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# Isometric Composition Operators Between Two Weighted Hardy Spaces

By

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**Abstract.** We study isometric composition operators  $C_\phi$  between two weighted Hardy spaces  $H^2(\nu)$  and  $H^2(\mu)$  when  $\nu$  is a radial measure. The isometric  $C_\phi$  is related to a moment sequence and such a  $\phi$  is studied by the Nevanlinna counting function of  $\phi$  when  $\mu$  is the normalized Lebesgue measure on the unit circle.

## §1. Introduction

Let  $D$  be the open unit disc in the complex plane  $\mathbb{C}$ . We denote by  $\mathcal{P}$  the set of all analytic polynomials and  $H$  the set of all analytic functions on  $D$ . Let  $\mu$  be a positive Borel measure on  $\overline{D}$  with  $\mu(\overline{D}) = 1$ .  $H^p(\mu)$  denotes the closure of all analytic polynomials in  $L^p(d\mu)$  for  $0 < p < \infty$ . If  $d\mu = d\theta/2\pi$ , then  $H^p(\mu) = H^p$  is the classical Hardy space. If  $d\mu = 2rdrd\theta/2\pi$ , then  $H^p(\mu) = L_a^p$  is the classical Bergman space.  $H^p$  and  $L_a^p$  can be embedded in  $H$ . In this paper, we assume that  $H^p(\mu)$  is embedded in  $H$  for a general  $\mu$ .  $H^\infty$  denotes the set of all bounded analytic functions on  $D$ . We also assume that  $H^\infty = H \cap L^\infty(d\mu)$ .

For an analytic self map  $\phi$  of  $D$ , the composition operator  $C_\phi$  is defined by  $(C_\phi f)(z) = f(\phi(z))$  ( $z \in D$ ) for  $f$  in  $H$ . Throughout this paper, we assume that  $\nu$  and  $\mu$  are positive Borel measures on  $\overline{D}$  with  $\nu(\overline{D}) = \mu(\overline{D}) = 1$ .  $\nu$  is called a radial measure if  $d\nu = d\nu_0(r)d\theta/2\pi$  for a positive Borel measure  $\nu_0$  on  $[0, 1]$ . Since  $d\theta/2\pi = d\delta_{r=1}d\theta/2\pi$ ,  $d\theta/2\pi$  is a radial measure.

In this paper, we studied isometric composition operators from  $H^2(\nu)$  into  $H^2(\mu)$  when  $\nu$  is a radial measure. As we show in the final section, our isometric composition operator  $C_\phi$  is related to an isometric operator  $T$  from  $H^p(\nu)$  into  $H^p(\mu)$  with  $T1 = 1$  when  $p \neq 2$ . We have a long history for such isometric operators (see [8]). The onto isometries on  $H^p$  or  $L_a^p$  for  $p \neq 2$  were described completely. Unfortunately into isometries have been known very little.

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**Problem 1.** For given measures  $\nu$  and  $\mu$ , does there exist an isometric composition operator  $C_\phi$  from  $H^2(\nu)$  into  $H^2(\mu)$  ? If there exists such a  $C_\phi$ , describe  $\phi$ .

A function  $F$  in  $H^2(\mu)$  is called an inner function in  $H^2(\mu)$  if

$$\int_{\overline{D}} f|F|^2 d\mu = \int_{\overline{D}} f d\mu \int_{\overline{D}} |F|^2 d\mu \quad (f \in \mathcal{P}).$$

If  $\phi^n$  is an inner function in  $H^2(\mu)$  with  $\int_{\overline{D}} \phi^n d\mu = 0$  for any  $n \geq 0$  then there exists a unique radial measure  $\nu$  such that  $C_\phi$  is isometric from  $H^2(\nu)$  into  $H^2(\mu)$  where  $d\nu = d\nu_0(r)d\theta/2\pi$  and  $1 \in \text{supp } \nu_0$ . This is not difficult to prove. However we don't know whether the converse is true.

**Problem 2.** If a composition operator  $C_\phi$  is isometric from  $H^2(\nu)$  into  $H^2(\mu)$  then is  $\phi^n$  an inner function in  $H^2(\mu)$  with  $\int_{\overline{D}} \phi^n d\mu = 0$  for any  $n \geq 0$  ?

A function  $\phi$  in  $H^\infty$  with  $\|\phi\|_\infty = 1$  is called a Rudin's orthogonal function in  $H^2(\mu)$  if  $\{\phi^n; n = 0, 1, 2, \dots\}$  is a set of orthogonal functions in  $H^2(\mu)$ . If  $\phi^n$  is an inner function in  $H^2(\mu)$  with  $\int_{\overline{D}} \phi^n d\mu = 0$  for any  $n \geq 0$  and  $\|\phi\|_\infty = 1$  then  $\phi$  is a Rudin's orthogonal function in  $H^2(\mu)$  because  $\mathcal{P}$  is dense in  $H^2(\mu)$  by its definition. We can ask whether the converse is true or not.

**Problem 3.** If  $\phi$  is a Rudin's orthogonal function in  $H^2(\mu)$  then is  $\phi^n$  an inner function in  $H^2(\mu)$  with  $\int_{\overline{D}} \phi^n d\mu = 0$  for any  $n \geq 0$  ?

