(L-1)t^2$
b)	even	both odd	odd	$1-t+(L-(3/4))t^2$
c)	even	not both odd	even	$1+(L-(1/2))t^2$
d)	even	not both odd	odd	$1-t+(L-(1/4))t^2$
e)	odd	both odd	even	$1+Lt^2$
f)	odd	both odd	odd	$1-t+(L+(1/4))t^2$
g)	odd	not both odd	even	$1+(L-(1/2))t^2$
h)	odd	not both odd	odd	$1-t+(L-(1/4))t^2$

By these Chern polynomials, we can show the following criterions for the freeness of B_2-type arrangements (recall a family $\{\mathcal{A}(k)\}$ is free if there exists an integer k_0 such that $\mathcal{A}(k)$ is free for all $k \geq k_0$).

Proposition 2.3
Let $\{\mathcal{A}(k)\}$ be the family of B_2-type arrangements. Assume that $a+d$ is odd. Then $\{\mathcal{A}(k)\}_{k=1}^{\infty}$ is a free family of arrangements if and only if one of the following holds:

(e-i)
$$c=f,\ 2a+b=2c+1,\ 2d+e=1.$$

(g-i)
$$c=f,\ 2a+b=2c+2,\ \text{or } 2c,\ 2d+e=2 \text{ or } 0.$$

(g-ii)
$$c=f+1,\ 2a+b=2c+1 \text{ or } 2c-1,\ 2d+e=0.$$

(g-iii)
$$c=f-1,\ 2a+b=2c+3 \text{ or } 2c+1,\ 2d+e=2.$$

(g-iv)
$$c=f+1,\ 2a+b=2c,\ 2d+e=1 \text{ or } -1.$$

(g-v)
$$c=f-1,\ 2a+b=2c+2,\ 2d+e=3 \text{ or } 1.$$

(h-i)
$$c=f,\ 2a+b=2c+2 \text{ or } 2c,\ 2d+e=1.$$

(h-ii)
$$c=f,\ 2a+b=2c+1,\ 2d+e=2 \text{ or } 0.$$

(h-iii)
$$c=f+1,\ 2a+b=2c,\ 2d+e=0.$$

(h-iv)
$$c=f-1,\ 2a+b=2c+2,\ 2d+e=2.$$

Moreover, in the above cases, $|\exp(\mathcal{A}(k))|=0$ or 1 for $k\gg 0$.

Proposition 2.4
Let $\{\mathcal{A}(k)\}$ be the family of B_2-type arrangements. Assume that $a+d$ is even. Then $\{\mathcal{A}(k)\}_{k=1}^{\infty}$ is a free family of arrangements if and only if one of the following holds:

(a-i)
$$c=f,\ 2a+b=2c+1,\ 2d+e=1.$$

(a-ii)
$$c = f,\ 2a + b = 2c + 3 \text{ or } 2c - 1,\ 2d + e = 1.$$

(a-iii)
$$c = f,\ 2a + b = 2c + 1,\ 2d + e = 3 \text{ or } -1.$$

(a-iv)
$$c = f + 2,\ 2a + b = 2c - 1,\ 2d + e = -1.$$

(a-v)
$$c = f - 2,\ 2a + b = 2c + 3,\ 2d + e = 3.$$

(b-i)
$$c = f + 1,\ 2a + b = 2c + 1 \text{ or } 2c - 1,\ 2d + e = 1 \text{ or } -1.$$

(b-ii)
$$c = f - 1,\ 2a + b = 2c + 3 \text{ or } 2c + 1,\ 2d + e = 3 \text{ or } 1.$$

(c-i)
$$c = f,\ 2a + b = 2c + 2, \text{ or } 2c,\ 2d + e = 2 \text{ or } 0.$$

(c-ii)
$$c = f + 1,\ 2a + b = 2c + 1 \text{ or } 2c - 1,\ 2d + e = 0.$$

(c-iii)
$$c = f - 1,\ 2a + b = 2c + 3 \text{ or } 2c + 1,\ 2d + e = 2.$$

(c-iv)
$$c = f + 1,\ 2a + b = 2c,\ 2d + e = 1 \text{ or } -1.$$

(c-v)
$$c = f - 1,\ 2a + b = 2c + 2,\ 2d + e = 3 \text{ or } 1.$$

(d-i)
$$c = f,\ 2a + b = 2c + 2 \text{ or } 2c,\ 2d + e = 1.$$

(d-ii)
$$c = f,\ 2a + b = 2c + 1,\ 2d + e = 2 \text{ or } 0.$$

(d-iii)
$$c = f + 1,\ 2a + b = 2c,\ 2d + e = 0.$$

(d-iv)
$$c = f - 1,\ 2a + b = 2c + 2,\ 2d + e = 2.$$

Moreover, in the above cases, $|\exp(\mathcal{A}(k))| = 2$ *when (a-i), and* 0 *or* 1 *otherwise.*

Outline of Proofs. Here, we show the conditions in the statement are necessary conditions. In the next section, we will show they are sufficient conditions by using the addition-deletion theorem. Assume that $\{\mathcal{A}(k)\}$ is a free family. Then the sheaf $\widetilde{D_0(\mathcal{A}(k))}$ is isomorphic to the direct sum of line bundles $\mathcal{O}(t_1) \oplus \mathcal{O}(t_2)$ $(t_1, t_2 \in \mathbb{Z})$. It is obvious that the normalization of a rank two splitting vector bundle is expressed as whether

$$\mathcal{O}(s) \oplus \mathcal{O}(-s)$$

or

$$\mathcal{O}(s) \oplus \mathcal{O}(-s-1).$$

Here s is a non-negative integer. Hence the second Chern class of the normalized $\widetilde{D_0(\mathcal{A}(k))}$ must be zero or a negative integer which is expressed as $-s^2$ or $-s^2 - s$ by some non-negative integer s. We know the normalized second Chern class of $\widetilde{D_0(\mathcal{A}(k))}$ by Lemma 2.2. Comparing these, we can determine the condition on (a, b, c, d, e, f) for $\{\mathcal{A}(k)\}$ to be free, and obtain all the necessary conditions as in the statement. Note that the conditions in Proposition 2.3 and 2.4 are equivalent to (a) in Theorem 0.2. $\square$

To consider the stability of 3-arrangements, we use the following criterions for the stability of arrangements.

Theorem 2.5
Let $\mathcal{A}$ *be a central, non-free arrangement in a three-dimensional vector space* V *with* $|\mathcal{A}| = 2u$ $(u \in \mathbb{Z}_{>0})$.

- *(a) Assume there exists a plane* $H \notin \mathcal{A}$ *in* V *such that the arrangement* $\mathcal{A} \cup H$ *is free. Then* $\widetilde{D_0(\mathcal{A})}$ *is stable if and only if* $\exp(\mathcal{A} \cup H) = (1, u, u)$.
- *(b) Assume there exists a plane* $H \in \mathcal{A}$ *in* V *such that the arrangement* $\mathcal{A} \setminus H$ *is free. Then* $\widetilde{D_0(\mathcal{A})}$ *is stable if and only if* $\exp(\mathcal{A} \cup H) = (1, u-1, u-1)$.

Proof. (a) We may assume that $H_0 := \{Z = 0\} \in \mathcal{A}$. Recall

$$D_0(\mathcal{A}) \simeq \{\theta \in D(\mathcal{A}) \mid \theta(Z) = 0\}.$$

We use this characterization of $D_0(\mathcal{A})$ in this proof. First, assume $\widetilde{D_0(\mathcal{A})}$ is stable and $\exp(\mathcal{A}\cup H)=(1,u-a,u+a)$ with $a>0$. This implies $\mathcal{O}(-u+a)\subset \widetilde{D_0(\mathcal{A})}$, so $H^0(\mathbf{P}^2,\widetilde{D_0(\mathcal{A})}(u-1))\neq 0$. Since $c_1(\widetilde{D_0(\mathcal{A})}(u-1))=-1$, Lemma 1.13 shows the result. Conversely, let us assume $\exp(\mathcal{A}\cup H)=(1,u,u)$. It is easy to see that $\exp(\mathcal{A}\cap H_0,m)=(u-1,u)$, where m is the canonically induced multiplicity on a simple arrangement $\mathcal{A}\cap H_0$. Put

$$\varphi : D_0(\mathcal{A})\to D(\mathcal{A}\cap H_0,m).$$

Consider the following sequence:

$$D_0(\mathcal{A}\cup H)\simeq S(u)^{\oplus 2}\subset D_0(\mathcal{A})\xrightarrow{\varphi} D(\mathcal{A}\cap H_0,m). \tag{2}$$

Note that

$$D(\mathcal{A}\cap H_0,m)\simeq S_{H_0}(u)\oplus S_{H_0}(u-1),$$

where $S_{H_0}:=S/S\cdot Z$. Let (e_1,e_2) be a homogeneous basis for $D_0(\mathcal{A}\cup H)$ and (θ_1,θ_2) be a basis for $D_0(\mathcal{A}\cap H_0,m)$ such that

$$\begin{aligned}\deg(\theta_1) &= u,\\ \deg(\theta_2) &= u-1.\end{aligned}$$

Note that $Z\not| \, e_1\wedge e_2$ and $\varphi(\theta)=\theta|_{Z=0}$ for all $\theta\in D_0(\mathcal{A})$. To show the stability, from Lemma 1.5 and Lemma 1.13, it suffices to show $H^0(\widetilde{D_0(\mathcal{A})}(u-1))\simeq D_0(\mathcal{A})_{u-1}=0$. Since $D_0(\mathcal{A})_n=0$ for all $n<u-1$, it holds that

$$D_0(\mathcal{A})_{u-1}\neq 0 \iff \theta_2\in \mathrm{Im}(\varphi).$$

Assume $D_0(\mathcal{A})_{u-1}\neq 0$. Then there exist elements $\alpha_i\in\mathbb{K}$, $\beta_i\in S_1$ $(i=1,2)$ such that

$$\varphi(e_i)=\alpha_i\theta_1+\beta_i\theta_2 \ (i=1,2).$$

If $\alpha_1\neq 0$, then we can see $\theta_1=(1/\alpha_1)(\varphi(e_1)-\beta_1\theta_2)\in\mathrm{Im}(\varphi)$. This implies that φ is surjective, which contradicts the non-freeness of $\mathcal{A}$. By the same way, we can see $\alpha_1=\alpha_2=0$. Then $Z|e_1\wedge e_2$, which is a contradiction.

