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ON A FREE BOUNDARY PROBLEM OF VISCOUS INCOMPRESSIBLE FLOWS

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ABSTRACT. We shall discuss a free boundary problem for viscous incompressible fluids which is considered as the relaxation of a two phase free boundary problem with surface tension on the interface. Our relaxation ensures the regularity of the interface, and we shall construst a unique time-local solution of the problem. One of the keys is to obtain the optimal regularity of the velocity in tangential direction to the interface.

1. INTRODUCTION AND FORMULATION

We are interested in a free boundary problem of viscous incompressible flows. We shall consider the Stokes systems:

$$\text{(FBP1)} \qquad \begin{cases} \partial_t u - \Delta u + \nabla p = \sigma_1 H \nu \mathcal{H}^{n-1}_{\llcorner \Gamma_t}, \ 0 < t \leq T, \ x \in \mathbb{R}^n, \\ \nabla \cdot u = 0, \ 0 < t \leq T, \ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^n, \end{cases}$$

where $u = (u_1, \cdots, u_n)$ and p are unknown velocity field and pressure field, respectively. The symbol Γ_t represents an unknown free interface evolving from the initial interface Γ_0 which is the boundary of a bounded domain Ω_0. The positive constant σ_1 represents the surface tension, and H, ν are the mean curvature, the exterior unit normal vector of Γ_t, respectively. The symbol $\mathcal{H}^{n-1}_{\llcorner \Gamma_t}$ means the $n-1$ dimensional Hausdorff measure restricted on Γ_t, i.e.,

$$(1.1) \qquad (f, \mathcal{H}^{n-1}_{\llcorner \Gamma_t}) = \int_{\Gamma_t} f(y) \mathcal{H}^{n-1}(dy), \ \forall f \in C_0(\mathbb{R}^n),$$

where $C_0(\mathbb{R}^n)$ is the class of continuous functions whose support are compact, and $(\ ,\ )$ is a coupling when we regard $\mathcal{H}^{n-1}_{\llcorner \Gamma_t}$ as the linear functional on $C_0(\mathbb{R}^n)$.

We assume that the free interface is given by $\Gamma_t = \{x(t,x_0) \in \mathbb{R}^n \ ; \ x_0 \in \Gamma_0\}$ where $x(t,x_0)$ is the solution of the ODE:

$$\text{(BC)} \qquad \begin{cases} \dfrac{dx(t)}{dt} = u(t,x(t)) + \sigma_2 H(t,x(t)) \nu(t,x(t)), \ 0 < t \leq T, \\ x(0) = x_0 \in \Gamma_0, \end{cases}$$

where σ_2 is a fixed positive constant.

The right hand side of the first equation in (FBP1) is the free boundary condition taken into account in weak sense. That is, the term $\sigma_1 H \nu \mathcal{H}^{n-1}_{\llcorner \Gamma_t}$ is formally equivalent to the free boundary condition

$$[(-p\delta_{ij} + \partial_j u_i + \partial_i u_j)_{1 \leq i,j \leq n}]_{\Gamma_t} \nu = \sigma_1 H \nu,$$

where $[\cdot]_{\Gamma_t}$ expresses the jump across the interface Γ_t.

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Our problem is closely related to the two phase free boundary problem for viscous incompressible flows. Indeed, the above problem is regarded as the relaxation of the following two phase Stokes flows problem (in weak form)

$$\text{(TP)} \qquad \begin{cases} \partial_t u - \nabla \cdot T(\kappa Du, p) = \sigma_1 H\nu \mathcal{H}^{n-1}_{\llcorner \Gamma_t},\ 0 < t \le T,\ x \in \mathbb{R}^n, \\ \nabla \cdot u = 0,\ 0 < t \le T,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x),\ x \in \mathbb{R}^n, \end{cases}$$

$$\text{(BC')} \qquad \begin{cases} \dfrac{dx(t)}{dt} = u(t, x(t)),\ 0 < t \le T,\ x(t) \in \Gamma_t, \\ x(0) = x_0 \in \Gamma_0, \end{cases}$$

where $T(\kappa Du, p) := 2\kappa_1 \chi_{\Omega_t} Du + 2\kappa_2(1 - \chi_{\Omega_t})Du - pI$ is the stress tensor, $2Du = (\partial_j u_i + \partial_i u_j)_{1 \le i,j \le n}$ is the deformation tensor, $\kappa_i > 0$ are viscosity coefficients of fluids, and Ω_t is a bounded domain with $\Gamma_t = \partial\Omega_t$.

Our problem relaxes the original problem (TP) in two points. First one is that the viscosities of the two fluids are assumed to be the same value. Second one is that we have a regularizing term $\sigma_2 H\nu$ in the kinematic boundary condition. Such relaxation for the kinematic boundary condition originates from the level set methods in numerical analysis; see Y. C. Chang, T. Y. Hou, B. Merriman and S. Osher [2]. The term $\sigma_2 H\nu$ in (BC) regularizes the interface, and especially, this overcomes the difficulties arising from the surface tension term in the free boundary condition.

Since u satisfies the divergence free condition in whole space, we have

$$\partial_t u - \Delta u = \mathbf{P}\sigma_1 H\nu \mathcal{H}^{n-1}_{\llcorner \Gamma_t}, \tag{1.2}$$

where $\mathbf{P} = (R_i R_j)_{1 \le i,j \le n} + I$ is the Helmholtz projection, and $R_j = \partial_j(-\Delta)^{-\frac{1}{2}}$ is the Riesz transformation. One can check that the term $\mathbf{P}\sigma_1 H\nu \mathcal{H}^{n-1}_{\llcorner \Gamma_t}$ is well-defined at least in the class of tempered distributions if the hypersurface Γ_t is a smooth boundary of a bounded domain.

In this paper, we shall construct the velocity field as the mild solution of the equation (1.2), that is, the integral equation associated with (1.2). Thus, we shall consider the system as follows.

$$\text{(FBP)} \qquad \begin{cases} u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \mathbf{P}\sigma_1 H\nu \mathcal{H}^{n-1}_{\llcorner \Gamma_s} ds, \\ (BC). \end{cases}$$

Here, $e^{t\Delta}$ is the heat semigroup; see Section 4 for details. We assume that u_0 belongs to the class of α-Hölder continuous functions ($= C^\alpha(\mathbb{R}^n)$) and Γ_0 is a $C^{2+\alpha}$ hypersurface for some $\alpha \in (0,1)$. Our aim is to construct the pair $(u, \{\Gamma_t\}_{0 \le t \le T})$ solving (FBP) with initial data (u_0, Γ_0).

We say that a family of hypersurfaces $\{\Gamma_t\}_{0 \le t \le T}$ belongs to $C^{1,2+\alpha}$ when the signed distance function of Γ_t belongs to $C^{1,2+\alpha}$ in a neighborhood of $\{\Gamma_t\}_{0 \le t \le T}$. Precise definition will be given in Section 3.

Now the main result of this paper is as follows.

Theorem 1.1 (Existence and uniqueness)**.**
Let $\alpha \in (0,1)$. Assume that $u_0 \in C^\alpha(\mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$ and Ω_0 is a bounded domain with $C^{2+\alpha}$ boundary. Let $\Gamma_0 = \partial\Omega_0$. Then, there exists a positive T such that there is a unique solution $(u, \{\Gamma_t\}_{0 \le t \le T})$ solving (FBP) with initial data (u_0, Γ_0) satisfying that $u \in C^{\frac{\alpha}{2},\alpha}([0,T] \times \mathbb{R}^n)$ and $\{\Gamma_t\}_{0 \le t \le T}$ belongs to $C^{1,2+\alpha}$.

Under the kinematic boundary condition (BC'), there are many literatures for the free boundary problems of viscous incompressible (Navier-Stokes) flows with or without surface tension.

I. Sh. Mogilevskii and V. A. Solonnikov [13] showed the local well-posedness in Hölder spaces for one phase flow problems; see also V. A. Solonnikov [21]. I. V. Denisova [3] and N. Tanaka [23] studied the two phase flows problems in the Sobolev-Slobodetskii spaces. It is known that the global solvability holds near the equilibrium states for one or two phase flows problems; see M. Padula and V. A. Solonnikov [17] and N. Tanaka [22].

In the papers listed above, the regular solutions are considered and Lagrangian coordinates are used in order to reduce the problem to the case of a fixed domain. But in our problem, such reduction is less useful because of the term $\sigma_2 H\nu$ in our kinematic boundary condition (BC). So we shall deal with the equation directly as in the formulation (FBP), and the free boundary condition appears in the term of the layer potential.

Let us comment on weak solutions of two phase flows problem. Y. Giga and S. Takahashi [6] studied two phase Stokes flows, and A. Nouri, F. Poupaud and Y. Demay [15] studied the multi-phase flows. Both papers deal with the case without surface tension. In P. I. Plotnikov [18], G. Nespoli and R. Salvi [14], and H. Abels [1], the case with surface tension is discussed. However, if surface tension is present, the existence of weak solutions is still open even for the Stokes flows, and only measure-valued varifold solutions or varifold solutions are obtained; see [18], [1] for details.

In numerical analysis, several methods are developed to study the free interface between two fluids. In [2], numerical experiments are presented by using the level set method in which the kinematic boundary condition is formulated as in (BC). The advantage of this method is that one can capture the interface even when it developes singularities such as merging and reconnection. This paper is motivated by these results, and our result of local well-posedness for regular solutions implies the validity of such formulation as a mathematical model.

Recently, the phase field approach is also established to capture the moving interface. Roughly speaking, in this approach, the equation for fluids is coupled with the kinematic boundary condition formulated by the Allen-Cahn equation or the Cahn-Hilliard equation; see C. Liu and J. Shen [10] and M. E. Gurtin, D. Polignore and J. Viñals [7].

Now let us state the main idea and the outline of the proof for the main theorem. As the first step, for a given u in the appropriate class of functions, we shall construct the family of hypersurfaces evolving by the equation in (BC). Since it is regarded as the mean curvature equation with the perturbation term u, we will follow the arguments of L. C. Evans and J. Spruck [4] (see also A. Lunardi [11] and Y. Giga and S. Goto [5]), which reduces the equation to the one for the signed distance function of interfaces; see Section 3.

Next, for a given family of hypersurfaces, we estimate the layer potential term in the integral equation in (FBP). The main difficulty is that we cannot expect high regularity for u in whole space (for example, we cannot expect $u(t) \in C^{1+\alpha}(\mathbb{R}^n)$ in general) because of the jump relation of the layer potential. However, in order to obtain the unique regular solution for the perturbed mean curvature equation in (BC), we need the regularity for the perturbation term u such as $u(t) \in C^{1+\alpha}(\mathbb{R}^n)$. To overcome these difficulties, we make use of the regularity for u in tangential directions to the interface. More precisely, if each interface has $C^{2+\alpha}$ regularity (and suitable regularity with respect to time), we have the optimal regularity for the layer potential term such as $C^{1+\alpha}$ in tangential directions. In order to establish this optimal regularity, we use the Hölder-Zygmund spaces; see Sect ion 4.1 for details. The desired result in the main theorem is obtained by constructing a suitable contraction mapping for velocity fields; see Section 5.

This paper is organized as follows. In Section 2, we give the definitions of function spaces which we use in our problem. In Section 3, we solve the mean curvature equation with a perturbation term. In Section 4.1, we establish the estimates for the layer potential term in (FBP). In Section 4.2, we remark on the mild solution of the Navier-Stokes equation with the term of the layer potential. In the section, we shall also give the outline of the proof for the local well-posedness of the Navier-Stokes equation when the layer potential term is given. Its proof is the usual contraction argument by T. Kato [8]. We will see that the velocity also has the fine regularity in tangential direction to Γ_t even in the case of the Navier-Stokes flow. In Section 5, we shall construct a suitable contraction mapping and obtain the desired results. In our strategy, the estimates for the layer potential by using local coordinate transforms play essential roles, but its proof will be a little lengthy. So some parts of the proof for the estimates are given in Appendix; see Section 6.

Remark 1.1. *When the fluid is described by the Navier-Stokes equation, the associated integral equation becomes*

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbf{P}\nabla\cdot u\otimes u ds + \int_0^t e^{(t-s)\Delta}\mathbf{P}\sigma_1 H\nu\mathcal{H}^{n-1}_{\llcorner\Gamma_s}ds. \tag{1.3}$$

Our result can be extended to this case, but its proof becomes complicated, especially when one constructs a contraction mapping for the free boundary problem. So in this paper, we consider the case of the Stokes flow for simplicity.

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2. Function spaces and embedding properties

First of all, we introduce several function spaces in which we deal with the problems. Let D be either $\mathbb{R}^n$ or an open set in $\mathbb{R}^n$ with uniformly C^2 boundary. Let $C(\overline{D})$ denote the Banach space of all continuous and bounded functions in $\overline{D}$, endowed with the sup norm. Let $C^m(\overline{D})$ denotes the set of all m times continuously differentiable functions in D, with derivatives up to the order m bounded and continuously extendable up to the boundary. The norm of $C^m(\overline{D})$ is defined as

$$||f||_{C^m(\overline{D})} := \sum_{0\le k\le m} ||\partial_x^k f||_{C(\overline{D})}$$

$$||\partial_x^k f||_{C(\overline{D})} := \sum_{|\theta|=k} ||\partial_x^\theta f||_{C(\overline{D})}.$$

Here, $\theta = (\theta_1,\cdots,\theta_n)$ is a multi-index. We recall that $C([a,b]\times\overline{D})$ is the space of all the continuous and bounded functions in $[a,b]\times\overline{D}$, endowed with the norm

$$||f||_{C([a,b]\times\overline{D})}(= ||f||_\infty) := \sup_{(t,x)\in[a,b]\times\overline{D}} |f(t,x)|.$$

For $0<\alpha<1$, we denote by $C^{0,\alpha}([a,b]\times\overline{D})$ (respectively, $C^{\frac{\alpha}{2},0}([a,b]\times\overline{D})$) the space of continuous functions that are α-Hölder continuous with respect to the space variables (respectively, $\frac{\alpha}{2}$-Hölder continuous with respect to time), i.e.,

$$C^{0,\alpha}([a,b]\times\overline{D}) := \{f\in C([a,b]\times\overline{D});\ f(t,\cdot)\in C^\alpha(\overline{D}),\ t\in[a,b]\},$$

$$||f||_{C^{0,\alpha}([a,b]\times\overline{D})}(=||f||_{C^{0,\alpha}}):=||f||_\infty+\sup_{t\in[a,b]}[f(t,\cdot)]_{C^\alpha(\overline{D})},$$

where

$$[g]_{C^\alpha(\overline{D})}:=\sup_{x,y\in\overline{D},x\neq y}\frac{|g(x)-g(y)|}{|x-y|^\alpha}$$

(respectively,

$$C^{\frac{\alpha}{2},0}([a,b]\times\overline{D}):=\{f\in C([a,b]\times\overline{D});\ f(\cdot,x)\in C^{\frac{\alpha}{2}}([a,b]),\ x\in\overline{D}\},$$
$$||f||_{C^{\frac{\alpha}{2},0}([a,b]\times\overline{D})}(=||f||_{C^{\frac{\alpha}{2},0}}):=||f||_\infty+\sup_{x\in\overline{D}}[f(\cdot,x)]_{C^{\frac{\alpha}{2}}([a,b])},$$

where

$$[h]_{C^{\frac{\alpha}{2}}([a,b])}:=\sup_{t,s\in[a,b],t>s}\frac{|h(t)-h(s)|}{(t-s)^{\frac{\alpha}{2}}}\ ).$$

Moreover, the function spaces $C^{\frac{\alpha}{2},\alpha}([a,b]\times\overline{D})$, $C^{1,2}([a,b]\times\overline{D})$, $C^{1,2+\alpha}([a,b]\times\overline{D})$, $C^{1+\frac{\alpha}{2},2+\alpha}([a,b]\times\overline{D})$ are defined as follows.

$$C^{\frac{\alpha}{2},\alpha}([a,b]\times\overline{D}):=C^{\frac{\alpha}{2},0}([a,b]\times\overline{D})\cap C^{0,\alpha}([a,b]\times\overline{D}),$$
$$||f||_{C^{\frac{\alpha}{2},\alpha}([a,b]\times\overline{D})}(=||f||_{C^{\frac{\alpha}{2},\alpha}}):=||f||_{C^{\frac{\alpha}{2},0}([a,b]\times\overline{D})}+||f||_{C^{0,\alpha}([a,b]\times\overline{D})}.$$

$$C^{1,2}([a,b]\times\overline{D}):=\{f\in C([a,b]\times\overline{D});\ \partial_t f,\ \partial_{ij}f\in C([a,b]\times\overline{D}),\ 1\le i,j\le n\},$$
$$||f||_{C^{1,2}([a,b]\times\overline{D})}(=||f||_{C^{1,2}}):=||f||_\infty+||\partial_x f||_\infty+||\partial_t f||_\infty+||\partial_x^2 f||_\infty.$$

$$C^{1,2+\alpha}([a,b]\times\overline{D}):=\{f\in C^{1,2}([a,b]\times\overline{D});\ \partial_t f,\partial_{ij}f\in C^{0,\alpha}([a,b]\times\overline{D}),\ 1\le i,j\le n\},$$
$$||f||_{C^{1,2+\alpha}([a,b]\times\overline{D})}(=||f||_{C^{1,2+\alpha}}):=||f||_\infty+||\partial_x f||_\infty+||\partial_t f||_{C^{0,\alpha}}+||\partial_x^2 f||_{C^{0,\alpha}}.$$

Let X be a Banach space endowed with the norm $||\cdot||_X$. We denote by $C^\alpha([a,b];X)$ the Hölder space such that

$$\begin{aligned}C^\alpha([a,b];X)\ &:=\ \{f\in C([a,b];X);\ [f]_{C^\alpha([a,b];X)}:=\sup_{t,s\in[a,b],t>s}\frac{||f(t)-f(s)||_X}{(t-s)^\alpha},\\ &\qquad ||f||_{C^\alpha([a,b];X)}:=\sup_{a\le t\le b}||f(t)||_X+[f]_{C^\alpha([a,b];X)}<\infty\}.\end{aligned}$$

Similarly,

$$\begin{aligned}Lip([a,b];X)\ &:=\ \{f\in C([a,b];X);\ [f]_{Lip([a,b];X)}:=\sup_{t,s\in[a,b],t>s}\frac{||f(t)-f(s)||_X}{t-s},\\ &\qquad ||f||_{Lip([a,b];X)}:=\sup_{a\le t\le b}||f(t)||_X+[f]_{Lip([a,b];X)}<\infty\}.\end{aligned}$$

Now we state the embedding properties of the Hölder spaces defined above. The following lemma will be used freely in this paper.

Lemma 2.1. *Let $0<\alpha<1$. Then there exists a positive constant K_α such that for any $f\in C^{1,2+\alpha}([a,b]\times\overline{D})$,*

$$\begin{aligned}&||f||_{C^{\frac{1}{2}}([a,b];C^{1+\alpha}(\overline{D}))}+||f||_{Lip([a,b];C^\alpha(\overline{D}))}+||\partial_x f||_{C^{1+\frac{\alpha}{2},0}}+||\partial_x^2 f||_{C^{\frac{\alpha}{2},0}}\\ &\le\ K_\alpha||f||_{C^{1,2+\alpha}}\end{aligned}\tag{2.1}$$

holds. Here, the constant K_α is independent of $b-a$ and f.

Proof. See A. Lunardi [11, Lemma 5.1.1].

