Chambers of Arrangements of Hyperplanes and Arrow's Impossibility Theorem

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Abstract

Let $\mathcal{A}$ be a nonempty real central arrangement of hyperplanes and $\mathbf{Ch}$ be the set of chambers of $\mathcal{A}$. Each hyperplane H makes a half-space H^+ and the other half-space H^-. Let $B = \{+, -\}$. For $H \in \mathcal{A}$, define a map $\epsilon_H^+ : \mathbf{Ch} \to B$ by $\epsilon_H^+(C) = +$ (if $C \subseteq H^+$) and $\epsilon_H^+(C) = -$ (if $C \subseteq H^-$). Define $\epsilon_H^- = -\epsilon_H^+$. Let $\mathbf{Ch}^m = \mathbf{Ch} \times \mathbf{Ch} \times \cdots \times \mathbf{Ch}$ (m times). Then the maps $\epsilon_H^\pm$ induce the maps $\epsilon_H^\pm : \mathbf{Ch}^m \to B^m$. We will study the admissible maps $\Phi : \mathbf{Ch}^m \to \mathbf{Ch}$ which are compatible with every $\epsilon_H^\pm$. Suppose $|\mathcal{A}| \geq 3$ and $m \geq 2$. Then we will show that $\mathcal{A}$ is indecomposable if and only if every admissible map is a projection to a component. When $\mathcal{A}$ is a braid arrangement, which is indecomposable, this result is equivalent to Arrow's impossibility theorem in economics. We also determine the set of admissible maps explicitly for every nonempty real central arrangement.

Key words: arrangement of hyperplanes, chambers, braid arrangements, Arrow's impossibility theorem.

1 Main Results

Let $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ be a nonempty real central arrangement of hyperplanes in $\mathbb{R}^\ell$. In other words, each hyperplane H_j goes through the origin of $\mathbb{R}^\ell$. In this note, we frequently refer [OT] for elementary facts about arrangements of hyperplanes, which are usually referred as **arrangements** for brevity. The connected components of the complement $\mathbb{R}^\ell \setminus \bigcup_{1 \leq j \leq n} H_j$ are called **chambers** of $\mathcal{A}$. Let $\mathbf{Ch} = \mathbf{Ch}(\mathcal{A})$ denote the set of chambers of $\mathcal{A}$. For each hyperplane $H_j \in \mathcal{A}$, fix a real linear form α_j such that $H_j = \ker(\alpha_j)$. The product $\prod_{j=1}^n \alpha_j$ is called a **defining polynomial** for $\mathcal{A}$. Define

$$H_j^+ = \{x \in \mathbb{R}^\ell \mid \alpha_j(x) > 0\}, \quad H_j^- = \{x \in \mathbb{R}^\ell \mid \alpha_j(x) < 0\} \ \ (j = 1, \ldots, n).$$

Throughout this note, let σ denote $+$ or $-$. Let $B = \{+, -\}$, which we frequently consider as a multiplicative group of order two in the natural way.

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Let $1 \le j \le n$. The maps $\epsilon_j^\sigma : \mathbf{Ch} \longrightarrow B$ are defined by $\epsilon_j^\sigma(C) = \sigma\tau$ if $C \subseteq H_j^\tau$ $(\sigma, \tau \in B)$. Let m be a positive integer. Consider the m-time direct products $\mathbf{Ch}^m$ and B^m. We let the same notation ϵ_j^σ also denote the map $\mathbf{Ch}^m \to B^m$ induced from $\epsilon_j^\sigma : \mathbf{Ch} \to B$:

$$\epsilon_j^\sigma(C_1, C_2, \ldots, C_m) = (\epsilon_j^\sigma(C_1), \epsilon_j^\sigma(C_2), \ldots, \epsilon_j^\sigma(C_m))$$

for $(C_1, C_2, \ldots, C_m) \in \mathbf{Ch}^m$.

Definition 1.1. *A map $\Phi : \mathbf{Ch}^m \longrightarrow \mathbf{Ch}$ is called an* **admissible map** *if there exists a family of maps φ_j^σ $(1 \le j \le n,\ \ \sigma \in B = \{+, -\})$ which satisfies the following two conditions:*

(1) *$\varphi_j^\sigma(+, +, \ldots, +) = +$, and*

(2) *the diagram*

$$\begin{tikzcd} \mathbf{Ch}^m \arrow[r, "\Phi"] \arrow[d, "\epsilon_j^\sigma"'] & \mathbf{Ch} \arrow[d, "\epsilon_j^\sigma"] \\ B^m \arrow[r, "\varphi_j^\sigma"] & B \end{tikzcd}$$

commutes for each $j, 1 \le j \le n$, and $\sigma \in B = \{+, -\}$.

Let $AM(\mathcal{A}, m)$ denote the set of all admissible maps determined by $\mathcal{A}$ and m.

As we will see in Proposition 2.5, when Φ is an admissible map, a family of maps φ_j^σ $(1 \le j \le n, \sigma \in B = \{+, -\})$ satisfying the conditions in Definition 1.1 is uniquely determined by Φ, $\mathcal{A}$ and m.

The main purpose of this note is to study the set $AM(\mathcal{A}, m)$ for all $\mathcal{A}$ and m.