(b) Instead of the sequence (2), we use the following sequence:

$$D_0(\mathcal{A}\setminus L)\simeq S(u-1)^{\oplus 2}\xrightarrow{\cdot\alpha_H} D_0(\mathcal{A})\to S_{H_0}(u)\oplus S_{H_0}(u-1). \tag{3}$$

Then the same argument as the above completes the proof. □

By using Theorem 2.5, we can construct a lot of stable, non-free arrangements from free arrangements which are inductively free or constructed from Theorem 1.7. e.g., consider the arrangement $\mathcal{A}:=\{Q(\mathcal{A})=XYZ(X+$

$Y + Z) = 0\}$ in a three dimensional vector space V. Since $\pi(\mathcal{A}, t) = (1+t)(1+3t+3t^2)$, this is not free. It is easy to see (e.g., by using Theorem 1.7) that the arrangement $\mathcal{A}' := \{XYZ = 0\}$ is free with exponents $(1, 1, 1)$. So Theorem 2.5 (b) shows $\mathcal{A}$ is stable, and Using Theorem 1.16, we can see that $\widetilde{D_0(\mathcal{A})} \simeq T_{\mathbf{P}^2}(-3)$.

Proposition 2.6
Let $\mathcal{A} \neq \emptyset$ be a central arrangement in a three-dimensional vector space V. Then the following holds:

(a) If $|\mathcal{A}|$ is even and there exists a hyperplane $H \in \mathcal{A}$ (resp : $H \notin \mathcal{A}$) such that $\mathcal{A} \setminus H$ (resp : $\mathcal{A} \cup H$) is stable, then $\mathcal{A}$ is also stable.

(b) If $|\mathcal{A}|$ is odd and there exists a hyperplane $H \in \mathcal{A}$ (resp : $H \notin \mathcal{A}$) such that $\mathcal{A} \setminus H$ (resp : $\mathcal{A} \cup H$) is stable, then $\mathcal{A}$ is semistable.

Proof. (a) First we show when $\mathcal{A} \setminus H$ is stable. Let us put $|\mathcal{A}| = 2u + 2$ and $c_1(\widetilde{D_0(\mathcal{A})}) = -2u - 1$ $(u \in \mathbb{Z}_{\geq 0})$. Define $\mathcal{B} := \mathcal{A} \setminus H$. Then $c_1(\widetilde{D_0(\mathcal{B})}) = -2u$. By the definition, there exists a canonical inclusion $\widetilde{D_0(\mathcal{A})} \subset \widetilde{D_0(\mathcal{B})}$. Since $\mathcal{B}$ is stable, it holds that $H^0(\widetilde{D_0(\mathcal{B})}(u)) = 0$, and this implies $H^0(\widetilde{D_0(\mathcal{A})}(u)) = 0$. Since $\widetilde{D_0(\mathcal{A})}(u)$ is normalized, $\mathcal{A}$ is also stable. Next, we show when $\mathcal{A} \cup H$ is stable. Define $\mathcal{C} := \mathcal{A} \cup H$. Then $\widetilde{D_0(\mathcal{C})} = -2u - 2$. By the multiplication of the defining linear form α_H, we obtain the injection $\widetilde{D_0(\mathcal{A})}(-1) \to \widetilde{D_0(\mathcal{C})}$. Since $\mathcal{C}$ is stable, it holds $H^0(\widetilde{D_0(\mathcal{C})}(u+1)) = 0$. This implies $H^0(\widetilde{D_0(\mathcal{A})}(u)) = 0$. Since $\widetilde{D_0(\mathcal{A})}(u)$ is normalized, $\mathcal{A}$ is also stable. We can show (b) by the same way as in (a), so leave it to the reader $\square$

From Proposition 2.3, Proposition 2.4 and Theorem 2.5, we obtain the following result.

Proposition 2.7
Let $\{\mathcal{A}(k)\}$ be the family of B_2-type arrangements. Then the normalization of the vector bundle $\widetilde{D_0(\mathcal{A}(k))}$ is isomorphic to $T_{\mathbf{P}^2}(-2)$ for sufficiently large k if one of the following holds:

(d-I), (h-I)

$$c = f + 2,\ 2a + b = 2c - 1,\ 2d + e = 0,\ \text{or}\ -2.$$

(d-II), (h-II)

$$c = f - 2,\ 2a + b = 2c + 3,\ 2d + e = 4\ \text{or}\ 2.$$

(d-III), (h-III)

$$c = f + 1,\ 2a + b = 2c,\ 2d + e = 2 \text{ or } -2.$$

(d-IV), (h-IV)

$$c = f - 1,\ 2a + b = 2c + 2,\ 2d + e = 4 \text{ or } 0.$$

(d-V), (h-V)

$$c = f + 2,\ 2a + b = 2c \text{ or } 2c - 2,\ 2d + e = -1.$$

(d-VI), (h-VI)

$$c = f - 2,\ 2a + b = 2c + 4 \text{ or } 2c + 2,\ 2d + e = 3.$$

(d-VII), (h-VII)

$$c = f + 1,\ 2a + b = 2c + 2 \text{ or } 2c - 2,\ 2d + e = 0.$$

(d-VIII), (h-VIII)

$$c = f - 1,\ 2a + b = 2c + 4, \text{ or } 2c,\ 2d + e = 2.$$

(d-IX), (h-IX)

$$c = f,\ 2a + b = 2c + 2 \text{ or } 2c,\ 2d + e = 3 \text{ or } -1.$$

(d-X), (h-X)

$$c = f,\ 2a + b = 2c + 3 \text{ or } 2c - 1,\ 2d + e = 2 \text{ or } 0.$$

(f-I)

$$c = f + 1,\ 2a + b = 2c + 1 \text{ or } 2c - 1,\ 2d + e = 1 \text{ or } -1.$$

(f-II)

$$c = f - 1,\ 2a + b = 2c + 3 \text{ or } 2c + 1,\ 2d + e = 3 \text{ or } 1.$$

Proof. In each case, we can find a plane $H \subset V$ satisfying the condition in Theorem 2.5. For example, let us consider the case (d-III) $c = f+1,\ 2a+b = 2c,\ 2d + e = 2$. Let us put

$$H_k := \{Y - X = (-k + d - 1)Z\}$$

and $\mathcal{B}(k) := \mathcal{A}(k) \cup H_k$. Then the integers (d, e) corresponding to $\mathcal{A}(k)$ are transformed to $(d-1, e+1)$ to $\mathcal{B}(k)$, so $2(d-1)+(e+1) = 2-1 = 1$ for $\mathcal{B}(k)$. Noting the condition $c = f + 1,\ 2a + b = 2c,\ 2(d - 1) + (e + 1) = 1$ is the freeness condition (c-iv) or (g-iv) in Proposition 2.3 and 2.4, the condition in Theorem 2.5 is satisfied. Since the normalized Chern polynomials of these cases are all $1 - t + t^2$, we have the proposition. Note that the conditions in this proposition are sufficient and necessary for the normalized Chern polynomial to be $1 - t + t^2$. $\square$

Now we have finished all the preparations for the proof of the main theorem.

Proof of Theorem 0.2. (a) and (b) are already proved in Proposition 2.3, 2.4 and Proposition 2.7. We prove (c') and (c). Let B denote the vector (B_1, B_2, B_3), $|B|$ the vector $(|B_1|, |B_2|, |B_3|)$ and $|\tilde{B}|$ the vector with $\{|B_1|, |B_2|, |B_3|\}$ ordered in the descending order. If we want to make clear to which family of arrangements these values or vectors are associated, we use the notation $B_i(\mathcal{A}(k)))$, $B(\mathcal{A}(k))$, $|\tilde{B}(\mathcal{A}(k))|$ and so on. For a family of B_2-type arrangements $\{\mathcal{A}(k)\}$, we call $(B_1(\mathcal{A}(k)), B_2(\mathcal{A}(k)), B_3(\mathcal{A}(k)))$ the B-*vector* of $\{\mathcal{A}(k)\}$ and its $|\tilde{B}|$ the *absolute descending vector* of $\{\mathcal{A}(k)\}$.

First, note that when $M = 5 \iff |\tilde{B}| = (2, 1, 0)$, Proposition 2.7 shows the stability, since the tangent bundle is stable (see [OSS]). Next we show the case $M = B_1^2 + B_2^2 + B_3^2 = 4$ and $a + d$ is odd. In this case $|\tilde{B}| = (2, 0, 0)$. From Lemma 2.2 the normalized Chern polynomial in this case is $1 + t^2$. Hence Lemma 1.14 shows the unstability. Since we can add a line to make the new absolute descending vector $(2, 1, 0)$ as we show in the proof of Proposition 2.7, Proposition 2.6 and 2.7 shows the semistability. When $M = 6$, we can express the vector $|\tilde{B}|$ as $(2, 1, 1)$. Since B_1 or B_2 is ± 1 and we can increase/decrease B_1 or B_2 by adding one line, we can see there exists a line $H_k \subset V$ such that $\{\mathcal{A}(k) \cup H_k\}$ has the absolute descending vector $(2, 1, 0)$, which is stable by (b). So Proposition 2.6 (b) shows this is semistable and Lemma 1.14 shows the unstability. By the same way we can see it is semistable when $M = 8$ and unstable when $a + d$ is even. Next, let us assume that the statement is proved when