3. Motion of hypersurfaces by perturbed mean curvature equations

In this section, we consider the hypersurfaces evolving in time via mean curvature with a perturbation term. Precisely, we shall construct a family of hypersurfaces $\{\Gamma_t\}_{0\le t\le T}$ such that for $0 \le t_0 \le t \le T$, $\Gamma_t = \{x(t,x_0); x_0 \in \Gamma_{t_0}\}$ satisfies the ODE

$$\begin{cases} \dfrac{dx(t)}{dt} & = -\dfrac{\sigma_2}{n-1}[\text{div } (\nu(t,x(t))]\nu(t,x(t)) + u(t,x(t)),\ t_0 \le t \le T, \\ x(t_0) & = x_0. \end{cases} \tag{3.1}$$

Here, $\nu(t,x)$ is the exterior unit normal vector of Γ_t, σ_2 is a positive constant, and $u(t,x)$ is a continuous function on $[0,T] \times \mathbb{R}^n$. The mean curvature $H(t,x)$ of the surface Γ_t is given by $H(t,x) = -\frac{1}{n-1}\text{div } \nu(t,x)$. So if $u \equiv 0$, the above equation is the well-known mean curvature flow equation. To construct an evolving hypersurfaces starting from a given smooth initial hypersurfaces, we will follow the arguments of L. C. Evans and J. Spruck [4]; see also A. Lunardi [11]. Let $\{\Gamma_t\}_{0\le t\le T}$ be the evolving hypersurfaces such that each Γ_t is the boundary of a bounded domain Ω_t. We reduce the equation to an equation for the signed distance function

$$d(t,x) = \begin{cases} \text{dist } (x,\Gamma_t), & x \in \mathbb{R}^n \setminus \overline{\Omega_t}, \\ -\text{dist } (x,\Gamma_t), & x \in \Omega_t. \end{cases} \tag{3.2}$$

If Γ_t is smooth, then the above $d(t,\cdot)$ is also smooth in the set

$$D^+ := \{x \in \mathbb{R}^n \ ;\ 0 \le d(t,x) < \delta_0\}$$

and

$$D^- := \{x \in \mathbb{R}^n \ ;\ -\delta_0 < d(t,x) \le 0\},$$

provided $\delta_0 > 0$ and $T > 0$ is small. Moreover, if δ_0 is sufficiently small, for each $x \in D^+$ there exists a unique $y \in \Gamma_t$ such that $d(t,x) = |y-x|$. The equation (3.1) implies that

$$\begin{aligned} d_t(t,x) &= < \frac{dy}{dt}, \frac{y-x}{|y-x|} > \\ &= < -\frac{\sigma_2}{n-1}[\text{div } \nu(t,y)]\nu(t,y) + u(t,y), \frac{y-x}{|y-x|} > \\ &= \frac{\sigma_2}{n-1}\text{div } \nu(t,y) - u(t, x - d\nabla_x d(t,x)) \cdot \nabla_x d(t,x). \end{aligned}$$

It is well-known that the eigenvalues of the Hessian $\nabla^2 d(t,x)$ are given by

$$\lambda_i = -\frac{\kappa_i(t,y)}{1-\kappa_i(t,y)d(t,x)},\ i = 1,\cdots,n-1,\ \lambda_n = 0, \tag{3.3}$$

where κ_i are the principal curvatures of the surface Γ_t. Since the mean curvature H is defined as $H = \frac{1}{n-1}\sum_{i=1}^{n-1}\kappa_i$, we have

$$d_t = \frac{\sigma_2}{n-1} f(d,\nabla^2 d) - u(t, x - d\nabla d) \cdot \nabla d, \tag{3.4}$$

where

$$f(s,q) = \sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_i s},\ s \in \mathbb{R},\ q \in \mathbb{R}^{n\times n}_S,\ \lambda_i s \ne 1. \tag{3.5}$$

Here, λ_i are the eigenvalues of the simmetric matrix q. The same equation can be deduced for $x \in D^-$. Since $|d|$ is a distance function, the spatial gradient ∇d should have modulus 1 at any point. This provides a nonlinear first order boundary condition for d. So the equation (3.1) is reduced to the following fully nonlinear parabolic problem

$$
\left\{
\begin{array}{l}
\partial_t v = \dfrac{\sigma_2}{n-1} f(v, \nabla^2 v) - u(t, x - v\nabla v)\cdot \nabla v,\ t \geq 0,\ x \in \overline{D}, \\
|\nabla v|^2 = 1,\ t \geq 0,\ x \in \partial D, \\
v(0,x) = d_0(x),\ x \in \overline{D},
\end{array}
\right. \tag{3.6}
$$

where $D = D^+ \cup D^- = \{x \in \mathbb{R}^n, -\delta_0 < d_0(x) < \delta_0\}$, d_0 is the signed distance function from Γ_0, and f is given as above. We choose δ_0 so small that $\lambda_i(\nabla^2 d_0)\delta_0 \neq 1$ for each i, so f is well-defined near the range of $(d_0(x), \nabla^2 d_0(x))$. Since $f(s,q) = \text{Tr }(q(I - sq)^{-1})$, f is analytic. Moreover, since $\text{Tr }(\frac{\partial f}{\partial q}(s,q)A) = \text{Tr }((I-sq)^{-2}A)$ for $A \in \mathbb{R}^{n\times n}$, we have for $\xi \in \mathbb{R}^n$,

$$
\begin{aligned}
\sum_{i,j=1}^{n} f_{q_{ij}}(s,q)\xi_i\xi_j &= \text{Tr}(\frac{\partial f}{\partial q}(s,q)\xi\otimes\xi) \\
&= \sum_{i=1}^{n} \frac{1}{(1-\lambda_i s)^2} < \xi, \bar{e_i} >^2,
\end{aligned}
$$

where $\{\bar{e_1}, \cdots, \bar{e_n}\}$ is an orthogonal basis in $\mathbb{R}^n$ such that each $\bar{e_i}$ is an eigenvector of q with eigenvalue λ_i. Thus, we have

$$
\sum_{i,j=1}^{n} f_{q_{ij}}(s,q)\xi_i\xi_j \geq \iota(s,q)|\xi|^2, \tag{3.7}
$$

with $\iota(s,q) = \min_{1\leq i\leq n}(1-\lambda_i s)^{-2}$.

Set $g(p) = p^2 - 1$. In order to solve the equation (3.6), we linearlize the principal term $f(v, \nabla^2 v)$ near the initial data $(d_0, \nabla^2 d_0)$ and $g(\nabla d_0)$ near (∇d_0). The existence and uniqueness results of the equation is proved by the general results for the linear parabolic equations and the usual contraction arguments. Let $B(d_0, \nabla^2 d_0)$ be a bounded open neighborhood of the set $\{(d_0(x), \nabla^2 d_0(x)) \in \mathbb{R}\times\mathbb{R}^{n\times n}\ ;\ x \in \overline{D}\}$ such that for each $(s,q) \in B(d_0, \nabla^2 d_0)$, the function $f(s,q)$ is well-defined. Set

$$
\iota \ := \ \inf\{\iota(s,q)\ ;\ (s,q) \in B(d_0, \nabla^2 d_0)\} > 0 \tag{3.8}
$$

$$
K_f \ := \ \sup\{|\frac{\partial^\beta f}{\partial s \partial q}(s,q)|\ ;\ (s,q) \in B(d_0, \nabla^2 d_0),\ |\beta| = 0,1,2\}. \tag{3.9}
$$

Fix $M > 0$. We assume that the perturbation term $u(t,x)$ belongs to $\mathcal{U}_M$, the closed subset of $C^{0,\alpha}([0,T]\times\mathbb{R}^n)$, defined as

$$
\begin{aligned}
\mathcal{U}_M \ := \ &\{u(t,x) \in C^{0,\alpha}([0,T]\times\mathbb{R}^n); u(t,\cdot) \in C^{1+\alpha}(\mathbb{R}^n), and \\
&\sup_{0<t<T} ||u(t,\cdot)||_{C^\alpha(\mathbb{R}^n)} + \sup_{0<t<T} t^{\frac{1-\alpha}{2}} ||\partial_x u(t,\cdot)||_{C(\mathbb{R}^n)} \\
&+ \sup_{0<t<T} t^{\frac{1}{2}} [\partial_x u(t,\cdot)]_{C^\alpha(\mathbb{R}^n)} \leq M\}
\end{aligned} \tag{3.10}
$$

The following proposotion states the existence and uniqueness of the equation (3.6).

Proposition 3.1. *Fix $M > 0$. Let $\alpha \in (0,1)$. Assume that Ω_0 is a bounded domain with uniformly $C^{2+\alpha}$ boundary and let d_0 be the signed distance function from $\Gamma_0 = \partial\Omega_0$. Then, there exist $T > 0$ such that for any $u \in \mathcal{U}_M$ there exists a unique $v \in C^{1,2+\alpha}([0,T]\times\overline{D})$, solution of (3.6).*

Proof. By considering the appropriate rescaling, we may assume that $\frac{\sigma_2}{n-1} = 1$ without loss of generality. For simplicity of notations, we write $||f||_{C^{\alpha,\beta}}, ||f||_\infty$ for

$||f||_{C^{\alpha,\beta}([0,T]\times\overline{D})}$, $||f||_{C([0,T]\times\overline{D})}$, respectively. Set

$$\begin{aligned}\mathcal{A}v(t,x) &:= \sum_{i,j=1}^{n} f_{q_{ij}}(d_0(x),\nabla^2 d_0(x))\partial_{ij}v(t,x) + f_s(d_0(x),\nabla^2 d_0(x))v(t,x),\\ \mathcal{B}v(t,x) &:= \sum_{i=1}^{n} g_{p_i}(\nabla d_0(x))\partial_i v(t,x).\end{aligned}$$

Let R be a positive number to be precised later. We will find the solution in the set

$$X := \{v \in C^{1,2+\alpha}([0,T]\times\overline{D})\ ;\ v(0,\cdot) = d_0,\ ||v-d_0||_{C^{1,2+\alpha}} \le R\} \tag{3.11}$$

as a fixed point of the operator Φ defined in X, $\Phi(v) = w$ is the solution of the equation

$$\begin{cases}\partial_t w(t,x) &= \mathcal{A}w + f(v,\nabla^2 v) - \mathcal{A}v - u(t,x-v\nabla v)\cdot\nabla v,\ 0\le t\le T,\ x\in\overline{D},\\ \mathcal{B}w(t,x) &= -g(\nabla v) + \mathcal{B}v,\ 0\le t\le T,\ x\in\partial D,\\ w(0,x) &= d_0(x),\ x\in\overline{D}.\end{cases} \tag{3.12}$$

Let K_α be the constant in Lemma 2.1. Then for all $v \in X$, we have

$$\sum_{i,j=1}^{n} ||\partial_{i,j}v - \partial_{i,j}d_0||_\infty + ||v-d_0||_\infty \le (K_\alpha T^{\frac{\alpha}{2}} + T)R. \tag{3.13}$$

So $f(v,\nabla^2 v)$ is well defined in X if we take T so small that every $v\in X$ satisfies $(v(t,x),\nabla^2 v(t,x)) \in B(d_0,\nabla^2 d_0)$ for $(t,x)\in[0,T]\times\overline{D}$. The general results for linear parabolic equations gurantee that the equation has a unique solution in $C^{1,2+\alpha}([0,T]\times\overline{D})$. We shall show that for every $v_1, v_2 \in X$,

$$||\Phi(v_1) - \Phi(v_2)||_{C^{1,2+\alpha}} \le C(R)T^{\frac{\alpha}{2}}||v_1-v_2||_{C^{1,2+\alpha}}. \tag{3.14}$$

Thus, if T is so small that $C(R)T^{\frac{\alpha}{2}} \le \frac{1}{2}$ then Φ is a contraction mapping. We also have for all $v\in X$,

$$\begin{aligned}||\Phi(v)-d_0||_{C^{1,2+\alpha}} &\le ||\Phi(v)-\Phi(d_0)||_{C^{1,2+\alpha}} + ||\Phi(d_0)-d_0||_{C^{1,2+\alpha}}\\ &\le \frac{R}{2} + ||\Phi(d_0)-d_0||_{C^{1,2+\alpha}}.\end{aligned}$$

The function $z = \Phi(d_0) - d_0$ is the solution of

$$\begin{cases}\partial_t z(t,x) &= \mathcal{A}z(t,x) + f(d_0,\nabla^2 d_0) - u(t,x-d_0\nabla d_0)\cdot\nabla d_0,\ 0\le t\le T,\ x\in\overline{D},\\ \mathcal{B}z(t,x) &= -g(\nabla d_0),\ 0\le t\le T,\ x\in\partial D,\\ z(0,x) &= 0,\ x\in\overline{D}.\end{cases} \tag{3.15}$$

So z satisfies the estimate

$$\begin{aligned}&||z||_{C^{1,2+\alpha}}\\ \le\ & C(||f(d_0,\nabla^2 d_0)||_{C^\alpha(\overline{D})} + ||u(\cdot,\cdot-d_0\nabla d_0)\cdot\nabla d_0||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)} + ||g(\nabla d_0)||_{C^{1+\alpha}})\\ \le\ & C(||d_0||_{C^{2+\alpha}}, M),\end{aligned}$$

hence,

$$||\Phi(v)-d_0||_{C^{1,2+\alpha}} \le \frac{R}{2} + C(||d_0||_{C^{2+\alpha}}, M). \tag{3.16}$$

Thus for $R = 2C(||d_0||_{C^{2+\alpha}}, M)$, Φ is a contraction mapping X into itself, which implies that Φ has a unique fixed point in X. The uniqueness of the solution in

$C^{1,2+\alpha}([0,T]\times\overline{D})$ follows easily, so we omit it. Now let us prove the key estimate (3.14). Set $w=\Phi(v_1)-\Phi(v_2)$. Then w is the solution of

(3.17)
$$\begin{cases} \partial_t w &= \mathcal{A}w + f(v_1,\nabla^2 v_1) - f(v_2,\nabla^2 v_2) - \mathcal{A}(v_1-v_2), \\ & \quad -u(t,x-v_1\nabla v_1)\cdot\nabla v_1 + u(t,x-v_2\nabla v_2)\cdot\nabla v_2,\ 0\le t\le T,\ x\in\overline{D}, \\ \mathcal{B}w &= \mathcal{B}(v_1-v_2) - g(\nabla v_1) + g(\nabla v_2),\ 0\le t\le T,\ x\in\partial D, \\ w(0,x) &= 0,\ x\in\overline{D}. \end{cases}$$

So w satisfies

$$||w||_{C^{1,2+\alpha}} \le C(||h_1||_{C^{0,\alpha}} + ||h_2||_{C^{0,\alpha}} + ||h_3||_{C^{\frac{1+\alpha}{2},1+\alpha}}), \tag{3.18}$$

where

$$\begin{aligned} & h_1(t,x) \\ := \ & f(v_1(t,x),\nabla^2 v_1(t,x)) - f(v_2(t,x),\nabla^2 v_2(t,x)) - \mathcal{A}(v_1(t,x)-v_2(t,x)), \\ & h_2(t,x) \\ := \ & u(t,x-v_1(t,x)\nabla v_1(t,x))\cdot\nabla v_1(t,x) - u(t,x-v_2(t,x)\nabla v_2(t,x))\cdot\nabla v_2(t,x), \\ & h_3(t,x) \\ := \ & \mathcal{B}(v_1(t,x)-v_2(t,x)) - g(\nabla v_1(t,x)) + g(\nabla v_2(t,x)). \end{aligned}$$

First we shall estimate h_1. By the definition of $\mathcal{A}$, we have

$$\begin{aligned} h_1(t,x) \ := \ & \int_0^1 (f_s(\eta_\tau(t,x)) - f_s(\eta_0(x)))d\tau(v_1(t,x)-v_2(t,x)) \\ & + \sum_{i,j=1}^n \int_0^1 (f_{q_{i,j}}(\eta_\tau(t,x)) - f_{q_{i,j}}(\eta_0(x))d\tau(\partial_{i,j}v_1(t,x) - \partial_{i,j}v_2(t,x)), \end{aligned}$$

where

$$\begin{aligned} \eta_\tau(t,x) &= \tau(v_1(t,x),\nabla^2 v_1(t,x)) + (1-\tau)(v_2(t,x),\nabla^2 v_2(t,x)) \in \mathbb{R}\times\mathbb{R}^{n\times n}, \\ \eta_0(x) &= (d_0(x),\nabla^2 v_0(x)) \in \mathbb{R}\times\mathbb{R}^{n\times n}. \end{aligned}$$

Clearly we have

$$|\eta_\tau(t,x)-\eta_0(x)| \le 2(T+K_\alpha T^{\frac{\alpha}{2}})R, \tag{3.19}$$
$$|\eta_\tau(t,x)-\eta_\tau(t,y)| \le C(||d_0||_{C^{2+\alpha}}+R)|x-y|^\alpha, \tag{3.20}$$
$$|\eta_0(x)-\eta_0(y)| \le C||d_0||_{C^{2+\alpha}}|x-y|^\alpha, \tag{3.21}$$

where C depends only on n and K_α is the constant in Lemma 2.1. From these estimates and regularity assumptions on f, it is not difficult to deduce the estimates

$$\begin{aligned} |h_1(t,x)| &\le C(T+K_\alpha T^{\frac{\alpha}{2}})RK_f T^{\frac{\alpha}{2}}||u-v||_{C^{1,2+\alpha}} \\ |h_1(t,x)-h_1(t,y)| &\le C(||d_0||_{C^{2+\alpha}}+R+K_\alpha)K_f T^{\frac{\alpha}{2}}||v_1-v_2||_{C^{1,2+\alpha}}|x-y|^\alpha, \end{aligned}$$

hence,

$$||h_1||_{C^{0,\alpha}} \le C(||d_0||_{C^{2+\alpha}}+R+K_\alpha)K_f T^{\frac{\alpha}{2}}||v_1-v_2||_{C^{1,2+\alpha}}, \tag{3.22}$$

where C depends only on n, and K_α, K_f are the constants of Lemma 2.1, (3.9), respectively. The estimate of h_3 is similar and we have

$$||h_3||_{C^{\frac{1+\alpha}{2},1+\alpha}} \le C(||d_0||_{C^{2+\alpha}},R,K_\alpha,K_f)T^{\frac{\alpha}{2}}||v_1-v_2||_{C^{1,2+\alpha}}, \tag{3.23}$$

the details are omitted.

Next we shall estimate h_2. By the mean value theorem, we have

$$
\begin{aligned}
(3.24)\qquad & h_2(t,x) \\
= & \int_0^t < -v_1\nabla v_1 + v_2\nabla v_2, (\nabla_x u)(t,\bar{\eta}_\tau(t,x)) > d\tau \cdot \nabla v_1 \\
& \quad +u(t,x-v_2\nabla v_2)\cdot(\nabla v_1 - \nabla v_2) \\
=: & \; h_{2,1}(t,x) + h_{2,2}(t,x),
\end{aligned}
$$

where

$$\bar{\eta}_\tau(t,x) := x - \tau v_1\nabla v_1(t,x) - (1-\tau)v_2\nabla v_2(t,x).$$

From the regularity assumption on u and Lemma 2.1, we have

$$
\begin{aligned}
|h_{2,1}(t,x)| &\le (t+K_\alpha t^{\frac{1+\alpha}{2}})(R+||d_0||_{C^{2+\alpha}})^2||v_1-v_2||_{C^{1,2+\alpha}}Mt^{-\frac{1-\alpha}{2}} \\
&\le M(R+||d_0||_{C^{2+\alpha}})^2T^{\frac{\alpha}{2}}||v_1-v_2||_{C^{1,2+\alpha}},
\end{aligned}
$$

and

$$|h_{2,2}(t,x)| \le MK_\alpha T^{\frac{1+\alpha}{2}}||v_1-v_2||_{C^{1,2+\alpha}}.$$

Since $|\bar{\eta}_\tau(t,x)-\bar{\eta}_\tau(t,y)| \le (1+4(R+||v_0||_{C^{2+\alpha}})^2)|x-y|$, we easily have

$$|h_{2,1}(t,x)-h_{2,1}(t,y)| \le C(M,K_\alpha,R,||d_0||_{C^{2+\alpha}}))T^{\frac{\alpha}{2}}||v_1-v_2||_{C^{1,2+\alpha}}|x-y|^\alpha,$$

and

$$|h_{2,2}(t,x)-h_{2,2}(t,y)| \le C(M,K_\alpha,R,||d_0||_{C^{2+\alpha}})T^{\frac{\alpha}{2}}||v_1-v_2||_{C^{1,2+\alpha}}|x-y|^\alpha.$$

Thus the estimate (3.14) follows, and the proof of Proposition 3.1 is completed.

Remark 3.1. *From (3.16) in the proof above, the solution v satisfies*

$$||v||_{C^{1,2+\alpha}([0,T]\times\overline{D})} \le ||d_0||_{C^{2+\alpha}} + 2C(||d_0||_{C^{2+\alpha}}, M). \tag{3.25}$$

We also see that the existence time of the solution does not depend on each $u \in \mathcal{U}_M$.

We set $\Gamma_t := \{x\in\overline{D}; v(t,x)=0\}$ when v is the solution of the equation (3.6) satisfying $v(0,x)=d_0(x)$. By arguing the same as in A. Lunardi [Proposition 8.5.9.], we see that the solution v has a regularity such as $\partial_i v \in C^{1,2+\alpha}([t_1,t_2]\times\overline{D'})$, $i=1,\cdots,n$, for any open set $D' \subset\subset D$, and $0<t_1<t_2\le T$. Under this regularity condition, we can show that $\{\Gamma_t\}_{0\le t\le T}$ is an evolving hypersurfaces satisfying the perturbed mean curvature equation (3.1). Presicely, we have the following proposition.

