Isometric Composition Operators Between Two Weighted Hardy Spaces

By

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Abstract. We study isometric composition operators C_ϕ between two weighted Hardy spaces $H^2(\nu)$ and $H^2(\mu)$ when ν is a radial measure. The isometric C_ϕ is related to a moment sequence and such a ϕ is studied by the Nevanlinna counting function of ϕ when μ is the normalized Lebesgue measure on the unit circle.

§1. Introduction

Let D be the open unit disc in the complex plane $\mathbb{C}$. We denote by $\mathcal{P}$ the set of all analytic polynomials and H the set of all analytic functions on D. Let μ be a positive Borel measure on $\overline{D}$ with $\mu(\overline{D}) = 1$. $H^p(\mu)$ denotes the closure of all analytic polynomials in $L^p(d\mu)$ for $0 < p < \infty$. If $d\mu = d\theta/2\pi$, then $H^p(\mu) = H^p$ is the classical Hardy space. If $d\mu = 2rdrd\theta/2\pi$, then $H^p(\mu) = L^p_a$ is the classical Bergman space. H^p and L^p_a can be embeded in H. In this paper, we assume that $H^p(\mu)$ is embeded in H for a general μ. H^∞ denotes the set of all bounded analytic functions on D. We also assume that $H^\infty = H \cap L^\infty(d\mu)$.

For an analytic self map ϕ of D, the composition operator C_ϕ is defined by $(C_\phi f)(z) = f(\phi(z)) \ \ (z \in D)$ for f in H. Throughout this paper, we assume that ν and μ are positive Borel measures on $\overline{D}$ with $\nu(\overline{D}) = \mu(\overline{D}) = 1$. ν is called a radial measure if $d\nu = d\nu_0(r)d\theta/2\pi$ for a positive Borel measure ν_0 on $[0, 1]$. Since $d\theta/2\pi = d\delta_{r=1}d\theta/2\pi$, $d\theta/2\pi$ is a radial measure.

In this paper, we studied isometric composition operators from $H^2(\nu)$ into $H^2(\mu)$ when ν is a radial measure. As we show in the final section, our isometric composition operator C_ϕ is related to an isometric operator T from $H^p(\nu)$ into $H^p(\mu)$ with $T1 = 1$ when $p \neq 2$. We have a long history for such isometric operators (see [8]). The onto isometries on H^p or L^p_a for $p \neq 2$ were described completely. Unfortunately into isometries have been known very little.

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Problem 1. *For given measures ν and μ, does there exist an isometric composition operator C_ϕ from $H^2(\nu)$ into $H^2(\mu)$? If there exists such a C_ϕ, describe ϕ.*

A function F in $H^2(\mu)$ is called an inner function in $H^2(\mu)$ if

$$\int_{\overline{D}} f|F|^2 d\mu = \int_{\overline{D}} f d\mu \int_{\overline{D}} |F|^2 d\mu \quad (f \in \mathcal{P}).$$

If ϕ^n is an inner function in $H^2(\mu)$ with $\int_{\overline{D}} \phi d\mu = 0$ for any $n \geq 0$ then there exists a unique radial measure ν such that C_ϕ is isometric from $H^2(\nu)$ into $H^2(\mu)$ where $d\nu = d\nu_0(r)d\theta/2\pi$ and $1 \in$ supp ν_0. This is not difficult to prove. However we don't know whether the converse is true.

Problem 2. *If a composition operator C_ϕ is isometric from $H^2(\nu)$ into $H^2(\mu)$ then is ϕ^n an inner function in $H^2(\mu)$ with $\int_{\overline{D}} \phi d\mu = 0$ for any $n \geq 0$?*

A function ϕ in H^∞ with $\|\phi\|_\infty = 1$ is called a Rudin's orthogonal function in $H^2(\mu)$ if $\{\phi^n; n = 0, 1, 2, \cdots\}$ is a set of orthogonal functions in $H^2(\mu)$. If ϕ^n is an inner function in $H^2(\mu)$ with $\int_{\overline{D}} \phi d\mu = 0$ for any $n \geq 0$ and $\|\phi\|_\infty = 1$ then ϕ is a Rudin's orthogonal function in $H^2(\mu)$ because $\mathcal{P}$ is dense in $H^2(\mu)$ by its definition. We can ask whether the converse is true or not.

Problem 3. *If ϕ is a Rudin's orthogonal function in $H^2(\mu)$ then is ϕ^n an inner function in $H^2(\mu)$ with $\int_{\overline{D}} \phi d\mu = 0$ for any $n \geq 0$?*

When $d\mu = d\theta/2\pi$, Problem 3 was studied by several people, for example, [2],[3],[5],[6] and [10]. C. Bishop [2] and C. Sundberg [10] gave counter examples. Hence there exists a Rudin's orthogonal function which is not an inner function in $H^2(d\theta/2\pi)$.