When  $d\mu = d\theta/2\pi$ , Problem 3 was studied by several people, for example, [2],[3],[5],[6] and [10]. C. Bishop [2] and C. Sundberg [10] gave counter examples. Hence there exists a Rudin's orthogonal function which is not an inner function in  $H^2(d\theta/2\pi)$ .

Problem 3 has a strong connection with Problem 2. In fact, if  $C_\phi$  is an isometric operator from  $H^2(\nu)$  into  $H^2(\mu)$  then by Theorem 1  $\phi$  is a Rudin's orthogonal function in  $H^2(\mu)$ . Conversely if  $\phi$  is a Rudin's orthogonal function in  $H^2(\mu)$  then by Proposition 8 there exists a unique radial measure  $\nu$  such that  $C_\phi$  is an isometric operator from  $H^2(\nu)$  into  $H^2(\mu)$ .

For each  $\phi$ , we will use two Borel measures  $\mu_\phi$  on  $\overline{D}$  and  $\mu_{|\phi|}$  on  $[0,1]$ . For a Borel set  $E$  in  $\overline{D}$   $\mu_\phi(E) = \mu(\{z \in \overline{D}; \phi(z) \in E\})$  and for a Borel set  $G$  in  $[0,1]$   $\mu_{|\phi|}(G) = \mu(\{z \in \overline{D}; |\phi(z)| \in G\})$ .

## §2. General case

In this section we assume that  $\nu$  is a radial measure,  $\mu$  is an arbitrary measure and  $\phi$  is an analytic selfmap with  $\|\phi\|_\infty = 1$ . We say that  $\{a_n\}$  is a moment sequence of  $\nu_0$ , a positive Borel measure on  $[0,1]$ , if  $a_n = \int_0^1 r^n d\nu_0$  ( $n = 0, 1, 2, \dots$ ).

**Theorem 1.** *Suppose  $d\nu = d\nu_0(r)d\theta/2\pi$ . Then  $C_\phi$  is an isometric operator from  $H^2(\nu)$  into  $H^2(\mu)$  if and only if  $\int_{\overline{D}} \phi^n \bar{\phi}^m d\mu = 0$  ( $n \neq m$ ) and  $\left\{ \int_{\overline{D}} |\phi|^n d\mu \right\}$  is a moment sequence of  $\nu_0$ .*

Proof. If  $C_\phi$  is isometric, by the polarization formula

$$\delta_{nm} \int_0^1 r^n r^m d\nu_0(r) = \int_{\overline{D}} z^n \bar{z}^m d\nu = \int_{\overline{D}} \phi^n \bar{\phi}^m d\mu$$

because  $\nu$  is a radial measure. Hence

$$\int_{\overline{D}} |\phi|^{2n} d\mu = \int_0^1 r^{2n} d\nu_0 \quad (n = 0, 1, 2, \dots).$$

It is elementary to see that  $x = \sqrt{1 - (1 - x^2)} = \sum_{n=0}^{\infty} a_n (1 - x^2)^n$  and  $\sum_{n=0}^{\infty} |a_n| (1 - x^2)^n < \infty$  ( $0 \leq x \leq 1$ ). Hence by Lebesgue's dominated convergence theorem

$$\begin{aligned} \int_{\overline{D}} |\phi| d\mu &= \int_{\overline{D}} \sum_{n=0}^{\infty} a_n (1 - |\phi|^2)^n d\mu = \sum_{n=0}^{\infty} a_n \int_{\overline{D}} (1 - |\phi|^2)^n d\mu \\ &= \sum_{n=0}^{\infty} a_n \int_0^1 (1 - r^2)^n d\nu_0 = \int_0^1 \sum_{n=0}^{\infty} a_n (1 - r^2)^n d\nu_0 = \int_0^1 r d\nu_0 \end{aligned}$$

because  $\left| \sum_{n=0}^k a_n (1 - |\phi|^2)^n \right| \leq \sum_{n=0}^{\infty} |a_n|$  and  $\left| \sum_{n=0}^k a_n (1 - r^2)^n \right| \leq \sum_{n=0}^{\infty} |a_n| < \infty$ . Similarly, as  $x^{2\ell+1} = \sqrt{1 - (1 - x^{4\ell+2})}$  we can show that  $\int_{\overline{D}} |\phi|^{2n+1} d\mu = \int_0^1 r^{2n+1} d\nu_0$  ( $n = 0, 1, 2, \dots$ ).

Thus  $\left\{ \int_{\overline{D}} |\phi|^n d\mu \right\}$  is a moment sequence of  $\nu_0$ .

Conversely if  $\int_{\overline{D}} \phi^n \bar{\phi}^m d\mu = 0$  ( $n \neq m$ ) and  $\left\{ \int_{\overline{D}} |\phi|^n d\mu \right\}$  is a moment sequence of  $\nu_0$ , then

$$\begin{aligned} \int_{\overline{D}} \left| \sum_{n=0}^k a_n \phi^n \right|^2 d\mu &= \sum_{n=0}^k |a_n|^2 \int_{\overline{D}} |\phi|^{2n} d\mu \\ &= \sum_{n=0}^k |a_n|^2 \int_0^1 r^{2n} d\nu_0 = \int_{\overline{D}} \left| \sum_{n=0}^k a_n z^n \right|^2 d\nu. \end{aligned}$$

Hence  $C_\phi$  is isometric.  $\square$

**Theorem 2.** *If  $d\nu = d\nu_0(r)d\theta/2\pi$  then the following conditions are equivalent.*

(1)  $C_\phi$  is an isometric operator from  $H^2(\nu)$  into  $H^2(\mu)$ .