Proposition 3.2. *Assume that the conditions in Proposition 3.1 are satisfied. Then, the first derivatives of the solution v saitsfy the estimate*

$$||\partial_x v||_{C^{1,2+\alpha}([t_1,t_2]\times\overline{D'})} \le C(\frac{(t_2-t_1)^{\frac{1}{2}}}{t_1} + t_1^{-\frac{1}{2}}), \tag{3.26}$$

where C depends only on n, ι, σ_2, α, K_α, $||d_0||_{C^{2+\alpha}(\overline{D})}$, M, K_f and dist $(D',\partial D)$. *Moreover, for each $t>0$, Γ_t is a $C^{3+\alpha}$ hypersurface and $\{\Gamma_t\}_{0\le t\le T}$ is a unique family of $C^{2+\alpha}$ hypersurfaces evolving by the perturbed mean curvature equation (3.1) starting from Γ_0.*

Proof. Again we may assume that $\frac{\sigma_2}{n-1}=1$. The asserition for the regularity of $\partial_x v$ follows by arguing as same as in A. Lunardi [11, Proposition 8.5.9]. We omit the details here. So we shall show the latter assertion of the proposition. First we shall prove that $|\partial_x v|\equiv 1$ for the solution v. Set $w=|\partial_x v|^2-1$. By the regularity results stated as above, the function w belongs to $C^{1,2+\alpha}([\epsilon,T]\times\overline{D'})$ for every open set $D' \subset\subset D$ and $0 < \epsilon < T$. Differentiating w with respect to time, we have for $(t,x) \in (0,T) \times D$,

$$
\begin{aligned}
&\partial_t w \\
= \ & 2\sum_{i=1}^n \partial_i v \partial_t \partial_i v \\
= \ & 2\sum_{i=1}^n \partial_i v \partial_i (f(v, \nabla^2 v) - u(t, x - v\nabla v)\cdot \nabla v) \\
= \ & 2\sum_{i,j,k=1}^n \partial_i v f_{q_{ij}}(v, \nabla^2 v)\partial_{ijk} v + 2\sum_{i=1}^n \partial_n v f_s(v, \nabla^2 v)\partial_i v \\
& -2\sum_{i=1}^n \partial_i v \sum_{j,k=1}^n \partial_i (x_j - v\partial_j v)\partial_j u_k(t, x - v\nabla v)\partial_k v - 2\sum_{i,k=1}^n \partial_i v \partial_{ik} v u_k(t, x - v\nabla v) \\
= \ & \sum_{i,j=1}^n f_{q_{ij}}(v, \nabla^2 v)\partial_{ij} w - 2\sum_{i,j,k=1}^n f_{q_{ij}}(v, \nabla^2 v)\partial_{ik} v \partial_{kj} v + 2 f_s(v, \nabla^2 v)|\partial_x v|^2 \\
& -2\sum_{i,j,k=1}^n \partial_i v \partial_k v(\delta_{ij} - \partial_i v \partial_j v - v\partial_{ij} v)\partial_j u_k(t, x - v\nabla v) - \sum_{i=1}^n u_k(t, x - v\nabla v)\partial_i w.
\end{aligned}
$$

Now we have

$$
\sum_{i,j,k=1}^n f_{q_{ij}}(v, \nabla^2 v)\partial_{ik} v \partial_{kj} v = \sum_{i=1}^n \frac{(\lambda_i(\nabla^2 v))^2}{1 - \lambda_i(\nabla^2 v)v)^2} = f_s(v, \nabla^2 v),
$$

and

$$
\begin{aligned}
&\sum_{i,j,k=1}^n \partial_i v \partial_k v \partial_i v \partial_j v \partial_j u_k(t, x - v\nabla v) = |\partial_x v|^2 \sum_{j,k=1}^n \partial_j v \partial_k v \partial_j u_k(t, x - v\nabla v) \\
= \ & (w+1)\sum_{j,k=1}^n \partial_j v \partial_k v \partial_j u_k(t, x - v\nabla v),
\end{aligned}
$$

$$
v\sum_{i,j,k=1}^n \partial_i v \partial_k v \partial_{ij} v \partial_j u_k(t, x - v\nabla v) = \frac{1}{2} v \sum_{j=1}^n \partial_j w \sum_{k=1}^n \partial_k v \partial_j u_k(t, x - v\nabla v).
$$

Thus,

$$
\begin{aligned}
\partial_t w \ = \ & \sum_{i,j=1}^n f_{q_{ij}}(v, \nabla^2 v)\partial_{ij} w \\
& + \sum_{i=1}^n \{u_i(t, x - v\nabla v) + l_{1,i}(t,x)\}\partial_i w + 2\{f_s(v, \nabla^2 v) + l_2(t,x)\} w,
\end{aligned}
$$

where

$$
\begin{aligned}
l_{1,i}(t,x) \ &:= \ v\sum_{k=1}^n \partial_k v \partial_i u_k(t, x - v\nabla v), \\
l_2(t,x) \ &:= \ \sum_{j,k=1}^n \partial_j v \partial_k v \partial_j u_k(t, x - v\nabla v).
\end{aligned}
$$

Since w vanishes on the parabolic boundary of $[0,T] \times \overline{D}$, then $w \equiv 0$ and $|\partial_x v| \equiv 1$ holds. This implies that each Γ_t, $t > 0$, is a hypersurface of class $C^{3+\alpha}$ and $\nu(t,x) = \nabla v(t,x)$ is a unit normal vector of Γ_t. In order to see that the family $\{\Gamma_t\}_{0 \le t \le T}$ is an evolving hypersurfaces satisfying the perturbed mean curvature equation (3.1), we consider the ODE

$$\begin{cases} \dfrac{dx}{dt} & = -f(v(t,x),\nabla^2 v(t,x))\nabla v(t,x) + u(t, x - v\nabla v(t,x)), \\ x(t_0) & = x_0,\ x_0 \in \Gamma_{t_0}. \end{cases} \tag{3.27}$$

for $0 \le t_0 \le T$. From the regularity conditions of v and u, the Lipschitz norm with respect to x of the right hand side of the above equation has a singularity near time zero. Neverthless, we can check that the problem is uniquely solvable and the solution $x(t)$ is at least in $C^1([0,T])$. Moreover, since $\frac{d}{dt}v(t,x(t)) = 0$ and $v(t_0, x(t_0)) = 0$, we have $v(t,x(t)) \equiv 0$. Hence, $x(t) \in \Gamma_t$. This completes the proof.

Let $u(t,x)$ and $\tilde{u}(t,x)$ be two functions in $\mathcal{U}_M$. Let $v,\ \tilde{v} \in C^{1,2+\alpha}([0,T]\times\overline{D})$ be solutions of the equation (3.6) with initial data $v(0,x) = \tilde{v}(0,x) = d_0(x)$ and with perturbation terms $u,\ \tilde{u}$, respectively. Remark that for fixed $M > 0$ and d_0, the above T can be taken uniformly in u belonging to $\mathcal{U}_M$.

Proposition 3.3. *Fix $M > 0$. Let $\alpha \in (0,1)$. Assume that Ω_0 be a bounded domain with uniformly $C^{2+\alpha}$ boundary and let d_0 be the signed distance function from $\Gamma_0 = \partial\Omega_0$. Let $u,\ \tilde{u},\ v,\ \tilde{v}$ be functions defined above. Then, it follows that*

$$||v - \tilde{v}||_{C^{1,2+\alpha}([0,T]\times\overline{D})} \le C||u - \tilde{u}||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)}, \tag{3.28}$$

where C depends only on n, α, ι, M, σ_2, K_f, K_α, and $||d_0||_{C^{2+\alpha}}$.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be operators defined in the proof of Proposition 3.1. Then, $w = v - \tilde{v}$ is a solution of the equation

$$\begin{cases} \partial_t w = \mathcal{A}w + f(v,\nabla^2 v) - f(\tilde{v},\nabla^2\tilde{v}) - \mathcal{A}w \\ \qquad\qquad -u(t, x - v\nabla v)\cdot\nabla v + \tilde{u}(t, x - \tilde{v}\nabla\tilde{v})\cdot\nabla\tilde{v},\ 0 \le t \le T,\ x \in \overline{D}, \\ \mathcal{B}w = -g(\nabla v) + g(\nabla\tilde{v}) + \mathcal{B}w,\ 0 \le t \le T,\ x \in \partial D, \\ w(0,x) = 0,\ x \in \overline{D}. \end{cases} \tag{3.29}$$

So the function w satisfies

$$||w||_{C^{1,2+\alpha}} \le C(||\bar{h}_1||_{C^{0,\alpha}} + ||\bar{h}_2||_{C^{0,\alpha}} + ||\bar{h}_3||_{C^{\frac{1+\alpha}{2},1+\alpha}}), \tag{3.30}$$

where

$$\begin{aligned} \bar{h}_1(t,x) &:= f(v,\nabla^2 v) - f(\tilde{v},\nabla^2\tilde{v}) - \mathcal{A}w, \\ \bar{h}_2(t,x) &:= u(t, x - v\nabla v)\cdot\nabla v - \tilde{u}(t, x - \tilde{v}\nabla\tilde{v})\cdot\nabla\tilde{v}, \\ \bar{h}_3(t,x) &:= -g(\nabla v) + g(\nabla\tilde{v}) + \mathcal{B}w. \end{aligned}$$

The estimate for $||\bar{h}_1||_{C^{0,\alpha}}$ and $||\bar{h}_3||_{C^{0,\alpha}}$ are the same as the ones of $||h_1||_{C^{0,\alpha}}$ and $||h_3||_{C^{0,\alpha}}$ in the proof of Proposition 3.1, respectively. So we have

$$||\bar{h}_1||_{C^{0,\alpha}} + ||\bar{h}_3||_{C^{0,\alpha}} \le CT^{\frac{\alpha}{2}}||v - \tilde{v}||_{C^{1,2+\alpha}}. \tag{3.31}$$

We decompose $\bar{h}_2$ as

$$\begin{aligned} \bar{h}_2(t,x) &= u(t, x - v\nabla v)\cdot\nabla v - u(t, x - \tilde{v}\nabla\tilde{v})\cdot\nabla\tilde{v} \\ &\quad +(u(t, x - \tilde{v}\nabla\tilde{v}) - \tilde{u}(t, x - \tilde{v}\nabla\tilde{v}))\cdot\nabla\tilde{v} \\ &=: \bar{h}_{2,1}(t,x) + \bar{h}_{2,2}(t,x). \end{aligned}$$

The estimate for $\bar{h}_{2,1}$ is the same as the one for h_2 in the proof of Proposition 3.1, so we obtain

$$||\bar{h}_{2,1}||_{C^{0,\alpha}} \le CT^{\frac{\alpha}{2}}||v - \tilde{v}||_{C^{1,2+\alpha}}.$$

The estimate for $\bar{h}_{2,2}$ is also easy and we have

$$||\bar{h}_{2,2}||_{C^{0,\alpha}} \leq C||u - \tilde{u}||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)}.$$

Thus, for sufficiently small $T' \leq T$, we obtain

$$||v - \tilde{v}||_{C^{1,2+\alpha}([0,T']\times\overline{D})} \leq C||u - \tilde{u}||_{C^{0,\alpha}([0,T']\times\mathbb{R}^n)}. \tag{3.32}$$

Repeating this argument, we have the desired estimate. The dependence of the constant is obvious. Now the proof is completed.

4. Mild solutions of the Stokes equations

In this section, we construct the mild solutions of the Stokes equations with the layer potential. Now we define the mild solution as the solution of the integral equation associated with the Stokes equations. Let $\mathcal{S}'(\mathbb{R}^n)$ be the class of tempered distributions.

Definition 4.1. *Let h be a bounded and continous function in $(0,T] \times \mathbb{R}^n$. Let $\{\Gamma_t\}_{0\leq t\leq T}$ be a family of hypersurfaces with suitable regularity. The function u is called a mild solution of the Stokes equation with the layer potential term $h\mathcal{H}^{n-1}_{\llcorner\Gamma_t}$ if there exists $u_0 \in \mathcal{S}'(\mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$ such that*

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\mathbf{P}h\mathcal{H}^{n-1}_{\llcorner\Gamma_s}ds \tag{4.1}$$

holds. Here, $e^{t\Delta}$ is the heat semigroup, and $\mathbf{P}$ is the Helmholtz projection.

Since both $e^{t\Delta}$ and $\mathbf{P}$ are convolution operators, we can regard $e^{t\Delta}\mathbf{P}$ as one convolution operator. More precisely, for $a \in (C_0(\mathbb{R}^n))^n$, the i-th component of the convolution $e^{t\Delta}\mathbf{P}a$ is expressed as

$$(e^{t\Delta}\mathbf{P}a)^{(i)} = \sum_{j=1}^n (\frac{1}{t^{\frac{n}{2}}}L_{i,j}(\frac{\cdot}{\sqrt{t}}) + G_t(\cdot)\delta_{ij}) * a_j. \tag{4.2}$$

Here, $G_t(x)$ is the Gauss kernel

$$G_t(x) := \frac{1}{(4\pi t)^{\frac{n}{2}}}\exp(-\frac{|x|^2}{4t})$$

and

$$L_{j,k} := -\mathcal{F}^{-1}(\frac{\xi_j\xi_k}{|\xi|^2}\exp(-|\xi|^2)),$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform

$$\mathcal{F}^{-1}g(x) := (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} g(\xi)e^{ix\cdot\xi}d\xi.$$

The pointwise estimates of the kernel function $K_{i,j}(t,x) := \frac{1}{t^{\frac{n}{2}}}L_{i,j}(\frac{\cdot}{\sqrt{t}}) + G_t(\cdot)\delta_{ij}$ are given as follows.

Lemma 4.1. *Let $K_{i,j}(t,x)$ be the function defined as above. Let $l = (l_1, \cdots, l_n)$ be a multi-index. Then, we have*

$$|\partial_x^l K_{i,j}(t,x)| \leq \frac{C}{t^{\frac{n+|l|}{2}}}(1 + \frac{|x|}{\sqrt{t}})^{-(n+|l|)}, \tag{4.3}$$

where C depends only on n and l.

Proof. This pointwise estimates were originally obtained by C. W. Oseen [16]. See also P. G. Lemarié-Rieusset [9], and Y. Shibata and S. Shimizu [20]. Simple proof is also obtained by the author and Y. Terasawa [12].

As for the heat semigroup $e^{t\Delta}$, we have the following estimates.

Lemma 4.2. *Let $0<\alpha<1$. Let $b(t,x):=e^{t\Delta}a$. Then, we have*

$$||b||_{C([0,T]\times\mathbb{R}^n)} \le C||a||_{C(\mathbb{R}^n)}, \tag{4.4}$$

$$[b]_{C^{\frac{\alpha}{2},\alpha}([0,T]\times\mathbb{R}^n)} \le C[a]_{C^\alpha(\mathbb{R}^n)}, \tag{4.5}$$

$$[b]_{C^{\frac{\alpha}{2},\alpha}([0,T]\times\mathbb{R}^n)} \le \frac{C}{t^{\frac{\alpha}{2}}}||a||_{C(\mathbb{R}^n)}. \tag{4.6}$$

Proof. These estmates are well known, so we omit the proof.

4.1. **Estimates for the layer potential.**
In this section, we shall estimate the term

$$F(t,x):=\int_0^t e^{(t-s)\Delta}\mathbf{P}h\mathcal{H}^{n-1}_{\llcorner\Gamma_s}ds, \tag{4.7}$$

which reflects the boundary condition on Γ_t when $h=H\nu$. First, we define the class of the evolving hypersurfaces which we deal with. Let Γ_0 be a boundary of a smooth bounded domain Ω_0. Let d_0 be the signed distance function of Γ_0

$$d_0(x)=\begin{cases} \text{dist }(x,\Gamma_0),\ x\in\mathbb{R}^n\setminus\overline{\Omega_0}, \\ -\text{dist }(x,\Gamma_0),\ x\in\Omega_0. \end{cases} \tag{4.8}$$

We set

$$D:=\{x\in\mathbb{R}^n\ ;\ -\delta_0<d_0(x)<\delta_0\} \tag{4.9}$$

for sufficiently small δ_0; see Section 3. We assume that Γ_0 is uniformly $C^{2+\alpha}$, that is, $d_0\in C^{2+\alpha}(\overline{D})$. Since d_0 is a distance function, we have $|\partial_x d_0(x)|\equiv 1$ on $x\in\overline{D}$, which implies that

$$\min_{x\in\overline{D}}\max_{1\le i\le n}|\partial_i d_0(x)|\ge\frac{1}{n}. \tag{4.10}$$

We set

$$r:=\min\{\frac{1}{n},\delta_0\}>0. \tag{4.11}$$

Remark that r depends only on n and Γ_0.

Definition 4.2. *Let $R\ge 1$ be a given number and $\alpha\in(0,1)$. We define the set $\mathcal{S}(\alpha,R,T,d_0)$ as the set of families of hypersurfaces $\{\Gamma_t\}_{0\le t\le T}$ such that each Γ_t is a boundary of a bounded domain $\Omega_t\subset\mathbb{R}^n$ and represented as*

$$\Gamma_t=\{x\in D;\ v(t,x)=0\} \tag{4.12}$$

by the signed distance function $v\in C^{1,2+\alpha}([0,T]\times\overline{D})$ satisfying $||v||_{C^{1,2+\alpha}([0,T]\times\overline{D})}\le R$ and $v(0,x)=d_0(x)$.

By the implicit function theorem, we can derive the properties of the local coordinate transforms of $\{\Gamma_t\}_{0\le t\le T}$ in $\mathcal{S}(\alpha,R,T,d_0)$; see Section 6.2.

Let $R\ge 1$, and let T_0 be a positive number given by Proposition 6.1 depending only on R and d_0. If $T\le T_0$ and $\{\Gamma_t\}_{0\le t\le T}$ is an evolving hypersurface belonging to $\mathcal{S}(\alpha,R,T,d_0)$, then the following statements hold from Proposition 6.1.

(a)There exsist a family of open sets $\{U_k\}_{k=1}^m$, $U_k \subset \overline{D}$, and a family of functions $\{\varphi_k(t,x)\}_{k=1}^m$, $\varphi_k(t,x) \in C^{1,2+\alpha}([0,T]\times\overline{U_k})$ satisfying following properties.

(b)For each $t \in [0,T]$, there exists an open set $U_j(t) \subset U_j$ such that:

(b-1)The functions $\varphi_k(t,\cdot) : \overline{U_k(t)} \to \overline{B}$, $B := \{y \in \mathbb{R}^n;\ |y| < 1\}$ are C^2-diffeomorphisms.

(b-2)For each $t \in [0,T]$,

$$\varphi_k(t,\{\overline{U_k(t)} \cap \Omega_t\}) = \{y \in \overline{B}; y_n > 0\}$$

and

$$\varphi_k(t,\{\overline{U_k(t)} \cap \Gamma_t\}) = \overline{B'},\ B' := \{y \in B;\ y_n = 0\}$$

.

(b-3)For some $\rho > 0$, there exist families of open balls $\{O_k\}_{k=1}^m$ and $\{\hat{O}_k\}_{k=1}^m$ such that

$$\hat{O}_k \subset\subset O_k, \tag{4.13}$$

$$O_k \subset\subset \cap_{0\le t\le T}\varphi_k^{-1}(t,B),\ 1 \le k \le m, \tag{4.14}$$

$$\cup_{0\le t\le T}(\Gamma_t)_\rho \subset\subset \cup_{k=1}^m \hat{O}_k, \tag{4.15}$$

where $(A)_\rho := \{x \in \mathbb{R}^n\ ;\ \text{dist}\ (x,A) < \rho\}$.

We can choose ρ, $\{U_k\}_{k=1}^m$, $\{O_k\}_{k=1}^m$, and $\{\hat{O}_k\}_{k=1}^m$ not depending on each evolving hypersurface belonging to $\cup_{0<T\le T_0}\mathcal{S}(\alpha,R,T,d_0)$. Especially, we can take $O_k = \{|x-\bar{x}_k| < 4\rho\}$, $\hat{O}_k = \{|x-\bar{x}_k| < 3\rho\}$, for some $\bar{x}_k \in \Gamma_0$. And if $\max_{1\le i\le n}|\partial_i d_0(\bar{x}_k)| = |\partial_{i_0} d_0(\bar{x}_k)|$, the local coordinate transforms $\varphi_k = (\varphi_k^{(1)},\cdots,\varphi_k^{(n)})$, $\psi_k = (\psi_k^{(1)},\cdots,\psi_k^{(n)})$ can be taken as

$$\varphi_k(t,x) = \Pi\hat{\varphi}_k(t,x), \tag{4.16}$$

$$\psi_k(t,y) = \hat{\psi}_k(t,\Pi^{-1}y), \tag{4.17}$$

respectively, where Π is a orthogonal matrix such that $\Pi(x^{(1)},\cdots,x^{(i_0)},\cdots,x^{(n)}) = (x^{(1)},\cdots,x^{(n)},\cdots,x^{(i_0)})$ for all $x = (x^{(1)},\cdots,x^{(n)})$ and

$$\hat{\varphi}_k^{(i)}(t,x) = \frac{64R^2}{r^2}(x^{(i)} - \bar{x}_k^{(i)}),\ i \ne i_0, \tag{4.18}$$

$$\hat{\varphi}_k^{(i_0)}(t,x) = \frac{64R^2}{r^2}(x^{(i_0)} - g_k(t,x^{(1)},\overset{i_0}{\check{\cdots}},x^{(n)})), \tag{4.19}$$

$$\hat{\psi}_k^{(i)}(t,y) = \frac{r^2}{64R^2}y^{(i)} + \bar{x}_k^{(i)},\ i \ne i_0, \tag{4.20}$$

$$\hat{\psi}_k^{(i_0)}(t,y) = \frac{r^2}{64R^2}y^{(i_0)} + g_k(t,\frac{r^2}{64R^2}y^{(1)} + \bar{x}_k^{(1)},\overset{i_0}{\check{\cdots}},\frac{r^2}{64R^2}y^{(n)} + \bar{x}_k^{(n)}). \tag{4.21}$$

Here, g_k is the function constructed in Proposition 6.1. For these local coordinate transforms, we have the following lemma. Let $y = (y', y^{(n)})$.

Lemma 4.3. *Let $\{\Gamma_t\}_{0\le t\le T}$ be an evolving hypersurface belonging to $\mathcal{S}(\alpha,R,T,d_0)$. Let $\{\varphi_k\}_{1\le k\le m}, \{\psi_k\}_{1\le k\le m}$ be the local coordinate transforms of $\{\Gamma_t\}_{0\le t\le T}$ above. Then, we have*

$$||\varphi_k||_{C^{1,2+\alpha}([0,T]\times\overline{U_k})}, ||\psi_k||_{C^{1,2+\alpha}([0,T]\times\overline{B})} \le C(1 + ||v||_{C^{1,2+\alpha}([0,T]\times\overline{D})}), \tag{4.22}$$

$$|\psi_k(t,y_1) - \psi_k(s,y_2)| \ge C|y_1' - y_2'|,\ \text{for } 0 \le s \le t \le T, \tag{4.23}$$

where C depends only on n, r and R.