Problem 3 has a strong conection with Problem 2. In fact, if C_ϕ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$ then by Theorem 1 ϕ is a Rudin's orthogonal function in $H^2(\mu)$. Conversely if ϕ is a Rudin's orthogonal function in $H^2(\mu)$ then by Proposition 8 there exists a unique radial measure ν such that C_ϕ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$.

For each ϕ, we will use two Borel measures μ_ϕ on $\overline{D}$ and $\mu_{|\phi|}$ on [0,1]. For a Borel set E in $\overline{D}$ $\mu_\phi(E) = \mu(\{z \in \overline{D}; \phi(z) \in E\})$ and for a Borel set G in [0,1] $\mu_{|\phi|}(G) = \mu(\{z \in \overline{D}; | \phi(z) | \in G\}$.

§2. General case

In this section we assume that ν is a radial measure, μ is an arbitrary measure and ϕ is an analytic selfmap with $\|\phi\|_\infty = 1$. We say that $\{a_n\}$ is a moment sequence of ν_0, a positive Borel measure on [0,1], if $a_n = \int_0^1 r^n d\nu_0 \quad (n = 0, 1, 2, \cdots)$.

Theorem 1. *Suppose $d\nu = d\nu_0(r)d\theta/2\pi$. Then C_ϕ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$ if and only if $\int_{\overline{D}} \phi^n \bar{\phi}^m d\mu = 0$ $(n \neq m)$ and $\left\{\int_{\overline{D}} |\phi|^n d\mu\right\}$ is a moment sequence of ν_0.*

Proof. If C_ϕ is isometric, by the polarization formula

$$\delta_{nm} \int_0^1 r^n r^m d\nu_0(r) = \int_{\overline{D}} z^n \bar{z}^m d\nu = \int_{\overline{D}} \phi^n \bar{\phi}^m d\mu$$

because ν is a radial measure. Hence

$$\int_{\overline{D}} |\phi|^{2n} d\mu = \int_0^1 r^{2n} d\nu_0 \quad (n = 0, 1, 2, \cdots).$$

It is elementary to see that $x = \sqrt{1 - (1 - x^2)} = \sum_{n=0}^{\infty} a_n (1 - x^2)^n$ and $\sum_{n=0}^{\infty} |a_n| (1 - x^2)^n < \infty \quad (0 \leq x \leq 1)$. Hence by Lebesgue's dominated convergence theorem

$$\begin{aligned} \int_{\overline{D}} |\phi| d\mu &= \int_{\overline{D}} \sum_{n=0}^{\infty} a_n (1 - |\phi|^2)^n d\mu = \sum_{n=0}^{\infty} a_n \int_{\overline{D}} (1 - |\phi|^2)^n d\mu \\ &= \sum_{n=0}^{\infty} a_n \int_0^1 (1 - r^2)^n d\nu_0 = \int_0^1 \sum_{n=0}^{\infty} a_n (1 - r^2)^n d\nu_0 = \int_0^1 r d\nu_0 \end{aligned}$$

because $\left|\sum_{n=0}^{k} a_n (1 - |\phi|^2)^n\right| \leq \sum_{n=0}^{\infty} |a_n|$ and $\left|\sum_{n=0}^{k} a_n (1 - r^2)^2\right| \leq \sum_{n=0}^{\infty} |a_n| < \infty$. Similarly, as $x^{2\ell+1} = \sqrt{1 - (1 - x^{4\ell+2})}$ we can show that $\int_{\overline{D}} |\phi|^{2n+1} d\mu = \int_0^1 r^{2n+1} d\nu_0 \quad (n = 0, 1, 2, \cdots)$. Thus $\left\{\int_{\overline{D}} |\phi|^n d\mu\right\}$ is a moment sequence of ν_0.

Conversely if $\int_{\overline{D}} \phi^n \bar{\phi}^m d\mu = 0$ $(n \neq m)$ and $\left\{\int_{\overline{D}} |\phi|^n d\mu\right\}$ is a moment sequence of ν_0, then

$$\begin{aligned} &\int_{\overline{D}} \left|\sum_{n=0}^{k} a_n \phi^n\right|^2 d\mu = \sum_{n=0}^{k} |a_n|^2 \int_{\overline{D}} |\phi|^{2n} d\mu \\ &= \sum_{n=0}^{k} |a_n|^2 \int_0^1 r^{2n} d\nu_0 = \int_{\overline{D}} \left|\sum_{n=0}^{k} a_n z^n\right|^2 d\nu. \end{aligned}$$

Hence C_ϕ is isometric. $\square$

Theorem 2. *If $d\nu = d\nu_0(r)d\theta/2\pi$ then the following conditions are equivalent.*
(1) C_ϕ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$.
(2) $\nu_0 = \mu_{|\phi|}$
(3) $\int_0^1 F(r)d\nu_0 = \int_{\overline{D}} F(|\phi|)d\mu$ for any Borel nonnegative function F on [0,1].