(2)  $\nu_0 = \mu|_{|\phi|}$

(3)  $\int_0^1 F(r)d\nu_0 = \int_{\bar{D}} F(|\phi|)d\mu$  for any Borel nonnegative function  $F$  on  $[0,1]$ .

Proof. (1)  $\Rightarrow$  (2) If  $G$  is a Borel set in  $[0,1]$ , then  $\nu_0(G) = \inf\{\nu_0(V) ; G \subset V, V \text{ is open in } [0,1]\}$  because  $\nu_0$  is a Borel measure. Hence there exists a sequence of continuous functions  $\{f_m\}$  such that  $f_m \rightarrow \chi_G$  a.e.  $\nu_0$  on  $[0,1]$  and  $\|f_m\|_\infty \leq \gamma < \infty$  ( $m = 1, 2, \dots$ ). By the Stone-Weierstrass theorem,

$$\int_0^1 f_m(r)d\nu_0 = \int_{\bar{D}} f_m(|\phi|)d\mu \quad (m = 1, 2, \dots)$$

because  $\int_0^1 r^n d\nu_0 = \int_{\bar{D}} |\phi|^n d\mu$  ( $n = 0, 1, 2, \dots$ ). Thus  $\nu_0(G) = \mu(\{z \in \bar{D} ; |\phi(z)| \in G\})$ . (2)  $\Rightarrow$  (3) is clear. (3)  $\Rightarrow$  (1) is a result of Theorem 1.  $\square$

The following theorem shows that we can solve Problem 2 in the Introduction when  $C_\phi$  is onto.

**Theorem 3.** *Suppose  $d\nu = d\nu_0(r)d\theta/2\pi$ . If  $C_\phi$  is an isometric operator from  $H^2(\nu)$  onto  $H^2(\mu)$  then  $\phi^n$  is an inner function in  $H^2(\mu)$  for any  $n \geq 0$ .*

Proof. Let  $F \in \mathcal{P}$  then there exists  $f \in H^2(\nu)$  such that  $F = f \circ \phi$ . Let  $f = \sum_{j=0}^{\infty} a_j z^j$ , since  $\sum_{j=0}^{\infty} |a_j|^2 \int_0^1 r^{2j} d\nu_0(r) < \infty$ ,  $F = \sum_{j=0}^{\infty} a_j \phi^j$  and  $\sum_{j=0}^{\infty} |a_j|^2 \int_{\bar{D}} |\phi|^{2j} d\mu < \infty$ . By Theorem 1, for any  $\ell \geq 0$

$$\int_{\bar{D}} F|\phi|^{2\ell} d\mu = a_0 \int_{\bar{D}} |\phi|^{2\ell} d\mu = \int_{\bar{D}} F d\mu \int_{\bar{D}} |\phi|^{2\ell} d\mu$$

because  $\int_{\bar{D}} \phi d\mu = 0$ . This implies that  $\phi^\ell$  is an inner function in  $H^2(\mu)$  for any  $\ell \geq 0$ .

When  $d\nu = d\nu_0(r)d\theta/2\pi$ , if  $C_\phi$  is an isometric operator from  $H^2(\nu)$  into  $H^2(\mu)$  then  $C_z$  is isometric from  $H^2(\nu)$  onto  $H^2(\mu_\phi)$ .

**Corollary 1.** *Suppose  $d\nu = d\nu_0(r)d\theta/2\pi$ . If  $C_z$  is an isometric operator then  $z^n$  is an inner function in  $H^2(\mu)$  for any  $n \geq 0$ . Moreover  $d\mu = d\nu_1(r)d\theta/2\pi + d\delta_{r=0}d\mu_1(\theta)$ , where  $\nu_1$  is a Borel measure on  $[0,1]$  and  $\mu_1$  is a Borel measure on  $\partial D$ . If  $\nu_0$  does not have point mass on  $\{r = 0\}$  then  $\nu = \mu$ .*

Proof. By the remark above,  $C_z$  is isometric from  $H^2(\nu)$  onto  $H^2(\mu)$  because  $\mu_z = \mu$ . By Theorem 3,  $z^n$  is inner in  $H^2(\mu)$  for any  $n \geq 0$ . Put  $C_0[0,1] = \{u; u \text{ is continuous on}$