Definition 1.2. *For $1 \le h \le m$, let*

$$\Phi = \textit{the projection to the } h\textit{-th component},$$
$$\varphi_j^\sigma = \textit{the projection to the } h\textit{-th component}.$$

Then it is easy to see that Φ is an admissible map with a family of maps φ_j^σ $(1 \le j \le n, \sigma \in B = \{+, -\})$. We call the admissible maps of this type **projective admissible maps.**

For a central arrangement $\mathcal{A}$, define

$$r(\mathcal{A}) = \operatorname{codim}_{\mathbb{R}^\ell} \bigcap_{1 \le j \le n} H_j.$$

Definition 1.3. *A central arrangement $\mathcal{A}$ is said to be* **decomposable** *if there exist nonempty arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ (disjoint) and $r(\mathcal{A}) = r(\mathcal{A}_1) + r(\mathcal{A}_2)$. In this case, write $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2$. A central arrangement $\mathcal{A}$ is said to be* **indecomposable** *if it is not decomposable.*

Note that $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2$ if and only if the defining polynomials for $\mathcal{A}_1$ and $\mathcal{A}_2$ have no common variables after an appropriate linear coordinate change.

Remark. It is also known [STV, Theorem 2.4 (2)] that $\mathcal{A}$ is decomposable if and only if its Poincaré polynomial [OT, Definition 2.48] $\pi(\mathcal{A}, t)$ is divisible by $(1+t)^2$.

We will see in Proposition 2.3 that any nonempty real central arrangement $\mathcal{A}$ can be uniquely (up to order) decomposed into nonempty indecomposable arrangements:

$$\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \cdots \uplus \mathcal{A}_r. \qquad (*)$$

The following two theorems completely determine the set $AM(\mathcal{A}, m)$ of admissible maps.

Theorem 1.4. *For a nonempty real central arrangement $\mathcal{A}$ with the decomposition $(*)$, there exists a natural bijection*

$$AM(\mathcal{A}, m) \simeq AM(\mathcal{A}_1, m) \times AM(\mathcal{A}_2, m) \times \cdots \times AM(\mathcal{A}_r, m)$$

for each positive integer m.

Theorem 1.5. *Let $\mathcal{A}$ be a nonempty indecomposable real central arrangement and m be a positive integer. Then,*

(1) *if $|\mathcal{A}| = 1$,*

$$AM(\mathcal{A}, m) = \{\Phi : \mathbf{Ch}^m \to \mathbf{Ch} \mid \Phi(C, C, \ldots, C) = C \textit{ for each chamber } C\},$$

(2) *if $|\mathcal{A}| \geq 3$, every admissible map is projective.*

(*Note that, if $|\mathcal{A}| = 2$, then $\mathcal{A}$ is decomposable.*)

Corollary 1.6. *Decompose a nonempty real central arrangement $\mathcal{A}$ into nonempty indecomposable arrangements as*

$$\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \cdots \uplus \mathcal{A}_a \uplus \mathcal{B}_1 \uplus \mathcal{B}_2 \uplus \cdots \uplus \mathcal{B}_b$$

with $|\mathcal{A}_p| = 1$ $(1 \leq p \leq a)$ and $|\mathcal{B}_q| \geq 3$ $(1 \leq q \leq b)$. Then, for each positive integer m,

$$|AM(\mathcal{A}, m)| = (2^{a(2^m - 2)}) m^b.$$

Remark. Theorem 1.5 can be regarded as a generalization of Kenneth Arrow's impossibility theorem ([A, M-CWG]) in economics:

In the impossibility theorem, we assume that a society of m people have ℓ policy options and that every individual has his/her own order of preferences on the ℓ policy options. A social welfare function can be interpreted as a voting system by which the individual preferences are aggregated into a single societal preference. We require the following two requirements for a reasonable social welfare function:

(A) the society prefers the option i to the option j if every individual prefers the option i to the option j (Pareto property), and (B) whether the society prefers the option i to the option j only depends which individuals prefer the option i to the option j (pairwise independence).

The conclusion of Arrow's impossibility theorem is striking: for $\ell \geq 3$, the only social welfare function satisfying the two requirements (A) and (B) is a dictatorship, that is, the societal preference has to be equal to the preference of one particular individual.

In Theorem 1.5, let $\mathcal{A}$ be a braid arrangement in $\mathbb{R}^\ell$ ($\ell \geq 3$), i. e.,

$$\mathcal{A} = \{H_{ij} \mid 1 \leq i < j \leq \ell\}, \text{ where } H_{ij} := \ker(x_i - x_j).$$

The braid arrangements are indecomposable as we will see in Example 2.2. Let $H_{ij}^+ = \{(x_1, x_2, \dots, x_\ell) \in \mathbb{R}^\ell \mid x_i > x_j\}$ and $H_{ij}^- = \{(x_1, x_2, \dots, x_\ell) \in \mathbb{R}^\ell \mid x_i < x_j\}$. Then each chamber of $\mathcal{A}$ can be uniquely expressed as

$$\{(x_1, x_2, \dots, x_\ell) \in \mathbb{R}^\ell \mid x_{\pi(1)} < x_{\pi(2)} < \cdots < x_{\pi(\ell)}\}$$

for a permutation π of $\{1, 2, \dots, \ell\}$. This gives a one-to-one correspondence between $\mathbf{Ch}(\mathcal{A})$ and the permutation group $\mathbb{S}_\ell$ of $\{1, 2, \dots, \ell\}$. Thus we can interpret an order of preferences on ℓ policy options as a chamber of a braid arrangement. Similarly, we interpret a social welfare function as the map Φ and the dictatorship by the h-th individual as the projection to the h-th component. The requirements (A) (Pareto property) and (B) (pairwise independence) correspond to the conditions (1) ($\varphi_j^\sigma(+, \dots, +) = +$) and (2) (commutativity) in Definition 1.1 respectively. So, in our terminology, Arrow's impossibility theorem can be formulated as:

> *If $\mathcal{A}$ is a braid arrangement with $\ell \geq 3$, then every admissible map is projective.*

Thanks to Theorems 1.4 and 1.5 we have the following necessary and sufficient condition for a nonempty real central arrangement to have the property that every admissible map is projective:

Corollary 1.7. *Let $\mathcal{A}$ be a nonempty real central arrangement and m be a positive integer. Every admissible map is projective if and only if*
(*case 1*) $m = 1$, *or*
(*case 2*) $\mathcal{A}$ *is indecomposable with* $|\mathcal{A}| \geq 3$.