$$M = 9,\ 10,\ 14.$$

Then from Proposition 2.6, the statement is true when

$$M = 11,\ 12,\ 13.$$

Here we only show the semistability when $M = 12$. So first, we prove the case $|\tilde{B}| = (2, 2, 1)$. To show it, we use Schenck's stability criterion (Theorem 1.15) for the triple $(\mathcal{A}(k), \mathcal{A}'(k), \mathcal{A}''(k))$. Here the line H_k with $\mathcal{A}(k) = \mathcal{A}'(k) \cup H_k$ is one of the exterior lines (for the definition, see Definition 2.1 in [A]). To show the stability of $\mathcal{A}(k)$, we need the stability of $\mathcal{A}'(k)$ and the numerical condition between $|\mathcal{A}(k)|$ and $|\mathcal{A}''(k)|$. By the calculation, the numerical condition (we write "NC" instead of "numerical condition") corresponding to each deleting of a line is as follows:

	deleting line	NC when $\lvert\mathcal{A}(k)\rvert$ is odd	NC when $\lvert\mathcal{A}(k)\rvert$ is even
i)	$Y+X=(k+a+b-1)Z$	$B_1>2$	$B_1>1$
ii)	$Y+X=(-k+a)Z$	$B_1<-2$	$B_1<-1$
iii)	$Y-X=(k+d+e-1)Z$	$B_2>2$	$B_2>1$
iv)	$Y-X=(-k+d)Z$	$B_2<-2$	$B_2<-1$
v)	$X=(k+c-1)Z$	$B_2-B_1>-B_3+2$	$B_2-B_1>-B_3+2$
vi)	$X=(-k+1)Z$	$B_2-B_1<B_3-2$	$B_2-B_1<B_3-2$
vii)	$Y=(k+f-1)Z$	$B_1+B_2<-B_3-2$	$B_1+B_2<-B_3-2$
viii)	$Y=(-k+1)Z$	$B_1+B_2>B_3+2$	$B_1+B_2>B_3+2$

Note these are the strongest conditions. When $|\mathcal{A}(k)|$ is odd, there are cases numerical conditions become weaker. Next we show the B-vector of $\mathcal{A}'(k)$ whose (semi)stability is needed to use Schenck's criterion. We can see $B(\mathcal{A}'(k))$ as follows (the numbers from i) to viii) correspond to that in the table above):

	B-vector after a line is deleted
i)	(B_1-1,B_2,B_3)
ii)	(B_1+1,B_2,B_3)
iii)	(B_1,B_2-1,B_3)
iv)	(B_1,B_2+1,B_3)
v)	(B_1+1,B_2-1,B_3-1)
vi)	(B_1-1,B_2+1,B_3-1)
vii)	(B_1+1,B_2+1,B_3+1)
viii)	(B_1-1,B_2-1,B_3+1)

i.e., $\mathcal{A}(k)$ is stable if one of conditions from i) to viii) in the two tables above is satisfied. For example, if we use i) in the tables above, a family $\{\mathcal{A}(k)\}$ is stable if $B_1(\mathcal{A}(k))>2$ and the family of B_2-type arrangements which has $(B_1(\mathcal{A}(k))-1,B_2(\mathcal{A}(k)),B_3(\mathcal{A}(k)))$ as its B-vector is stable. Now let us prove the case $|\tilde{B}|=(2,2,1)$. By the proper choice, we can find the line H_k such that $\mathcal{A}(k)\setminus H_k$ has the absolute descending vector $(2,1,1)$. Since the cardinality of arrangements is even in this case, the semistability when $M=6$ and the table above complete the proof. Next we show the case $M=10$, i.e., $|\tilde{B}|=(3,1,0)$. If $|B_1|=3$ or $|B_2|=3$, then the tables from i) to iv) and the stability of $M=5$ show the stability. Let us assume $|B_3|=3$, i.e., B=$(\pm1,0,\pm3)$ or $(0,\pm1,\pm3)$. First we use, for each case, the condition from v) to viii) and obtain the new B-vector as follows:

B	used condition	B-vector after a line is deleted
$(1,0,3)$	vi)	$(0,1,2)$
$(-1,0,3)$	v)	$(0,-1,2)$
$(0,1,3)$	v)	$(1,0,2)$
$(0,-1,3)$	vi)	$(-1,0,2)$
$(1,0,-3)$	viii)	$(0,-1,-2)$
$(-1,0,-3)$	vii)	$(0,1,-2)$
$(0,1,-3)$	viii)	$(-1,0,-2)$
$(0,-1,-3)$	vii)	$(1,0,-2)$

So the stability of $M = 5$ shows the stability of $M = 10$. Moreover, Proposition 2.6 and the case $M = 10$ shows the stability of the case $|\tilde{B}| = (3,0,0)$. Next we show the case $M = 14$, i.e., $|\tilde{B}| = (3,2,1)$. When $|B_1| = 3$ or $|B_2| = 3$, we can show its stability by the same way as above. So let us assume $|B_3| = 3$. By the same way as when $M = 10$, we use the condition from v) to viii) and obtain the new B-vector as follows:

B	used condition	B-vector after a line is deleted
$(1,2,3)$	v)	$(2,1,2)$
$(-1,2,3)$	v)	$(0,1,2)$
$(1,-2,3)$	vi)	$(0,-1,2)$
$(-1,-2,3)$	vi)	$(-2,-1,2)$
$(2,1,3)$	vi)	$(1,2,2)$
$(-2,1,3)$	v)	$(-1,0,2)$
$(2,-1,3)$	vi)	$(1,0,2)$
$(-2,-1,3)$	v)	$(-1,-2,2)$
$(1,2,-3)$	viii)	$(0,1,-2)$
$(-1,2,-3)$	viii)	$(-2,1,-2)$
$(1,-2,-3)$	vii)	$(2,-1,-2)$
$(-1,-2,-3)$	vii)	$(0,-1,-2)$
$(2,1,-3)$	viii)	$(1,0,-2)$
$(-2,1,-3)$	vii)	$(-1,2-2)$
$(2,-1,-3)$	viii)	$(1,-2,-2)$
$(-2,-1,-3)$	vii)	$(-1,0,-2)$

So the stability of $|\tilde{B}| = (2,2,1)$ and $(2,1,0)$ shows the stability.

When $M = 16 \iff |\tilde{B}| = (4,0,0)$, Schenck's criterion and the stability of $(3,0,0)$ and $(3,1,1)$ show its stability. When $M = 17$, Proposition 2.6 and the cases $M = 14,\ 16$ show the stability. When $M = 19,\ 22$ or $|\tilde{B}| = (3,3,0),\ (3,3,3)$, we can show their stability by the same way as above. Now we have classified the stability and freeness of families whose $|\tilde{B}|$ is, if ordered in the lexicographic order, less than $(4,0,0)$. So for the rest of this theorem, we may assume the maximal value of $|B_i|$ $(i=1,2,3)$ is more than or equal to 4. Note that we know if $|B_i|=3$ for some i, then the family is stable. So we can show the stability by the induction on M. If $|B_1|$ or $|B_2|$ is the maximal value, then Schenck's criterion shows its stability immediately. Moreover, if $|B_1|$ or $|B_2|$ is more than or equal to 3, then also Schenck's criterion shows the stability. So we may assume $|B_3|$ is the maximal value and $|B_3|-|B_i|>1$ for $i=1,2$. In this case, one of the inequality of numerical conditions v), vi), vii), or viii) holds for some line $H(k)\in\mathcal{A}(k)$. Let us put

$$M' := B_1(\mathcal{A}(k)\setminus H(k))^2 + B_2(\mathcal{A}(k)\setminus H(k))^2 + B_3(\mathcal{A}(k)\setminus H(k))^2.$$

Recall the family is stable if $|B_i|=3$ for some i. Hence we may assume, as the induction hypothesis, families of B_2-type arrangements are stable if $\sum_{i=1}^3 B_i^2 < M$ with $|B_i|\geq 3$ for some i. Then the following table makes the induction work:

inequality	$M-M'$
v)	$M-((B_1+1)^2+(B_2-1)^2+(B_3-1)^2 = 2(-B_1+B_2+B_3))-3>0$
vi)	$M-((B_1-1)^2+(B_2+1)^2+(B_3-1)^2 = 2(B_1-B_2+B_3))-3>0$
vii)	$M-((B_1+1)^2+(B_2+1)^2+(B_3+1)^2 = 2(-B_1-B_2-B_3))-3>0$
viii)	$M-((B_1-1)^2+(B_2-1)^2+(B_3+1)^2 = 2(B_1+B_2-B_3))-3>0$

At last, we show the stability when $M=8,\ 12$ and $a+d$ is odd. First we show when $M=8$. In this case the normalized Chern polynomial is $1+2t^2$. Since $a+d$, b and e are odd in this case, the numerical condition (NC) of Schenck's criterion becomes weaker as follows:

number	deleting line	NC
i)	$Y+X=(k+a+b-1)Z$	$B_1>0$
ii)	$Y+X=(-k+a)Z$	$B_1<0$
iiii)	$Y-X=(k+d+e-1)Z$	$B_2>0$
iv)	$Y-X=(-k+d)Z$	$B_2<0$

Since it holds that $|B_1|=2$ or $|B_2|=2$, the stability when $M=5$ shows the stability of this case. The stability when $M=12$ and $a+d$ is odd can be shown by the same argument as above with the stability when $M=9$. Hence the theorem is proved. □

Remark 2.1
To show the stability and semistability are determined by the combinatorics, it suffices to show the combinatorics when $M=12$ and $a+d$ is even is

different from those of other case when the normalized Chern polynomial is $1+2t^2$. It is easy to see when $M=10$ or $M=8$ and $a+d$ is odd, there do not exist any plane $H(k)$ such that $\{\mathcal{A}(k)\cup H(k)\}$ or $\{\mathcal{A}(k)\setminus H(k)\}$ is a free family of B_2-type arrangements. On the other hand, when $M=12$ and $a+d$ is even, we can find such $H(k)$, for each B-vector, as follows:

B-vector	add/delete a line	B-vector after the addition-deletion
$(2,2,2)$	add $Y=(k+f)Z$	$(1,1,1)$
$(-2,2,2)$	delete $X=(k+c-1)Z$	$(-1,1,1)$
$(2,-2,2)$	delete $X=(-k+1)Z$	$(1,-1,1)$
$(2,2,-2)$	delete $Y=(-k+1)Z$	$(1,1,-1)$
$(-2,-2,2)$	add $Y=-kZ$	$(-1,-1,1)$
$(-2,2,-2)$	add $X=-kZ$	$(-1,1,-1)$
$(2,-2,-2)$	add $X=(k+c)Z$	$(1,-1,-1)$
$(-2,-2,-2)$	delete $Y=(k+f-1)Z$	$(-1,-1,-1)$

Since $a+d$ is even after adding/deleting these lines, we have the statement. Then it is immediate to see the combinatorics of these are different.

