Isometric Composition Operators Between Two Weighted Hardy Spaces

By

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Abstract. We study isometric composition operators $C_\phi$ between two weighted Hardy spaces $H^2(\nu)$ and $H^2(\mu)$ when $\nu$ is a radial measure. The isometric $C_\phi$ is related to a moment sequence and such a $\phi$ is studied by the Nevanlinna counting function of $\phi$ when $\mu$ is the normalized Lebesgue measure on the unit circle.

§1. Introduction

Let $D$ be the open unit disc in the complex plane $\mathbb{C}$. We denote by $\mathcal{P}$ the set of all analytic polynomials and $H$ the set of all analytic functions on $D$. Let $\mu$ be a positive Borel measure on $\overline{D}$ with $\mu(\overline{D}) = 1$. $H^p(\mu)$ denotes the closure of all analytic polynomials in $L^p(d\mu)$ for $0 < p < \infty$. If $d\mu = d\theta/2\pi$, then $H^p(\mu) = H^p$ is the classical Hardy space. If $d\mu = 2rdrd\theta/2\pi$, then $H^p(\mu) = L^p_a$ is the classical Bergman space. $H^p$ and $L^p_a$ can be embedded in $H$. In this paper, we assume that $H^p(\mu)$ is embedded in $H$ for a general $\mu$. $H^\infty$ denotes the set of all bounded analytic functions on $D$. We also assume that $H^\infty = H \cap L^\infty(d\mu)$.

For an analytic self map $\phi$ of $D$, the composition operator $C_\phi$ is defined by $(C_\phi f)(z) = f(\phi(z))$ ($z \in D$) for $f$ in $H$. Throughout this paper, we assume that $\nu$ and $\mu$ are positive Borel measures on $\overline{D}$ with $\nu(\overline{D}) = \mu(\overline{D}) = 1$. $\nu$ is called a radial measure if $d\nu = d\nu_0(r)d\theta/2\pi$ for a positive Borel measure $\nu_0$ on $[0,1]$. Since $d\theta/2\pi = d\delta_{r=1}d\theta/2\pi$, $d\theta/2\pi$ is a radial measure.

In this paper, we studied isometric composition operators from $H^2(\nu)$ into $H^2(\mu)$ when $\nu$ is a radial measure. As we show in the final section, our isometric composition operator $C_\phi$ is related to an isometric operator $T$ from $H^p(\nu)$ into $H^p(\mu)$ with $T1 = 1$ when $p \neq 2$. We have a long history for such isometric operators (see [8]). The onto isometries on $H^p$ or $L^p_a$ for $p \neq 2$ were described completely. Unfortunately into isometries have been known very little.

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Problem 1. For given measures $\nu$ and $\mu$, does there exist an isometric composition operator $C_\phi$ from $H^2(\nu)$ into $H^2(\mu)$? If there exists such a $C_\phi$, describe $\phi$.

A function $F$ in $H^2(\mu)$ is called an inner function in $H^2(\mu)$ if

$$
\int_{\mathbb{D}} |F|^2 d\mu = \int_{\mathbb{D}} f d\mu \int_{\mathbb{D}} |F|^2 d\mu \quad (f \in \mathcal{P}).
$$

If $\phi^n$ is an inner function in $H^2(\mu)$ with $\int_{\mathbb{D}} \phi d\mu = 0$ for any $n \geq 0$ then there exists a unique radial measure $\nu$ such that $C_\phi$ is isometric from $H^2(\nu)$ into $H^2(\mu)$ where $d\nu = dv_0(r) d\theta/2\pi$ and $1 \in \text{supp } \nu_0$. This is not difficult to prove. However we don’t know whether the converse is true.

Problem 2. If a composition operator $C_\phi$ is isometric from $H^2(\nu)$ into $H^2(\mu)$ then is $\phi^n$ an inner function in $H^2(\mu)$ with $\int_{\mathbb{D}} \phi d\mu = 0$ for any $n \geq 0$?

A function $\phi$ in $H^\infty$ with $\|\phi\|_\infty = 1$ is called a Rudin’s orthogonal function in $H^2(\mu)$ if $\{\phi^n; n = 0, 1, 2, \ldots\}$ is a set of orthogonal functions in $H^2(\mu)$. If $\phi^n$ is an inner function in $H^2(\mu)$ with $\int_{\mathbb{D}} \phi d\mu = 0$ for any $n \geq 0$ and $\|\phi\|_\infty = 1$ then $\phi$ is a Rudin’s orthogonal function in $H^2(\mu)$ because $\mathcal{P}$ is dense in $H^2(\mu)$ by its definition. We can ask whether the converse is true or not.