2 Proof of Theorem 1.4

Let $\mathcal{A} = \{H_1, H_2, \dots, H_n\}$ be a nonempty real central arrangement in $\mathbb{R}^\ell$. Let $\mathcal{B}$ be a subarrangement of $\mathcal{A}$, in other words, $\mathcal{B} \subseteq \mathcal{A}$. We say that $\mathcal{B}$ is **dependent** if

$$r(\mathcal{B}) = \operatorname{codim}_{\mathbb{R}^\ell}\Big(\bigcap_{H \in \mathcal{B}} H\Big) < |\mathcal{B}|.$$

A subarrangement $\mathcal{B}$ of $\mathcal{A}$ is called **independent** if it is not dependent. If $\mathcal{B}$ is a minimally dependent subset, then $\mathcal{B}$ is called a **circuit.** If $\mathcal{B}$ is a maximally independent subset in $\mathcal{A}$, then $\mathcal{B}$ is called a **basis** for $\mathcal{A}$.

We introduce a graph $\Gamma(\mathcal{A})$ associated with $\mathcal{A}$. The set of vertices of $\Gamma(\mathcal{A})$ is $\mathcal{A}$. Two vertices $H_{j_1}, H_{j_2} \in \mathcal{A}$ $(j_1 \neq j_2)$ are connected by an edge if and only if there exists a circuit (in $\mathcal{A}$) containing $\{H_{j_1}, H_{j_2}\}$.

Lemma 2.1. *A nonempty real central arrangement $\mathcal{A}$ is indecomposable if and only if the graph $\Gamma(\mathcal{A})$ is connected.*

Proof. If $\Gamma(\mathcal{A})$ is disconnected, then decompose $\mathcal{A}$ as $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ so that $\mathcal{A}_1 \neq \emptyset$, $\mathcal{A}_2 \neq \emptyset$, and $\{H_{j_1}, H_{j_2}\}$ is not contained in any circuit whenever $H_{j_p} \in \mathcal{A}_p$ $(p = 1, 2)$. Choose a basis $\mathcal{B}_p$ of $\mathcal{A}_p$ $(p = 1, 2)$. Then $\mathcal{B}_1 \cup \mathcal{B}_2$ is also independent because it does not contain any circuit. Thus

$$r(\mathcal{A}) = |\mathcal{B}_1 \cup \mathcal{B}_2| = |\mathcal{B}_1| + |\mathcal{B}_2| = r(\mathcal{A}_1) + r(\mathcal{A}_2),$$

which implies $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2$. So $\mathcal{A}$ is decomposable.

Conversely assume that $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2$ with $\mathcal{A}_1 \neq \emptyset$, $\mathcal{A}_2 \neq \emptyset$. We may assume, after an appropriate linear coordinate change, that the defining polynomials for $\mathcal{A}_1$ and $\mathcal{A}_2$ have no common variables. Let $H_{j_p} \in \mathcal{A}_p$ $(p = 1, 2)$. Suppose that there exists a circuit $\mathcal{B}$ containing H_{j_1} and H_{j_2}. Then $\mathcal{B} \cap \mathcal{A}_1$ and $\mathcal{B} \cap \mathcal{A}_2$ are both independent. This implies that $\mathcal{B}$ is also independent, which is a contradiction. $\square$

Example 2.2. *Let $\mathcal{A}$ be a braid arrangement in $\mathbb{R}^\ell (\ell \geq 2)$:*

$$\mathcal{A} = \{H_{ij} \mid 1 \leq i < j \leq \ell\},$$

where $H_{ij} = \ker(x_i - x_j)$. If $\ell = 2$, then $|\mathcal{A}| = 1$ and $\mathcal{A}$ is indecomposable. Let $\ell \geq 3$. Then $\{H_{ij}, H_{jk}, H_{ik}\}$ for $1 \leq i < j < k \leq \ell$ is a circuit. Thus it is easy to check that $\mathcal{A}$ is indecomposable by applying Lemma 2.1.

By Lemma 2.1, we immediately have

Proposition 2.3. *Any nonempty real central arrangement $\mathcal{A}$ can be uniquely (up to order) decomposed into nonempty indecomposable arrangements*

$$\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \cdots \uplus \mathcal{A}_r.$$

Let m be a positive integer. For $S \subseteq \{1, \ldots, m\}$, define $S_+ = (\sigma_1, \ldots, \sigma_m) \in B^m$ with

$$\sigma_i = \begin{cases} + \text{ if } i \in S, \\ - \text{ if } i \notin S. \end{cases}$$

Then $(S^c)_+ = (-\sigma_1, \ldots, -\sigma_m) = -S_+$.