Proof. (1) $\Rightarrow$ (2) If G is a Borel set in [0,1], then $\nu_0(G) = \inf\{\nu_0(V)\ ;\ G \subset V,\ V$ is open in [0,1]$\}$ because ν_0 is a Borel measure. Hence there exists a sequence of continuous functions $\{f_m\}$ such that $f_m \to \chi_G$ *a.e.* ν_0 on [0,1] and $\|f_m\|_\infty \le \gamma < \infty$ $(m = 1, 2, \cdots)$. By the Stone-Weierstrass theorem,

$$\int_0^1 f_m(r)d\nu_0 = \int_{\overline{D}} f_m(|\phi|)d\mu \quad (m = 1, 2, \cdots)$$

because $\int_0^1 r^n d\nu_0 = \int_{\overline{D}} |\phi|^n d\mu \quad (n = 0, 1, 2, \cdots)$. Thus $\nu_0(G) = \mu(\{z \in \overline{D}\ ;\ |\phi(z)| \in G\})$.
(2) $\Rightarrow$ (3) is clear. (3) $\Rightarrow$ (1) is a result of Theorem 1. $\square$

The following theorem shows that we can solve Problem 2 in the Introduction when C_ϕ is onto.

Theorem 3. *Suppose $d\nu = d\nu_0(r)d\theta/2\pi$. If C_ϕ is an isometric operator from $H^2(\nu)$ onto $H^2(\mu)$ then ϕ^n is an inner function in $H^2(\mu)$ for any $n \ge 0$.*

Proof. Let $F \in \mathcal{P}$ then there exists $f \in H^2(\nu)$ such that $F = f \circ \phi$. Let $f = \sum_{j=0}^\infty a_j z^j$, since $\sum_{j=0}^\infty |a_j|^2 \int_0^1 r^{2j} d\nu_0(r) < \infty$, $F = \sum_{j=0}^\infty a_j \phi^j$ and $\sum_{j=0}^\infty |a_j|^2 \int_{\overline{D}} |\phi|^{2j} d\mu < \infty$. By Theorem 1, for any $\ell \ge 0$

$$\int_{\overline{D}} F|\phi|^{2\ell} d\mu = a_0 \int_{\overline{D}} |\phi|^{2\ell} d\mu = \int_{\overline{D}} F d\mu \int_{\overline{D}} |\phi|^{2\ell} d\mu$$

because $\int_{\overline{D}} \phi d\mu = 0$. This implies that ϕ^ℓ is an inner function in $H^2(\mu)$ for any $\ell \ge 0$.

When $d\nu = d\nu_0(r)d\theta/2\pi$, if C_ϕ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$ then C_z is isometric from $H^2(\nu)$ onto $H^2(\mu_\phi)$.

Corollary 1. *Suppose $d\nu = d\nu_0(r)d\theta/2\pi$. If C_z is an isometric operator then z^n is an inner function in $H^2(\mu)$ for any $n \ge 0$. Moreover $d\mu = d\nu_1(r)d\theta/2\pi + d\delta_{r=0}d\mu_1(\theta)$. where ν_1 is a Borel measure on [0,1] and μ_1 is a Borel measure on ∂D. If ν_0 does not have point mass on $\{r = 0\}$ then $\nu = \mu$.*

Proof. By the remark above, C_z is isometric from $H^2(\nu)$ onto $H^2(\mu)$ because $\mu_z = \mu$. By Theorem 3, z^n is inner in $H^2(\mu)$ for any $n \ge 0$. Put $C_0[0, 1] = \{u; u$ is continuous on

[0,1] and $u(0)=0\}$ and $C_0(\partial D)=\{f; f$ is continuous on ∂D and $f(1)=0\}$. Since $r^n d\mu$ annihilates $z\mathcal{P}+\bar{z}\bar{\mathcal{P}}$ for any $n\geq 0$, for any $j\neq 0$, $d\mu\perp\{r^{2n+|j|}e^{ij\theta}; n=0,1,2,\cdots\}$. By the Müntz-Szasz theorem [6], $d\mu\perp C_0[0,1]e^{ij\theta}$ for any $j\neq 0$ and so $d\mu\perp C_0[0,1]\otimes C_0(\partial D)$. This implies that $d\mu = d\nu_1(r)d\theta/2\pi + d\delta_{r=0}d\mu_1(\theta)$ where ν_1 is a Borel measure on [0,1] and μ_1 is a Borel measure on T. If ν_0 does not have point mass on $\{r=0\}$ then we may assume that $\mu_1=0$ and so $d\mu = d\nu_1(r)d\theta/2\pi$. By Theorem 2 $\nu_0=\mu_{|z|}$ and $\mu_{|z|}=\nu_1$ because $d\mu=d\nu_1(r)d\theta/2\pi$. $\square$