Remark 2.2
In the arrangement theory, there is a famous conjecture by Terao as follows:

Conjecture 2.8 (Terao conjecture)
The freeness of an arrangement depends only on its combinatorics.

This is an open problem, and there are some results which support this conjecture. For example, Terao conjecture is true for free arrangements constructed by the addition-deletion theorem, so is for B_2-type arrangements. Moreover, Terao conjecture is true for A_2-type arrangements (see [A]). However, it is uncertain whether Terao conjecture is true or not. On the other hand, the results in [DK], [A] and in this article show the examples of arrangements whose (semi)stability depend only on their combinatorics. Hence we can pose the following conjecture:

Conjecture 2.9
The (semi)stability of an arrangement depends only on its combinatorics.

Remark 2.3
Let $\{\mathcal{A}(k)\}$ be a family of B_2-type arrangements. Then by Lemma 2.1, it holds that

$$c_t(D_0(\widetilde{\mathcal{A}(k+1)}))=c_t(D_0(\widetilde{\mathcal{A}(k)})\otimes\mathcal{O}(-4))\ (k\gg 0).$$

Based on some other evidences, Yoshinaga conjectured there are a family of isomorphisms

$$D_0(\widetilde{\mathcal{A}(k+1)}) \simeq D_0(\widetilde{\mathcal{A}(k)}) \otimes \mathcal{O}(-4) \ (k \gg 0).$$

We call this conjecture the 4-*shift problem, since the shift in the isomorphism and the Coxeter number of the root system of type* B_2 *are both* 4*. We have formulated, by the same manner, the* 3-*shift problem in [A]. By the main theorem, we can see the* 4-*shift problem is true when*

$$M = 0,\ 1,\ 2,\ 3,\ 5,$$

or

$$M = 4 \text{ and } a + d \text{ is even}.$$

Moreover, the (semi)stability of B_2*-type arrangements (Theorem 0.2) also supports this conjecture.*

3 Proof of the freeness criterions

We prove Proposition 2.3 and 2.4 in this section. The notation is the same as in Section 2. To show them, we use two arguments. One is the induction on c and the other is to reduce to some other free conditions. Both of them are completed by the addition-deletion theorem (Theorem 1.7). To make the counting easy, we often consider 3-arrangements in $\mathbf{P}^2$, or their deconing with respect to the infinite line $\{Z = 0\}$. So we use the terms "line" and "plane" interchangeably in this section. First we show some freeness conditions for induction.

Proposition 3.1
Let $\{\mathcal{A}(k)\}$ *be the family of* B_2*-type arrangements. If* $a + d$ *is odd, then* $\{\mathcal{A}(k)\}_{k=1}^{\infty}$ *is a free family of arrangements if one of the following holds:*

(e-i)

$$c = f = 0,\ 2a + b = 1,\ 2d + e = 1.$$

(g-i)

$$c = f = 0,\ 2a + b = 2 \text{ or } 0,\ 2d + e = 2 \text{ or } 0.$$

(g-ii)

$$c = 1,\ f = 0,\ 2a + b = 2,\ 2d + e = 1 \text{ or } -1.$$

(h-i)

$$c = f = 0,\ 2a + b = 2 \text{ or } 0,\ 2d + e = 1.$$

(h-ii)
$$c = f = 0,\ 2a + b = 1,\ 2d + e = 2 \text{ or } 0.$$

Moreover, in the cases above, $|\exp(\mathcal{A}(k))| = 0$ *or* 1 *for* $k \gg 0$.

Proof. First, note that the arrangement defined by

$$\begin{aligned} X &= (-k+1)Z, (-k+2)Z, \ldots, (k-1)Z, \\ Y &= (-k+1)Z, (-k+2)Z, \ldots, (k-1)Z, \\ Y+X &= (-k+1)Z, (-k+2)Z, \ldots, (k-1)Z, \\ Y-X &= (-k+1)Z, (-k+2)Z, \ldots, (k-1)Z, \\ Z &= 0, \end{aligned}$$

is free with exponents $(1, 4k-3, 4k-1)$. This is shown by Lemma 2.1, Theorem 1.9, and Theorem 1.1 in [T]. We call this arrangement $\mathcal{A}_0(k)$.

(e-i) $c = f = 0,\ 2a + b = 1,\ 2d + e = 1$. We prove by induction on $a + d$ and using the addition-deletion theorem. The condition implies $a \le 1$ and $d \le 1$. Hence $a + d$ is maximal when $(a, b, d, e) = (1, -1, 0, 1)$ or $(0, 1, 1, -1)$. In this case $a + d = 1$. First we consider the former case. At first we add the plane
$$H_1(k) := \{Y - X = kZ\}$$
to $\mathcal{A}_0(k)$ and secondly

$$H_2(k) := \{Y - X = -kZ\}$$

to $\mathcal{A}_0(k) \cup H_1(k)$. Then it is easy to see that

$$|\mathcal{A}_0(k) \cap H_1(k)| = 1 + (k) + (2k - 1 - (-k+1) + 1) = 4k,$$

so the addition-deletion theorem shows the family is free with

$$\exp(\mathcal{A}_0(k) \cup H_1(k)) = (1, 4k-2, 4k-1)$$

when $(a, b, c, d, e, f) = (1, -1, 0, 0, 1, 0)$. Similarly, we can see that

$$|(\mathcal{A}_0(k) \cup H_1(k)) \cap H_2(k)| = 4k,$$

so the addition-deletion theorem shows

$$\exp(\mathcal{A}_0(k) \cup H_1(k) \cup H_2(k)) = (1, 4k-1, 4k-1).$$

For the rest of this article we express the above process as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $H_1(k)$	$4k$	$(1, 4k-2, 4k-1)$
add $H_2(k)$	$4k$	$(1, 4k-1, 4k-1)$

The second case $(a, b, d, e) = (0, 1, 1, -1)$ can be proved by the same way, so the first step of induction is completed. Let us assume that

$$\exp(\mathcal{A}(k)) = (1, 4k-a-d, 4k-a-d)$$

for $\mathcal{A}(k)$ defined by (e-i). Since $a+d$ is odd, it suffices to show the freeness is invariant under the following three transforms of (a, d):

(P1) $(a, d) \mapsto (a-2, d)$.

(P2) $(a, d) \mapsto (a, d-2)$.

(P3) $(a, d) \mapsto (a-1, d-1)$.

Noting that the plane $Y + X = (k+a+b-1)Z$ is equal to $Y + X = (k-a)Z$ in this case, (P1) is shown as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (-k+a-1)Z$	$4k-a-d+1$	$(1, 4k-a-d, 4k-a-d+1)$
add $Y + X = (k-a+1)Z$	$4k-a-d+1$	$(1, 4k-a-d, 4k-a-d+2)$
add $Y + X = (-k+a-2)Z$	$4k-a-d+3$	$(1, 4k-a-d+1, 4k-a-d+2)$
add $Y + X = (k-a+2)Z$	$4k-a-d+3$	$(1, 4k-a-d+2, 4k-a-d+2)$

(P2) is as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (-k+d-1)Z$	$4k-a-d+1$	$(1, 4k-a-d, 4k-a-d+1)$
add $Y - X = (k-d+1)Z$	$4k-a-d+1$	$(1, 4k-a-d, 4k-a-d+2)$
add $Y - X = (-k+d-2)Z$	$4k-a-d+3$	$(1, 4k-a-d+1, 4k-a-d+2)$
add $Y - X = (k-d+2)Z$	$4k-a-d+3$	$(1, 4k-a-d+2, 4k-a-d+2)$

(P3) is as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (-k+a-1)Z$	$4k-a-d+1$	$(1, 4k-a-d, 4k-a-d+1)$
add $Y - X = (-k+d-1)Z$	$4k-a-d+2$	$(1, 4k-a-d+1, 4k-a-d+1)$
add $Y + X = (k-a+1)Z$	$4k-a-d+2$	$(1, 4k-a-d+1, 4k-a-d+2)$
add $Y - X = (k-d+1)Z$	$4k-a-d+3$	$(1, 4k-a-d+2, 4k-a-d+2)$

These tables show (e-i) is free. For the rest of the proof, we use the same notation as in this proof.

(g-i) $c = f = 0,\ 2a+b = 2$ or $0,\ 2d+e = 2$ or 0. First, we consider when $c = f = 0,\ 2a+b = 2$ and $2d+e = 2$. We show by the descending induction on $a+d$. The $a+d$ is maximal when

$$(a, b, d, e) = (1, 0, 0, 2) \text{ or } (0, 2, 1, 0).$$

Since these are symmetric, it suffices to show the former case. We start from the arrangement $\mathcal{A}_0(k)$ and add lines as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = -kZ$	$4k$	$(1, 4k-2, 4k-1)$
add $Y - X = kZ$	$4k$	$(1, 4k-1, 4k-1)$
add $Y - X = (k+1)Z$	$4k$	$(1, 4k-1, 4k)$
add $Y + X = kZ$	$4k+1$	$(1, 4k, 4k)$

So the case when $a+d$ is maximal is proved. Let us assume that

$$\exp(\mathcal{A}(k)) = (1, 4k-a-d+1, 4k-a-d+1).$$

We show the freeness by the same way as in the proof of (e-i) in this proposition. (P1) is shown as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (-k+a-1)Z$	$4k-a-d+2$	$(1, 4k-a-d+1, 4k-a-d+2)$
add $Y + X = (k-a+2)Z$	$4k-a-d+3$	$(1, 4k-a-d+2, 4k-a-d+2)$
add $Y + X = (-k+a-2)Z$	$4k-a-d+3$	$(1, 4k-a-d+2, 4k-a-d+3)$
add $Y + X = (k-a+3)Z$	$4k-a-d+4$	$(1, 4k-a-d+3, 4k-a-d+3)$