$[0,1]$  and  $u(0) = 0$  and  $C_0(\partial D) = \{f; f \text{ is continuous on } \partial D \text{ and } f(1) = 0\}$ . Since  $r^n d\mu$  annihilates  $z\mathcal{P} + \bar{z}\bar{\mathcal{P}}$  for any  $n \geq 0$ , for any  $j \neq 0$ ,  $d\mu \perp \{r^{2n+|j|}e^{ij\theta}; n = 0, 1, 2, \dots\}$ . By the Müntz-Szasz theorem [6],  $d\mu \perp C_0[0, 1]e^{ij\theta}$  for any  $j \neq 0$  and so  $d\mu \perp C_0[0, 1] \otimes C_0(\partial D)$ . This implies that  $d\mu = d\nu_1(r)d\theta/2\pi + d\delta_{r=0}d\mu_1(\theta)$  where  $\nu_1$  is a Borel measure on  $[0,1]$  and  $\mu_1$  is a Borel measure on  $T$ . If  $\nu_0$  does not have point mass on  $\{r = 0\}$  then we may assume that  $\mu_1 = 0$  and so  $d\mu = d\nu_1(r)d\theta/2\pi$ . By Theorem 2  $\nu_0 = \mu_{|z|}$  and  $\mu_{|z|} = \nu_1$  because  $d\mu = d\nu_1(r)d\theta/2\pi$ .  $\square$

### §3. Radial measure

In this section we assume that  $\nu$  and  $\mu$  are radial measures, that is,  $d\nu = d\nu_0(r)d\theta/2\pi$  and  $d\mu = d\mu_0(r)d\theta/2\pi$ . Proposition 1 solves Problem 2 when  $\nu = \mu$ . By Theorem 2, if  $C_\phi$  is isometric from  $H^2(\nu)$  into  $H^2(\mu)$ , then for some positive integer  $k$

$$\int_0^1 \log r d\nu_0 \leq k \int_0^1 \log r d\mu_0 + \int_0^{2\pi} \log |\phi(e^{i\theta})| d\theta$$

as  $F(t) = \log t$ , using the inner outer factorization of  $\phi$ . Proposition 2 gives an exact formula for this.

**Proposition 1.** *Suppose  $\nu$  is a radial measure. If  $C_\phi$  is an isometric operator from  $H^2(\nu)$  into  $H^2(\nu)$ , then  $\phi^n$  is an inner function in  $H^2(\nu)$  for any  $n \geq 0$ .*

Proof. By Theorem 1,  $\phi(0) = 0$  because  $\nu$  is a radial measure and so by Schwarz's lemma,  $|\phi(z)| \leq |z|$  ( $z \in D$ ). Since  $\int_D |\phi(z)|^2 d\nu = \int_D |z|^2 d\nu$ ,  $|\phi(z)| = |z|$  a.e.  $\nu$ . For  $f \in \mathcal{P}$ ,

$$\int_D f |\phi|^{2n} d\nu = \int_D f |z|^{2n} d\nu = f(0) \int_0^1 r^{2n} d\nu_0 = \int_D f d\nu \int_D |\phi|^{2n} d\nu. \square$$

**Proposition 2.** *Suppose  $\nu$  and  $\mu$  are radial measures, that is,  $d\nu = d\nu_0(r)d\theta/2\pi$  and  $d\mu = d\mu_0(r)d\theta/2\pi$ . Let  $\phi = z^k BQh$  where  $k$  is a positive integer,  $B$  is a Blaschke product with  $B(0) \neq 0$ ,  $Q(z) = \exp - \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\lambda(t)$  is a singular inner function and  $h$  is an outer function. If  $C_\phi$  is an isometric operator from  $H^2(\nu)$  into  $H^2(\mu)$ , then*

$$\int_0^1 \log r d\nu_0 = k \int_0^1 \log r d\mu_0 + \int_0^1 d\mu_0 \int_0^r n(s, B) \frac{ds}{s} + \log |B(0)| - \mu_0([0, 1])\lambda([0, 2\pi]) + \int_0^{2\pi} \log |\phi(e^{i\theta})| d\theta/2\pi$$

where  $n(s, B)$  is the number of zeros of  $B$  on the closed disc  $\{z \in \mathbb{C}; |z| \leq r\}$ .

Proof. Let  $n(s, B) = n(s, BQh)$  is the number of zeros of  $BQh$  on the closed disc  $\{z \in \mathbb{C} ; |z| \leq r\}$ . Then, by Theorem 2 and [1, §2 of Chapter 5]

$$\begin{aligned}
& \int_0^1 \log r d\nu_0 \\
&= \int_0^{1-} d\mu_0 \int_0^{2\pi} \log |\phi(re^{i\theta})| d\theta/2\pi + \mu_0(\{1\}) \int_0^{2\pi} \log |\phi(e^{i\theta})| d\theta/2\pi \\
&= \int_0^{1-} d\mu_0 \left\{ \log r^k + \int_0^r n(s, B) \frac{ds}{s} \right\} + \mu_0([0, 1)) \log |B(0)Q(0)h(0)| \\
&\quad + \mu_0(\{1\}) \int_0^{2\pi} \log |\phi(e^{i\theta})| d\theta/2\pi \\
&= k \int_0^1 \log r d\mu_0 + \int_0^1 d\mu_0 \int_0^r n(s, B) \frac{ds}{s} + \log |B(0)| \\
&\quad - \mu_0([0, 1)) \lambda([0, 2\pi]) + \int_0^{2\pi} \log |\phi(e^{i\theta})| d\theta/2\pi
\end{aligned}$$

because  $\mu_0(\{1\}) \int_0^1 n(s, B) \frac{ds}{s} = -\mu_0(\{1\}) \log |B(0)|$ .  $\square$

#### §4. Special cases

In this section we assume that  $\nu$  or  $\mu$  is the normalized Lebesgue measure or the normalized area measure. Proposition 3 solves Problems 1 and 2 when  $\nu$  is the normalized Lebesgue measure on the circle and  $\mu$  is a radial measure. Proposition 5 solves Problem 2 when  $\nu$  is a radial measure or the Lebesgue measure on the circle. Corollary 3 solves Problem 2 negatively when  $d\nu = 2rdrd\theta/2\pi$  and  $d\mu = d\theta/2\pi$ .