Proposition 2.4. *Assume $\sigma \in B = \{+, -\}$ and $1 \leq j \leq n$. Then the map $\epsilon_j^\sigma : \mathbf{Ch}^m \to B^m$ is surjective.*

Proof. An arbitrary element of B^m can be expressed as S_+ for some $S \subseteq \{1, 2, \ldots, m\}$. Suppose that C and C' are chambers such that $C \subseteq H_j^+$ and $C' \subseteq H_j^-$. Define $\mathcal{C} = (C_1, C_2, \ldots, C_m) \in \mathbf{Ch}^m$ by

$$C_i = \begin{cases} C \text{ if } i \in S, \\ C' \text{ if } i \notin S. \end{cases}$$

Then we have $\epsilon_j^+(\mathcal{C}) = S_+$. Let $-\mathcal{C} = (-C_1, -C_2, \ldots, -C_m) \in \mathbf{Ch}^m$, where $-C_i$ implies the antipodal chamber of C_i. Then $\epsilon_j^-(-\mathcal{C}) = -(S^c)_+ = S_+$. $\square$

Proposition 2.5. *When Φ is an admissible map, a family of maps φ_j^σ $(1 \le j \le n,\ \sigma \in B = \{+, -\})$ satisfying the the conditions in Definition 1.1 is uniquely determined.*

Proof. It is obvious because of Proposition 2.4. $\square$

Proposition 2.6. *When Φ is an admissible map, $\Phi(C, C, \ldots, C) = C$ for any chamber $C \in \mathbf{Ch}$.*

Proof. By Definition 1.1, two chambers $\Phi(C, C, \ldots, C)$ and C are on the same side of every $H_j \in \mathcal{A}$. Thus $\Phi(C, C, \ldots, C) = C$. $\square$

Suppose that $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2$ with $\mathcal{A}_1 \neq \emptyset$ and $\mathcal{A}_2 \neq \emptyset$. We may assume that the defining polynomials for $\mathcal{A}_1$ and $\mathcal{A}_2$ have no common variables. Then the following lemma is obvious:

Lemma 2.7. *The map*

$$\alpha : \mathbf{Ch}(\mathcal{A}_1)^m \times \mathbf{Ch}(\mathcal{A}_2)^m \longrightarrow \mathbf{Ch}(\mathcal{A}_1 \uplus \mathcal{A}_2)^m,$$

given by

$$\alpha(C_1, \ldots, C_m, D_1, \ldots, D_m) = (C_1 \cap D_1, \ldots, C_m \cap D_m)$$

for $C_i \in \mathbf{Ch}(\mathcal{A}_1), D_i \in \mathbf{Ch}(\mathcal{A}_2)$ $(i = 1, \ldots, m)$, is bijective.

Lemma 2.8. *Let $p \in \{1, 2\}$. For $H_j \in \mathcal{A}_p$, the diagram*

$$\begin{tikzcd}
\mathbf{Ch}(\mathcal{A}_1) \times \mathbf{Ch}(\mathcal{A}_2) \arrow[r, "\alpha"] \arrow[d, "\pi_p"'] & \mathbf{Ch}(\mathcal{A}_1 \uplus \mathcal{A}_2) \arrow[d, "\epsilon_j^\sigma"] \\
\mathbf{Ch}(\mathcal{A}_p) \arrow[r, "\epsilon_{j,p}^\sigma"] & B
\end{tikzcd}$$

is commutative, where π_p is the projection to the p-th component, and $\epsilon_{j,p}^\sigma$ is the map ϵ_j^σ for $\mathcal{A}_p$.

Proof. Let $p = 1$ for simplicity. Then

$$\epsilon_j^\sigma \circ \alpha(C, D) = \epsilon_j^\sigma (C \cap D) = \epsilon_{j,1}^\sigma (C) = \epsilon_{j,1}^\sigma \circ \pi_1(C, D)$$

for $C \in \mathbf{Ch}(\mathcal{A}_1)$, $D \in \mathbf{Ch}(\mathcal{A}_2)$, and $H_j \in \mathcal{A}_1$. $\square$

From now on, identify $\mathbf{Ch}(\mathcal{A}_1)^m \times \mathbf{Ch}(\mathcal{A}_2)^m$ and $\mathbf{Ch}(\mathcal{A}_1 \uplus \mathcal{A}_2)^m$ by the bijection α in Lemma 2.7. Then Lemma 2.8 can be stated as

$$\epsilon_{j,p}^\sigma \circ \pi_p = \epsilon_j^\sigma \quad (p \in \{1, 2\}, \sigma \in B, H_j \in \mathcal{A}_p).$$

Proposition 2.9. *There exists a natural bijection between $AM(\mathcal{A}_1 \uplus \mathcal{A}_2)$ and $AM(\mathcal{A}_1) \times AM(\mathcal{A}_2)$.*

Proof. Suppose that Φ is an admissible map for $\mathcal{A}_1 \uplus \mathcal{A}_2$ and that a family of maps φ_j^σ $(H_j \in \mathcal{A}_1 \uplus \mathcal{A}_2,\ \sigma \in B)$ satisfies the conditions in Definition 1.1. Fix $p \in \{1, 2\}$ and $H_j \in \mathcal{A}_p$. Consider the following diagram:

$$\begin{tikzcd}
\mathbf{Ch}(\mathcal{A}_1)^m \times \mathbf{Ch}(\mathcal{A}_2)^m \arrow[r, "\Phi"] \arrow[d, "\pi_p"'] & \mathbf{Ch}(\mathcal{A}_1) \times \mathbf{Ch}(\mathcal{A}_2) \arrow[d, "\pi_p"'] \\
\mathbf{Ch}(\mathcal{A}_p)^m \arrow[r, dashed, "\Phi_p"] \arrow[d, "\epsilon_{j,p}^\sigma"'] & \mathbf{Ch}(\mathcal{A}_p) \arrow[d, "\epsilon_{j,p}^\sigma"'] \\
B^m \arrow[r, "\varphi_j^\sigma"] & B\ .
\end{tikzcd}$$

By Lemma 2.8, we have

$$\epsilon_{j,p}^\sigma \circ \pi_p \circ \Phi = \epsilon_j^\sigma \circ \Phi = \varphi_j^\sigma \circ \epsilon_j^\sigma = \varphi_j^\sigma \circ \epsilon_{j,p}^\sigma \circ \pi_p \ (p \in \{1, 2\}, \sigma \in B).$$