Moreover, let $\{\tilde{\Gamma}_t\}_{0\le t\le T}$ be another evolving hypersurface belonging to $\mathcal{S}(\alpha, R, T, d_0)$ and let $\{\tilde{\varphi}_k\}_{1\le k\le m}, \{\tilde{\psi}_k\}_{1\le k\le m}$ be the local coordinate transforms of $\{\tilde{\Gamma}_t\}_{0\le t\le T}$ given by (4.16)-(4.21). Assume that $l_1,\ l_2 \in \mathbb{N}$, $\{t_i\}_{i=1}^{l_1+l_2} \subset [0,T]$, $\{y_i\}_{i=1}^{l_1+l_2} \subset B$, and $\{\tau_i\}_{i=1}^{l_1+l_2} \subset \mathbb{R}$ with $\sum_{i=1}^{l_1+l_2} \tau_i = 0$. Then, we have

$$
\begin{aligned}
(4.24) \qquad & ||\varphi_k - \tilde{\varphi}_k||_{C^{1,2+\alpha}([0,T]\times\overline{U_k})} + ||\psi_k - \tilde{\psi}_k||_{C^{1,2+\alpha}([0,T]\times\overline{B})} \\
\le \;& C||v - \tilde{v}||_{C^{1,2+\alpha}([0,T]\times\overline{D})}),
\end{aligned}
$$

$$
(4.25) \qquad |\sum_{i=1}^{l_1} \tau_i \psi_k(t_i, y_i) + \sum_{i=l_1+1}^{l_1+l_2} \tau_i \tilde{\psi}_k(t_i, y_i)| \ge C|\sum_{i=1}^{l_1+l_2} \tau_i y_i'|,
$$

where C depends only on n, r and R.

Proof. The estimates (4.22) and (4.24) are obvious by Proposition 6.1. The estimate (4.23) follows from (4.25). So we shall prove only (4.25). From the definition of ψ_k and $\tilde{\psi}_k$, we have

$$
\begin{aligned}
|\sum_{i=1}^{l_1} \tau_i \psi_k(t_i, y_i) + \sum_{i=l_1+1}^{l_1+l_2} \tau_i \tilde{\psi}_k(t_i, y_i)| \;&\ge\; \{\sum_{j\ne i_0} (\sum_{i=1}^{l_1} \tau_i \psi_k^{(j)}(t_i, y_i) + \sum_{i=l_1+1}^{l_1+l_2} \tau_i \tilde{\psi}_k^{(i)}(t_i, y_i))^2\}^{\frac{1}{2}} \\
&=\; \{\sum_{j\ne i_0} [\sum_{i=1}^{l_1+l_2} \tau_i (\frac{r^2}{64R^2} (\Pi^{-1} y_i)^{(j)} + \bar{x}_k^{(j)})]^2\}^{\frac{1}{2}} \\
&=\; \{\sum_{j\ne i_0} [\sum_{i=1}^{l_1+l_2} \frac{r^2}{64R^2} \tau_i (\Pi^{-1} y_i)^{(j)}]^2\}^{\frac{1}{2}} \\
&=\; \frac{r^2}{64R^2} \{\sum_{j=1}^{n-1} [\sum_{i=1}^{l_1+l_2} \tau_i y_i^{(j)}]^2\}^{\frac{1}{2}} = \frac{r^2}{64R^2} |\sum_{i=1}^{l_1+l_2} \tau_i y_i'|.
\end{aligned}
$$

This completes the proof.

Let $K_{i,j}(t,x) := t^{-\frac{n}{2}} L_{i,j}(\frac{x}{\sqrt{t}}) + G_t(x)\delta_{ij}$ be the (i,j)-component of the kernel of the matrix convolution operator $e^{t\Delta}\mathbf{P}$. Combining the above estimate (4.25) with the pointwise estimate (4.3), we have

$$
\begin{aligned}
(4.26) \qquad & |(\partial_x^\theta K_{i,j})(t, \sum_{i=1}^{l_1} \tau_i \psi_k(t_i, y_i) + \sum_{i=l_1+1}^{l_1+l_2} \tau_i \tilde{\psi}_k(t_i, y_i))| \\
\le \;& C t^{-\frac{n+|\theta|}{2}} (1 + C\frac{|\sum_{i=1}^{l_1+l_2} \tau_i y_i'|}{\sqrt{t}})^{-n-|\theta|},
\end{aligned}
$$

for any multi-index $\theta = (\theta_1, \cdots, \theta_n)$.

Let $\{a_k\}_{k=1}^m$ be a partition of unity for $\cup_{1\le k\le m} \hat{O}_k$ subordinate to $\{O_k\}_{k=1}^m$, i.e., $\{a_k(x)\}_{k=1}^m$ satisfies that: for every k, $a_k(x)$ is smooth and $0 \le a_k \le 1$; for every k, supp $a_k \subset O_k$; for every $x \in \cup_{1\le k\le m} \hat{O}_k$, we have $\sum_{k=1}^m a_k(x) = 1$. Note that we can take $\{a_k\}_{k=1}^m$ not depending on each evolving hypersurface belonging to $\cup_{0<T\le T_0} \mathcal{S}(\alpha, R, T, d_0)$. We set

$$
(4.27) \qquad \delta_1 := \min_{1\le k\le m} \mathrm{dist}(O_k, \mathrm{supp}\ a_k) > 0.
$$

We may assume that $\delta_1 \ge \frac{7}{2}\rho$. Next proposition play essential roles in this paper.

Proposition 4.1. *Let $p \in (1,\infty]$, $\alpha, \beta \in (0,1)$. Assume that $R \ge 1$ is a given number and Γ_0 is a given $C^{2+\alpha}$ hypersurface. Let d_0 be the signed distance function, T_0 be a positive number in Proposition 6.1, and r be the number defined by (4.11). Let*

$\{\Gamma_t\}_{0\le t\le T}$ be an evolving hypersurface belonging to $\mathcal{S}(\alpha,R,T,d_0)$ for some $T\le T_0$. Then, the function $F(t,x)$ given as (4.7) satisfies the following estimates.

$$||F||_{C^{\frac{\beta}{2},\beta}([0,T]\times\mathbb{R}^n)}\le C_1T^{\frac{1-\beta}{2}}||h||_{C([0,T]\times\overline{D})}, \tag{4.28}$$

$$\sup_{0\le t\le T}||F(t)||_{L^p(\mathbb{R}^n)}\le C_2T^{\frac12}||h||_{C([0,T]\times\overline{D})}, \tag{4.29}$$

$$\sup_{0\le t\le T}||F(t)||_{C^{1+\alpha}(\Gamma_t)}\le C_3||h||_{C^{0,\alpha}([0,T]\times\overline{D})}, \tag{4.30}$$

where $C_1=C_1(n,\beta,r,R)$, $C_2=C_2(n,p,r,R)$, and $C_3=C_3(n,\alpha,r,R)$.

Proof. For simplicity of notations, we write $||h||_C$, $||h||_{C^{0,\alpha}}$ instead of $||h||_{C([0,T]\times\overline{D})}$, $||h||_{C^{0,\alpha}([0,T]\times\overline{D})}$, respectively. First, we shall prove the estimate (4.28). Remark that

$$\begin{aligned}
||F(t,\cdot)||_{C(\mathbb{R}^n)} &\le \int_0^t ||e^{(t-s)\Delta}\mathbf{P}h\mathcal{H}^{n-1}_{\llcorner\Gamma_s}||_{C(\mathbb{R}^n)}ds,\\
[F(t,\cdot)]_{C^\beta(\mathbb{R}^n)} &\le C\int_0^t (t-s)^{-\frac{\beta}{2}}||e^{\frac{t-s}{2}\Delta}\mathbf{P}h\mathcal{H}^{n-1}_{\llcorner\Gamma_s}||_{C(\mathbb{R}^n)}ds,
\end{aligned}$$

and

$$\begin{aligned}
|F(t,x)-F(\tau,x)| &\le \int_\tau^t ||e^{(t-s)\Delta}\mathbf{P}h\mathcal{H}^{n-1}_{\llcorner\Gamma_s}||_{C(\mathbb{R}^n)}ds\\
&\quad+\int_0^\tau |(e^{(t-\tau)\Delta}-I)e^{(\tau-s)\Delta}\mathbf{P}h\mathcal{H}^{n-1}_{\llcorner\Gamma_s}|ds,\\
&\le \int_\tau^t ||e^{(t-s)\Delta}\mathbf{P}h\mathcal{H}^{n-1}_{\llcorner\Gamma_s}||_{C(\mathbb{R}^n)}ds\\
&\quad+C(t-\tau)^{\frac{\beta}{2}}\int_0^\tau (\tau-s)^{-\frac{\beta}{2}}||e^{\frac{\tau-s}{2}\Delta}\mathbf{P}h\mathcal{H}^{n-1}_{\llcorner\Gamma_s}||_{C(\mathbb{R}^n)}ds.
\end{aligned}$$

Thus it suffices to estimate

$$||e^{(t-s)\Delta}\mathbf{P}h\mathcal{H}^{n-1}_{\llcorner\Gamma_s}||_{C(\mathbb{R}^n)}.$$

The i-th component of $e^{(t-s)\Delta}\mathbf{P}h\mathcal{H}^{n-1}_{\llcorner\Gamma_s}$ is given as

$$\begin{aligned}
&\sum_{j=1}^n\int_{\Gamma_s}K_{i,j}(t-s,x-y)h_j(s,y)\mathcal{H}^{n-1}(dy)\\
=&\ \sum_{j=1}^n\sum_{k=1}^m\int_{\Gamma_s\cap O_k}a_k(y)K_{i,j}(t-s,x-y)h_j(s,y)\mathcal{H}^{n-1}(dy)\\
=:&\ \sum_{j=1}^n\sum_{k=1}^m\mathcal{I}_{i,j,k}(t,s,x).
\end{aligned} \tag{4.31}$$

So we shall estimate $\mathcal{I}_{i,j,k}(t,s,x)$. By the area formula, we have

$$\mathcal{I}_{i,j,k}(t,s,x) = \int_{\mathbb{R}^{n-1}}K_{i,j}(t-s,x-\psi_k(s,\xi',0))\Upsilon_1(s,\xi')d\xi',$$

where

$$\Upsilon_1(s,\xi')=a_k(\psi_k(s,\xi',0))h_j(s,\psi_k(s,\xi',0))J_{\psi_k}(s,\xi'). \tag{4.32}$$

Here

$$J_{\psi_k}(s,\xi') = \Big[\sum_{l=1}^{n}\Big\{\frac{\partial(\psi_k^{(1)},\overset{l}{\check{\cdots}},\psi_k^{(n)})}{\partial(\xi^{(1)},\cdot,\xi^{(n-1)})}\Big\}^{\frac{1}{2}}\Big]\Bigg|_{t=s,\xi^{(n)}=0}.$$

Note that the integrand in the above integration is naturally extended by zero to $\mathbb{R}^{n-1}$ thanks to the inclusion $B'\cap \mathrm{supp}a_k(\psi_k(s,\cdot))\subset B'\cap\varphi_k(s,O_k)$.

It is easy to obtain the estimate $|\Upsilon_1(s,\xi')|\le C$ where C depends only on n, r, and R. If $x\in O_k$, then there exists a point $\bar\xi(s,x)\in B$ such that $x=\psi_k(s,\bar\xi)$ for each $s\in[0,T]$. Thus, from (4.26),

$$\begin{aligned}
&|\int_{\mathbb{R}^{n-1}} K_{i,j}(t-s,\psi_k(s,\bar\xi)-\psi_k(s,\xi',0))\Upsilon_1(s,\xi')d\xi'|\\
\le\ & C(t-s)^{-\frac{n}{2}}\int_{\mathbb{R}^{n-1}}(1+C\frac{|\bar\xi'-\xi'|}{(t-s)^{\frac{1}{2}}})^{-n}d\xi'||h||_C\\
=\ & (t-s)^{-\frac{1}{2}}\int_{\mathbb{R}^{n-1}}(1+C|z'|)^{-n}dz'||h||_C\\
=\ & C(t-s)^{-\frac{1}{2}}||h||_C.
\end{aligned}$$

In the second last integration, we changed the variable as $\bar\xi'-\xi'=(t-s)^{\frac{1}{2}}z'$.

If $x\notin O_k$, then $|x-y|\ge\delta_1>0$ for all $y\in\mathrm{supp}\ a_k$. Hence,

$$\begin{aligned}
&|\int_{\mathbb{R}^{n-1}} K_{i,j}(t-s,x-\psi_k(s,\xi',0))\Upsilon_1(s,\xi')d\xi'|\\
\le\ & C(t-s)^{-\frac{n}{2}}\int_{B'\cap\mathrm{supp}a_k(\psi_k(s,\cdot))}\big(1+\frac{|x-\psi_k(s,\xi',0)|}{(t-s)^{\frac{1}{2}}}\big)^{-n}d\xi'||h||_C\\
\le\ & C(t-s)^{-\frac{n}{2}}(1+\frac{\delta_1}{(t-s)^{\frac{1}{2}}})^{-n}\int_{B'}d\xi'||h||_C\le C||h||_C.
\end{aligned}$$

Collecting these estimates, we obtain

$$||\mathcal{I}_{i,j,k}(t,s,\cdot)||_{C(\mathbb{R}^n)}\le C(t-s)^{-\frac{1}{2}}||h||_C, \tag{4.33}$$

where C depends only on n, r and R (Remark that we can take δ_1 depending only on r and R). The estimate (4.28) immediately follows from above.

Next we shall show the estimate (4.29). Note again that in (4.13)-(4.15), we can take $O_k=\{|x-\bar x_k|<4\rho\}$ for some $\bar x_k\in\Gamma_0$ from Proposition 6.1. Set $D_k:=\{|x-\bar x_k|<8\rho\}$. By the estimate (4.33), we have

$$\begin{aligned}
&||\mathcal{I}_{i,j,k}(t,s,\cdot)||_{L^p(D_k)}\le|D_k|^{\frac{1}{p}}||\mathcal{I}_{i,j,k}(t,s,\cdot)||_{L^\infty(D_k)}\\
\le\ & C(t-s)^{-\frac{1}{2}}||h||_C.
\end{aligned}$$

For $x\notin D_k$, we see that for all $y\in O_k$,

$$|x-y|\ \ \ge\ \ \frac{1}{4}|x-\bar x_k|+2\rho.$$

Thus, from the estimate (4.26),

$$\begin{aligned}
||\mathcal{I}_{i,j,k}(t,s,\cdot)||_{L^p(\mathbb{R}^n\setminus D_k)}\ \le\ & C(t-s)^{-\frac{n}{2}}||\int_{B'}(1+\frac{|x-\psi_k(s,\xi',0)|}{(t-s)^{\frac{1}{2}}})^{-n}d\xi'||_{L^p_x(\mathbb{R}^n\setminus D_k)}||h||_C\\
\le\ & C(t-s)^{-\frac{n}{2}}||\int_{\mathbb{B}'}\big(1+\frac{2\rho}{(t-s)^{\frac{1}{2}}}+\frac{|x-\bar x_k|}{4(t-s)^{\frac{1}{2}}}\big)^{-n}d\xi'||_{L^p_x(\mathbb{R}^n)}||h||_C\\
\le\ & C||h||_C.
\end{aligned}$$

Combining these, we obtain the estimate (4.29).

Finally, we shall prove the estimate (4.30). Since

$$||F(t)||_{C^{1+\alpha}(\Gamma_t)} := \sup_{1\le h\le m} ||F(t,\psi_h(t,\cdot,0)||_{C^{1+\alpha}(\overline{B'})},$$

we shall estimate $F(t,\psi_h(t,\zeta',0))$, or, by (4.31),

$$\mathcal{J}(t,\zeta') := \int_0^t \mathcal{I}_{i,j,k,h}(t,s,\zeta')ds, \tag{4.34}$$

where $\mathcal{I}_{i,j,k,h}(t,s,\zeta') := \mathcal{I}_{i,j,k}(t,s,\psi_h(t,\zeta',0))$.

Let $\eta \in (0,1)$. In order to obtain the optimal regularity, we devide $\mathcal{J}(t,\zeta')$ into $\mathcal{J}_1(t,\zeta')$ and $\mathcal{J}_2(t,\zeta')$ where

$$\mathcal{J}_1(t,\zeta') := \int_{(1-\eta)t}^t \mathcal{I}_{i,j,k,h}(t,s,\zeta')ds, \tag{4.35}$$

$$\mathcal{J}_2(t,\zeta') := \int_0^{(1-\eta)t} \mathcal{I}_{i,j,k,h}(t,s,\zeta')ds. \tag{4.36}$$

We shall show that

$$||\mathcal{J}(t,\cdot)||_{C^\alpha(B')} \le Ct^{\frac12}||h||_{C^{0,\alpha}}, \tag{4.37}$$

$$||\mathcal{J}_1(t,\cdot)||_{C^\alpha(B')} \le C(\eta t)^{\frac12}||h||_{C^{0,\alpha}}, \tag{4.38}$$

$$||\mathcal{J}_2(t,\cdot)||_{C^{2+\alpha}(B')} \le C(\eta t)^{-\frac12}||h||_{C^{0,\alpha}}. \tag{4.39}$$

First, we set

$$B_1' := B' \cap \varphi_h(t, O_k \cap U_h(t)), \tag{4.40}$$

$$B_2' := B' \cap \varphi_h(t, \{x \in U_h(t);\ \mathrm{dist}(x, \mathrm{supp} a_k) > \frac{\delta_1}{2}\}) \ne \emptyset. \tag{4.41}$$

Then, both B_1' and B_2' are relatively open in $\mathbb{R}^{n-1}$, so from now we regard B', B_1', and B_2' as open sets in $\mathbb{R}^{n-1}$. For all $y' \in B'$ there exists a unique $x \in U_h(t)$ such that $y = (y',0) = \varphi_h(t,x)$. If $x \notin O_k$, then $\mathrm{dist}(x,\mathrm{supp} a_k) \ge \delta_1$. Hence, we have $y' \in B_2'$, which implies that $B' = B_1' \cup B_2'$.

Moreover, if f is a continuous function on $\overline{B'}$ and satisfies $f \in C^{l+\alpha}(\overline{B_1'}) \cap C^{l+\alpha}(\overline{B_2'})$ for some $l \in \mathbb{N}\cup\{0\}$, then $f \in C^{l+\alpha}(\overline{B'})$. To see this, we may assume that $B_1' \ne \emptyset$. Clearly, f is l- times differentiable in $B' = B_1' \cup B_2'$ and

$$\sup_{y'\in B'} |\partial_{y'}^{\theta'} f(y')| \le ||f||_{C^l(\overline{B_1'})} + ||f||_{C^l(\overline{B_2'})}. \text{ for } 0 \le |\theta'| \le l.$$

Next we consider the C^α norm of $\partial_{y'}^l f$. It suffices to consider the case $y_1' \in B_1' \setminus B_2'$, $y_2' \in B_2' \setminus B_1'$ if they are exist. In this case, there exists a positive constant C depending only on n, r and R, such that

$$|y_1' - y_2'| \ge C\delta_1.$$

Indeed, $y_1' \in B_1' \setminus B_2'$ implies that there exists a point $x_1 \in O_k \cap U_h(t)$ such that $y_1 := (y_1',0) = \varphi_h(t,x_1)$ and $\mathrm{dist}(x_1, \mathrm{supp}\ a_k) \le \frac{\delta_1}{2}$. On the other hand, $y_2' \in B_2' \setminus B_1'$ implies that there exists a point $x_2 \in U_h(t) \setminus O_k$ such that $y_2 := (y_2',0) = \varphi_h(t,x_2)$. Since $x_2 \notin O_k$, we have $\mathrm{dist}(x_2, \mathrm{supp}\ a_k) \ge \delta_1$.

Thus,

$$\begin{aligned} |y_1' - y_2'| &= |y_1 - y_2| = |\varphi_h(t,x_1) - \varphi_h(t,x_2)| \\ &\ge C|x_1 - x_2| \\ &\ge C(|x_2 - z| - |x_1 - z|), \end{aligned}$$

for all $z \in \mathrm{supp}\ a_k$. Especially, we can take z as satisfying $|x_1 - z| = \mathrm{dist}(x_1, \mathrm{supp}\ a_k)$. Hence,

$$|y_1' - y_2'| \geq C(\delta_1 - \frac{\delta_1}{2}) = \frac{C}{2}\delta_1.$$

This implies that $f \in C^{l+\alpha}(\overline{B'})$ and

$$||f||_{C^{l+\alpha}(\overline{B'})} \leq C(||f||_{C^{l+\alpha}(\overline{B_1'})} + ||f||_{C^{l+\alpha}(\overline{B_2'})}), \tag{4.42}$$

where C depends only on n, r, and R.

Now we return to the estimates for $\mathcal{J},\ \mathcal{J}_1,\ \mathcal{J}_2$.

(i)Estimates on B_2'.