**Proposition 3.** *Let  $\mu$  be a radial measure.  $C_\phi$  is an isometric operator from  $H^2$  into  $H^2(\mu)$  if and only if  $\phi^n$  is an inner function with  $\int_D \phi^n d\mu = 0$  in  $H^2(\mu)$  for any  $n \geq 1$  and  $H^2(d\mu) = H^2$ .*

Proof. If  $C_\phi$  is isometric, by Theorem 1  $\int_D \phi^n \bar{\phi}^m d\mu = 0$  ( $n \neq m$ ) and we have

$$1 = \int_0^{2\pi} |z|^2 d\theta/2\pi = \int_D |\phi|^2 d\mu \leq 1.$$

Hence  $|\phi(z)| = 1$  a.e.  $\mu$  and so  $\text{supp } \mu \subset \partial D$ . This implies that  $d\mu = d\delta_{r=1}d\theta/2\pi$  because  $\mu$  is a radial measure. The converse is clear.  $\square$

**Proposition 4.** *Suppose  $d\nu = d\nu_0(r)d\theta/2\pi$  and  $C_\phi$  is an isometric operator from  $H^2(\nu)$  into  $H^2$ .*

(1)  $\nu_0(\{a\}) > 0$  for  $0 \leq a \leq 1$  if and only if  $d\theta/2\pi(\{e^{i\theta}; |\phi(e^{i\theta})| = a\}) > 0$ .

(2)  $d\nu_0 = d\delta_{r=1}$  if and only if  $\phi$  is an inner function in  $H^2$ .

(3)  $\nu_0$  is a discrete measure if and only if  $|\phi| = \sum_{n=1}^{\infty} a_n \chi_{E_n}$  where  $0 \leq a_n \leq 1$ , and  $d\theta/2\pi(E_n) = \nu_0(\{a_n\})$  ( $n = 1, 2, \dots$ ).

Proof. Since  $\nu_0(G) = d\theta/2\pi\{e^{i\theta}; |\phi(e^{i\theta})| \in G\}$  for a Borel set  $G$  in  $[0,1]$  by Theorem 2, it is easy to see.  $\square$

Proof. This is just (2) of Proposition 4.

Now we consider when  $d\nu = r dr d\theta/\pi$  or  $d\mu = r dr d\theta/\pi$ .

**Proposition 5.** *If  $C_\phi$  is an isometric operator from  $L_a^2$  into  $H^2(\mu)$ , then  $\mu(\{z \in \bar{D}; |\phi| = b\}) = 0$  and  $\int_{\bar{D}} (b - |\phi|)^{-1} d\mu = \infty$  for any  $0 \leq b \leq 1$ .*

Proof. It is clear by Theorem 2.  $\square$

**Corollary 2.** *If  $C_\phi$  is an isometric operator from  $L_a^2$  into  $H^2$ , then  $\phi$  is not inner in  $H^2$ .*

**Proposition 6.** *Suppose  $d\nu = d\nu_0(r)d\theta/2\pi$ . If  $C_\phi$  is an isometric operator from  $H^2(\nu)$  into  $L_a^2$ , then  $\int_0^1 \log r d\nu_0 = -\frac{k}{4} + \int_0^1 2r dr \int_0^r n(s, B) \frac{ds}{s} + \log |B(0)| - \lambda([0, 2\pi]) + \int_0^{2\pi} \log |\phi(e^{i\theta})| d\theta/2\pi$ , where the inner part of  $\phi$  is  $z^k BQ$ ,  $B$  is a Blaschke product with  $B(0) \neq 0$ ,  $Q(z) = \exp - \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\lambda$  is a singular inner function. Hence if  $\phi$  is a shrict function, then  $\int_0^1 \log r d\nu_0 = -\frac{1}{4} + \int_0^{2\pi} \log |\phi(e^{i\theta})| d\theta/2\pi$ .*

Proof. It is clear by Proposition 2.  $\square$

## §5. Nevanlinna counting function

Suppose  $\nu$  or  $\mu$  is the normalized Lebesgue measure or the normalized area measure. We assume that  $\phi$  is a non-constant function in  $H^\infty$  with  $\|\phi\|_\infty = 1$ . The Nevanlinna counting function of  $\phi$ ,  $N_\phi$ , is defined on  $D \setminus \{\phi(0)\}$  by

$$N_\phi(w) = \sum_{\phi(z)=w} \log \frac{1}{|z|},$$

where multiplicities are counted and  $N_\phi(w)$  is taken to be zero if  $w$  is not in the range of  $\phi$ . Corollary 4 seems to be interesting in spite of Corollary 3.