Assume $p = 1$ for simplicity. Let $C_i \in \mathbf{Ch}(\mathcal{A}_1), D_i \in \mathbf{Ch}(\mathcal{A}_2)$ for $1 \le i \le m$. Then

$$\begin{aligned}
&\epsilon_{j,1}^\sigma \circ \pi_1 \circ \Phi(C_1, C_2, \dots, C_m, D_1, D_2, \dots, D_m) \\
&= \varphi_j^\sigma \circ \epsilon_{j,1}^\sigma \circ \pi_1(C_1, C_2, \dots, C_m, D_1, D_2, \dots, D_m) \\
&= \varphi_j^\sigma \circ \epsilon_{j,1}^\sigma (C_1, C_2, \dots, C_m)
\end{aligned}$$

for each $H_j \in \mathcal{A}_1$. Thus the chamber

$$\pi_1 \circ \Phi(C_1, C_2, \dots, C_m, D_1, D_2, \dots, D_m) \in \mathbf{Ch}(\mathcal{A}_1)$$

is independent of $D_1, D_2, \dots, D_m$. Therefore we can express

$$\Phi_1(C_1, C_2, \dots, C_m) = \pi_1 \circ \Phi(C_1, C_2, \dots, C_m, D_1, D_2, \dots, D_m)$$

for some map

$$\Phi_1 : \mathbf{Ch}(\mathcal{A}_1)^m \to \mathbf{Ch}(\mathcal{A}_1).$$

Then Φ_1 is an admissible map for $\mathcal{A}_1$ because the diagram above, including Φ_1, is commutative for each $H_j \in \mathcal{A}_1$. Simililarly we can define

$$\Phi_2 : \mathbf{Ch}(\mathcal{A}_2)^m \to \mathbf{Ch}(\mathcal{A}_2)$$

so that Φ_2 is an admissible map for $\mathcal{A}_2$. The construction so far gives a natural map

$$F : AM(\mathcal{A}_1 \uplus \mathcal{A}_2) \to AM(\mathcal{A}_1) \times AM(\mathcal{A}_2).$$

Conversely suppose that Φ_p is an admissible map for $\mathcal{A}_p$ and that a family of maps φ_j^σ $(H_j \in \mathcal{A}_p,\ \sigma \in B)$ satisfies the conditions in Definition 1.1. Define

$$\Phi := \Phi_1 \times \Phi_2 : \mathbf{Ch}(\mathcal{A}_1)^m \times \mathbf{Ch}(\mathcal{A}_2)^m \longrightarrow \mathbf{Ch}(\mathcal{A}_1) \times \mathbf{Ch}(\mathcal{A}_2).$$

Then Φ is an admissible map for $\mathcal{A}_1 \uplus \mathcal{A}_2$ because the family of maps φ_j^σ $(H_j \in \mathcal{A}_1 \uplus \mathcal{A}_2,\ \sigma \in B)$ satisfies the conditions in Definition 1.1. This construction gives a map

$$G : AM(\mathcal{A}_1) \times AM(\mathcal{A}_2) \to AM(\mathcal{A}_1 \uplus \mathcal{A}_2).$$

It is easy to check that F and G are inverses of each other. □

Now we have proved Theorem 1.4 by applying Propositions 2.3 and 2.9.

3 Proof of Thorem 1.5

In this section we assume that $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ is a nonempty real central *indecomposable* arrangement. We assume $n \neq 2$ because any arrangement $\mathcal{A} = \{H_1, H_2\}$ is decomposable:

$$\mathcal{A} = \{H_1\} \uplus \{H_2\}.$$

Lemma 3.1. *Let m be a positive integer. Suppose $\mathcal{A}$ is an arrangement with only one hyperplane H_1. Let H_1^+ and H_1^- be the two chambers. Then*

(1) *an arbitrary admissible map is given by*

$$\Phi(C_1, C_2, \ldots, C_m) = \begin{cases} H_1^+ & \text{if } C_i = H_1^+ \text{ for all } i, \\ H_1^- & \text{if } C_i = H_1^- \text{ for all } i, \\ \text{either } H_1^+ \text{ or } H_1^- & \text{otherwise}, \end{cases}$$

(2) *the number of admissible maps is equal to 2^{2^m-2}, and*

(3) *every admissible map is projective if and only if $m = 1$.*

Proof. (1) Note that the map $\epsilon_1^\sigma : \mathbf{Ch}(\mathcal{A}) \to B$ is a bijection. So the commutativity condition in Definition 1.1 can be ignored and we simply consider a map $\Phi : \mathbf{Ch}^m \to \mathbf{Ch}$ satisfying $\Phi(H_1^\sigma, H_1^\sigma, \ldots, H_1^\sigma) = H_1^\sigma$ $(\sigma \in B = \{+, -\})$.

(2) We have two choices for each element of the set

$$\mathbf{Ch}^m \setminus \{(H_1^+, H_1^+, \ldots, H_1^+), (H_1^-, H_1^-, \ldots, H_1^-)\}$$

whose cardinality is equal to $2^m - 2$.

(3) When $m = 1$, by Proposition 2.6, the only admissible map is the identity map, which is projective. For $m \geq 2$, the number of admissible maps, which is equal to 2^{2^m-2}, exceeds the number of projective ones, which is m. □

Therefore we have proved Theorem 1.5 (1). Let us concentrate on Theorem 1.5 (2).

Assume that $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ is indecomposable with $n = |\mathcal{A}| \geq 3$. Let m be a positive integer. We will show that every admissible map of $\mathcal{A}$ is projective. Suppose that Φ is an admissible map and that a family of maps φ_j^σ $(H_j \in \mathcal{A},\ \sigma \in B)$ satisfies the conditions in Definition 1.1.