We first estimate $\mathcal{I}_{i,j,k,h}(t,s,\zeta')$ on B_2'. By the definition of B_2', we see that $|\psi_h(t,\zeta',0) - \psi_k(s,\xi',0)| > \frac{\delta_1}{2}$ for all $\zeta' \in B_2'$, if $\psi_k(s,\xi',0) \in \operatorname{supp} a_k$. So we have, for any multi-index θ with $|\theta| = 0,1,2$, from the estimate (4.26),

$$\begin{aligned}
& |\partial_{\zeta'}^{\theta}\theta_{i,j,k,h}(t,s,\zeta')| \\
\leq\ & \int_{B'} |\partial_{\zeta'}^{\theta} K_{i,j}(t-s, \psi_h(t,\zeta',0) - \psi_k(s,\xi',0)| |\Upsilon_1(s,\xi')d\xi' \\
\leq\ & \sum_{0\leq|\theta'|\leq|\theta|} C(1 + ||\psi_h(t,\cdot)||_{C^{2+\alpha}(\overline{B})})(t-s)^{-\frac{n+|\theta'|}{2}} \\
& \qquad \cdot \int_{B'} (1 + \frac{|\psi_h(t,\zeta',0) - \psi_k(s,\xi',0)|}{(t-s)^{\frac{1}{2}}})^{-n-|\theta'|} d\xi' ||h||_C \\
\leq\ & C(1 + ||v||_{C^{1,2+\alpha}}) \sum_{0\leq|\theta'|\leq|\theta|} (t-s)^{-\frac{n+|\theta'|}{2}} (1 + \frac{\delta_1}{2(t-s)^{\frac{1}{2}}})^{-n-|\theta'|} d\xi' ||h||_C, \\
\leq\ & C||h||_C
\end{aligned}$$

where C depends only on n, r, and R. Similarly, we easily have

$$[\partial_{\zeta'}^{\theta}\mathcal{I}_{i,j,k,h}(t,s,\cdot)]_{C^{\alpha}(\overline{B_2'})} \leq C||h||_C.$$

In particular, we have

$$||\mathcal{J}(t,\cdot)||_{C^{\alpha}(\overline{B_2'})} \leq Ct^{\frac{1}{2}}||h||_C, \tag{4.43}$$

$$||\mathcal{J}_1(t,\cdot)||_{C^{\alpha}(\overline{B_2'})} \leq C(\eta t)^{\frac{1}{2}}||h||_C, \tag{4.44}$$

$$||\mathcal{J}_2(t,\cdot)||_{C^{2+\alpha}(\overline{B_2'})} \leq C(\eta t)^{-\frac{1}{2}}||h||_C. \tag{4.45}$$

(ii)Estimates on B_1'.

In order to obtain the desired estimates, it suffices to show that

$$||\mathcal{I}_{i,j,k,h}(t,s,\cdot)||_{C^{\alpha}(\overline{B_1'})} \leq C(t-s)^{-\frac{1}{2}}||h||_{C^{0,\alpha}}, \tag{4.46}$$

$$||\mathcal{I}_{i,j,k,h}(t,s,\cdot)||_{C^{2+\alpha}(\overline{B_1'})} \leq C(t-s)^{-\frac{3}{2}}||h||_{C^{0,\alpha}}. \tag{4.47}$$

If $\zeta' \in B_1$, then there exists a unique $w' \in B'$ such that $\psi_h(t,\zeta',0) = \psi_k(t,w',0)$, or, $w = (w',0) = \varphi_k(t,\psi_h(t,\zeta',0))$. Remark that w' is a function of t and ζ'. But for simplicity of notation, we just write w', or, $w'(\zeta')$. From the estimate (4.22), we have

$$||w'||_{C^{1,2+\alpha}([0,T]\times\overline{B'})} \leq C(1 + ||v||^2_{C^{1,2+\alpha}}). \tag{4.48}$$

As same as in (i), for any multi-index θ with $|\theta| = 0, 1, 2$, we have from (4.26),

$$
\begin{aligned}
& |\partial_{\zeta'}^{\theta} \mathcal{I}_{i,j,k,h}(t,s,\zeta')| \\
\leq \ & \int_{\mathbb{R}^{n-1}} |\partial_{\zeta'}^{\theta} K_{i,j}(t-s, \psi_h(t,\zeta',0) - \psi_k(s,\xi',0)||\Upsilon_1(s,\xi')| d\xi' \\
\leq \ & C(1 + ||\psi_h(t,\cdot)||_{C^{2+\alpha}(\overline{B'})}) \sum_{|\theta'|=0}^{|\theta|} \int_{\mathbb{R}^{n-1}} |(\partial_x^{\theta'} K_{i,j})(t-s, \psi_k(t,w',0) - \psi_k(s,\xi',0)| d\xi' ||h||_C \\
\leq \ & C(1 + ||v||_{C^{1,2+\alpha}}) \sum_{|\theta'|=0}^{|\theta|} (t-s)^{-\frac{n+|\theta'|}{2}} \int_{\mathbb{R}^{n-1}} (1 + C\frac{|w'-\xi'|}{(t-s)^{\frac{1}{2}}})^{-n-|\theta'|} d\xi' ||h||_C \\
\leq \ & C(1 + ||v||_{C^{1,2+\alpha}})(t-s)^{-\frac{1+|\theta|}{2}} ||h||_C.
\end{aligned}
$$

So we obtain

$$
||\mathcal{I}_{i,j,k,h}(t,s,\cdot)||_{C(\overline{B_1'})} \leq C(t-s)^{-\frac{1}{2}} ||h||_C,
$$

and

$$
||\mathcal{I}_{i,j,k,h}(t,s,\cdot)||_{C^2(\overline{B_1'})} \leq C(t-s)^{-\frac{3}{2}} ||h||_C.
$$

Thus it suffices to estimate $[\mathcal{I}_{i,j,k,h}(t,s,\cdot)]_{C^{\alpha}(\overline{B_1'})}$ and $[\partial_{\zeta'}^2 \mathcal{I}_{i,j,k,h}(t,s,\cdot)]_{C^{\alpha}(\overline{B_1'})}$. We give the proof only for the estimate of $[\mathcal{I}_{i,j,k,h}(t,s,\cdot)]_{C^{\alpha}(\overline{B_1'})}$.

The estimate of $[\partial_{\zeta'}^2 \mathcal{I}_{i,j,k,h}(t,s,\cdot)]_{C^{\alpha}(\overline{B_1'})}$ is similarly established.

In order to obtain the above estimates, we make use of the α-Hölder continuity of Υ_1. For this purpose, by changing variable as $w' - \xi' = z'$, we rewrite $\mathcal{I}_{i,j,k,h}(t,s,\zeta')$ as

(4.49)
$$
\mathcal{I}_{i,j,k,h}(t,s,\zeta') = \int_{\mathbb{R}^{n-1}} K_{i,j}(t-s, \psi_k(t,w',0) - \psi_k(s,w'-z',0))\Upsilon_1(s,w'-z') dz'.
$$

For ζ_1', $\zeta_2' \in B_1'$, we set $w_1' = w_1(\zeta')$, $w_2' = w_2'(\zeta_2')$, respectively. We also set

$$
\Upsilon_2(t,s,w',z') := \psi_k(t,w',0) - \psi_k(s,w'-z') \tag{4.50}
$$

Then,

$$
\begin{aligned}
& |\Upsilon_2(t,s,w_1',z') - \Upsilon_2(t,s,w_2',z')| \\
= \ & |\int_0^1 < w_1' - w_2', (\nabla_{w'}\Upsilon_2)(t,s,\tau w_1' + (1-\tau)w_2', z') >' d\tau| \\
\leq \ & C|\zeta_1' - \zeta_2'|((t-s)^{\frac{1+\alpha}{2}} + |z'|).
\end{aligned}
$$

So from the estimate (4.26), we have

$$\begin{aligned}
&|\mathcal{I}_{i,j,k,h}(t,s,\zeta_1') - \mathcal{I}_{i,j,k,h}(t,s,\zeta_2')| \\
\le\ & \int_{\mathbb{R}^{n-1}} |K_{i,j}(t-s,\Upsilon_2(t,s,w_1',z')) - K_{i,j}(t-s,\Upsilon_2(t,s,w_2',z'))||\Upsilon_1(s,w_1'-z')|dz' \\
&\quad + \int_{\mathbb{R}^{n-1}} |K_{i,j}(t-s,\Upsilon_2(t,s,w_2',z'))||\Upsilon_1(s,w_1'-z') - \Upsilon_1(s,w_2'-z')|dz' \\
\le\ & C||h||_C \int_{\mathbb{R}^{n-1}} \int_0^1 | < \Upsilon_2(t,s,w_1',z') - \Upsilon_2(t,s,w_2',z'), \\
&\quad (\nabla_x K_{i,j})(t-s,\tau\Upsilon_2(t,s,w_1',z') + (1-\tau)\Upsilon_2(t,s,w_2',z')) > |d\tau dz' \\
&\quad + C||h||_{C^{0,\alpha}} |w_1'-w_2'|^\alpha (t-s)^{-\frac{n}{2}} \int_{\mathbb{R}^{n-1}} (1 + C\frac{|z'|}{(t-s)^{\frac12}})^{-n} dz' \\
\le\ & C||h||_C |\zeta_1'-\zeta_2'|(t-s)^{-\frac{n+1}{2}} \int_{\mathbb{R}^{n-1}} ((t-s)^{\frac{1+\alpha}{2}} + |z'|)(1 + C\frac{|z'|}{(t-s)^{\frac12}})^{-n-1} dz' \\
&\quad + C||h||_{C^{0,\alpha}} |\zeta_1'-\zeta_2'|^\alpha (t-s)^{-\frac12} \\
\le\ & C|\zeta_1'-\zeta_2'|^\alpha (t-s)^{-\frac12} ||h||_{C^{0,\alpha}},
\end{aligned}$$

which is the desired estimate. We have just established the estimates (4.37)-(4.39).

We are now in position to show the optimal regularity $||\mathcal{J}(t,\cdot)||_{C^{1+\alpha}(\overline{B'})}$. We use the relation $C^{1+\alpha}(\overline{B'}) = \mathcal{C}^{1+\alpha}(\overline{B'})$ (equivalent norms), where $\mathcal{C}^l(\overline{B'})$, $0 < l < 3$, are the Hölder-Zygmund spaces defined as follows.

$$\mathcal{C}^l(\overline{B'}) := \{f \in C(\overline{B'});\ ||f||_{\mathcal{C}^l(\overline{B'})} := ||f||_{C(\overline{B'})} + [f]_{\mathcal{C}^\alpha(\overline{B'})} < \infty\},$$

where

$$[f]_{\mathcal{C}^l(\overline{B'})} := \sup_{x,y\in\overline{B'}, x\neq y} \frac{|f(x) - 3f(\frac{2x+y}{3}) + 3f(\frac{x+2y}{3}) - f(y)|}{|x-y|^l}.$$

The advantage of the Hölder-Zygmund spaces is that derivatives do not appear in the definition of its norms. We have the relation $C^l(\overline{B'}) = \mathcal{C}^l(\overline{B'})$ with equivalent norms if l is not an integer; see A. Lunardi [11] or T. Runst and W. Sickel [19].

It suffices to estimate $[\mathcal{J}(t,\cdot)]_{\mathcal{C}^{1+\alpha}}$. Let $\zeta_1', \zeta_2' \in B'$ with $\zeta_1' \neq \zeta_2'$. We first consider the case such that $|\zeta_1'-\zeta_2'| \le \frac12 t^{\frac12}$. Since $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$, we have from (4.38) and (4.39),

$$\begin{aligned}
&|\mathcal{J}(t,\zeta_1') - 3\mathcal{J}(t,\frac{2\zeta_1'+\zeta_2'}{3}) + 3\mathcal{J}(t,\frac{\zeta_1'+2\zeta_2'}{3}) - \mathcal{J}(t,\zeta_2')| \\
\le\ & [\mathcal{J}_1(t,\cdot)]_{\mathcal{C}^\alpha} |\zeta_1'-\zeta_2'|^\alpha + [\mathcal{J}_2(t,\cdot)]_{\mathcal{C}^{2+\alpha}} |\zeta_1'-\zeta_2'|^{2+\alpha} \\
\le\ & C[\mathcal{J}_1(t,\cdot)]_{C^\alpha} |\zeta_1'-\zeta_2'|^\alpha + C[\mathcal{J}_2(t,\cdot)]_{C^{2+\alpha}} |\zeta_1'-\zeta_2'|^{2+\alpha} \\
\le\ & C||h||_{C^{0,\alpha}} (\eta t)^{\frac12} |\zeta_1'-\zeta_2'|^\alpha + C||h||_{C^{0,\alpha}} (\eta t)^{-\frac12} |\zeta_1'-\zeta_2'|^{2+\alpha}
\end{aligned}$$

for $\eta \in (0,1)$. So if we set $\eta = \frac{|\zeta_1'-\zeta_2'|^2}{t}$, then we have

$$|\mathcal{J}(t,\zeta_1') - 3\mathcal{J}(t,\frac{2\zeta_1'+\zeta_2'}{3}) + 3\mathcal{J}(t,\frac{\zeta_1'+2\zeta_2'}{3}) - \mathcal{J}(t,\zeta_2')| \le C||h||_{C^{0,\alpha}} |\zeta_1'-\zeta_2'|^{1+\alpha},$$

for any $\zeta_1',\ \zeta_2' \in B'$ with $|\zeta_1'-\zeta_2'| \le \frac12 t^{\frac12}$. If $|\zeta_1'-\zeta_2'| \ge \frac12 t^{\frac12}$, then, from (4.37),

$$\begin{aligned}
&|\mathcal{J}(t,\zeta_1') - 3\mathcal{J}(t,\frac{2\zeta_1'+\zeta_2'}{3}) + 3\mathcal{J}(t,\frac{\zeta_1'+2\zeta_2'}{3}) - \mathcal{J}(t,\zeta_2')| \\
\le\ & [\mathcal{J}(t,\cdot)]_{\mathcal{C}^\alpha} |\zeta_1'-\zeta_2'|^\alpha \\
\le\ & C||h||_{C^{0,\alpha}} t^{\frac12} |\zeta_1'-\zeta_2'|^\alpha \\
\le\ & C||h||_{C^{0,\alpha}} |\zeta_1'-\zeta_2'|^{1+\alpha}.
\end{aligned}$$

Hence, we have $\mathcal{J} \in \mathcal{C}^{1+\alpha}(\overline{B'})$, equivalently, $\mathcal{J} \in C^{1+\alpha}(\overline{B'})$. This completes the proof.

4.2. **Remark on the mild solution of the Navier-Stokes equation.**
In this section, we shall construct the mild solution of the Navier-Stokes equation with initial velocity $u_0 \in C^\alpha(\mathbb{R}^n)$ and with a term of the layer potential $h\mathcal{H}^{n-1}_{\llcorner\Gamma_t}$ for convenience to reader. Thanks to the estimates for the layer potential term established in the previous section, we can obtain the appropriate regularity for solutions in tangential directions to Γ_t. We recall that the mild solution of the Navier-Stokes equation which we consider here is the solution of the integral equation

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbf{P}\nabla \cdot u \otimes u ds + \int_0^t e^{(t-s)\Delta}\mathbf{P}h\mathcal{H}^{n-1}_{\llcorner\Gamma_s}ds.$$

Now let $\alpha \in (0,1)$. Assume that $u_0 \in C^\alpha(\mathbb{R}^n)$ satisfies $\nabla \cdot u_0 = 0$ and d_0 is the distance function of a $C^{2+\alpha}$ hypersurface Γ_0. Let $R \geq 1$ be a given number and T_0 (< 1) be the number given by Proposition 6.1 depending only on n, d_0, and R. Let $T_1 \in (0, T_0]$ and $\{\Gamma_t\}_{0\leq t\leq T_1}$ be an evolving hypersurfaces belonging to $\mathcal{S}(\alpha, R, T_1, d_0)$. Then, we have the following proposition.

Proposition 4.2. *There exists a $T \leq T_1$ such that the mild solution u belonging to $C^{\frac{\alpha}{2},\alpha}([0,T] \times \mathbb{R}^n)$ uniquely exists. The existence time T can be taken uniformly in $\mathcal{S}(\alpha, R, T_1, d_0)$. Moreover, this solution satisfies the following estimates.*

$$\begin{aligned}
(4.51)\qquad & ||u||_{C^{\frac{\alpha}{2},\alpha}([0,T\times\mathbb{R}^n)} \\
\leq\ & C||u_0||_{C^\alpha(\mathbb{R}^n)} + C_1 T^{\frac{1-\alpha}{2}}||h||_{C([0,T]\times\overline{D})}, \\
(4.52)\qquad & \sup_{0<t<T} t^{\frac{1-\alpha}{2}}||u(t,\cdot)||_{C^1(\Gamma_t)} + \sup_{0<t<T} t^{\frac{1}{2}}||u(t,\cdot)||_{C^{1+\alpha}(\Gamma_t)} \\
\leq\ & C||u_0||_{C^\alpha(\mathbb{R}^n)} + C_2 T^{\frac{1-\alpha}{2}}||h||_{C^{0,\alpha}([0,T]\times\overline{D})},
\end{aligned}$$

where $C = C(n,\alpha)$, $C_1(n,\alpha,d_0,R)$, and $C_2(n,\alpha,d_0,R)$.

. *Proof.* We will follow the contraction argument by T. Kato [8]. Since this argument is well-known, we state only the outline of the proof. Set

$$\begin{aligned}
&F_0 := e^{t\Delta}u_0, \\
&B(f,g) := \int_0^t e^{(t-s)\Delta}\mathbf{P}\nabla \cdot f \otimes g ds,\ \ f,\ g \in C^{0,\alpha}([0,T]\times\mathbb{R}^n), \\
&F := \int_0^t e^{(t-s)\Delta}\mathbf{P}h\mathcal{H}^{n-1}_{\llcorner\Gamma_s}ds.
\end{aligned}$$

From the pointwise estimate of the kernel $e^{t\Delta}\mathbf{P}\nabla\cdot$ in Lemma 4.1, it is not difficult to deduce the estimate

$$(4.53)\qquad ||B(f,g)||_{C^{\frac{\alpha}{2},\alpha}([0,T]\times\mathbb{R}^n)} \leq CT^{\frac{1}{2}}||f||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)}||g||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)}$$

where C depends only on n and α. Combining the estimates in Lemma 4.2 and Proposition 4.1, we easily see that

$$(4.54)\qquad G(w) := F_0 - B(w,w) + F$$

is a contraction mapping from the usual closed ball in $C^{\frac{\alpha}{2},\alpha}([0,T]\times\mathbb{R}^n)$ with radius $2(||F_0||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)} + ||F||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)})$ into itself for sufficiently small $T \leq T_1$. This implies the time-local existence of the unique solution u. Note that the existence time can be taken uniformly in $\mathcal{S}(\alpha, R, T_1, d_0)$ since the constants in Proposition 4.1 do not depend on each family of hypersurfaces in $\mathcal{S}(\alpha, R, T_1, d_0)$. The estimate (4.51) is obvious. We shall show the estimate (4.52). Note that we have

$$||\partial_x F_0(t,\cdot)||_{C(\mathbb{R}^n)} \le Ct^{-\frac{1-\alpha}{2}}||u_0||_{C^\alpha},$$
$$[\partial_x F_0(t,\cdot)]_{C^\alpha(\mathbb{R}^n)} \le Ct^{-\frac{1}{2}}||u_0||_{C^\alpha}.$$

As for the nonlinear term $B(f,g)$, we have, from the maximal regularity estimates for the heat equation (see A. Lunardi [11]),

$$||B(f,g)||_{C^{0,1+\alpha}([0,T]\times\mathbb{R}^n)} \le C||\mathbf{P}f\otimes g||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)}.$$

Since the Helmholtz projection $\mathbf{P}$ is bounded in $C^\alpha(\mathbb{R}^n)$, we see

$$||B(f,g)||_{C^{0,1+\alpha}([0,T]\times\mathbb{R}^n)} \le C||f||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)}||g||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)}. \tag{4.55}$$

Combining the above estimates with the estimate for F in tangential direction to Γ_t established in Proposition 4.1, we have the desired estimates. This completes the proof.