§3. Radial measure

In this section we assume that ν and μ are radial measures, that is, $d\nu = d\nu_0(r)d\theta/2\pi$ and $d\mu=d\mu_0(r)d\theta/2\pi$. Proposition 1 solves Problem 2 when $\nu=\mu$. By Theorem 2, if C_ϕ is isometric from $H^2(\nu)$ into $H^2(\mu)$, then for some positive integer k

$$\int_0^1 \log r\, d\nu_0 \leq k\int_0^1 \log r\, d\mu_0 + \int_0^{2\pi}\log|\phi(e^{i\theta})|d\theta$$

as $F(t)=\log t$, using the inner outer factorization of ϕ. Proposition 2 gives an exact formula for this.

Proposition 1. *Suppose ν is a radial measure. If C_ϕ is an isometric operator from $H^2(\nu)$ into $H^2(\nu)$, then ϕ^n is an inner function in $H^2(\nu)$ for any $n\geq 0$.*

Proof. By Theorem 1, $\phi(0)=0$ because ν is a radial measure and so by Schwarz's lemma, $|\phi(z)|\leq|z|\ \ (z\in D)$. Since $\int_{\bar{D}}|\phi(z)|^2 d\nu = \int_{\bar{D}}|z|^2 d\nu$, $|\phi(z)|=|z|$ *a.e.* ν. For $f\in\mathcal{P}$,

$$\int_{\bar{D}} f|\phi|^{2n}d\nu = \int_{\bar{D}} f|z|^{2n}d\nu = f(0)\int_0^1 r^{2n}d\nu_0 = \int_{\bar{D}} f d\nu\int_{\bar{D}}|\phi|^{2n}d\nu. \ \square$$

Proposition 2. *Suppose ν and μ are radial measures, that is, $d\nu = d\nu_0(r)d\theta/2\pi$ and $d\mu = d\mu_0(r)d\theta/2\pi$. Let $\phi = z^k BQh$ where k is a positive integer, B is a Blaschke product with $B(0)\neq 0$, $Q(z)=\exp -\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}d\lambda(t)$ is a singular inner function and h is an outer function. If C_ϕ is an isometric oprator from $H^2(\nu)$ into $H^2(\mu)$, then*

$$\int_0^1 \log r\, d\nu_0 = k\int_0^1\log r\, d\mu_0 + \int_0^1 d\mu_0\int_0^r n(s,B)\frac{ds}{s} + \log|B(0)| - \mu_0([0,1))\lambda([0,2\pi]) + \int_0^{2\pi}\log|\phi(e^{i\theta})|d\theta/2\pi$$

where $n(s,B)$ is the number of zeros of B on the closed disc $\{z\in\mathbb{C}\ ;\ |z|\leq r\}$.

Proof. Let $n(s,B)=n(s,BQh)$ is the number of zeros of BQh on the closed disc $\{z\in\mathbb{C}\ ;\ |z|\le r\}$. Then, by Theorem 2 and [1, §2 of Chapter 5]

$$\begin{aligned}
&\int_0^1 \log r d\nu_0\\
&\quad= \int_0^{1-} d\mu_0 \int_0^{2\pi} \log|\phi(re^{i\theta})|d\theta/2\pi + \mu_0(\{1\})\int_0^{2\pi}\log|\phi(e^{i\theta})|d\theta/2\pi\\
&\quad= \int_0^{1-} d\mu_0 \left\{\log r^k + \int_0^r n(s,B)\frac{ds}{s}\right\} + \mu_0([0,1))\log|B(0)Q(0)h(0)|\\
&\qquad + \mu_0(\{1\})\int_0^{2\pi}\log|\phi(e^{i\theta})|d\theta/2\pi\\
&\quad= k\int_0^1 \log r d\mu_0 + \int_0^1 d\mu_0\int_0^r n(s,B)\frac{ds}{s} + \log|B(0)|\\
&\qquad - \mu_0([0,1))\lambda([0,2\pi]) + \int_0^{2\pi}\log|\phi(e^{i\theta})|d\theta/2\pi
\end{aligned}$$

because $\mu_0(\{1\})\int_0^1 n(s,B)\frac{ds}{s} = -\mu_0(\{1\})\log|B(0)|$. $\square$

§4. Special cases

In this section we assume that ν or μ is the normalized Lebesgue measure or the normalized area measure. Proposition 3 solves Problems 1 and 2 when ν is the normalized Lebesgue measure on the circle and μ is a radial measure. Proposition 5 solves Problem 2 when ν is a radial measure or the Lebesgue measure on the circle. Corollary 3 solves Problem 2 negatively when $d\nu = 2rdrd\theta/2\pi$ and $d\mu = d\theta/2\pi$.