Lemma 3.2. *Assume $1 \leq j \leq n$ and $S \subseteq \{1, 2, \ldots, m\}$. Then $\varphi_j^+(S_+) = -\varphi_j^-(-S_+)$. In particular, $\varphi_j^-(-, -, \ldots, -) = -$.*

Proof. By Proposition 2.4, we may choose $\mathcal{C} \in \mathbf{Ch}^m$ so that $\epsilon_j^+(\mathcal{C}) = S_+$. Then

$$\begin{aligned}\varphi_j^+(S_+) = + &\Longleftrightarrow \epsilon_j^+ \circ \Phi(\mathcal{C}) = \varphi_j^+ \circ \epsilon_j^+(\mathcal{C}) = + \Longleftrightarrow \Phi(\mathcal{C}) \subseteq H_j^+ \\ &\Longleftrightarrow - = \epsilon_j^- \circ \Phi(\mathcal{C}) = \varphi_j^- \circ \epsilon_j^-(\mathcal{C}) = \varphi_j^-((S^c)_+) = \varphi_j^-(-S_+).\end{aligned}$$

□

Define $\delta_{\mathcal{A}}^\sigma : \mathbf{Ch}(\mathcal{A}) \longrightarrow B^n$, for $\sigma \in B = \{+, -\}$, by

$$\delta_{\mathcal{A}}^\sigma(C) = (\epsilon_1^\sigma(C), \epsilon_2^\sigma(C), \ldots, \epsilon_n^\sigma(C)).$$

Then $\delta_{\mathcal{A}}^\sigma$ is injective. We frequently suppress the subscript $\mathcal{A}$ in $\delta_{\mathcal{A}}^\sigma$ when there is no fear of confusion. Note that $\delta^+(-C) = -\delta^+(C) = \delta^-(C)$, where $-C$ is the antipodal chamber of C. Thus $\delta^- = -\delta^+$.

Lemma 3.3. *Let $\mathcal{B} = \{H_1, H_2, \ldots, H_\nu\} \subseteq \mathcal{A}$ be a circuit with $3 \leq \nu \leq n$. Then*
(1) *$|\mathbf{Ch}(\mathcal{B})| = 2^\nu - 2$, and*
(2) *there exists $\tau = (\tau_1, \tau_2, \ldots, \tau_\nu) \in B^\nu$ such that*

$$\operatorname{im} \delta_{\mathcal{B}}^\sigma = B^\nu \setminus \{\tau, -\tau\}.$$

Proof. (1) Since the intersection lattice [OT, Definition 2.1] $L(\mathcal{B})$ of $\mathcal{B}$ is the same as that of the ν-dimensional Boolean arrangement (= the arrangement of the ν coordinate hyperplanes) in $\mathbb{R}^\nu$ up to the rank $\nu - 1$, the Poincaré polynomial $\pi(\mathcal{B}, t)$ coincides with the Poincaré polynomial of the ν-dimensional Boolean arrangement up to degree $\nu - 1$. The Poincaré polynomial of the ν-dimensional Boolean arrangement is equal to $(1+t)^\nu$ [OT, Example 2.49]. Since $\deg \pi(\mathcal{B}, t) = r(\mathcal{B}) = \nu - 1$ and $\pi(\mathcal{B}, -1) = 0$, $\pi(\mathcal{B}, t) = (1+t)^\nu - t^\nu - t^{\nu-1}$. By [Z] [OT, Theorem 2.68], one has $|\mathbf{Ch}(\mathcal{B})| = \pi(\mathcal{B}, 1) = 2^\nu - 2$.

(2) By (1),

$$| B^\nu \setminus \operatorname{im} \delta^+_{\mathcal{B}} | = | B^\nu | - | \operatorname{im} \delta^+_{\mathcal{B}} | = | B^\nu | - | \mathbf{Ch}(\mathcal{B}) | = 2^\nu - (2^\nu - 2) = 2.$$

Since $\delta^+_{\mathcal{B}}(-C) = -\delta^+_{\mathcal{B}}(C)$ for $C \in \mathbf{Ch}(\mathcal{B})$, the set $\operatorname{im} \delta^+_{\mathcal{B}}$ is closed under the operation $\tau \mapsto -\tau$. Thus the set $B^\nu \setminus \operatorname{im} \delta^+_{\mathcal{B}}$ is expressed as $\{\tau, -\tau\}$ for some $\tau \in B^\nu$. □

Define

$$K^\sigma_j := \{S \subseteq \{1, 2, \dots, m\} \mid \varphi^\sigma_j(S_+) = +\} \quad (1 \le j \le n,\ \sigma \in B = \{+, -\}).$$

Lemma 3.4. *Suppose that $\mathcal{A}$ is indecomposable and $n = |\mathcal{A}| \ge 3$. Then the maps φ^σ_j do not depend upon j or σ.*

Proof. Choose a circuit $\mathcal{B} \subseteq \mathcal{A}$. We may assume that $\mathcal{B} = \{H_1, H_2, \dots, H_\nu\}$ and $3 \le \nu \le n$. By Lemma 3.3, there exists $\tau = (\tau_1, \tau_2, \dots, \tau_\nu) \in B^\nu$ such that

$$B^\nu = (\operatorname{im} \delta^+_{\mathcal{B}}) \cup \{\tau, -\tau\} \text{ (disjoint).}$$

Let $1 \le p \le \nu$, $1 \le q \le \nu$, $p \ne q$. Since neither of $(\tau_1, \dots, -\tau_q, \dots, \tau_\nu)$ nor $(\tau_1, \dots, -\tau_p, \dots, \tau_\nu)$ lies in $\{\tau, -\tau\}$, they both lie in $\operatorname{im} \delta^+_{\mathcal{B}}$. Choose $C, C' \in \mathbf{Ch}(\mathcal{B})$ such that

$$\delta^+_{\mathcal{B}}(C) = (\tau_1, \dots, -\tau_p, \dots, \tau_\nu),\ \ \delta^+_{\mathcal{B}}(C') = (\tau_1, \dots, -\tau_q, \dots, \tau_\nu).$$