(P2) is as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (-k+d-1)Z$	$4k-a-d+2$	$(1, 4k-a-d+1, 4k-a-d+2)$
add $Y - X = (k-d+2)Z$	$4k-a-d+3$	$(1, 4k-a-d+2, 4k-a-d+2)$
add $Y - X = (-k+d-2)Z$	$4k-a-d+3$	$(1, 4k-a-d+2, 4k-a-d+3)$
add $Y - X = (k-d+3)Z$	$4k-a-d+4$	$(1, 4k-a-d+3, 4k-a-d+3)$

(P3) is as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (-k+a-1)Z$	$4k-a-d+2$	$(1, 4k-a-d+1, 4k-a-d+2)$
add $Y - X = (-k+d-1)Z$	$4k-a-d+3$	$(1, 4k-a-d+2, 4k-a-d+2)$
add $Y + X = (k-a+2)Z$	$4k-a-d+3$	$(1, 4k-a-d+2, 4k-a-d+3)$
add $Y - X = (k-d+2)Z$	$4k-a-d+4$	$(1, 4k-a-d+3, 4k-a-d+3)$

So this case is proved. Next, we show when $c = f = 0,\ 2a+b = 2,\ 2d+e = 0$. We add two lines to reduce to the case above $c = f = 0,\ 2a+b = 2,\ 2d+e = 2$ with $|\exp| = 0$ as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (k-d)Z$	$4k - a - d + 1$	$(1, 4k-a-d, 4k-a-d+1)$
add $Y - X = (k-d+1)Z$	$4k - a - d + 2$	$(1, 4k-a-d+1, 4k-a-d+1)$

So this case is proved. Next, we show when $c = f = 0,\ 2a+b = 0,\ 2d+e = 2$. We add two lines to reduce to the case above $c = f = 0,\ 2a+b = 2,\ 2d+e = 2$ with $|\exp| = 0$ as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (k-a)Z$	$4k - a - d + 1$	$(1, 4k-a-d, 4k-a-d+1)$
add $Y + X = (k-a+1)Z$	$4k - a - d + 2$	$(1, 4k-a-d+1, 4k-a-d+1)$

So this case is proved. Next, we show when $c = f = 0,\ 2a+b = 0,\ 2d+e = 0$. We delete four lines to reduce to the case above $c = f = 0,\ 2a+b = 2,\ 2d+e = 2$ with $|\exp| = 0$ as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (k-d)Z$	$4k - a - d$	$(1, 4k-a-d-1, 4k-a-d)$
add $Y - X = (k-d+1)Z$	$4k - a - d + 1$	$(1, 4k-a-d, 4k-a-d)$
add $Y + X = (k-a)Z$	$4k - a - d + 1$	$(1, 4k-a-d, 4k-a-d+1)$
add $Y + X = (k-a+1)Z$	$4k - a - d + 2$	$(1, 4k-a-d+1, 4k-a-d+1)$

So (g-i) is proved.

(g-ii) $c = 1,\ f = 0,\ 2a+b = 2,\ 2d+e = 1$ or -1. First we show the case $c = 1,\ f = 0,\ 2a+b = 2,\ 2d+e = 1$ is a free arrangement with exponents $(1, 4k-a-d+1, 4k-a-d+1)$. We add/delete four lines to reduce to the case (g-i) $c = f = 0,\ 2a+b = 2,\ 2d+e = 0$, which is free with $|\exp| = 0$ as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (-k+d-1)Z$	$4k - a - d + 2$	$(1, 4k-a-d+1, 4k-a-d+2)$
add $Y - X = (-k+d-2)Z$	$4k - a - d + 3$	$(1, 4k-a-d+2, 4k-a-d+2)$
add $Y - X = (k-d+1)Z$	$4k - a - d + 3$	$(1, 4k-a-d+2, 4k-a-d+3)$
delete $X = kZ$	$4k - a - d + 3$	$(1, 4k-a-d+2, 4k-a-d+2)$

So this case is proved. Next, we consider the case $c = 1,\ f = 0,\ 2a+b = 2,\ 2d+e = -1$. We add/delete lines to reduce to (g-i) $c = f = 0,\ 2a+b = 2,\ 2d+e = 0$ in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (k-d-1)Z$	$4k-a-d+1$	$(1, 4k-a-d, 4k-a-d+1)$
delete $X = kZ$	$4k-a-d+1$	$(1, 4k-a-d, 4k-a-d)$

So (g-ii) is proved.

(h-i) $c = f = 0,\ 2a+b = 2$ or $0,\ 2d+e = 1$. We show the case $c = f = 0,\ 2a+b = 2,\ 2d+e = 1$ is free with exponents $(1, 4k-a-d, 4k-a-d+1)$. We add a line to reduce to the case (g-i) $c = f = 0,\ 2a+b = 2,\ 2d+e = 2$ such that $|\exp| = 0$ as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (k-d+1)Z$	$4k-a-d+2$	$(1, 4k-a-d+1, 4k-a-d+1)$

So this case is proved. Next we show the case $c = f = 0,\ 2a+b = 0,\ 2d+e = 1$ is free with exponents $(1, 4k-a-d-1, 4k-a-d)$. We add a line to reduce to the case (e-i) $c = f = 0,\ 2a+b = 1,\ 2d+e = 1$ such that $|\exp| = 0$ as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (k-a)Z$	$4k-a-d+1$	$(1, 4k-a-d, 4k-a-d)$

So (h-i) is proved.

(h-ii) $c = f = 0,\ 2a+b = 1,\ 2d+e = 2$ or 0. Since this is symmetric to the case (h-i), we omit the proof. Hence Proposition 3.1 is proved. $\square$

By the same way as above, we show the following classification for freeness.

Proposition 3.2
Let $\{\mathcal{A}(k)\}$ be the family of B_2-type arrangements. If $a+d$ is even, then $\{\mathcal{A}(k)\}_{k=1}^{\infty}$ is a free family of arrangements if one of the following holds:

(a-i)
$$c = f = 0,\ 2a+b = 1,\ 2d+e = 1.$$

(a-ii)
$$c = f = 0,\ 2a+b = 3 \text{ or } -1,\ 2d+e = 1.$$

(a-iii)
$$c = 2,\ f = 0,\ 2a+b = 3,\ 2d+e = -1.$$

(c-i)
$$c = f = 0,\ 2a+b = 2 \text{ or } 0,\ 2d+e = 2 \text{ or } 0.$$

Moreover, in the cases above, $|\exp(\mathcal{A}(k))| = 2$ *when (a-i), and* 0 *or* 1 *otherwise.*

Proof. We prove by the same way as that of Proposition 3.1.

(a-i) $c = f = 0,\ 2a + b = 1,\ 2d + e = 1$. We show its exponents are $(1, 4k - a - d - 1, 4k - a - d + 1)$. We add a line to reduce to (h-i) $c = f = 0,\ 2a + b = 0,\ 2d + e = 1$ of Proposition 3.1 such that $|\exp| = 1$. The process is as follows:

add/delete a line	$\mathcal{A} \cap \{$the line$\}$	exponents
add $Y + X = (-k + a - 1)Z$	$4k - a - d + 2$	$(1, 4k - a - d, 4k - a - d + 1)$

So $|\exp| = 2$ in this case.

(a-ii) $c = f = 0,\ 2a + b = 3$ or $-1,\ 2d + e = 1$. First consider the case $c = f = 0,\ 2a + b = 3,\ 2d + e = 1$. We show the exponents are $(1, 4k - a - d + 1, 4k - a - d + 1)$. We add a line to reduce to (h-i) $c = f = 0,\ 2a + b = 2,\ 2d + e = 1$ in Proposition 3.1 such that $|\exp| = 1$. The process is as follows:

add/delete a line	$\mathcal{A} \cap \{$the line$\}$	exponents
add $Y + X = (-k + a - 1)Z$	$4k - a - d + 2$	$(1, 4k - a - d + 1, 4k - a - d + 2)$

So this case is proved. Next we show the case $c = f = 0,\ 2a+b = -1,\ 2d+e = 1$. We show the exponents are $(1, 4k - a - d - 1, 4k - a - d - 1)$. We add two lines to reduce to the case (a-i) $c = f = 0,\ 2a + b = 1,\ 2c + d = 1$ with $|\exp| = 2$ in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{$the line$\}$	exponents
add $Y + X = (k - a - 1)Z$	$4k - a - d$	$(1, 4k - a - d - 1, 4k - a - d - 1)$
add $Y + X = (k - a)Z$	$4k - a - d$	$(1, 4k - a - d - 1, 4k - a - d + 1)$

So (a-ii) is proved.

(a-iii) $c = 2,\ f = 0,\ 2a + b = 3,\ 2d + e = -1$. We show the exponents are $(1, 4k - a - d + 1, 4k - a - d + 1)$. We add/delete two lines to reduce to (g-ii) $c = 1,\ f = 0,\ 2a + b = 2,\ 2d + e = -1$ in Proposition 3.1 such that $|\exp| = 0$. The process is as follows:

add/delete a line	$\mathcal{A} \cap \{$the line$\}$	exponents
delete $X = (k + 1)Z$	$4k - a - d + 2$	$(1, 4k - a - d, 4k - a - d + 1)$
add $Y + X = (-k + a - 1)Z$	$4k - a - d + 2$	$(1, 4k - a - d + 1, 4k - a - d + 1)$

So (a-iii) is proved.