Proposition 3. *Let μ be a radial measure. C_ϕ is an isometric operator from H^2 into $H^2(\mu)$ if and only if ϕ^n is an inner function with $\int_{\overline{D}}\phi d\mu = 0$ in $H^2(\mu)$ for any $n\ge 1$ and $H^2(d\mu)=H^2$.*

Proof. If C_ϕ is isometric, by Theorem 1 $\int_{\overline{D}}\phi^n\bar{\phi}^m d\mu = 0$ $(n\ne m)$ and we have

$$1 = \int_0^{2\pi}|z|^2 d\theta/2\pi = \int_{\overline{D}}|\phi|^2 d\mu \le 1.$$

Hence $|\phi(z)|=1$ *a.e.* μ and so supp $\mu\subset\partial D$. This implies that $d\mu = d\delta_{r=1}d\theta/2\pi$ because μ is a radial measure. The converse is clear. $\square$

Proposition 4. *Suppose $d\nu = d\nu_0(r)d\theta/2\pi$ and C_ϕ is an isometric operator from $H^2(\nu)$ into H^2.*

(1) $\nu_0(\{a\})>0$ for $0\le a\le 1$ if and only if $d\theta/2\pi(\{e^{i\theta};|\phi(e^{i\theta})|=a\})>0$.

(2) $d\nu_0 = d\delta_{r=1}$ *if and only if* ϕ *is an inner function in* H^2*.*

(3) ν_0 *is a discrete measure if and only if* $|\phi| = \sum_{n=1}^{\infty} a_n \chi_{E_n}$ *where* $0 \leq a_n \leq 1$*, and* $d\theta/2\pi(E_n) = \nu_0(\{a_n\})$ $(n = 1, 2, \cdots)$.

Proof. Since $\nu_0(G) = d\theta/2\pi\{e^{i\theta}; |\phi(e^{i\theta})| \in G\})$ for a Borel set G in [0,1] by Theorem 2, it is easy to see. □

Proof. This is just (2) of Proposition 4.

Now we consider when $d\nu = rdrd\theta/\pi$ or $d\mu = rdrd\theta/\pi$.

Proposition 5. *If* C_ϕ *is an isometric operator from* L_a^2 *into* $H^2(\mu)$*, then* $\mu(\{z \in \overline{D}; |\phi| = b\} = 0$ *and* $\int_{\overline{D}} (b - |\phi|)^{-1} d\mu = \infty$ *for any* $0 \leq b \leq 1$*.*

Proof. It is clear by Theorem 2. □

Corollary 2. *If* C_ϕ *is an isometric operator from* L_a^2 *into* H^2*, then* ϕ *is not inner in* H^2*.*

Proposition 6. *Suppose* $d\nu = d\nu_0(r)d\theta/2\pi$*. If* C_ϕ *is an isometric operator from* $H^2(\nu)$ *into* L_a^2*, then* $\int_0^1 \log r d\nu_0 = -\frac{k}{4} + \int_0^1 2rdr \int_0^r n(s, B)\frac{ds}{s} + \log|B(0)| - \lambda([0, 2\pi]) + \int_0^{2\pi} \log|\phi(e^{i\theta})| d\theta/2\pi$*, where the inner part of* ϕ *is* $z^k BQ$*,* B *is a Blaschke procuct with* $B(0) \neq 0$*,* $Q(z) = \exp - \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\lambda$ *is a singular inner function. Hence if* ϕ *is a shricht function, then* $\int_0^1 \log r d\nu_0 = -\frac{1}{4} + \int_0^{2\pi} \log|\phi(e^{i\theta})| d\theta/2\pi$*.*

Proof. It is clear by Proposition 2. □

§5. Nevanlinna counting function

Suppose ν or μ is the normalized Lebesgue measure or the normalized area measure. We assume that ϕ is a non-constant function in H^∞ with $\|\phi\|_\infty = 1$. The Nevanlinna counting function of ϕ, N_ϕ, is defined on $D \backslash \{\phi(0)\}$ by

$$N_\phi(w) = \sum_{\phi(z)=w} \log \frac{1}{|z|},$$

where multiplicities are counted and $N_\phi(w)$ is taken to be zero if w is not in the range of ϕ. Corollary 4 seems to be interesting in spite of Corollary 3.