**Theorem 4.** *Suppose  $d\nu = d\nu_0(r)d\theta/2\pi$ . Then,  $C_\phi$  is an isometric operator from  $H^2(\nu)$  into  $H^2$  if and only if*

$$N_\phi(z) = \int_{|z|}^1 \log \frac{r}{|z|} d\nu_0(r)$$

for nearly all  $z$  in  $D$ .

Proof. The ‘only if’ part was proved in [6, Lemma 3]. If  $N_\phi(z) = \int_{|z|}^1 \log \frac{r}{|z|} d\nu_0(r)$  for nearly all  $z$  in  $D$ , by the Littlewood-Paley theorem (see [3]),

$$\begin{aligned} & \int_0^{2\pi} \phi^n(e^{i\theta}) \bar{\phi}^m(e^{i\theta}) d\theta / 2\pi \\ &= 2nm \int_{\overline{D}} z^{n-1} \bar{z}^{m-1} N_\phi(|z|) dA(z) \\ &= 4nm \delta_{nm} \int_0^1 r^{n+m-1} \left( \int_r^1 \log \frac{s}{r} d\nu_0(s) \right) dr \\ &= 4nm \delta_{nm} \int_0^1 d\nu_0(s) \int_0^s r^{m+n-1} \left( \log \frac{s}{r} \right) dr \\ &= \frac{4nm}{(n+m)^2} \delta_{nm} \int_0^1 s^{n+m} d\nu_0(s). \end{aligned}$$

When  $n = m$ ,  $\int_0^{2\pi} |\phi(e^{i\theta})|^{2n} d\theta / 2\pi = \int_0^1 s^{2n} d\nu_0(s)$  for  $n = 0, 1, 2, \dots$ . Hence by Theorem 1 and its proof,  $C_\phi$  is an isometric operator from  $H^2(\nu)$  into  $H^2$ .

**Lemma.**  *$D \setminus \{z \in D ; \phi'(z) = 0\}$  can be decomposed into an at most countable disjoint collection  $\{R_n\}$  of “semi-closed” polar rectangles, on each of which  $\phi$  is schlicht.*

Proof. It is known in [9, p186].  $\square$

**Corollary 3.** *Suppose  $\phi$  is a finite-to-one map. Then  $C_\phi$  is not an isometric operator from  $L_a^2$  into  $H^2$ .*

Proof. By Lemma, there exists the inverse  $\psi_n$  of the restriction of  $\phi$  to  $R_n$ . Let  $w \in \phi(R_{j_1})$ . If  $\phi$  is an  $\ell$  to 1 map, then there exist  $j_2, \dots, j_\ell$  such that  $\psi_{j_1}(z) = \psi_{j_2}(z) = \dots = \psi_{j_\ell}(z) = w$ . Hence there exists a small disc  $\Delta$  in  $\phi(R_{j_1})$  such that

$$N_\phi(z) = \sum_{z=\phi(w)} \log \frac{1}{|w|} = \sum_{t=1}^{\ell} \log \frac{1}{|\psi_{j_t}(z)|}$$

for all  $w \in \Delta$ . Therefore there exists a subdisc  $\Delta_0$  in  $\Delta$  such that  $N_\phi(z)$  is harmonic on  $\Delta_0$ . On the other hand, by Proposition 5

$$N_\phi(z) = 2 \int_{|z|}^1 \left( \log \frac{r}{|z|} \right) r dr = \frac{|z|^2 - 1}{2} + \log \frac{1}{|z|}.$$

This contradicts that  $N_\phi(z)$  is harmonic on  $\Delta_0$ .  $\square$

**Theorem 5.** *Suppose  $\phi$  is a contractive function in  $H^\infty$  such that  $\phi$  is a finite-to-one map and  $|\phi| = \sum_{j=1}^{\ell} a_j \chi_{E_j}$  where  $0 < a_j < a_{j+1}$ ,  $\sum_{j=1}^{\ell} \chi_{E_j} = 1$  and  $E_j$  is a measurable set in  $\partial D$  where  $1 \leq \ell \leq \infty$ . If the inner part of  $z - \phi$  is a Blaschke product for each  $z \in D$ , then  $C_\phi$  is not an isometric operator from  $H^2(\nu)$  into  $H^2$  for any  $d\nu = d\nu_0(r)d\theta/2\pi$  if  $\ell \neq 1$ .*

Proof. Suppose  $C_\phi$  is an isometric operator from  $H^2(\nu)$  into  $H^2$  for some  $d\nu = d\nu_0(r)d\theta/2\pi$ . By Proposition 4,  $\nu_0$  is a discrete measure and  $d\theta/2\pi(E_j) = \nu_0(\{a_j\})$  ( $j = 1, 2, \dots$ ). Since  $\phi(0) = 0$ , by Lemma 2 in [6] and Proposition 7

$$\begin{aligned} N_\phi(z) &= \int_0^{2\pi} \log |z - \phi(e^{i\theta})| d\theta/2\pi + \log \frac{1}{|z|} \\ &= \int_{|z|}^1 \log \frac{r}{|z|} d\nu_0(r) \end{aligned}$$

for  $z \in D \setminus \{0\}$ . If  $|z| \leq a_1$ , then

$$\begin{aligned} \int_{|z|}^1 \log \frac{r}{|z|} d\nu_0(r) &= \sum_{j=1}^{\infty} \left( \log \frac{a_j}{|z|} \right) \nu_0(\{a_j\}) \\ &= \sum_{j=1}^{\infty} \nu_0(\{a_j\}) \log \frac{1}{|z|} + \sum_{j=1}^{\infty} \nu_0(\{a_j\}) \log a_j \\ &= \log \frac{1}{|z|} + \sum_{j=1}^{\infty} \nu_0(\{a_j\}) \log a_j. \end{aligned}$$