(c-i) $c = f = 0,\ 2a + b = 2$ or $0,\ 2d + e = 2$ or 0. First we show the case $c = f = 0,\ 2a + b = 2,\ 2d + e = 2$ is free with exponents $(1, 4k - a - d + 1, 4k - a - d + 1)$. We add lines to reduce to the case (h-i) $c = f = 0,\ 2a + b = 2,\ 2d + e = 1$ in Proposition 3.1 such that $|\exp| = 1$. The process is as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (-k + d - 1)Z$	$4k - a - d + 2$	$(1, 4k - a - d + 1, 4k - a - d + 2)$

So this case is proved. Next we show the case $c = f = 0,\ 2a+b = 2,\ 2d+e = 0$ is free with exponents $(1, 4k - a - d, 4k - a - d)$. We add a line to reduce to the case (h-ii) $c = f = 0,\ 2a + b = 1,\ 2d + e = 0$ such that $|\exp| = 1$ as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (-k + a - 1)Z$	$4k - a - d + 1$	$(1, 4k - a - d, 4k - a - d + 1)$

So this case is proved. Next we show the case $c = f = 0,\ 2a+b = 0,\ 2d+e = 2$ is free with exponents $(1, 4k - a - d, 4k - a - d)$. We add a line to reduce to the case (h-i) $c = f = 0,\ 2a + b = 0,\ 2d + e = 1$ such that $|\exp| = 1$ as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (-k + d - 1)Z$	$4k - a - d + 1$	$(1, 4k - a - d, 4k - a - d + 1)$

So this case is proved. At last, we show the case $c = f = 0,\ 2a + b = 0,\ 2d + e = 0$ is free with exponents $(1, 4k - a - d - 1, 4k - a - d - 1)$. We add two lines to reduce to the case above, $c = f = 0,\ 2a + b = 0,\ 2d + e = 2$ such that $|\exp| = 0$ as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (k - d)Z$	$4k - a - d$	$(1, 4k - a - d - 1, 4k - a - d)$
add $Y - X = (k - d + 1)Z$	$4k - a - d + 1$	$(1, 4k - a - d, 4k - a - d)$

So (c-i) is, hence Proposition 3.2 is proved. □

Proof of Proposition 2.3. We use the same notation and argument as in the proof of Proposition 3.1 and 3.2.

(e-i) $c = f,\ 2a + b = 2c + 1,\ 2d + e = 1$. We show by the induction on $c \geq 0$. When $c = f = 0$, it was shown in (e-i) Proposition 3.1 . Let us assume the statement is true for $c-1 \geq 0$. We show the exponents when c are $(1, 4k-a-d+2c, 4k-a-d+2c)$. Let us put

$$A := 4k - a - d + 2c.$$

We add/delete four lines to/from the case when c to reduce to the case $c-1$ as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y+X=(-k+a-1)Z$	$A+1$	$(1, A, A+1)$
delete $Y=(k+c-1)Z$	$A+1$	$(1, A, A)$
delete $X=(k+c-1)Z$	$A+1$	$(1, A-1, A)$
add $Y+X=(-k+a-2)Z$	$A+1$	$(1, A, A)$

i.e., we made the change of variables $(a, b, c) \mapsto (a-2, b+2, c-1)$. By the condition on a and b, this kind of addition/deletion can be completed without restriction. So (e-i) is proved. Note that we use this kind of the addition-deletion repeatedly in the proof.

(g-i) $c = f,\ 2a+b = 2c+2$ or $2c,\ 2d+e = 2$ or 0. First we consider the case $c = f,\ 2a+b = 2c+2,\ 2d+e = 2$. When $c = f = 0$, it is proved in (g-i) Proposition 3.1. Assume the statement is true when $c-1 \geq 0$. We show the exponents when c are $(1,\ A+1,\ A+1)$. We add/delete four lines to reduce to the case $c-1$ as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y+X=(-k+a-1)Z$	$A+2$	$(1, A+1, A+2)$
add $Y+X=(-k+a-2)Z$	$A+3$	$(1, A+2, A+2)$
delete $X=(k+c-1)Z$	$A+3$	$(1, A+1, A+2)$
delete $Y=(k+c-1)Z$	$A+2$	$(1, A+1, A+1)$

So this case is proved. Next we show the case $c = f,\ 2a+b = 2c,\ 2d+e = 2$ is free with $|\exp| = 0$. We delete two lines from the above case as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
delete $Y+X=(k-a+2c+1)Z$	$A+2$	$(1, A, A+1)$
delete $Y+X=(k-a+2c)Z$	$A+1$	$(1, A, A)$

So this case is proved. Next we show the case $c = f,\ 2a+b = 2c+2,\ 2d+e = 0$ is free with $|\exp| = 0$. By the same way as above, we can show the freeness of this case as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
delete $Y-X=(k-d+1)Z$	$A+2$	$(1, A, A+1)$
delete $Y-X=(k-d)Z$	$A+1$	$(1, A, A)$

At last we show the case $c = f,\ 2a+b = 2c,\ 2d+e = 0$ is free with $|\exp| = 0$. We delete four lines from the case $c = f,\ 2a + b = 2c + 2,\ 2d + e = 2$ above with $|\exp| = 0$ as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
delete $Y - X = (k - d + 1)Z$	$A + 2$	$(1, A, A + 1)$
delete $Y - X = (k - d)Z$	$A + 1$	$(1, A, A)$
delete $Y + X = (k - a + 2c + 1)Z$	$A + 1$	$(1, A - 1, A)$
delete $Y + X = (k - a + 2c)Z$	A	$(1, A - 1, A - 1)$

So (g-i) is proved.

(g-ii) $c = f + 1,\ 2a + b = 2c + 1$ or $2c - 1,\ 2d + e = 0$. First we show the case $c = f + 1,\ 2a + b = 2c + 1,\ 2d + e = 0$ is free with the exponents $(1, A-1, A-1)$. We add two lines to reduce to (g-i) in this proposition with $|\exp| = 0$ as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y = (k + c - 1)Z$	A	$(1, A - 1, A)$
add $Y + X = (k - a + 2c + 1)Z$	$A + 1$	$(1, A, A)$

So this case is proved. Next we show the case $c = f + 1,\ 2a + b = 2c - 1,\ 2d + e = 0$ is free with the exponents $(1, A - 2, A - 2)$. We add two lines to reduce to the case above as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (k - a + 2c - 1)Z$	$A - 1$	$(1, A - 2, A - 1)$
add $Y + X = (k - a + 2c)Z$	A	$(1, A - 1, A - 1)$

So (g-ii) is proved.

(g-iii) $c = f - 1,\ 2a + b = 2c + 3$ or $2c + 1,\ 2d + e = 2$. First we consider the case $c = f - 1,\ 2a + b = 2c + 3,\ 2d + e = 2$. We show this is a free arrangement with exponents $(1, A+2, A+2)$ by induction on c. When $c = 0$ and $f = 1$ we can show its freeness by deleting the lines $Y+X = (k-a+2)Z$ at first, and $Y = kZ$ secondly from this case to reduce to (g-i) Proposition 3.1. Let us assume the statement is true when $c-1 \geq 0$. We add/delete four lines from the case c to reduce to the case $c - 1$ as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (-k + a - 1)Z$	$A + 3$	$(1, A + 2, A + 3)$
add $Y + X = (-k + a - 2)Z$	$A + 4$	$(1, A + 3, A + 3)$
delete $Y = (k + c)Z$	$A + 4$	$(1, A + 2, A + 3)$
delete $X = (k + c - 1)Z$	$A + 3$	$(1, A + 2, A + 2)$

So this case is proved. Next we consider the case $c = f - 1,\ 2a + b = 2c + 1,\ 2d + e = 2$. We show this is free with exponents $(1, A+1, A+1)$. We add two lines to reduce to the case above as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (k - a + 2c + 1)Z$	$A + 2$	$(1, A+1, A+2)$
add $Y + X = (k - a + 2c + 2)Z$	$A + 3$	$(1, A+2, A+2)$

So (g-iii) is proved.

(g-iv) $c = f + 1,\ 2a + b = 2c,\ 2d + e = 1$ or -1. First we consider the case $c = f + 1,\ 2a + b = 2c,\ 2d + e = 1$. We show this is a free arrangement with exponents $(1, A-1, A-1)$ by induction on c. When $c = 1$ and $f = 0$ this is (g-ii) in Proposition 3.1. We show the case when c is free with exponents $(1, A-1, A-1)$. We add/delete four lines to reduce to the case $c - 1$ as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (-k + a - 1)Z$	A	$(1, A-1, A)$
delete $X = (k + c - 1)Z$	$A + 1$	$(1, A-2, A)$
delete $Y = (k + c - 2)Z$	$A - 1$	$(1, A-2, A-1)$
add $Y + X = (-k + a - 2)Z$	A	$(1, A-1, A-1)$

So this case is proved. Next we consider the case $c = f + 1,\ 2a + b = 2c,\ 2d + e = -1$. We show this is free with exponents $(1, A-2, A-2)$. We add two lines to reduce to the case above as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (k - d - 1)Z$	$A - 1$	$(1, A-2, A-1)$
add $Y - X = (k - d)Z$	A	$(1, A-1, A-1)$

So (g-iv) is proved.

(g-v) $c = f - 1,\ 2a + b = 2c + 2,\ 2d + e = 3$ or 1. First we consider the case $c = f - 1,\ 2a + b = 2c + 2,\ 2d + e = 3$. We use the induction on c. When $c = 0$, we can show it by adding lines $Y = kZ$ at first, and $Y - X = (k - d + 2)Z$ secondly to (g-i) in Proposition 3.1. Assume the statement is true when $c - 1 \geq 0$. We show the case when c is a free arrangement with exponents $(1, A+2, A+2)$. We add/delete four lines to reduce to the case $c - 1$ as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (-k + a - 1)Z$	$A + 3$	$(1, A+2, A+3)$
delete $Y = (k + c)Z$	$A + 3$	$(1, A+2, A+2)$
delete $X = (k + c - 1)Z$	$A + 3$	$(1, A+1, A+2)$
add $Y + X = (-k + a - 2)Z$	$A + 3$	$(1, A+2, A+2)$

So this case is proved. Next we consider the case $c = f - 1,\ 2a + b = 2c + 2,\ 2d + e = 1$. We show this is free with exponents $(1, A+1, A+1)$. We add two lines to reduce to the case above as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (k - d + 1)Z$	$A + 2$	$(1, A+1, A+2)$
add $Y - X = (k - d + 2)Z$	$A + 3$	$(1, A+2, A+2)$

So (g-v) is proved.