Theorem 4. *Suppose $d\nu = d\nu_0(r)d\theta/2\pi$. Then, C_ϕ is an isometric operator from $H^2(\nu)$ into H^2 if and only if*

$$N_\phi(z) = \int_{|z|}^{1} \log \frac{r}{|z|} d\nu_0(r)$$

for nearly all z in D.

Proof. The 'only if' part was proved in [6, Lemma 3]. If $N_\phi(z) = \int_{|z|}^{1} \log \frac{r}{|z|} d\nu_0(r)$ for nearly all z in D, by the Littlewood-Paley theorem (see [3]),

$$\begin{aligned}
&\int_0^{2\pi} \phi^n(e^{i\theta})\bar{\phi}^m(e^{i\theta})d\theta/2\pi \\
&= 2nm \int_D z^{n-1}\bar{z}^{m-1} N_\phi(|z|)dA(z) \\
&= 4nm\delta_{nm} \int_0^1 r^{n+m-1} \left(\int_r^1 \log \frac{s}{r} d\nu_0(s) \right) dr \\
&= 4nm\delta_{nm} \int_0^1 d\nu_0(s) \int_0^s r^{m+n-1} \left(\log \frac{s}{r} \right) dr \\
&= \frac{4nm}{(n+m)^2} \delta_{nm} \int_0^1 s^{n+m} d\nu_0(s).
\end{aligned}$$

When $n = m$, $\int_0^{2\pi} |\phi(e^{i\theta})|^{2n} d\theta/2\pi = \int_0^1 s^{2n} d\nu_0(s)$ for $n = 0, 1, 2, \cdots$. Hence by Theorem 1 and its proof, C_ϕ is an isometric operator from $H^2(\nu)$ into H^2.

Lemma. *$D \setminus \{z \in D \ ; \ \phi'(z) = 0\}$ can be decomposed into an at most countable disjoint collection $\{R_n\}$ of "semi-closed" polar rectangles, on each of which ϕ is schricht.*

Proof. It is known in [9, p186]. $\square$

Corollary 3. *Suppose ϕ is a finite-to-one map. Then C_ϕ is not an isometric operator from L_a^2 into H^2.*

Proof. By Lemma, there exists the inverse ψ_n of the restriction of ϕ to R_n. Let $w \in \phi(R_{j_1})$. If ϕ is an ℓ to 1 map, then there exist $j_2, \cdots, j_\ell$ such that $\psi_{j_1}(z) = \psi_{j_2}(z) = \cdots = \psi_{j_\ell}(z) = w$. Hence there exists a small disc $\triangle$ in $\phi(R_{j_1})$ such that

$$N_\phi(z) = \sum_{z=\phi(w)} \log \frac{1}{|w|} = \sum_{t=1}^{\ell} \log \frac{1}{|\psi_{j_t}(z)|}$$

for all $w \in \triangle$. Therefore there exists a subdisc $\triangle_0$ in $\triangle$ such that $N_\phi(z)$ is harmonic on $\triangle_0$. On the other hand, by Proposition 5

$$N_\phi(z) = 2\int_{|z|}^{1} \left(\log \frac{r}{|z|} \right) rdr = \frac{|z|^2 - 1}{2} + \log \frac{1}{|z|}.$$

This contradicts that $N_\phi(z)$ is harmonic on $\triangle_0$. $\square$

Theorem 5. *Suppose ϕ is a contractive function in H^∞ such that ϕ is a finite-to-one map and $|\phi| = \sum_{j=1}^{\ell} a_j \chi_{E_j}$ where $0 < a_j \lneq a_{j+1}$, $\sum_{j=1}^{\ell} \chi_{E_j} = 1$ and E_j is a measurable set in ∂D where $1 \le \ell \le \infty$. If the inner part of $z - \phi$ is a Blaschke product for each $z \in D$, then C_ϕ is not an isometric operator from $H^2(\nu)$ into H^2 for any $d\nu = d\nu_0(r)d\theta/2\pi$ if $\ell \ne 1$.*