Hence if  $|z| \leq a_1$  then

$$\int_0^{2\pi} \log |z - \phi(e^{i\theta})| d\theta/2\pi = \sum_{j=1}^{\infty} \nu_0(\{a_j\}) \log a_j = \alpha.$$

If  $a_1 < |z| \leq a_2$ , then

$$\begin{aligned} \int_{|z|}^1 \log \frac{r}{|z|} d\nu_0(r) &= \sum_{j=2}^{\infty} \left( \log \frac{a_j}{|z|} \right) \nu_0(\{a_j\}) \\ &= \sum_{j=2}^{\infty} \nu_0(\{a_j\}) \log \frac{1}{|z|} + \sum_{j=2}^{\infty} \nu_0(\{a_j\}) \log a_j \end{aligned}$$

$$\begin{aligned} \int_0^{2\pi} \log |z - \phi(e^{i\theta})| d\theta/2\pi &= -\nu_0(\{a_1\}) \log \frac{1}{|z|} + \sum_{j=2}^{\infty} \nu_0(\{a_j\}) \log a_j \\ &= \beta \log \frac{1}{|z|} + \gamma. \end{aligned}$$

where  $\beta \neq 0$

For each  $z \in D$ , put

$$z - \phi(\zeta) = q_z(\zeta)h_z(\zeta) \quad (\zeta \in D)$$

where  $q_z(\zeta)$  is inner and  $h_z(\zeta)$  is outer. Since  $\phi$  is a finite-to-one map,  $q_z$  is a finite Blaschke product by hypothesis and so

$$q_{\phi(t)}(\zeta) = \prod_{j=1}^n \frac{\zeta - b_j(t)}{1 - \overline{b_j(t)}\zeta} \quad (t \in D).$$

Then, since  $\phi(0) = 0$ ,

$$\phi(t) = (-1)^n \left( \prod_{j=1}^n b_j(t) \right) h_{\phi(t)}(0) \quad (t \in D).$$

Put  $D_r = \{t \in \mathbb{C}; |t| \leq r\}$  for  $0 < r < 1$ . If both  $\phi$  and  $\phi'$  have no zeros on  $\partial D_r$  then there is a division  $\{D_r^j\}_{1 \leq j \leq n}$  of  $D_r$  such that  $\phi$  is one-to-one on  $D_r^j$  for  $1 \leq j \leq n$ . For,  $\phi$  is conformal in a neighborhood of each point on  $\partial D_r$  and so  $\arg \phi$  is increasing on,  $\partial D_r$ . Put  $\phi_j = \phi | D_r^j$  and  $b_j(t) = \phi_j^{-1}(\phi(t))$  for  $1 \leq j \leq n$ . Then  $b_j(t)$  is analytic except  $\phi'(t) = 0$  when  $\phi(t)$  in  $\phi(D_r)$ . Hence  $h_{\phi(t)}(0)$  is analytic except  $\phi'(t) = 0$  and  $\bigcup_{j=1}^n \{t \in D; b_j(t) = 0\}$  when  $\phi(t)$  in  $\phi(D_r)$ . Since  $\phi(0) = 0$ ,  $\{t \in D; |\phi(t)| < a_1\}$  is a nonempty open set. We can choose  $r$  such that  $\{t \in D; |\phi(t)| < a_1\} \cap \phi(D_r) \neq \emptyset$ . If  $|\phi(t)| \leq a_1$ , by what was proved above,

$$\begin{aligned} \alpha &= \int_0^{2\pi} \log |\phi(t) - \phi(e^{i\theta})| d\theta / 2\pi \\ &= \int_0^{2\pi} \log |h_{\phi(t)}(e^{i\theta})| d\theta / 2\pi = \log |h_{\phi(t)}(0)|. \end{aligned}$$

Hence  $|h_{\phi(t)}(0)| = e^\alpha$ . and so  $h_{\phi(t)}(0)$  is constant on  $D_r$ . If  $a_1 < |\phi(t)| \leq a_2$ , by what was proved above,

$$\begin{aligned} \beta \log \frac{1}{|\phi(t)|} + \gamma &= \int_0^{2\pi} \log |\phi(t) - \phi(e^{i\theta})| d\theta / 2\pi \\ &= \int_0^{2\pi} \log |h_{\phi(t)}(e^{i\theta})| d\theta / 2\pi = \log |h_{\phi(t)}(0)| \end{aligned}$$

and so  $|h_{\phi(t)}(0)| = e^\gamma |\phi(t)|^\beta$ . Since there exists  $0 < r < 1$  such that  $\{t \in D; a_1 < |\phi(t)| < a_2\} \cap \phi(D_r) \neq \emptyset$ , this implies that  $|\phi(t)|$  is constant there and so  $\phi$  is constant on  $D$ . This contradicts that  $\phi$  is a finite-to-one map. Therefore  $C_\phi$  is not isometric.  $\square$

If  $\phi$  is a one-to-one map then it is known [4, Theorem 3.17] that the inner part of  $z - \phi$  is a Blaschke product for each  $z \in D$ . Hence we need not such a hypothesis in

Theorem 5. Unfortunately we could not prove it in general, that is, for a finite-to-one map.