(h-i) $c = f,\ 2a + b = 2c + 2$ or $2c,\ 2d + e = 1$. First we consider the case $c = f,\ 2a + b = 2c + 2,\ 2d + e = 1$. We show this is a free arrangement with exponents $(1, A, A+1)$. We delete a line to reduce to the case (e-i) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
delete $Y + X = (k - a + 2c + 1)Z$	$A + 1$	$(1, A, A)$

So this case is proved. Next we consider the case $c = f,\ 2a+b = 2c,\ 2d+e = 1$. We show this is free with exponents $(1, A-1, A)$. We add a line to reduce to (e-i) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (k - a + 2c)Z$	$A + 1$	$(1, A, A)$

So (h-i) is proved.

(h-ii) $c = f,\ 2a + b = 2c + 1,\ 2d + e = 2$ or 0. First we consider the case $c = f,\ 2a + b = 2c + 1,\ 2d + e = 2$. We show this is a free arrangement with exponents $(1, A, A+1)$. We delete a line to reduce to the case (e-i) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
delete $Y - X = (k - d + 1)Z$	$A + 1$	$(1, A, A)$

So this case is proved. Next we consider the case $c = f,\ 2a+b = 2c+1,\ 2d+e = 0$. We show this is free with exponents $(1, A-1, A)$. We add a line to reduce to the case (e-i) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (k - d)Z$	$A + 1$	$(1, A, A)$

So (h-ii) is proved.

(h-iii) $c = f + 1,\ 2a + b = 2c,\ 2d + e = 0$. We show this is a free arrangement with exponents $(1, A-2, A-1)$. We add a line to reduce to the case (g-i) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y = (k+c-1)Z$	A	$(1, A-1, A-1)$

So (h-iii) is proved.

(h-iv) $c = f-1,\ 2a+b = 2c+2,\ 2d+e = 2$. We show this is a free arrangement with exponents $(1, A+1, A+2)$. We delete a line to reduce to the case (g-v) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
delete $Y - X = (k-d+1)Z$	$A+2$	$(1, A+1, A+1)$

So (h-iv) is, and hence Proposition 2.3 is proved. □

Proof of Proposition 2.4. We show by the same way as in Proposition 2.3.

(a-i) $c = f,\ 2a+b = 2c+1,\ 2d+e = 1$. We show by the induction on $c \geq 0$. When $c = f = 0$, it was shown in (a-i) Proposition 3.2. Let us assume the statement is true for $c-1 \geq 0$. We show the exponents when c are $(1, 4k-a-d+2c-1, 4k-a-d+2c+1)$. Let us put

$$A := 4k - a - d + 2c.$$

We add/delete four lines to reduce to the case $c-1$ as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (-k+a-1)Z$	$A+2$	$(1, A, A+1)$
add $Y + X = (-k+a-2)Z$	$A+2$	$(1, A+1, A+1)$
delete $Y = (k+c-1)Z$	$A+2$	$(1, A, A+1)$
delete $X = (k+c-1)Z$	$A+2$	$(1, A-1, A+1)$

So (a-i) is proved.

(a-ii) $c = f,\ 2a+b = 2c+3$ or $2c-1,\ 2d+e = 1$. First we consider the case $c = f,\ 2a+b = 2c+3,\ 2d+e = 1$. We show this is free with exponents $(1, A+1, A+1)$. We add two lines to reduce to (a-i) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (-k+a-1)Z$	$A+2$	$(1, A+1, A+2)$
add $Y + X = (-k+a-2)Z$	$A+2$	$(1, A+1, A+3)$

So this case is proved. Next we consider the case $c = f,\ 2a+b = 2c-1,\ 2d+e = 1$. We show this is free with exponents $(1, A-1, A-1)$. We add two lines to reduce to (a-i) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (k - a + 2c - 1)Z$	A	$(1, A - 1, A)$
add $Y + X = (k - a + 2c)Z$	A	$(1, A - 1, A + 1)$

So (a-ii) is proved.

(a-iii) $c = f,\ 2a + b = 2c + 1,\ 2d + e = 3$ or -1. First we consider the case $c = f,\ 2a + b = 2c + 1,\ 2d + e = 3$. We show this is free with exponents $(1, A + 1, A + 1)$. We add two lines to reduce to (a-i) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (-k + d - 1)Z$	$A + 2$	$(1, A + 1, A + 2)$
add $Y - X = (-k + d - 2)Z$	$A + 2$	$(1, A + 1, A + 3)$

So this case is proved. Next we consider the case $c = f,\ 2a + b = 2c + 1,\ 2d + e = -1$. We show this is free with exponents $(1, A - 1, A - 1)$. We add two lines to reduce to (a-i) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (k - d - 1)Z$	A	$(1, A - 1, A)$
add $Y - X = (k - d)Z$	A	$(1, A - 1, A + 1)$

So (a-iii) is proved.

(a-iv) $c = f + 2,\ 2a + b = 2c - 1,\ 2d + e = -1$. When $c = 2$, this is (a-iii) in Proposition 3.2. Let us assume the statement is true for $c - 1 \geq 0$. We show the exponents when c are $(1, A - 3, A - 3)$. We add/delete four lines to reduce to the case when $c - 1$ as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
delete $X = (k + c - 1)Z$	$A - 2$	$(1, A - 4, A - 3)$
add $Y + X = (-k + a - 1)Z$	$A - 2$	$(1, A - 3, A - 3)$
add $Y + X = (-k + a - 2)Z$	$A - 2$	$(1, A - 3, A - 2)$
delete $Y = (k + c - 3)Z$	$A - 2$	$(1, A - 3, A - 3)$

So (a-iv) is proved.

(a-v) $c = f - 2,\ 2a + b = 2c + 3,\ 2d + e = 3$. When $c = 0$, we can show the freeness by deleting four lines from this case, i.e., first $Y = (k + 1)Z$, secondly $Y - X = (k - d + 2)Z$, thirdly $Y - X = (k - d + 1)Z$ and fourthly $Y = kZ$ to reduce to (a-ii) Proposition 3.2. Let us assume the statement is true for $c - 1 \geq 0$. We show the exponents when c are $(1, A + 3, A + 3)$. We add/delete four lines to reduce to the case when $c - 1$ as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
delete $Y = (k+c+1)Z$	$A+4$	$(1, A+2, A+3)$
add $Y + X = (-k+a-1)Z$	$A+4$	$(1, A+3, A+3)$
add $Y + X = (-k+a-2)Z$	$A+4$	$(1, A+3, A+4)$
delete $X = (k+c-1)Z$	$A+4$	$(1, A+3, A+3)$

So (a-v) is proved.

(b-i) $c = f+1,\ 2a+b = 2c+1$ or $2c-1,\ 2d+e = 1$ or -1. First we consider the case $c = f+1,\ 2a+b = 2c+1,\ 2d+e = 1$. In this case, we show the exponents are $(1, A-1, A)$. We add a line to reduce to (a-i) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y = (k+c-1)Z$	A	$(1, A-1, A+1)$

So this case is proved. Next we consider the case $c = f+1,\ 2a+b = 2c+1,\ 2d+e = -1$. In this case, we show the exponents are $(1, A-2, A-1)$. We add a line to reduce to (a-iii) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y = (k+c-1)Z$	A	$(1, A-1, A-1)$

So this case is proved. Next we consider the case $c = f+1,\ 2a+b = 2c-1,\ 2d+e = 1$. In this case, we show the exponents are $(1, A-2, A-1)$. We add a line to reduce to (a-ii) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y = (k+c-1)Z$	A	$(1, A-1, A-1)$

So this case is proved. At last we show the case $c = f+1,\ 2a+b = 2c-1,\ 2d+e = -1$. We show this is free with exponents $(1, A-3, A-2)$. We delete a line to reduce to (a-iv) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
delete $Y = (k+c-2)Z$	$A-2$	$(1, A-3, A-3)$

So (b-i) is proved.

(b-ii) $c = f-1,\ 2a+b = 2c+3$ or $2c+1,\ 2d+e = 3$ or 1. First we consider the case $c = f-1,\ 2a+b = 2c+3,\ 2d+e = 3$. In this case, we show the exponents are $(1, A+2, A+3)$. We add a line to reduce to (a-v) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y = (k+c+1)Z$	$A+4$	$(1, A+3, A+3)$

So this case is proved. Next we consider the case $c = f - 1,\ 2a + b = 2c + 3,\ 2d + e = 1$. In this case, we show the exponents are $(1, A+1, A+2)$. We delete a line to reduce to (a-ii) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
delete $Y = (k+c)Z$	$A+2$	$(1, A+1, A+1)$

So this case is proved. Next we consider the case $c = f - 1,\ 2a + b = 2c + 1,\ 2d + e = 3$. In this case, we show the exponents are $(1, A+1, A+2)$. We delete a line to reduce to (a-iii) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
delete $Y = (k+c)Z$	$A+2$	$(1, A+1, A+1)$

So this case is proved. At last we show the case $c = f - 1,\ 2a + b = 2c + 1,\ 2d + e = 1$. We show this is free with exponents $(1, A, A+1)$. We delete a line to reduce to (a-i) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
delete $Y = (k+c)Z$	$A+2$	$(1, A-1, A+1)$

So (b-ii) is proved.