5. Construction of the solution for free boundary problem

Now we return to the problem (FBP). In this section, we shall prove the main theorem. Let u_0 be a function in $C^\alpha(\mathbb{R}^n)$ and satisfy $\nabla\cdot u_0 = 0$. Let Γ_0 be a $C^{2+\alpha}$ hypersurface which is a boundary of a bounded domain Ω_0 and let d_0 be the signed distance function of Γ_0. We set $F_0(t,\cdot) = e^{t\Delta}u_0$ and

(5.1)
$$M := 2(||F_0||_{C^{0,\alpha}([0,\infty)\times\mathbb{R}^n)} + \sup_{t>0} t^{\frac{1-\alpha}{2}}||\partial_x F_0(t,\cdot)||_{C(\mathbb{R}^n)} + \sup_{t>0} t^{\frac{1}{2}}[\partial_x F_0(t,\cdot)]_{C^\alpha(\mathbb{R}^n)}).$$

Recall that $\mathcal{U}_M$ is the closed subset of $C^{0,\alpha}([0,T]\times\mathbb{R}^n)$ defined as

$$\begin{aligned}\mathcal{U}_M = \{u(t,x) \in C^{0,\alpha}([0,T]\times\mathbb{R}^n); u(t,\cdot)\in C^{1+\alpha}(\mathbb{R}^n),\\ L_u := \sup_{0<t<T}||u(t,\cdot)||_{C^\alpha(\mathbb{R}^n)} + \sup_{0<t<T} t^{\frac{1-\alpha}{2}}||\partial_x u(t,\cdot)||_{C(\mathbb{R}^n)}\\ + \sup_{0<t<T} t^{\frac{1}{2}}[\partial_x u(t,\cdot)]_{C^\alpha(\mathbb{R}^n)} \le M\}\end{aligned} \tag{5.2}$$

From Proposition 3.1, there exists a positive T_1 such that for any $u\in\mathcal{U}_M$, there exists a unique family of hypersurfaces $\{\Gamma_t^u\}_{0\le t\le T_1}$ evolving via perturbed mean curvature equation (3.1) starting from Γ_0. Moreover, this $\{\Gamma_t^u\}_{0\le t\le T_1}$ belongs to $\mathcal{S}(\alpha, R, T_1, d_0)$ with $R = ||d_0||_{C^{2+\alpha}} + 2C(||d_0||_{C^{2+\alpha}(\overline{D})}, K_f, M)$; see (3.16). Let v be the signed distance function of $\{\Gamma_t^u\}_{0\le t\le T_1}$. From Proposition 3.2, we have

$$\sup_{0<t<T_1} t^{\frac{1}{2}}||\partial_x^3 v(t,\cdot)||_{C^\alpha(\overline{D'})} < \infty,$$

for any open set $D' \subset\subset D$. Let $T_2 := \min(T_1, T_0)$, where T_0 is the number given by Proposition 6.1. We set

$$C_4 := \sup_{0<t<T_2} t^{\frac{1}{2}}||\partial_x^3 v(t,\cdot)||_{C^\alpha(\cup_{1\le k\le m}\overline{O_k})}, \tag{5.3}$$

where $\{O_k\}_{k=1}^m$ is the family of open sets given by Proposition 6.1. Remark that C_4 is bounded uniformly in each function belonging to $\mathcal{U}_M$. Set

$$F^u(t,\cdot) := \int_0^t e^{(t-s)\Delta}\mathbf{P}\sigma_1 H^u\nu^u\mathcal{H}_{\llcorner\Gamma_s^u}ds, \tag{5.4}$$

where H^u, ν^u are the mean curvature and the exterior unit normal vector of Γ_t^u, respectively. We remark that with the signed distance function v, the exterior unit normal vector $\nu^u(t,x)$ and the mean curvature $H^u(t,x)$ of the surface Γ_t are given by

$$\nu^u(t,x) = \nabla_x v(t,x), \tag{5.5}$$

$$H^u(t,x) = -\frac{1}{n-1}\text{div } \nu(t,x) = -\frac{1}{n-1}\Delta v(t,x). \tag{5.6}$$

Since v is a function on $[0,T]\times\overline{D}$, ν and H^u can be also regarded as functions on $[0,T]\times\overline{D}$. Especially, if $\{\Gamma_t\}_{0\le t\le T}$ is an evolving hypersurface belonging to $\mathcal{S}(\alpha,R,T,d_0)$, then the mean curvature vector $H^u\nu^u$ belongs to $C^{0,\alpha}([0,T]\times\overline{D})$ as the function on $[0,T]\times\overline{D}$. Moreover, if the above v satisfies $\sup_{0<t<T} t^{\frac{1}{2}}||\partial_x v^3(t,\cdot)||_{C^\alpha(\overline{D'})} < \infty$ for an open set $D'\subset\subset D$, then

$$\sup_{0<t<T} t^{\frac{1}{2}}||\partial_x H^u(t,\cdot)||_{C^\alpha(\overline{D'})} \le C \sup_{0<t<T} t^{\frac{1}{2}}||\partial_x^3 v(t,\cdot)||_{C^\alpha(\overline{D'})}. \tag{5.7}$$

From Proposition 4.1, the function F^u satisfies

$$||F^u||_{C^{\frac{\alpha}{2},\alpha}([0,T_2]\times\overline{D})} \le \sigma_1 C T^{\frac{1-\alpha}{2}}, \tag{5.8}$$

$$\sup_{0\le t\le T_2} ||F^u(t)||_{C^{1+\alpha}(\Gamma_t)} \le \sigma_1 C, \tag{5.9}$$

where C depends only on n, α, R, and Γ_0. Since $F^u(t,\cdot)$ belongs to $C^{1+\alpha}(\Gamma_t^u)$ for each $t\in(0,T_2]$, we can construct the function in $C^{1+\alpha}(\mathbb{R}^n)$ as the extension of $\gamma_{\Gamma_t^u}F^u(t,\cdot)$, where $\gamma_{\Gamma_t^u}$ is the restriction operator on Γ_t^u. We fix the way of the extension as follows. Let $\{\varphi_k\}_{k=1}^m$, $\{\psi_k\}_{k=1}^m$ be local coordinate transforms of $\{\Gamma_t^u\}_{0\le t\le T_2}$ and $\{\hat{O}_k\}_{k=1}^m$ be the family of open sets given by Proposition 6.1. Let $\{a_k\}_{k=1}^m$ be a partition of unity for $\cup_{1\le k\le m}\hat{O}_k$ subordinate to $\{O_k\}_{k=1}^n$. Set

$$E_{1,k}^v(F^u)(t,y) := F^u(t,\psi_k(t,y',0)),\ y=(y',y^{(n)})\in\mathbb{R}^n,\ |y|\le 1,$$

$$E_{2,k}^v(F^u)(t,x) := \begin{cases} a_k(x)E_{1,k}^v(F^u)(t,\varphi_k(t,x)),\ x\in O_k, \\ 0,\ x\notin O_k. \end{cases}$$

And we set

$$E^v(F^u)(t,x) := \sum_{k=1}^m E_{2,k}^u(F^u)(t,x). \tag{5.10}$$

Then, obviously, we have $E^v(F^u)(t,x) = F^u(t,x)$ for all $x\in\Gamma_t$ and

$$||E^v(F^u)||_{C^{\frac{\alpha}{2},\alpha}([0,T_2]\times\overline{D})} \le C_5\sigma_1 C T^{\frac{1-\alpha}{2}}, \tag{5.11}$$

$$\sup_{0\le t\le T_2} ||E^v(F^u)(t)||_{C^{1+\alpha}(\mathbb{R}^n)} \le C_5\sigma_1 C, \tag{5.12}$$

where C_5 depends only on n, R, and Γ_0. Note that the partition of unity $\{a_k\}_{k=1}^m$ can be taken uniformly in $\cup_{0<T\le T_2}\mathcal{S}(\alpha,R,T,d_0)$. Moreover, although the extension E^v depends on $\{\Gamma_t^u\}_{0\le t\le T_2}$, the constant C_5 is independent of each evolving hypersurface belonging to $\cup_{0<T\le T_2}\mathcal{S}(\alpha,R,T,d_0)$. Let $\Psi_0(u)$ be the unique mild solution with initial velocity u_0 and with the layer potential $\sigma_1 H^u\nu^u\mathcal{H}^{n-1}_{\llcorner\Gamma_t}$, i.e., $\Psi_0(u)$ satisfies

$$\Psi_0(u) = F_0 + F^u. \tag{5.13}$$

Finally, we set

$$\Psi(u)(t,x) := F_0 + E^v(F^u). \tag{5.14}$$

Clearly, Ψ is a mapping on $\mathcal{U}_M$. We shall show that Ψ is a contraction mapping $\mathcal{U}_M$ into itself. From the estimates (5.11), (5.12) and the definition of M, we have

$$L_{\Psi(u)} \leq \frac{M}{2} + C_5\sigma_1 CT^{\frac{1-\alpha}{2}}. \tag{5.15}$$

Since each constant above is independent of T, we see that for sufficiently small $T \leq T_2$, Ψ maps $\mathcal{U}_M$ into itself. Next we shall show that Ψ is a contraction mapping. Let u, $\tilde{u} \in \mathcal{U}_M$, and v, $\tilde{v}$ be the signed distance functions of $\{\Gamma_t^u\}_{0\leq t\leq T}$, $\{\Gamma_t^{\tilde{u}}\}_{0\leq t\leq T}$, respectively. From the construction of the map Ψ, we see

$$\begin{aligned} & \Psi(u)(t,x) - \Psi(\tilde{u})(t,x) \\ = \ & E^v(F^u)(t,x) - E^{\tilde{v}}(F^{\tilde{u}})(t,x) \\ = \ & \sum_{k=1}^m a_k(x)\{F^u(t,\psi_k(t,\varphi_k'(t,x),0)) - F^{\tilde{u}}(t,\tilde{\psi}_k(t,\tilde{\varphi}_k'(t,x),0))\}. \end{aligned}$$

Thus it suffices to estimate

$$||F^u(\cdot,\psi_k(\cdot,\varphi_k'(\cdot,\cdot),0)) - F^{\tilde{u}}(\cdot,\tilde{\psi}_k(\cdot,\tilde{\varphi}_k'(\cdot,\cdot),0))||_{C^{0,\alpha}([0,T]\times\overline{C_k})}. \tag{5.16}$$

By using the local coordinate transforms $\{\psi_k\}_{k=1}^m$, we see that (5.16) is equivalent to

$$\sup_{0<t<T} ||F^u(t,\psi_k(t,y',0)) - F^{\tilde{u}}(t,\tilde{\psi}_k(t,w_k'(t,y),0))||_{C_y^\alpha(\overline{B})}, \tag{5.17}$$

where $w_k'(t,y)$ is defined as

$$w_k'(t,y) := (w_k^{(1)}(t,y),\cdots,w_k^{(n-1)}(t,y)), \tag{5.18}$$

$$w_k^{(i)}(t,y) := \tilde{\varphi}_k^{(i)}(t,\psi_k(t,y)). \tag{5.19}$$

In fact, we have the following proposition.

Proposition 5.1. *Let T_0 be the positive number obtained in Proposition 6.1. Then, it follows that for all $T \in (0,T_0]$,*

$$\begin{aligned} & \sup_{0<t<T}\max_{1\leq k\leq m} ||F(t,\psi_k(t,y',0)) - \tilde{F}(t,\tilde{\psi}_k(t,w_k'(t,y),0))||_{C_y^\alpha(\overline{B})} \\ \leq \ & CT^{\frac{1}{2}}(1+\sup_{0<t<T} t^{\frac{1}{2}}||\partial_x^3\tilde{v}(t,\cdot)||_{C^\alpha(\cup_{1\leq k\leq m}\overline{O_k})})||v-\tilde{v}||_{C^{1,2+\alpha}([0,T]\times\cup_{1\leq k\leq m}\overline{O_k})}, \end{aligned} \tag{5.20}$$

where C depends only on n, α, r and R.

The proof of the above proposition will be given in Appendix; see Section 6.1. From this proposition, we have

$$||\Psi(u)-\Psi(\tilde{u})||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)} \quad \leq \quad \sigma_1 CT^{\frac{1}{2}}(1+C_4)||v-\tilde{v}||_{C^{1,2+\alpha}([0,T]\times\overline{D})},$$

where C depends only on n, α, Γ_0, and M. On the other hand, from Proposition 3.3, we have

$$||v-\tilde{v}||_{C^{1,2+\alpha}([0,T]\times\overline{D})} \leq C||u-\tilde{u}||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)}, \tag{5.21}$$

where C depends only on n, α, σ_2, Γ_0, and M. Collecting the above two estimates, we see that the map Ψ is a contraction on $\mathcal{U}_M$ for sufficiently small T.

Since Ψ is a contraction on $\mathcal{U}_M$, there exists a unique fixed point $u^* \in \mathcal{U}_M$ of Ψ. Since $u^* = \Psi(u^*)$, we have from the defition of Ψ_0,

$$\begin{aligned} \Psi_0(u^*)(t,x) &= F_0(t,x) + F^{u^*}(t,x) \\ &= F_0(t,x) + E^{v^*}(F^{u^*})(t,x) \\ &= \Psi(u^*)(t,x) \\ &= u^*(t,x) \end{aligned}$$

for any $(t,x) \in \cup_{0\le t\le T}\{t\}\times \Gamma_t^{u^*}$. Hence, $\{\Gamma_t^{u^*}\}_{0\le t\le T}$ evolves by the equation

$$\begin{cases} \dfrac{dx}{dt} & = \sigma_2 H^*(t,x)\nu^*(t,x) + u^*(t,x) \\ & = \sigma_2 H^*(t,x)\nu^*(t,x) + \Psi_0(u^*)(t,x), \\ x(0) & = x_0 \in \Gamma_0, \end{cases}$$

that is, the pair $(\Psi_0(u^*), \{\Gamma_t^{u^*}\}_{0\le t\le T})$ is a solution of our free boundary problem.

Although the above mapping Ψ depends on the particular way of the extension, we can see that the solution, in fact, does not depend on such extension and is unique in the class stated in the main theorem. To see this, let $(u, \{\Gamma_t\}_{0\le t\le T})$ be another pair of the solution for (FBP). Let v be the signed distance function of $\{\Gamma_t\}_{0\le t\le T}$. Then, v belongs to $C^{1,2+\alpha}([0,T]\times\overline{D})$ where $D = \{x\in\mathbb{R}^n \ ; \ -\delta < d_0(x) < \delta\}$ for sufficiently small $\delta > 0$. Since $\{\Gamma_t\}_{0\le t\le T}$ evolves by the equation in (BC), v satisfies the equation (3.6) in Section 3. The important fact is that for any $x \in \overline{D}$, the point $x - v(t,x)\nabla_x v(t,x)$ must belong to Γ_t by the definition of the signed distance function. This implies that for any $\tilde{u}$ satisfying $\tilde{u} = u$ on $\cup_{0\le t\le T}\{t\}\times\Gamma_t$, the function v is also the solution of the equation (3.6) with $\tilde{u}$ instead of u. This concludes that the solution $(u, \{\Gamma_t\}_{0\le t\le T})$ does not depend on the particular extension, and the above $(\Psi_0(u^*), \{\Gamma_t^{u^*}\}_{0\le t\le T})$ is the unique solution solving (FBP) in the class stated in the theorem. Now the proof of the main theorem is completed.

6. Appendix

6.1. **Proof of Proposition 5.1.**

In this section, we shall give the proof of the Proposition 5.1. Its proof is just direct calculation and essentially same as the proof of Proposition 4.1. For simplicity of notation, we write $||v-\tilde{v}||_{C^{1,2+\alpha}}$, $||\partial_x\tilde{v}(t,\cdot)||_{C^{2+\alpha}}$ instead of $||v-\tilde{v}||_{C^{1,2+\alpha}([0,T]\times\cup_{1\le k\le m}\overline{O_k})}$, $||\partial_x\tilde{v}(t,\cdot)||_{C^{2+\alpha}(\cup_{1\le k\le m}\overline{O_k})}$, respectively. We also write $w_h'(y)$ instead of $w_h'(t,y)$. First note that by (4.31), it suffices to estimate

$$\begin{aligned} &\mathcal{I}_{i,j,k}(t,s,\psi_h(t,y',0)) - \tilde{\mathcal{I}}_{i,j,k}(t,s,\tilde{\psi}_h(t,w_h'(y),0)) \\ = \ & \int_{\mathbb{R}^{n-1}} K_{i,j}(t-s,\psi_h(t,y',0)-\psi_k(s,\xi',0))\Upsilon_1(s,\xi')d\xi' \\ & - \int_{\mathbb{R}^{n-1}} K_{i,j}(t-s,\tilde{\psi}_h(t,w_h'(y),0)-\tilde{\psi}_k(s,\xi',0))\tilde{\Upsilon}_1(s,\xi')d\xi', \end{aligned} \tag{6.1}$$

where

$$\Upsilon_1(s,\xi') = a_k(\psi_k(s,\xi',0))(H\nu_j)(s,\psi_k(s,\xi',0))J_{\psi_k}(s,\xi') \tag{6.2}$$

$$\tilde{\Upsilon}_1(s,\xi') = a_k(\tilde{\psi}_k(s,\xi',0))(\tilde{H}\tilde{\nu}_j)(s,\tilde{\psi}_k(s,\xi',0))J_{\tilde{\psi}_k}(s,\xi'). \tag{6.3}$$

Set

$$B_1 := \{y = (y', y^{(n)}) \in B; \ (y',0)\in B_1'\}, \tag{6.4}$$

$$B_2 := \{y = (y', y^{(n)}) \in B; \ (y',0)\in B_2'\}. \tag{6.5}$$

Here, B_1' and B_2' are subsets of $B' = \{(y',0) \ ; \ |y'|<1\}$ defined by (4.40) and (4.41). Then, we have $B = B_1\cup B_2$ and, as in the proof of Proposition 4.1, we see for $f\in C(\overline{B})$,

$$||f||_{C^\alpha(\overline{B})} \le C(||f||_{C^\alpha(\overline{B_1})} + ||f||_{C^\alpha(\overline{B_2})}), \tag{6.6}$$

where C depends only on n, δ_1, r and R.

Hence, we shall estimate

$$||\mathcal{I}_{i,j,k}(t,s,\psi_h(t,y',0)) - \tilde{\mathcal{I}}_{i,j,k}(t,s,\tilde{\psi}_h(t,w_h'(y),0))||_{C_y^\alpha(\overline{B_1})}$$

and

$$||\mathcal{I}_{i,j,k}(t,s,\psi_h(t,y',0)) - \tilde{\mathcal{I}}_{i,j,k}(t,s,\tilde{\psi}_h(t,w_h'(y),0))||_{C_y^\alpha(\overline{B_2})}.$$

If $y \in B_1$, then there exists a point $(\zeta',0) = (\zeta'(y'),0) \in B$ such that $\psi_h(t,y',0) = \psi_k(t,\zeta',0)$. In this case, since $\psi_h(t,y',0) \in O_k$, we see that $\tilde{\psi}_h(t,w_h'(y),0) \in \tilde{\psi}_k(t,B)$ for all $t \in [0,T]$ if $||v-\tilde{v}||_{C^{1,2+\alpha}}$ is sufficiently small. Indeed, note that since $\psi_h(0,y) = \tilde{\psi}_h(0,y)$, by the results of Proposition 6.1 in Section 6.2, we have

$$\begin{aligned}
(6.7)\qquad & |\psi_h(t,y',0) - \tilde{\psi}_h(t,w_h'(y),0)| \\
\leq\ & |\psi_h(t,y',0) - \tilde{\psi}_h(t,y',0)| + |\tilde{\psi}_h(t,y',0) - \tilde{\psi}_h(t,w_h'(y),0)| \\
\leq\ & ||\partial_t\psi_h - \partial_t\tilde{\psi}_h||_{C([0,T]\times\overline{B})}T + ||\partial_y\tilde{\psi}_h||_{C([0,T]\times\overline{B})}|y' - w_h'(y)| \\
\leq\ & CT||v-\tilde{v}||_{C^{1,2+\alpha}} + C|y'-w_h'(y)|,
\end{aligned}$$

where C depends only on n, r, and R. We also have

$$\begin{aligned}
(6.8)\qquad |y'-w_h'(y)| &= |\varphi_h'(t,\psi_h(t,y)) - \tilde{\varphi}_h'(t,\psi_h(t,y))| \\
&\leq ||\varphi_h(t,\cdot) - \tilde{\varphi}_h(t,\cdot)||_{C(\overline{O_h})} \\
&\leq T||\partial_t\varphi_h - \partial_t\tilde{\varphi}_h||_{C([0,T]\times\overline{O_h})} \\
&\leq CT||v-\tilde{v}||_{C^{1,2+\alpha}}.
\end{aligned}$$

Combining these with the fact $O_k \subset\subset \cap_{0\leq t\leq T}\tilde{\psi}_k(t,B)$, we have that $\tilde{\psi}_h(t,w_h'(y),0) \in \tilde{\psi}_k(t,B)$ for $t \in [0,T]$ if $||v-\tilde{v}||_{C^{1,2+\alpha}} \leq \epsilon_1$, ϵ_1 is sufficiently small. Remark that this ϵ_1 can be taken depending only on n, r and R. We omit the details.