Choose $\hat{C} \in \mathbf{Ch}(\mathcal{A})$ and $\hat{C}' \in \mathbf{Ch}(\mathcal{A})$ so that $\hat{C} \subseteq C'$ and $\hat{C}' \subseteq C'$. Let $S \subseteq \{1, 2, \dots, m\}$. Define $\mathcal{C} = (C_1, C_2, \dots, C_m) \in \mathbf{Ch}(\mathcal{A})^m$ by

$$C_i = \begin{cases} \hat{C}' \text{ if } i \in S, \\ \hat{C} \text{ if } i \notin S. \end{cases}$$

Then

$$\epsilon^{\tau_p}_p(\mathcal{C}) = \epsilon^{-\tau_q}_q(\mathcal{C}) = S_+,\ \ \epsilon^{\tau_r}_r(\mathcal{C}) = (+, +, \dots, +)\ \ (1 \le r \le \nu, r \notin \{p, q\}).$$

Suppose $S \in K^{\tau_p}_p$, i. e., $\varphi^{\tau_p}_p(S_+) = +$. Then

$$\epsilon^{\tau_p}_p \circ \Phi(\mathcal{C}) = \varphi^{\tau_p}_p \circ \epsilon^{\tau_p}_p(\mathcal{C}) = \varphi^{\tau_p}_p(S_+) = +.$$

This implies that $\Phi(\mathcal{C}) \subseteq H^{\tau_p}_p$. Similarly we have $\Phi(\mathcal{C}) \subseteq H^{\tau_r}_r$ when $1 \le r \le \nu, r \notin \{p, q\}$, because $\varphi^{\tau_r}_r \circ \epsilon^{\tau_r}_r(\mathcal{C}) = \varphi^{\tau_r}_r(+, +, \dots, +) = +$. Note that

$$\bigcap_{j=1}^{\nu} H^{\tau_j}_j = \emptyset$$

because $\tau \notin \operatorname{im} \delta^+_{\mathcal{B}}$. Therefore

$$\Phi(\mathcal{C}) \subseteq \bigcap_{j \ne q}^{\nu} H^{\tau_j}_j \subseteq H^{-\tau_q}_q.$$

Thus

$$\varphi_q^{-\tau_q}(S_+) = \varphi_q^{-\tau_q} \circ \epsilon_q^{-\tau_q}(\mathcal{C}) = \epsilon_q^{-\tau_q} \circ \Phi(\mathcal{C}) = +,$$

which implies $S \in K_q^{-\tau_q}$. Therefore $K_p^{\tau_p} \subseteq K_q^{-\tau_q}$.

Smilarly one can show $K_p^{\tau_p} \supseteq K_q^{-\tau_q}$, and thus $K_p^{\tau_p} = K_q^{-\tau_q}$ if $p \neq q$. Since $\nu \geq 3$, we can conclude that K_j^σ does not depend upon j, $1 \leq j \leq \nu$, or $\sigma \in B$. So φ_j^σ does not depend upon j, $1 \leq j \leq \nu$, or $\sigma \in B$. Apply Lemma 2.1, and we know φ_j^σ does not depend upon j, $1 \leq j \leq n$, or $\sigma \in B$. □

Because of Lemma 3.4, write $\varphi = \varphi_j^\sigma$ for j, $1 \leq j \leq n$, and $\sigma \in B$. Let

$$K = \{S \subseteq \{1, 2, \ldots, m\} \mid \varphi(S_+) = +\}.$$

Lemma 3.5. (1) $\{1, \ldots, m\} \in K$, (2) $S \in K$ *if and only if* $S^c \notin K$, (3) $S_1 \cap S_2 \in K$ *if* $S_1 \in K$ *and* $S_2 \in K$.

Proof. (1) is obvious because $\varphi(+, +, \ldots, +) = +$.

(2) By Lemma 3.2

$$\begin{aligned} S \in K = K_1^+ &\iff \varphi_1^+(S_+) = + \iff \varphi_1^-((S^c)_+) = \varphi_1^-(-S_+) = -\varphi_1^+(S_+) = - \\ &\iff \varphi_1^-((S^c)_+) = - \iff S^c \notin K_1^- = K. \end{aligned}$$

(3) Choose a circuit $\mathcal{B} \subseteq \mathcal{A}$. We may assume $\mathcal{B} = \{H_1, H_2, \ldots, H_\nu\}$ with $3 \leq \nu \leq n$. By Lemma 3.3, there exists $\tau = (\tau_1, \tau_2, \ldots, \tau_\nu) \in B^\nu$ such that

$$B^\nu = (\operatorname{im} \delta^+) \cup \{\tau, -\tau\} \text{ (disjoint).}$$

There exist four chambers $C, C', C'', C''' \in \mathbf{Ch}(\mathcal{B})$ such that

$$\begin{aligned} \delta_\mathcal{B}^+(C) &= (\tau_1, \tau_2, -\tau_3, \tau_4, \ldots, \tau_\nu), \;\; \delta_\mathcal{B}^+(C') = (\tau_1, -\tau_2, \tau_3, \tau_4, \ldots, \tau_\nu), \\ \delta_\mathcal{B}^+(C'') &= (-\tau_1, \tau_2, \tau_3, \tau_4, \ldots, \tau_\nu), \;\; \delta_\mathcal{B}^+(C''') = (-\tau_1, -\tau_2, \tau_3, \tau_4, \ldots, \tau_\nu). \end{aligned}$$