(c-i) $c = f,\ 2a + b = 2c + 2$ or $2c,\ 2d + e = 2$ or 0. First we consider the case $c = f,\ 2a + b = 2c + 2,\ 2d + e = 2$. When $c = f = 0$, this is the case (c-i) Proposition 3.2. We show by the induction on c. Assume that the case when $c - 1 \geq 0$ is free with $|\exp| = 0$. We show the exponents when c are $(1, A+1, A+1)$. We add/delete four lines to reduce to the case when $c - 1$ as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (-k + a - 1)Z$	$A+2$	$(1, A+1, A+2)$
add $Y + X = (-k + a - 2)Z$	$A+3$	$(1, A+2, A+2)$
delete $X = (k + c - 1)Z$	$A+3$	$(1, A+1, A+2)$
delete $Y = (k + c - 1)Z$	$A+2$	$(1, A+1, A+1)$

So this case is proved. Next we consider the case $c = f,\ 2a+b = 2c+2,\ 2d+e = 0$. In this case, we show the exponents are $(1, A, A)$. We add two lines to reduce to the case above as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (k - d)Z$	$A+1$	$(1, A, A+1)$
add $Y - X = (k - d + 1)Z$	$A+2$	$(1, A+1, A+1)$

So this case is proved. Next we consider the case $c = f,\ 2a+b = 2c,\ 2d+e = 2$. In this case, we show the exponents are $(1, A, A)$. We add two lines to reduce to the case above as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (k - a + 2c)Z$	$A + 1$	$(1, A, A + 1)$
add $Y + X = (k - a + 2c + 1)Z$	$A + 2$	$(1, A + 1, A + 1)$

So this case is proved. At last we show the case $c = f,\ 2a+b = 2c,\ 2d+e = 0$. We show this is free with exponents $(1, A - 1, A - 1)$. We add four lines to reduce to the case above as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (k - d)Z$	A	$(1, A - 1, A)$
add $Y - X = (k - d + 1)Z$	$A + 1$	$(1, A, A)$
add $Y + X = (k - a + 2c)Z$	$A + 1$	$(1, A, A + 1)$
add $Y + X = (k - a + 2c + 1)Z$	$A + 2$	$(1, A + 1, A + 1)$

So (c-i) is proved.

(c-ii) $c = f + 1,\ 2a + b = 2c + 1$ or $2c - 1,\ 2d + e = 0$. First we consider the case $c = f + 1,\ 2a + b = 2c + 1,\ 2d + e = 0$. We show that the exponents are $(1, A - 1, A - 1)$. We add a line to reduce to (b-i) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (k - d)Z$	A	$(1, A - 1, A)$

So this case is proved. Next we consider the case $c = f + 1,\ 2a + b = 2c - 1,\ 2d + e = 0$. In this case, we show the exponents are $(1, A - 2, A - 2)$. We add two lines to reduce to the case above as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (k - a + 2c - 1)Z$	$A - 1$	$(1, A - 2, A - 1)$
add $Y + X = (k - a + 2c)Z$	A	$(1, A - 1, A - 1)$

So (c-ii) is proved.

(c-iii) $c = f - 1,\ 2a + b = 2c + 3$ or $2c + 1,\ 2d + e = 2$. First we consider the case $c = f - 1,\ 2a + b = 2c + 3,\ 2d + e = 2$. We show that the exponents are $(1, A + 2, A + 2)$. We add a line to reduce to (b-ii) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (k - d + 2)Z$	$A + 3$	$(1, A + 2, A + 3)$

So this case is proved. Next we consider the case $c = f - 1,\ 2a + b = 2c + 1,\ 2d + e = 2$. In this case, we show the exponents are $(1, A + 1, A + 1)$. We add two lines to reduce to the case above as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (k - a + 2c + 1)Z$	$A + 2$	$(1, A + 1, A + 2)$
add $Y + X = (k - a + 2c + 2)Z$	$A + 3$	$(1, A + 2, A + 2)$

So (c-iii) is proved.

(c-iv) $c = f + 1,\ 2a + b = 2c,\ 2d + e = 1$ or -1. First we consider the case $c = f + 1,\ 2a + b = 2c,\ 2d + e = 1$. We show that the exponents are $(1, A - 1, A - 1)$. We add a line to reduce to (b-i) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (k - a + 2c)Z$	A	$(1, A - 1, A)$

So this case is proved. Next we consider the case $c = f + 1,\ 2a + b = 2c,\ 2d + e = -1$. In this case, we show the exponents are $(1, A - 2, A - 2)$. We add two lines to reduce to the case above as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (k - d - 1)Z$	$A - 1$	$(1, A - 2, A - 1)$
add $Y - X = (k - d)Z$	A	$(1, A - 1, A - 1)$

So (c-iv) is proved.

(c-v) $c = f - 1,\ 2a + b = 2c + 2,\ 2d + e = 3$ or 1. First we consider the case $c = f - 1,\ 2a + b = 2c + 2,\ 2d + e = 3$. We show that the exponents are $(1, A + 2, A + 2)$. We add a line to reduce to (b-ii) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (k - a + 2c + 2)Z$	$A + 3$	$(1, A + 2, A + 3)$

So this case is proved. Next we consider the case $c = f - 1,\ 2a + b = 2c + 2,\ 2d + e = 1$. In this case, we show the exponents are $(1, A + 1, A + 1)$. We add two lines to reduce to the case above as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y - X = (k - d + 1)Z$	$A + 2$	$(1, A + 1, A + 2)$
add $Y - X = (k - d + 2)Z$	$A + 3$	$(1, A + 2, A + 2)$

So (c-v) is proved.

(d-i) $c = f,\ 2a + b = 2c + 2$ or $2c,\ 2d + e = 1$. First we consider the case $c = f,\ 2a + b = 2c + 2,\ 2d + e = 1$. We show that the exponents are $(1, A, A + 1)$. We add a line to reduce to (a-ii) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (k - a + 2c + 2)Z$	$A + 2$	$(1, A + 1, A + 1)$

So this case is proved. Next we consider the case $c = f,\ 2a+b = 2c,\ 2d+e = 1$. In this case, we show the exponents are $(1, A-1, A)$. We add a line to reduce to (a-i) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (k - a + 2c)Z$	A	$(1, A-1, A+1)$

So (d-i) is proved.

(d-ii) $c = f,\ 2a + b = 2c + 1,\ 2d + e = 2$ or 0. First we consider the case $c = f,\ 2a + b = 2c + 1,\ 2d + e = 2$. We show that the exponents are $(1, A, A+1)$. We delete a line to reduce to (a-i) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
delete $Y - X = (k - d + 1)Z$	$A + 2$	$(1, A-1, A+1)$

So this case is proved. Next we consider the case $c = f,\ 2a+b = 2c+1,\ 2d+e = 0$. In this case, we show the exponents are $(1, A-1, A)$. We add a line to reduce to (c-i) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y + X = (k - a + 2c + 1)Z$	$A + 1$	$(1, A, A)$

So (d-ii) is proved.

(d-iii) $c = f + 1,\ 2a + b = 2c,\ 2d + e = 0$. We show that the exponents are $(1, A-2, A-1)$. We add a line to reduce to (c-i) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
add $Y = (k + c - 1)Z$	A	$(1, A-1, A-1)$

So (d-iii) is proved.

(d-iv) $c = f-1,\ 2a+b = 2c+2,\ 2d+e = 2$. We show that the exponents are $(1, A+1, A+2)$. We delete a line to reduce to (c-i) in this proposition as follows:

add/delete a line	$\mathcal{A} \cap \{\text{the line}\}$	exponents
delete $Y = (k + c)Z$	$A + 2$	$(1, A+1, A+1)$

So (d-iv) is, and hence Proposition 2.3 is proved. $\square$

By these free arrangements in a three-dimensional vector space, we can obtain the multi-exponents of 2-multiarrangements consist of four lines.

Corollary 3.3
Let V be a two-dimensional vector space over an algebraically closed field $\mathbb{K}$ with $S := Sym(V^*) \simeq \mathbb{K}[X, Y]$. Let $(\mathcal{A}, m)$ be a 2-multiarrangement such that

$$\mathcal{A} = \{X = 0,\ Y = 0,\ X + Y = 0,\ Y - X = 0\}$$

and that

$$\begin{aligned} m(X = 0) &= n, \\ m(Y = 0) &= n + \alpha, \\ m(Y + X = 0) &= m, \\ m(Y - X = 0) &= l \end{aligned}$$

with $\alpha = 0,\ \pm 1$ or ± 2. If n is sufficiently large, then it holds that

- *i) If $\alpha = 0,\ 2n + m + l \equiv 0 \pmod 4$ and $l \equiv m \equiv 1 \pmod 2$, then $|\exp(\mathcal{A})| = 2$.*
- *ii) If $\alpha = 0$ but i) does not hold, then $|\exp(\mathcal{A})| = 0$ or 1 otherwise.*
- *iii) If $\alpha = \pm 1$, then $|\exp(\mathcal{A})| = 0$ or 1.*
- *iv) If $\alpha = \pm 2$ and $l \equiv m \equiv 1 \pmod 2$, then $|\exp(\mathcal{A})| = 0$.*

References

[A] T. Abe, The stability of the family of A_2-type arrangements, preprint in Kyoto University **05**, 2006, to appear in Journal of Mathematics of Kyoto University.

[AY] T. Abe and M. Yoshinaga, Splitting criterion for reflexive sheaves, math.AG/0503710.

[DK] I. Dolgachev and M. Kapranov, Arrangements of hyperplanes and vector bundles on $\mathbf{P}^n$, Duke. Math. J. **71** (1993), no. 3, 633–664

[H] R. Hartshorne, Stable reflexive sheaves. Math. Ann. **254** (1980), 121-176.

[MS] M. Mustaţă and H. Schenck, The module of logarithmic p-forms of a locally free arrangement. J. Algebra **241** (2001), no. 2, 699–719.

[OSS] C. Okonek, M. Schneider and H. Spindler, Vector bundles on complex projective spaces. Progress in Mathematics, **3**. Birkhäuser, Boston, Mass., 1980.

[OT] P. Orlik and H. Terao, Arrangements of hyperplanes. Grundlehren der Mathematischen Wissenschaften, **300**. Springer-Verlag, Berlin, 1992.

[PS] A. Postnikov and R. P. Stanley, Deformations of Coxeter hyperplane arrangements. J. Combin. Theory Ser. A **91** (2000), no. 1-2, 544–597.

[Sch] H. Schenck, Elementary modifications and line configurations in $\mathbf{P}^2$. Comment. Math. Helv. **78** (2003), no. 3, 447–462.

[T] H. Terao, Multiderivations of Coxeter arrangements, Invent. Math. **148** (2002), 659–674.

[Y] M. Yoshinaga, On the freeness of 3-arrangements. Bull. London. Math. Soc. **37** (2005), no. 1, 126–134.

[Z] G. M. Ziegler, Multiarrangements of hyperplanes and their freeness. in *Singularities* (Iowa City, IA, 1986), 345–359, Contemp. Math., **90**, Amer. Math. Soc., Providence, RI, 1989.