The fact $\tilde{\psi}_h(t,w_h'(y),0) \in \tilde{\psi}_k(t,B)$ implies that there exists a point $(\eta',0) = (\eta'(y),0) \in B$ such that $\tilde{\psi}_h(t,w_h'(y),0) = \tilde{\psi}_k(t,\eta',0)$. Thus, if $y \in B_1$, then, we can decompose as

$$\begin{aligned}
(6.9)\qquad & \mathcal{I}_{i,j,k}(t,s,\psi_h(t,y',0)) - \tilde{\mathcal{I}}_{i,j,k}(t,s,\tilde{\psi}_h(t,w_h'(y),0)) \\
:=\ & \mathcal{O}_1(t,s,y) + \mathcal{O}_2(t,s,y),
\end{aligned}$$

where

$$\begin{aligned}
(6.10)\qquad & \mathcal{O}_1(t,s,y) \\
:=\ & \mathcal{I}_{i,j,k}(t,s,\psi_h(t,y',0)) - \tilde{\mathcal{I}}_{i,j,k}(t,s,\tilde{\psi}_k(t,\zeta'(y'),0)) \\
=\ & \mathcal{I}_{i,j,k}(t,s,\psi_k(t,\zeta'(y'),0)) - \tilde{\mathcal{I}}_{i,j,k}(t,s,\tilde{\psi}_k(t,\zeta'(y'),0)) \\
(6.11)\qquad & \mathcal{O}_2(t,s,y) \\
:=\ & \tilde{\mathcal{I}}_{i,j,k}(t,s,\tilde{\psi}_k(t,\zeta'(y'),0)) - \tilde{\mathcal{I}}_{i,j,k}(t,s,\tilde{\psi}_h(t,w_h'(y),0)) \\
=\ & \tilde{\mathcal{I}}_{i,j,k}(t,s,\tilde{\psi}_k(t,\zeta'(y'),0)) - \tilde{\mathcal{I}}_{i,j,k}(t,s,\tilde{\psi}_k(t,\eta'(y),0)).
\end{aligned}$$

We shall estimate $\mathcal{O}_1$. By the definition of $\mathcal{I}_{i,j,k}$, the function $\mathcal{O}_1$ is expressed as

$$\begin{aligned}
&\mathcal{O}_1(t,s,y) = \Upsilon_7(t,s,y) + \Upsilon_8(t,s,y), \\
&\Upsilon_7(t,s,y) := \int_{\mathbb{R}^{n-1}} K_{i,j}(t-s,\psi_k(t,\zeta'(y'),0) - \psi_k(s,\xi',0))(\Upsilon_1 - \tilde{\Upsilon}_1)(s,\xi')d\xi', \\
&\Upsilon_8(t,s,y) := \int_{\mathbb{R}^{n-1}} \Upsilon_9(t,s,y,\xi')\tilde{\Upsilon}_1(s,\xi')d\xi',
\end{aligned}$$

where

$$\begin{aligned}
& \Upsilon_9(t,s,y,\xi') \\
:=\ & K_{i,j}(t-s,\psi_k(t,\zeta'(y'),0) - \psi_k(s,\xi',0)) - K_{i,j}(t-s,\tilde{\psi}_k(t,\zeta'(y'),0) \\
=\ & \int_0^1 < \Upsilon_{10}(t,s,y,\xi'), \Upsilon_{11}(t,s,y,\xi',\tau_1) > d\tau_1,
\end{aligned}$$

with

$$\begin{aligned}
&\Upsilon_{10}(t,s,y,\xi') \\
:=\ &\psi_k(t,\zeta'(y'),0)-\psi_k(s,\xi',0)-\tilde{\psi}_k(t,\zeta'(y'),0)+\tilde{\psi}_k(s,\xi',0), \\
&\Upsilon_{11}(t,s,y,\xi',\tau_1) \\
:=\ &(\nabla_x K_{i,j})(t-s,\tau_1\Upsilon_{10}(t,s,y,\xi')+\tilde{\psi}_k(t,\zeta'(y'),0)-\tilde{\psi}_k(s,\xi',0)).
\end{aligned}$$

First we shall estimate Υ_8. Using the mean value theorem, we see

$$\begin{aligned}
&|\Upsilon_{10}(t,s,y,\xi')| \\
\leq\ &|\psi_k(t,\zeta'(y'),0)-\psi_k(s,\zeta'(y'),0)-(\tilde{\psi}_k(t,\zeta'(y'),0)-\tilde{\psi}_k(s,\zeta'(y'),0))| \\
&\quad +|\psi_k(s,\zeta'(y'),0)-\psi_k(s,\xi',0)-(\tilde{\psi}_k(s,\zeta'(y'),0)-\tilde{\psi}_k(s,\xi',0))| \\
\leq\ &||\partial_t\psi_k-\partial_t\tilde{\psi}_k||_{C([0,T]\times\overline{B})}(t-s)+||\partial_y\psi_k-\partial_y\tilde{\psi}_k||_{C([0,T]\times\overline{B})}|\zeta'(y')-\xi'| \\
\leq\ &C||v-\tilde{v}||_{C^{1,2+\alpha}}(t-s+T^{\frac{1+\alpha}{2}}|\zeta'(y')-\xi'|),
\end{aligned}$$

and for $y_1,y_2\in B_1$,

$$|\Upsilon_{10}(t,s,y_1,\xi')-\Upsilon_{10}(t,s,y_2,\xi')| \quad\leq\quad CT^{\frac{1+\alpha}{2}}||v-\tilde{v}||_{C^{1,2+\alpha}}|y_1-y_2|.$$

From Lemma 4.1 and (4.25), we have

$$\begin{aligned}
&|\Upsilon_{11}(t,s,y,\xi',\tau_1)| \\
\leq\ &C(t-s)^{-\frac{n+1}{2}}\Big(1+\frac{|\tau_1\Upsilon_{10}(t,s,y,\xi')+\tilde{\psi}_k(t,\zeta'(y'),0)-\tilde{\psi}_k(s,\xi',0)|}{(t-s)^{\frac{1}{2}}}\Big)^{-(n+1)} \\
\leq\ &C(t-s)^{-\frac{n+1}{2}}\Big(1+\frac{r^2|\zeta'(y')-\xi'|}{64R^2(t-s)^{\frac{1}{2}}}\Big)^{-(n+1)}.
\end{aligned}$$

For $y_1,y_2\in B_1$, it follows that

$$\begin{aligned}
&|\Upsilon_{11}^{(l)}(t,s,y_1,\xi',\tau_1)-\Upsilon_{11}^{(l)}(t,s,y_2,\xi',\tau_1)| \\
=\ &\Big|\int_0^1 <\Upsilon_{12}(t,s,y_1,y_2,\xi',\tau_1),\Upsilon_{13}(t,s,y_1,y_2,\xi',\tau_1,\tau_2)> d\tau_2\Big|,
\end{aligned}$$

where

$$\begin{aligned}
&\Upsilon_{12}(t,s,y_1,y_2,\xi',\tau_1) \\
:=\ &\tau_1\Upsilon_{10}(t,s,y_1,\xi')+\tilde{\psi}_k(t,\zeta'(y_1'),0)-\tilde{\psi}_k(s,\xi',0) \\
&\quad -\big(\tau_1\Upsilon_{10}(t,s,y_2,\xi')+\tilde{\psi}_k(t,\zeta'(y_2'),0)-\tilde{\psi}_k(s,\xi',0)\big) \\
=\ &\tau_1(\psi_k(t,\zeta'(y_1'),0)-\psi_k(t,\zeta'(y_2'),0)) \\
&\quad +(1-\tau_1)(\tilde{\psi}_k(t,\zeta'(y_1'),0)-\tilde{\psi}_k(t,\zeta'(y_2'),0)), \\
&\Upsilon_{13}(t,s,y_1,y_2,\xi',\tau_1,\tau_2) \\
:=\ &(\nabla_x\partial_{x_l}K_{i,j})(t-s,\Upsilon_{14}(t,s,y_1,y_2,\xi',\tau_1,\tau_2)), \\
&\Upsilon_{14}(t,s,y_1,y_2,\xi',\tau_1,\tau_2) \\
:=\ &\tau_2\Upsilon_{12}(t,s,y_1,y_2,\xi',\tau_1)+\tau_1\Upsilon_{10}(t,s,y_2,\xi')+\tilde{\psi}_k(t,\zeta'(y_2'),0)-\tilde{\psi}_k(s,\xi',0).
\end{aligned}$$

We have

$$|\Upsilon_{12}(t,s,y_1,y_2,\xi',\tau_1)|\leq C|y_1-y_2|,$$

and from (4.23),

$$|\Upsilon_{14}(t,s,y_1,y_2,\xi',\tau_1,\tau_2)|\geq\frac{r^2}{64R^2}|\tau_2\zeta'(y_1')+(1-\tau_2)\zeta'(y_2')-\xi'|.$$

Thus, from Lemma 4.1, we have

$$\begin{aligned}
&|\Upsilon_{11}^{(l)}(t,s,y_1,\xi',\tau_1)-\Upsilon_{11}^{(l)}(t,s,y_2,\xi',\tau_1)|\\
\leq\ & C|y_1-y_2|(t-s)^{-\frac{n}{2}-1}\int_0^1(1+\frac{|\Upsilon_{14}(t,s,y_1,y_2,\xi',\tau_1,\tau_2)|}{(t-s)^{\frac{1}{2}}})^{-(n+2)}d\tau_2\\
\leq\ & C|y_1-y_2|(t-s)^{-\frac{n}{2}-1}\int_0^1(1+\frac{r^2|\tau_2\zeta'(y_1')+(1-\tau_2)\zeta'(y_2')-\xi'|}{64R^2(t-s)^{\frac{1}{2}}})^{-(n+2)}d\tau_2,
\end{aligned}$$

where C depends only on n, r, and R.

Collecting these above, we have

$$\begin{aligned}
&|\Upsilon_9(t,s,y,\xi')|\\
\leq\ & \int_0^1|\Upsilon_{10}(t,s,y,\xi')||\Upsilon_{11}(t,s,y,\xi',\tau_1)|d\tau_1\\
\leq\ & C||v-\tilde{v}||_{C^{1,2+\alpha}}(t-s+T^{\frac{1+\alpha}{2}}|\zeta'(y')-\xi'|)(t-s)^{-\frac{n+1}{2}}(1+\frac{r^2|\zeta'(y')-\xi'|}{64R^2(t-s)^{\frac{1}{2}}})^{-(n+1)},
\end{aligned}$$

so

$$||\Upsilon_8(t,s,\cdot)||_{C(\overline{B_1})}\leq C(1+T^{\frac{1+\alpha}{2}}(t-s)^{-\frac{1}{2}})||v-\tilde{v}||_{C^{1,2+\alpha}}, \tag{6.12}$$

where C depends only on n, r, and R. And for $y_1,y_2\in B_1$ with $|y_1-y_2|\leq(t-s)^{\frac{1}{2}}$, we have

$$\begin{aligned}
&|\Upsilon_9(t,s,y_1,\xi')-\Upsilon_9(t,s,y_2,\xi')|\\
\leq\ & \int_0^1|<\Upsilon_{10}(t,s,y_1,\xi')-\Upsilon_{10}(t,s,y_2,\xi'),\Upsilon_{11}(t,s,y_1,\xi',\tau_1)>|d\tau_1\\
&+\int_0^1|<\Upsilon_{10}(t,s,y_2,\xi'),\Upsilon_{11}(t,s,y_1,\xi',\tau_1)-\Upsilon_{11}(t,s,y_2,\xi',\tau_1)>|d\tau_1\\
\leq\ & CT^{\frac{1+\alpha}{2}}||v-\tilde{v}||_{C^{1,2+\alpha}}|y_1-y_2|(t-s)^{-\frac{n+1}{2}}(1+\frac{r^2|\zeta'(y_1')-\xi'|}{64R^2(t-s)^{\frac{1}{2}}})^{-(n+1)}\\
&+C||v-\tilde{v}||_{C^{1,2+\alpha}}(t-s+T^{\frac{1+\alpha}{2}}|\zeta'(y_2')-\xi'|)|y_1-y_2|(t-s)^{-\frac{n}{2}-1}\\
&\qquad\times\int_0^1(1+\frac{r^2|\tau_2\zeta'(y_1')+(1-\tau_2)\zeta'(y_2')-\xi'|}{64R^2(t-s)^{\frac{1}{2}}})^{-(n+2)}d\tau_2,
\end{aligned}$$

using the inequality

$$\begin{aligned}
|\zeta'(y_2')-\xi'|\ &\leq\ |\tau_2(\zeta'(y_2')-\zeta'(y_1'))|+|\tau_2\zeta'(y_1')+(1-\tau_2)\zeta'(y_2')-\xi'|\\
&\leq\ C|y_1-y_2|+|\tau_2\zeta'(y_1')+(1-\tau_2)\zeta'(y_2')-\xi'|\\
&\leq\ C(t-s)^{\frac{1}{2}}+|\tau_2\zeta'(y_1')+(1-\tau_2)\zeta'(y_2')-\xi'|,
\end{aligned}$$

we have,

$$\begin{aligned}
&|\Upsilon_9(t,s,y_1,\xi')-\Upsilon_9(t,s,y_2,\xi')|\\
&\leq C||v-\tilde{v}||_{C^{1,2+\alpha}}|y_1-y_2|\Big\{T^{\frac{1+\alpha}{2}}(t-s)^{-\frac{n+1}{2}}(1+\frac{r^2|\zeta'(y_1')-\xi'|}{64R^2(t-s)^{\frac{1}{2}}})^{-(n+1)}\\
&+(t-s)^{-\frac{n}{2}}\int_0^1(1+\frac{r^2|\tau_2\zeta'(y_1')+(1-\tau_2)\zeta'(y_2')-\xi'|}{64R^2(t-s)^{\frac{1}{2}}})^{-(n+2)}d\tau_2\\
&+T^{\frac{1+\alpha}{2}}(t-s)^{-\frac{n}{2}-1}\int_0^1(C(t-s)^{\frac{1}{2}}+|\tau_2\zeta'(y_1')+(1-\tau_2)\zeta'(y_2')-\xi'|)\\
&\qquad\cdot(1+\frac{r^2|\tau_2\zeta'(y_1')+(1-\tau_2)\zeta'(y_2')-\xi'|}{64R^2(t-s)^{\frac{1}{2}}})^{-(n+2)}d\tau_2\Big\}.
\end{aligned}$$

Hence, using Fubini's thorem, we have for $y_1, y_2 \in B_1$ with $|y_1 - y_2| \le (t-s)^{\frac{1}{2}}$,

$$\begin{aligned}
&|\Upsilon_8(t,s,y_1) - \Upsilon_8(t,s,y_2)| \\
\le\ & \int_{\mathbb{R}^{n-1}} |\Upsilon_9(t,s,y_1,\xi') - \Upsilon_9(t,s,y_2,\xi')||\tilde{\Upsilon}_1(s,\xi')|d\xi' \\
\le\ & C||v-\tilde{v}||_{C^{1,2+\alpha}}|y_1 - y_2|\{T^{\frac{1+\alpha}{2}}(t-s)^{-1} + (t-s)^{-\frac{1}{2}}\},
\end{aligned}$$

where C depends only on n, r and R. On the other hand, for $y_1, y_2 \in B_1$ with $|y_1 - y_2| > (t-s)^{\frac{1}{2}}$, we have, from (6.12),

$$\begin{aligned}
|\Upsilon_8(t,s,y_1) - \Upsilon_8(t,s,y_2)| &\le 2||\Upsilon_8(t,s,\cdot)||_{C(\overline{B_1})} \\
&\le C||v-\tilde{v}||_{C^{1,2+\alpha}}(1 + T^{\frac{1+\alpha}{2}}(t-s)^{-\frac{1}{2}}) \\
&\le C||v-\tilde{v}||_{C^{1,2+\alpha}}((t-s)^{-\frac{1}{2}} + T^{\frac{1+\alpha}{2}}(t-s)^{-1})|y_1 - y_2|.
\end{aligned}$$

Thus, for all $y_1, y_2 \in B_2$, we obtain

$$\begin{aligned}
&|\Upsilon_8(t,s,y_1) - \Upsilon_8(t,s,y_2)| \\
\le\ & C||v-\tilde{v}||_{C^{1,2+\alpha}}((t-s)^{-\frac{1}{2}} + T^{\frac{1+\alpha}{2}}(t-s)^{-1})|y_1 - y_2|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&|\Upsilon_8(t,s,y_1) - \Upsilon_8(t,s,y_2)| \\
\le\ & C||v-\tilde{v}||_{C^{1,2+\alpha}}((t-s)^{-\frac{\alpha}{2}} + T^{\frac{1+\alpha}{2}}(t-s)^{-\frac{1+\alpha}{2}})|y_1 - y_2|^{\alpha},
\end{aligned}$$

or,

$$[\Upsilon_8(t,s,\cdot)]_{C^{\alpha}(\overline{B_1})} \le C||v-\tilde{v}||_{C^{1,2+\alpha}}((t-s)^{-\frac{\alpha}{2}} + T^{\frac{1+\alpha}{2}}(t-s)^{-\frac{1+\alpha}{2}}), \tag{6.13}$$

where C depends only on n, r and R.

Next we shall estimate Υ_7. By the definition of Υ_1, $\tilde{\Upsilon}_1$, we have from Lemma 2.1,

$$|(\Upsilon_1 - \tilde{\Upsilon}_1)(s,\xi')| \le CT^{\frac{\alpha}{2}}(1 + \sup_{0<t<T} t^{\frac{1}{2}}||\partial_x^3 \tilde{v}(t,\cdot)||_{C^{\alpha}})||v-\tilde{v}||_{C^{1,2+\alpha}},$$

where C depends only on n, r and R. Thus, together with the pointwise estimate in Lemma 4.1 and (4.23), we have

$$||\Upsilon_7(t,s,\cdot)||_{C^{\alpha}(\overline{B_1})} \le CT^{\frac{\alpha}{2}}(t-s)^{-\frac{1+\alpha}{2}}(1 + \sup_{0<t<T} t^{\frac{1}{2}}||\partial_x^3 \tilde{v}(t,\cdot)||_{C^{\alpha}})||v-\tilde{v}||_{C^{1,2+\alpha}},$$

where C depends only on n, r and R.

The estimate for $\mathcal{O}_2$ is also obtained by direct calculations as above, and we have

$$\begin{aligned}
&||\mathcal{O}_2(t,s,\cdot)||_{C^{\alpha}(\overline{B_1})} \\
\le\ & CT^{\frac{1+\alpha}{2}}(t-s)^{-\frac{1}{2}}(1 + s^{-\frac{1}{2}} \sup_{0<t<T} t^{\frac{1}{2}}||\partial_x^3 \tilde{v}(t,\cdot)||_{C^{\alpha}})||v-\tilde{v}||_{C^{1,2+\alpha}}.
\end{aligned} \tag{6.14}$$

Next we shall estimate

$$||\mathcal{I}_{i,j,k}(t,s,\psi_h(t,y',0)) - \tilde{\mathcal{I}}_{i,j,k}(t,s,\tilde{\psi}_h(t,w_h'(y),0))||_{C_y^{\alpha}(\overline{B_2})}.$$

Let $\epsilon_2 \in (0, \epsilon_1]$ be a small positive number to be precised later.

From (6.1) we decompose as

$$\mathcal{I}_{i,j,k}(t,s,\psi_h(t,y',0)) - \tilde{\mathcal{I}}_{i,j,k}(t,s,\tilde{\psi}_h(t,w_h'(y),0)) =: \mathcal{O}_3(t,s,y) + \mathcal{O}_4(t,s,y),$$

where

$$\begin{aligned} & \mathcal{O}_3(t,s,y) \\ := & \int_{\mathbb{R}^{n-1}} K_{i,j}(t-s,\psi_h(t,y',0)-\psi_k(s,\xi',0))(\Upsilon_1-\tilde{\Upsilon}_1)(s,\xi')d\xi', \\ & \mathcal{O}_4(t,s,y) \\ := & \int_{\mathbb{R}^{n-1}} \big(K_{i,j}(t-s,\psi_h(t,y',0)-\psi_k(s,\xi',0)) \\ & \qquad -K_{i,j}(t-s,\tilde{\psi}_h(t,w_h'(y),0)-\tilde{\psi}_k(s,\xi',0))\big)\tilde{\Upsilon}_1(s,\xi')d\xi'. \end{aligned}$$

Let $||v-\tilde{v}||_{C^{1,2+\alpha}} \leq \epsilon_2$. Since $T \leq T_0 < 1$, if $\tilde{\psi}_k(s,\xi',0) \in \operatorname{supp} a_k$, then we have for all $y \in B_2$, then

$$\begin{aligned} |\psi_h(t,y',0)-\psi_k(s,\xi',0)| & \geq |\psi_h(t,y',0)-\tilde{\psi}_k(s,\xi',0)| - |\psi_k(s,\xi',0)-\tilde{\psi}_k(s,\xi',0)| \\ & \geq \frac{\delta_1}{2} - CT||v-\tilde{v}||_{C^{1,2+\alpha}} \\ & \geq \frac{\delta_1}{2} - C\epsilon_2 \geq \frac{\delta_1}{4}, \end{aligned}$$

for sufficiently small ϵ_2. Thus, from Lemma 4.1 and the estimate (6.14), we have for all $y \in B_2$,

$$\begin{aligned} & |\mathcal{O}_3(t,s,y)| \\ \leq & C\int_{B'} (t-s)^{-\frac{n}{2}}(1+\frac{|\psi_h(t,y',0)-\psi_k(s,\xi',0)|}{(t-s)^{\frac{1}{2}}})^{-n}|(\Upsilon_1-\tilde{\Upsilon}_1)(s,\xi')|d\xi' \\ \leq & C\int_{B'} (t-s)^{-\frac{n}{2}}(1+\frac{\delta_1}{4(t-s)^{\frac{1}{2}}})^{-n}d\xi'(1+s^{-\frac{1}{2}}\sup_{0<t<T} t^{\frac{1}{2}}||\partial_x^3\tilde{v}(t,\cdot)||_{C^\alpha})||v-\tilde{v}||_{C^{1,2+\alpha}} \\ \leq & C(1+s^{-\frac{1}{2}}\sup_{0<t<T} t^{\frac{1}{2}}||\partial_x^3\tilde{v}(t,\cdot)||_{C^\alpha})||v-\tilde{v}||_{C^{1,2+\alpha}}, \end{aligned}$$

where C depends only on n, r and R.