## §6. Rudin's orthogonal function

In this section, we study Rudin's orthogonal functions. By Theorem 1, if  $C_\phi$  is an isometric operator from  $H^2(\nu)$  into  $H^2(\mu)$  then  $\phi$  is a Rudin's orthogonal function. Proposition 7 implies the converse. This was proved by the author [6] when  $d\mu = d\theta/2\pi$ . The proof is valid for an arbitrary  $\mu$ . However we give a new proof due to K. Izuchi.

**Proposition 7.** *If  $\phi$  is a Rudin's orthogonal function in  $H^2(\mu)$  then there exists a unique radial measure  $\nu$  such that  $C_\phi$  is an isometric operator from  $H^2(\nu)$  into  $H^2(\mu)$  where  $d\nu = d\nu_0(r)d\theta/2\pi$  and  $1 \in \text{supp } \nu_0$ .*

Proof. Put  $\nu_0 = \mu_{|\phi|}$  and  $d\nu = d\nu_0 d\theta/2\pi$ , then Theorems 1 and 2 imply the proposition.  $\square$

**Corollary 4.** *Suppose  $\phi$  is a finite-to-one map and  $\phi$  is a Rudin's orthogonal function. If the inner part of  $z - \phi$  is a Blaschke product for each  $z \in D$  and  $|\phi| = \sum_{j=1}^{\ell} a_j \chi_{E_j}$*

*where  $0 \leq a_j < a_{j+1}$ ,  $\sum_{j=1}^{\ell} \chi_{E_j} = 1$  and  $E_j$  is a measurable set in  $\partial D$  where  $1 \leq \ell \leq \infty$ , then  $|\phi| = 1$  and so  $\phi$  is a finite Blaschke product.*

Proof. If  $\phi$  is a Rudin's orthogonal function, then by Proposition 7 and Theorem 5,  $\ell = 1$  and so  $\phi$  is a finite Blaschke product.  $\square$

In Corollary 4, if  $\phi$  is one-to-one map then the inner part of  $z - \phi$  is a Blaschke product (see [4, Theorem 3.17]). Hence we can take off such a condition. However in such a case Corollary 4 is not new. In fact, P. S. Bourdon [3] showed that if  $\phi$  is univalent and a Rudin's orthogonal function then  $\phi$  is just the coordinate function  $z$ .

## §7. Final remark

The research in this paper gives more general one. Suppose  $0 < p < \infty$  and  $p \neq 2$ .  $T$  is an isometric operator from  $H^p(\nu)$  into  $H^p(\mu)$  with  $T1 = 1$  if and only if  $T = C_\phi$  for some  $\phi$  in  $H^\infty$  with  $\|\phi\|_\infty = 1$  and  $C_\phi$  is an isometric operator from  $H^p(\nu)$  into  $H^p(\mu)$ . For the 'if' part is trivial. For the 'only if' part, if  $T$  is isometric and  $T1 = 1$ , then by [5, Theorem 7.5.3]  $T(fg) = Tf \cdot Tg$  a.e.  $\mu$  and  $\|Tf\|_\infty = \|f\|_\infty$  for

all  $f \in \mathcal{P}$ ,  $g \in \mathcal{P}$ . Hence if  $\phi = Tz$  then  $\phi$  belongs to  $H^\infty$  and  $\|\phi\|_\infty = 1$ . Therefore  $Tf = C_\phi f$  ( $f \in \mathcal{P}$ ) and so  $Tf = C_\phi f$  ( $f \in H^p(\nu)$ ). When  $p \neq 2$ , if  $C_\phi$  is an isometric operator from  $H^p(\nu)$  into  $H^p(\mu)$ , then  $C_\phi$  is an isometric operator from  $H^2(\nu)$  into  $H^2(\mu)$ . For by [5, Theorem 8.5.3], for all  $f \in \mathcal{P}$  and  $g \in \mathcal{P}$

$$\int_{\overline{D}} C_\phi f \cdot \overline{C_\phi g} d\mu = \int_{\overline{D}} f \bar{g} d\mu$$

and  $\|C_\phi f\|_\infty = \|f\|_\infty$ . This implies that  $C_\phi$  is an isometric operator from  $H^2(\nu)$  into  $H^2(\mu)$ .

We give two open problems :

(1) Are there any isometric  $C_\phi$  from  $L_a^2$  into  $H^2$  ?

(2) When  $\nu_0$  is a discrete measure and not a dirac measure, are there any isometric  $C_\phi$  from  $H^2(\nu)$  to  $H^2$  where  $d\nu = d\nu_0(r)d\theta/2\pi$  ?

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