Choose four chambers $\hat{C}, \hat{C'}, \hat{C''}, \hat{C'''} \in \mathbf{Ch}(\mathcal{A})$ such that

$$\hat{C} \subseteq C, \;\; \hat{C'} \subseteq C', \;\; \hat{C''} \subseteq C'', \;\; \hat{C'''} \subseteq C'''.$$

Assume that $S_1, S_2 \in K$. Define $\mathcal{C} = (C_1, C_2, \ldots, C_m) \in \mathbf{Ch}(\mathcal{A})^m$ by

$$C_i = \begin{cases} \hat{C} \text{ if } i \in S_1 \cap S_2, \\ \hat{C'} \text{ if } i \in S_1 \setminus S_2, \\ \hat{C''} \text{ if } i \in S_2 \setminus S_1, \\ \hat{C'''} \text{ if } i \notin S_1 \cup S_2. \end{cases}$$

Then

$$\begin{aligned} &\epsilon_1^{\tau_1}(\mathcal{C}) = (S_1)_+, \;\; \epsilon_2^{\tau_2}(\mathcal{C}) = (S_2)_+, \;\; \epsilon_3^{-\tau_3}(\mathcal{C}) = (S_1 \cap S_2)_+, \\ &\epsilon_j^{\tau_j}(\mathcal{C}) = (+, +, \ldots, +) \;\; (4 \leq j \leq \nu). \end{aligned}$$

Thus we have

$$\begin{aligned}
\epsilon_1^{\tau_1} \circ \Phi(\mathcal{C}) &= \varphi \circ \epsilon_1^{\tau_1}(\mathcal{C}) = \varphi((S_1)_+) = +,\\
\epsilon_2^{\tau_2} \circ \Phi(\mathcal{C}) &= \varphi \circ \epsilon_2^{\tau_2}(\mathcal{C}) = \varphi((S_2)_+) = +,\\
\epsilon_j^{\tau_j} \circ \Phi(\mathcal{C}) &= \varphi \circ \epsilon_j^{\tau_j}(\mathcal{C}) = \varphi(+,+,\ldots,+) = + \ \ (4 \le j \le \nu),
\end{aligned}$$

which implies

$$\Phi(\mathcal{C}) \subseteq H_1^{\tau_1} \cap H_2^{\tau_2} \cap H_4^{\tau_4} \cap \cdots \cap H_p^{\tau_\nu} \subseteq H_3^{-\tau_3}.$$

Therefore

$$\varphi((S_1 \cap S_2)_+) = \varphi \circ \epsilon_3^{-\tau_3}(\mathcal{C}) = \epsilon_3^{-\tau_3} \circ \Phi(\mathcal{C}) = +$$

and $S_1 \cap S_2 \in K$. □

Now we are ready to prove the following statement, which is Theorem 1.5 (2).

Let $\mathcal{A}$ be a real central indecomposable arrangement with $|\mathcal{A}| \ge 3$. Then every admissible map is projective.

Proof. Define $S_0 = \bigcap_{S\in K} S$. By Lemma 3.5 (3), $S_0 \in K$. By Lemma 3.5 (1) and (2), we have $\emptyset \notin K$. Thus $S_0 \neq \emptyset$. Let $h \in S_0$. Since $S_0 \setminus \{h\} \notin K$, $(S \setminus S_0) \cup \{h\} \in K$ by Lemma 3.5 (2). By Lemma 3.5 (3),

$$\{h\} = ((S \setminus S_0) \cup \{h\}) \cap S_0 \in K.$$

Thus $S_0 = \{h\}$. Note that, by Lemma 3.5 (2),

$$S \in K \Rightarrow h \in S \Leftrightarrow h \notin S^c \Rightarrow S^c \notin K \Leftrightarrow S \in K.$$

Therefore, $S \in K$ if and only if $h \in S$:

$$K = \{S \subseteq \{1,2,\ldots,m\} \mid h \in S\}.$$

This implies that φ is equal to the projection to the h-th component. Let $\mathcal{C} \in \mathbf{Ch}^m$. Then

$$\epsilon_j^{\sigma} \circ \Phi(\mathcal{C}) = \varphi \circ \epsilon_j^{\sigma}(\mathcal{C}) = \varphi(\epsilon_j^{\sigma}(C_1), \epsilon_j^{\sigma}(C_2), \ldots, \epsilon_j^{\sigma}(C_m)) = \epsilon_j^{\sigma}(C_h).$$

Since $\Phi(\mathcal{C})$ and C_h lie on the same side of every hyperplane $H_j \in \mathcal{A}$, $\Phi(\mathcal{C}) = C_h$. Therefore Φ is the projection to the h-th component. □

Decompose a nonempty real central arrangement $\mathcal{A}$ into nonempty indecomposable arrangements as

$$\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \cdots \uplus \mathcal{A}_a \uplus \mathcal{B}_1 \uplus \mathcal{B}_2 \uplus \cdots \uplus \mathcal{B}_b, \qquad (**)$$

where $|\mathcal{A}_p| = 1$ $(1 \le p \le a)$ and $|\mathcal{B}_q| \ge 3$ $(1 \le q \le b)$. Then, by Lemma 3.1, Theorems 1.4 and 1.5, the number of admissible maps for $\mathcal{A}$ is equal to

$$\left(2^{2^m-2}\right)^a m^b.$$

This proves Corollary 1.6.

Next we will prove Corollary 1.7: If $m = 1$, then, by Proposition 2.6, the only admissible map is the identity map $\mathbf{Ch} \to \mathbf{Ch}$, which is projective. Assume $m \geq 2$. Then, by Lemma 3.1, Theorems 1.4 and 1.5, every admissible map is projective if and only if $a = 0$ and $b = 1$ in the decomposition $(**)$ above.

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