Proof. Suppose C_ϕ is an isometric operator from $H^2(\nu)$ into H^2 for some $d\nu = d\nu_0(r)d\theta/2\pi$. By Proposition 4, ν_0 is a discrete measure and $d\theta/2\pi(E_j) = \nu_0(\{a_j\})$ $(j = 1, 2, \cdots)$. Since $\phi(0) = 0$, by Lemma 2 in [6] and Proposition 7

$$\begin{aligned} N_\phi(z) &= \int_0^{2\pi} \log|z - \phi(e^{i\theta})| d\theta/2\pi + \log\frac{1}{|z|} \\ &= \int_{|z|}^1 \log\frac{r}{|z|} d\nu_0(r) \end{aligned}$$

for $z \in D\backslash\{0\}$. If $|z| \le a_1$, then

$$\begin{aligned} \int_{|z|}^1 \log\frac{r}{|z|} d\nu_0(r) &= \sum_{j=1}^{\infty} \left(\log\frac{a_j}{|z|}\right) \nu_0(\{a_j\}) \\ &= \sum_{j=1}^{\infty} \nu_0(\{a_j\}) \log\frac{1}{|z|} + \sum_{j=1}^{\infty} \nu_0(\{a_j\}) \log a_j \\ &= \log\frac{1}{|z|} + \sum_{j=1}^{\infty} \nu_0(\{a_j\}) \log a_j. \end{aligned}$$

Hence if $|z| \le a_1$ then

$$\int_0^{2\pi} \log|z - \phi(e^{i\theta})| d\theta/2\pi = \sum_{j=1}^{\infty} \nu_0(\{a_j\}) \log a_j = \alpha.$$

If $a_1 < |z| \le a_2$, then

$$\begin{aligned} \int_{|z|}^1 \log\frac{r}{|z|} d\nu_0(r) &= \sum_{j=2}^{\infty} \left(\log\frac{a_j}{|z|}\right) \nu_0(\{a_j\}) \\ &= \sum_{j=2}^{\infty} \nu_0(\{a_j\}) \log\frac{1}{|z|} + \sum_{j=2}^{\infty} \nu_0(\{a_j\}) \log a_j \end{aligned}$$

$$\begin{aligned} \int_0^{2\pi} \log|z - \phi(e^{i\theta})| d\theta/2\pi &= -\nu_0(\{a_1\}) \log\frac{1}{|z|} + \sum_{j=2}^{\infty} \nu_0(\{a_j\}) \log a_j \\ &= \beta \log\frac{1}{|z|} + \gamma. \end{aligned}$$

where $\beta \neq 0$

For each $z \in D$, put

$$z - \phi(\zeta) = q_z(\zeta) h_z(\zeta) \quad (\zeta \in D)$$

where $q_z(\zeta)$ is inner and $h_z(\zeta)$ is outer. Since ϕ is a finite-to-one map, q_z is a finite Blaschke product by hypothesis and so

$$q_{\phi(t)}(\zeta) = \prod_{j=1}^{n} \frac{\zeta - b_j(t)}{1 - \overline{b_j(t)}\zeta} \quad (t \in D).$$

Then, since $\phi(0) = 0$,

$$\phi(t) = (-1)^n \left(\prod_{j=1}^{n} b_j(t) \right) h_{\phi(t)}(0) \quad (t \in D).$$

Put $D_r = \{t \in \mathbb{C}; |t| \leq r\}$ for $0 < r < 1$. If both ϕ and ϕ' have no zeros on ∂D_r then there is a division $\{D_r^j\}_{1 \leq j \leq n}$ of D_r such that ϕ is one-to-one on D_r^j for $1 \leq j \leq n$. For, ϕ is conformal in a neighborhood of each point on ∂D_r and so $\arg \phi$ is increasing on, ∂D_r. Put $\phi_j = \phi \mid D_r^j$ and $b_j(t) = \phi_j^{-1}(\phi(t))$ for $1 \leq j \leq n$. Then $b_j(t)$ is analytic except $\phi'(t) = 0$ when $\phi(t)$ in $\phi(D_r)$. Hence $h_{\phi(t)}(0)$ is analytic except $\phi'(t) = 0$ and $\displaystyle\bigcup_{j=1}^{n} \{t \in D; b_j(t) = 0\}$ when $\phi(t)$ in $\phi(D_r)$. Since $\phi(0) = 0, \{t \in D; |\phi(t)| < a_1\}$ is a nonempty open set. We can choose r such that $\{t \in D; |\phi(t)| < a_1\} \cap \phi(D_r) \neq \emptyset$. If $|\phi(t)| \leq a_1$, by what was proved above,

$$\begin{aligned} \alpha &= \int_0^{2\pi} \log |\phi(t) - \phi(e^{i\theta}) d\theta / 2\pi \\ &= \int_0^{2\pi} \log |h_{\phi(t)}(e^{i\theta})| d\theta / 2\pi = \log |h_{\phi(t)}(0)|. \end{aligned}$$