For $y_1, y_2 \in B_2$ with $|y_1-y_2| \leq \epsilon_2$, we see for all $\xi' \in B'$ satisfying $\psi_k(s,\xi',0) \in \operatorname{supp} a_k$ or $\tilde{\psi}_k(s,\xi',0) \in \operatorname{supp} a_k$,

$$\begin{aligned} & |\tau_1(\psi_h(t,y_1',0)-\psi_k(s,\xi',0)) + (1-\tau_1)(\psi_h(t,y_2',0)-\psi_k(s,\xi',0))| \\ \geq & |\psi_h(t,y_2',0)-\psi_k(s,\xi',0)| - |\psi_h(t,y_1',0)-\psi_h(t,y_2',0)| \\ \geq & \frac{\delta_1}{4} - C|y_1-y_2|, \end{aligned}$$

where C depends on n, r and R. Hence, if ϵ_2 is sufficiently small, then, for $y_1, y_2 \in B_2$ with $|y_1-y_2| \leq \epsilon_2$,

$$\begin{aligned} & |\mathcal{O}_3(t,s,y_1)-\mathcal{O}_3(t,s,y_2)| \\ = & |\int_{B'}\int_0^1 <\psi_h(t,y_1',0)-\psi_h(t,y_2',0), \\ & \quad (\nabla_x K_{i,j})(t-s,\tau_1\psi_h(t,y_1',0)+(1-\tau_1)\psi_h(t,y_2',0)-\psi_k(s,\xi',0))> d\tau_1 \\ & \qquad \times(\Upsilon_1-\tilde{\Upsilon}_1)(s,\xi')d\xi'| \\ \leq & C|y_1-y_2|(t-s)^{-\frac{n+1}{2}}(1+\frac{\delta_1}{8(t-s)^{\frac{1}{2}}})^{-(n+1)}(1+s^{-\frac{1}{2}}\sup_{0<t<T} t^{\frac{1}{2}}||\partial_x^3\tilde{v}(t,\cdot)||_{C^\alpha})||v-\tilde{v}||_{C^{1,2+\alpha}} \\ \leq & C|y_1-y_2|(1+s^{-\frac{1}{2}}\sup_{0<t<T} t^{\frac{1}{2}}||\partial_x^3\tilde{v}(t,\cdot)||_{C^\alpha})||v-\tilde{v}||_{C^{1,2+\alpha}}, \end{aligned}$$

where C depends on n, r and R. If $|y_1 - y_2| > \epsilon_2$, then,

$$\begin{aligned}|\mathcal{O}_3(t,s,y_1) - \mathcal{O}_3(t,s,y_2)| &\leq 2||\mathcal{O}_3(t,s,\cdot)||_{C(\overline{B_2})} \\ &\leq C(1 + s^{-\frac{1}{2}} \sup_{0<t<T} t^{\frac{1}{2}}||\partial_x^3 \tilde{v}(t,\cdot)||_{C^\alpha})||v - \tilde{v}||_{C^{1,2+\alpha}} \\ &\leq C(1 + s^{-\frac{1}{2}} \sup_{0<t<T} t^{\frac{1}{2}}||\partial_x^3 \tilde{v}(t,\cdot)||_{C^\alpha})||v - \tilde{v}||_{C^{1,2+\alpha}} \frac{|y_1 - y_2|}{\epsilon_2}\end{aligned}$$

where C depends on n, r and R. Similarly, if $||v - \tilde{v}||_{C^{1,2+\alpha}} \leq \epsilon_2$, it is not difficult to see

$$||\mathcal{O}_4(t,s,\cdot)||_{C^\alpha(\overline{B_2})} \leq C||v - \tilde{v}||_{C^{1,2+\alpha}},$$

where C depends on n, r and R.

Collecting these, if $||v - \tilde{v}||_{C^{1,2+\alpha}} \leq \epsilon_2$, we have

$$\begin{aligned}&||\mathcal{I}_{i,j,k}(t,s,\psi_h(t,y',0)) - \tilde{\mathcal{I}}_{i,j,k}(t,s,\tilde{\psi}_h(t,w'_h(y),0))||_{C_y^\alpha(\overline{B_2})} \\ &\leq C(1 + s^{-\frac{1}{2}} \sup_{0<t<T} t^{\frac{1}{2}}||\partial_x^3 \tilde{v}(t,\cdot)||_{C^\alpha})||v - \tilde{v}||_{C^{1,2+\alpha}}\end{aligned} \tag{6.15}$$

where C depends on n, r and R. Note that ϵ_2 depends only on n, r, and R. By the results of Proposition 4.1, we easily see that

$$||\mathcal{I}_{i,j,k}(t,s,\psi_h(t,y',0))||_{C_y^\alpha(\overline{B_2})} + ||\tilde{\mathcal{I}}_{i,j,k}(t,s,\tilde{\psi}_h(t,w'_h(y),0))||_{C_y^\alpha(\overline{B_2})} \leq C,$$

where C depends on n, r and R. Hence, the estimate (6.15) holds for all $v, \tilde{v}$ with a constatnt C depending only on n, r and R. This completes the proof.

6.2. **Implicit function theorem.**

Let Γ_0 be the boundary of a smooth bounded domain Ω_0. Let d_0 be the signed distance function and we set

$$D = \{x \in \mathbb{R}^n \ ; \ -\delta_0 < d_0(x) < \delta_0\}.$$

In order to estimate the term

$$\int_0^t e^{(t-s)\Delta}\mathbf{P}H\nu\mathcal{H}^{n-1}_{\llcorner\Gamma_s}ds,$$

we need some informations of the local coordinate transforms $\{\varphi_j\}$ associated with $\{\Gamma_t\}_{0\leq t\leq T}$ belonging to $\mathcal{S}(\alpha, R, T, d_0)$. Let r be the number given by (4.11). For $A \subset \mathbb{R}^n$ and $\rho > 0$, we set

$$(A)_\rho := \{x \in \mathbb{R}^n \ ; \ \text{dist}\ (x, A) < \rho\}.$$

Proposition 6.1. *Let $R \geq 1$ be a given number. Let $\alpha \in (0,1)$. Then, there exists a positive T_0 depending only on r and R such that for all $T \in (0, T_0]$ and $\{\Gamma_t\}_{0\leq t\leq T} \in \mathcal{S}(\alpha, R, T, d_0)$, the following statements hold.*

(i)There exsist a family of open sets $\{U_j\}_{j=1}^m$, $U_j \subset \overline{D}$, and a family of functions $\{\varphi_j(t,x)\}_{j=1}^m$, $\varphi_j(t,x) \in C^{1,2+\alpha}([0,T] \times \overline{U_j})$ satisfying following properties.

(i-1)For each $t \in [0,T]$, there exists an open set $U_j(t) \subset U_j$ such that:

(i-1-1)The functions $\varphi_j(t,\cdot) : \overline{U_j(t)} \to \overline{B}$ ($B := \{y \in \mathbb{R}^n; \ |y| < 1\}$ are C^2-diffeomorphisms.

(i-1-2)For each $t \in [0,T]$,

$$\varphi_j(t, \{\overline{U_j(t)} \cap \Omega_t\}) = \{y \in \overline{B}; y_n > 0\}$$

and

$$\varphi_j(t, \{\overline{U_j(t)} \cap \Gamma_t\}) = \overline{B'} \ (B' := \{y \in B; \ y_n = 0\}$$

. *(i-1-3)For some $\rho > 0$, there exist families of open balls $\{O_j\}_{j=i}^m$ and $\{\hat{O}_j\}_{j=i}^m$ such that*

$$\hat{O}_j \subset\subset O_j, \tag{6.16}$$

$$O_j \subset\subset \cap_{0\le t\le T}\varphi_j^{-1}(t,B),\ 1\le j\le m, \tag{6.17}$$

$$\cup_{0\le t\le T}(\Gamma_t)_\rho \subset\subset \cup_{j=1}^m \hat{O}_j. \tag{6.18}$$

The above ρ, $\{U_j\}_{j=1}^m$, $\{O_j\}_{j=1}^m$, and $\{\hat{O}_j\}_{j=1}^m$ are taken independently with respect to each evolving hypersurface belonging to $\cup_{0<T\le T_0}\mathcal{S}(\alpha,R,T,d_0)$. Especially, we can take $\rho = \frac{r^2}{(32R)^2(1+\frac{R}{r})}$, $\hat{O}_j = \{|x-\bar{x}_j < 3\rho\}$, and $O_j = \{|x-\bar{x}_{k\rfloor}| < 4\rho\}$ for some $\bar{x}_j \in \Gamma_0$.

(i-2)Set $\psi_j(t,y) := \varphi_j^{-1}(t,y) : \overline{B} \to \overline{U_j(t)}$. Then there exists a positive constant C depending only on n, r and R such that

$$||\varphi_j||_{C^{1,2+\alpha}([0,T]\times\overline{U_j})},\ ||\psi_j||_{C^{1,2+\alpha}([0,T]\times\overline{B})} \le C(1+||v||_{C^{1,2+\alpha}([0,T]\times\overline{D})}).$$

Proof. (i) The assertions essentially follow from the implicit function theorem. However, we shall state the proof for convenience to the reader.

For any $\bar{x}\in\Gamma_0$, there exists $i\in\{1,\cdots,n\}$ such that

$$\max_{1\le i\le n}|\partial_i v(0,\bar{x})| = |\partial_{i_0}v(0,\bar{x})| \ge r.$$

Without loss of generality we may assume that $i_0 = n$ and $\partial_n v(0,\bar{x}) \ge r > 0$. We set $\eta := \frac{r^2}{32R}$ and $V_{\bar{x}} := \{|x'-\bar{x}'| < \eta\}$, $W_{\bar{x}^{(n)}} := \{|x^{(n)}-\bar{x}^{(n)}| < \frac{4\eta}{r}\}$. Then it is easy to see that we have $U_{\bar{x}} := V_{\bar{x}'}\times W_{\bar{x}^{(n)}} \subset D$ and

$$\partial_n v(t,x) \ge \frac{r}{2},\ (t,x)\in[0,T]\times\overline{U}_{\bar{x}},$$

if T is sufficiently small.

Now consider the function of $x^{(n)}$ defined as $v^{(0,\bar{x}')}(x^{(n)}) := v(0,\bar{x}',x^{(n)})$. Then $v^{(0,\bar{x}')}$ is strictly monotone increasing and $v^{(0,\bar{x}')}(\bar{x}^{(n)}) = 0$. Thus, for $y\in\overline{W}_{\bar{x}^{(n)}}$,

$$y > \bar{x}^{(n)} \Leftrightarrow v(0,\bar{x}',y) = v^{(0,\bar{x}')}(y) > 0.$$

$$y < \bar{x}^{(n)} \Leftrightarrow v(0,\bar{x}',y) = v^{(0,\bar{x}')}(y) < 0.$$

Espetially, we have

$$\alpha_1 := v(0,\bar{x}',\bar{x}^{(n)}+\frac{4\eta}{r}) > 0 > v(0,\bar{x}',\bar{x}^{(n)}-\frac{4\eta}{r}) =: \alpha_2.$$

Each $|\alpha_i|$ is estimated from below as $|\alpha_i| \ge 4\eta$. Indeed,

$$\begin{aligned} v(0,\bar{x}',\bar{x}^{(n)}+\frac{4\eta}{r}) &= v(0,\bar{x}',\bar{x}^{(n)}+\frac{4\eta}{r}) - v(0,\bar{x}',\bar{x}^{(n)}) \\ &= \int_0^{\frac{4\eta}{r}} \partial_n v(0,\bar{x}',\bar{x}^{(n)}+s)ds \\ &\ge 4\eta. \end{aligned}$$

The estimate for α_2 is similarly obtained. From this, we can see that we have, for $(t,x')\in[0,T]\times\overline{V}_{\bar{x}'}$,

$$v(t,x',\bar{x}^{(n)}+\frac{4\eta}{r}) \ge 2\eta > 0 > -2\eta \ge v(t,x',\bar{x}^{(n)}-\frac{4\eta}{r}),$$

for sufficienrly small T.

Since the function $v^{(t,x')}(x^{(n)}) := v(t,x',\bar{x}^{(n)})$ is also strictly monotone increasing on $\overline{W}_{\bar{x}^{(n)}}$, for any $(t,x')\in[0,T]\times\overline{V}_{\bar{x}'}$ there exists a unique $x^{(n)}\in\overline{W}_{\bar{x}^{(n)}}$ such that

$$v(t,x',x^{(n)}) = 0.$$

We write this coorespodence as $x^{(n)} = g_{\bar{x}}(t, x')$. Note that

$$y > x^{(n)} \quad \Leftrightarrow \quad v(t, x', y) > 0, \tag{6.19}$$

$$y < x^{(n)} \quad \Leftrightarrow \quad v(t, x', y) < 0. \tag{6.20}$$

By definition, $\bar{x}^{(n)} = g_{\bar{x}}(0, x')$ and $v(t, x', g_{\bar{x}}(t, x')) = 0$ for $(t, x') \in [0, T] \times \overline{V}_{\bar{x}'}$. Inversely, if $(t, x', x^{(n)}) \in [0, T] \times \overline{V}_{\bar{x}'} \times \overline{W}_{\bar{x}^{(n)}}$ and $v(t, x', x^{(n)}) = 0$, then, since $v^{(t,x')}(x^{(n)})$ is strictly monotone increasing on $\overline{W}_{\bar{x}^{(n)}}$, we must have $x^{(n)} = g_{\bar{x}}(t, x')$. It is not difficult to check that $g_{\bar{x}}(t, x') \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T] \times \overline{U}_{\bar{x}})$. In fact, the first derivatives of $g_{\bar{x}}$ are given by

$$\partial_t g_{\bar{x}}(t, x') = -\frac{\partial_t v(t, x', g_{\bar{x}}(t, x'))}{\partial_n v(t, x', g_{\bar{x}}(t, x'))}, \tag{6.21}$$

$$\partial_i g_{\bar{x}}(t, x') = -\frac{\partial_i v(t, x', g_{\bar{x}}(t, x'))}{\partial_n v(t, x', g_{\bar{x}}(t, x'))}, \; 1 \le i \le n-1, \tag{6.22}$$

and these are estimated as

$$||\partial_t g||_{C([0,T]\times \overline{V}_{\bar{x}_j})}, ||\partial_x g||_{C([0,T]\times \overline{V}_{\bar{x}_j})} \le \frac{2R}{r}.$$

Let $N \ge 1$ be a sufficiently large number to be precised later. Since Γ_0 is compact, there exists a sequence $\{\bar{x}_j\}_{j=1}^m \subset \Gamma_0$ such that

$$\Gamma_0 \subset \cup_{j=1}^m \{|x - \bar{x}_j| < \frac{\eta}{2N}\} \subset \cup_{j=1}^m U_{\bar{x}_j}.$$

Moreover, there exists a positive number $\rho > 0$ such that

$$(\Gamma_0)_\rho \subset \cup_{j=1}^m \{|x - \bar{x}_j| < \frac{3\eta}{2N}\}, \tag{6.23}$$

where $(\Gamma_0)_\rho = \{x \in D; \; \mathrm{dist}(x, \Gamma_0) < \rho\}$.

In fact, we can take $\rho = \frac{\eta}{N}$. To see this, we take any $x \in (\Gamma_0)_\rho$, ρ is given as above. Then, there exists a point $z \in \Gamma_0$ such that

$$|x - z| = \mathrm{dist}(x, \Gamma_0).$$

Since $\Gamma_0 \subset \cup_{j=1}^m \{|x - \bar{x}_j| < \frac{\eta}{2N}\}$, for some j, we have $z \in \{|x - \bar{x}_j| < \frac{\eta}{2N}\}$. Hence,

$$|x - \bar{x}_j| \le |x - z| + |z - \bar{x}_j| < \frac{3\eta}{2N},$$

and the claim follows.

We also have for sufficiently small T,

$$\cup_{0 \le t \le T} \Gamma_t \subset (\Gamma_0)_\rho. \tag{6.24}$$

To see this, note that since $\overline{D \backslash (\Gamma_0)_\rho}$ is compact, we have

$$r' := \min\{|v(0, x)|; \; x \in \overline{D \backslash (\Gamma_0)_\rho}\} > 0.$$

So it follows that

$$\{x \in \overline{D}; \; |v(0, x)| < r'\} \subset (\Gamma_0)_\rho.$$

Since each $x \in \Gamma_t$ satisfies $v(t, x) = 0$, we have

$$|v(0, x)| = |v(0, x) - v(t, x)| \le RT < r',$$

if T is sufficiently small. This proves that $\cup_{0 \le t \le T} \Gamma_t \subset (\Gamma_0)_\rho$.

Next we shall show that for all $t \in (0, T]$, it follows that

$$(\Gamma_t)_\rho \subset\subset \cup_{j=1}^m \hat{O}_j, \tag{6.25}$$

where $\hat{O}_j := \{|x - \bar{x}_j| < \frac{3\eta}{N}\}$. Indeed, for $x \in (\Gamma_t)_\rho$, we have $z \in \Gamma_t$ such that $|x - z| = \mathrm{dist}(x, \Gamma_t) < \rho$. Since $\Gamma_t \subset (\Gamma_0)_\rho \subset \cup_{j=1}^m \{|x - \bar{x}_j| < \frac{3\eta}{2N}\}$, for some j, we have

$$|x - \bar{x}_j| \leq |x - z| + |z - \bar{x}_j| < \frac{5\eta}{2N},$$

which shows the above claim.

Now let $\psi_j(t, y) = (\psi_j^{(1)}(t, y), \cdots, \psi_j^{(n)}(t, y))$ be a function on $[0, T] \times \overline{B}$ defined as follows.

$$\psi_j^{(i)}(t, y) := \frac{\eta}{2R} y^{(i)} + \bar{x}_j^{(i)}, \ 1 \leq i \leq n - 1, \tag{6.26}$$

$$\psi_j^{(n)}(t, y) := \frac{\eta}{2R} y^{(n)} + g_{\bar{x}_j}(t, \frac{\eta}{2R} y' + \bar{x}_j'). \tag{6.27}$$

Remark that since $\frac{\eta}{2R} y' + \bar{x}_j' \in V_{\bar{x}_j'}$ for $|y'| \leq 1$, ψ_j is well-defined. From now on, we write g_j instead of $g_{\bar{x}_j}$. Since we have

$$\begin{aligned} |\psi_j^{(n)}(t, y) - \bar{x}_j^{(n)}| &\leq \frac{\eta}{2R} |y^{(n)}| + |g_j(t, \frac{\eta}{2R} y' + \bar{x}_j') - g_j(0, \bar{x}_j')| \\ &\leq \frac{\eta}{2R} + ||\partial_x g_j||_{C([0,T] \times \overline{V}_{\bar{x}_j})} \frac{\eta}{2R} + T ||\partial_t g_j||_{C([0,T] \times \overline{V}_{\bar{x}_j})} \\ &\leq \frac{\eta}{2R} + \frac{2R}{r} (\frac{\eta}{2R} + T), \end{aligned}$$

if we take $T \leq \frac{\eta}{2R}$, then $\psi_j^{(n)}(t, y) \in W_{\bar{x}_j^{(n)}}$ for all $y \in B$. Thus $\psi_j(t, B) \subset U_{\bar{x}_j}$ and we have the inverse function of ψ_j which is given as

$$\varphi_j^{(i)}(t, x) := \frac{2R}{\eta} (x^{(i)} - \bar{x}_j^{(i)}), \ 1 \leq i \leq n - 1, \tag{6.28}$$

$$\varphi_j^{(n)}(t, x) := \frac{2R}{\eta} (x^{(n)} - g_j(t, x')). \tag{6.29}$$

Note that the function φ_j can be defined on $[0, T] \times \overline{U}_{\bar{x}_j}$. Obviously $\varphi_j : \overline{U_j(t)} := \psi_j(t, \overline{B}) \to \overline{B}$ is C^2-diffeomorphism. Now we claim that if N is sufficiently large and T is sufficiently small, then we have $O_j := \{x;\ |x - \bar{x}_j| < \frac{4\eta}{N}\} \subset\subset \{x;\ |\varphi_j(t, x)| \leq 1\} (= \psi_j(t, \overline{B}))$ for all $t \in [0, T]$. Indeed, if $x \in O_j$, then

$$\begin{aligned} |\varphi_j(t, x)| &\leq \frac{2R}{\eta} |x' - \bar{x}_j'| + \frac{2R}{\eta} |x^{(n)} - g_j(t, x')| \\ &< \frac{2R}{\eta} \cdot \frac{4\eta}{N} \\ &\quad + \frac{2R}{\eta} (|x^{(n)} - \bar{x}_j^{(n)}| + |\bar{x}_j^{(n)} - g_j(t, \bar{x}_j')| + |g_j(t, \bar{x}_j') - g_j(t, x')|) \\ &\leq \frac{16R}{N} + \frac{8R ||\partial_x g_j||_{C([0,T] \times \overline{V}_{\bar{x}_j'})}}{N} + \frac{2RT ||\partial_t g_j||_{C([0,T] \times \overline{V}_{\bar{x}_j^{(n)}})}}{\eta} \\ &\leq \frac{16R}{N} (1 + \frac{R}{r}) + \frac{4R^2 T}{r\eta}, \end{aligned}$$

which proves the claim with $N = 32R(1 + \frac{R}{r})$ and small T. Combining these above, we can see that Proposition 6.1 (i) holds. The estimates for φ_j and ψ_j follow from (6.21), (6.22), (6.26), (6.27), (6.28) and (6.29).

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