Hence $|h_{\phi(t)}(0)| = e^{\alpha}$. and so $h_{\phi(t)}(0)$ is constant on D_r. If $a_1 < |\phi(t)| \leq a_2$, by what was proved above,

$$\begin{aligned} &\beta \log \frac{1}{|\phi(t)|} + \gamma = \int_0^{2\pi} \log |\phi(t) - \phi(e^{i\theta})| d\theta / 2\pi \\ &= \int_0^{2\pi} \log |h_{\phi(t)}(e^{i\theta})| d\theta 2\pi = \log |h_{\phi(t)}(0)| \end{aligned}$$

and so $|h_{\phi(t)}(0)| = e^{\gamma} |\phi(t)|^{\beta}$. Since there exists $0 < r < 1$ such that $\{t \in D; a_1 < |\phi(t)| < a_2\} \cap \phi(D_r) \neq \emptyset$, this implies that $|\phi(t)|$ is constant there and so ϕ is constant on D. This contradicts that ϕ is a finite-to-one map. Therefore C_ϕ is not isometric. $\square$

If ϕ is a one-to-one map then it is known [4, Theorem 3.17] that the inner part of $z - \phi$ is a Blaschke product for each $z \in D$. Hence we need not such a hypothesis in

Theorem 5. Unfortunately we could not prove it in general, that is, for a finite-to-one map.

§6. Rudin's orthogonal function

In this section, we study Rudin's orthogonal functions. By Theorem 1, if C_ϕ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$ then ϕ is a Rudin's orthogonal function. Proposition 7 implies the converse. This was proved by the author [6] when $d\mu = d\theta/2\pi$. The proof is valid for an arbitrary μ. However we give a new proof due to K. Izuchi.

Proposition 7. *If ϕ is a Rudin's orthogonal function in $H^2(\mu)$ then there exists a unique radial measure ν such that C_ϕ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$ where $d\nu = d\nu_0(r)d\theta/2\pi$ and $1 \in$ supp ν_0.*

Proof. Put $\nu_0 = \mu_{|\phi|}$ and $d\nu = d\nu_0 d\theta/2\pi$, then Theorems 1 and 2 imply the proposition. □

Corollary 4, *Suppose ϕ is a finite-to-one map and ϕ is a Rudin's orthogonal function. If the inner part of $z-\phi$ is a Blashke product for each $z \in D$ and $|\phi| = \sum_{j=1}^{\ell} a_j \chi_{E_j}$ where $0 \leq a_j \lneq a_{j+1}$, $\sum_{j=1}^{\ell} \chi_{E_j} = 1$ and E_j is a measurable set in ∂D where $1 \leq \ell \leq \infty$, then $|\phi| = 1$ and so ϕ is a finite Blaschke product.*

Proof. If ϕ is a Rudin's orthogonal function, then by Proposition 7 and Theorem 5, $\ell = 1$ and so ϕ is a finite Blashke product. □

In Corollary 4, if ϕ is one-to-one map then the inner part of $z - \phi$ is a Blaschke product (see [4.Theorem 3.17]). Hence we can take off such a condition. However in such a case Corollary 4 is not new. In fact, P. S. Bourdon [3] showed that if ϕ is univalent and a Rudin's orthogonal function then ϕ is just the coordinate function z.

§7. Final remark

The research in this paper gives more general one. Suppose $0 < p < \infty$ and $p \neq 2$. T is an isometric operator from $H^p(\nu)$ into $H^p(\mu)$ with $T1 = 1$ if and only if $T = C_\phi$ for some ϕ in H^∞ with $\|\phi\|_\infty = 1$ and C_ϕ is an isometric operator from $H^p(\nu)$ into $H^p(\mu)$. For the 'if' part is trivial. For the 'only if' part, if T is isometric and $T1 = 1$, then by [5, Theorem 7.5.3] $T(fg) = Tf \cdot Tg$ *a.e.* μ and $\|Tf\|_\infty = \|f\|_\infty$ for all $f \in \mathcal{P}$, $g \in \mathcal{P}$. Hence if $\phi = Tz$ then ϕ belongs to H^∞ and $\|\phi\|_\infty = 1$. Therefore $Tf = C_\phi f \ \ (f \in \mathcal{P})$ and so $Tf = C_\phi f \ \ (f \in H^p(\nu))$. When $p \neq 2$, if C_ϕ is an isometric operator from $H^p(\nu)$ into $H^p(\mu)$, then C_ϕ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$. For by [5, Theorem 8.5.3], for all $f \in \mathcal{P}$ and $g \in \mathcal{P}$

$$\int_{\overline{D}} C_\phi f \cdot \overline{C_\phi g} d\mu = \int_{\overline{D}} f \bar{g} d\mu$$

and $\|C_\phi f\|_\infty = \|f\|_\infty$. This implies that C_ϕ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$.

We give two open problems :

(1) Are there any isometric C_ϕ from L_a^2 into H^2 ?

(2) When ν_0 is a discrete measure and not a dirac measure, are there any isometric C_ϕ from $H^2(\nu)$ to H^2 where $d\nu = d\nu_0(r)d\theta/2\pi$?

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