Problem 3. If $\phi$ is a Rudin’s orthogonal function in $H^2(\mu)$ then is $\phi^n$ an inner function in $H^2(\mu)$ with $\int_{\mathbb{D}} \phi d\mu = 0$ for any $n \geq 0$?

When $d\mu = d\theta/2\pi$, Problem 3 was studied by several people, for example, [2],[3],[5],[6] and [10]. C. Bishop [2] and C. Sundberg [10] gave counter examples. Hence there exists a Rudin’s orthogonal function which is not an inner function in $H^2(d\theta/2\pi)$.

Problem 3 has a strong connection with Problem 2. In fact, if $C_\phi$ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$ then by Theorem 1 $\phi$ is a Rudin’s orthogonal function in $H^2(\mu)$. Conversely if $\phi$ is a Rudin’s orthogonal function in $H^2(\mu)$ then by Proposition 8 there exists a unique radial measure $\nu$ such that $C_\phi$ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$.

For each $\phi$, we will use two Borel measures $\mu_\phi$ on $\overline{\mathbb{D}}$ and $\mu_{|\phi|}$ on $[0,1]$. For a Borel set $E$ in $\overline{\mathbb{D}}$ $\mu_\phi(E) = \mu(\{z \in \overline{\mathbb{D}}; \phi(z) \in E\})$ and for a Borel set $G$ in $[0,1]$ $\mu_{|\phi|}(G) = \mu(\{z \in \overline{\mathbb{D}}; |\phi(z)| \in G\})$. 

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§2. General case

In this section we assume that \( \nu \) is a radial measure, \( \mu \) is an arbitrary measure and \( \phi \) is an analytic selfmap with \( \| \phi \|_\infty = 1 \). We say that \( \{ a_n \} \) is a moment sequence of \( \nu_0 \), a positive Borel measure on \([0,1]\), if \( a_n = \int_0^1 r^n d\nu_0 \) \( (n = 0, 1, 2, \cdots) \).

**Theorem 1.** Suppose \( d\nu = d\nu_0(\theta)d\theta/2\pi \). Then \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2(\mu) \) if and only if \( \int_{\mathcal{D}} \phi^n \bar{\phi}^m d\mu = 0 \) \( (n \neq m) \) and \( \{ \int_{\mathcal{D}} |\phi|^n d\mu \} \) is a moment sequence of \( \nu_0 \).

Proof. If \( C_\phi \) is isometric, by the polarization formula

\[
\delta_{nm} \int_0^1 r^n r^m d\nu_0 = \int_{\mathcal{D}} z^n \bar{z}^m d\nu = \int_{\mathcal{D}} \phi^n \bar{\phi}^m d\mu
\]

because \( \nu \) is a radial measure. Hence

\[
\int_{\mathcal{D}} |\phi|^{2n} d\mu = \int_0^1 r^{2n} d\nu_0 \quad (n = 0, 1, 2, \cdots).
\]

It is elementary to see that \( x = \sqrt{1 - (1 - x^2)} = \sum_{n=0}^\infty a_n(1 - x^2)^n \) and \( \sum_{n=0}^\infty |a_n|(1 - x^2)^n < \infty \) \( (0 \leq x \leq 1) \). Hence by Lebesgue’s dominated convergence theorem

\[
\int_{\mathcal{D}} |\phi|^{2n} d\mu = \int_0^1 \sum_{n=0}^\infty a_n(1 - |\phi|^2)^n d\mu = \sum_{n=0}^\infty a_n \int_0^1 (1 - |\phi|^2)^n d\mu = \sum_{n=0}^\infty a_n \int_0^1 (1 - r^2)^n d\nu_0 = \int_0^1 \sum_{n=0}^\infty a_n (1 - r^2)^n d\nu_0 = \int_0^1 r d\nu_0
\]

because \( \left| \sum_{n=0}^k a_n(1 - |\phi|^2)^n \right| \leq \sum_{n=0}^\infty |a_n| \) and \( \left| \sum_{n=0}^k a_n(1 - r^2)^2 \right| \leq \sum_{n=0}^\infty |a_n| < \infty \). Similarly, as \( x^{2\ell+1} = \sqrt{1 - (1 - x^{2\ell+2})} \) we can show that \( \int_{\mathcal{D}} |\phi|^{2n+1} d\mu = \int_0^1 r^{2n+1} d\nu_0 \) \( (n = 0, 1, 2, \cdots) \).

Thus \( \{ \int_{\mathcal{D}} |\phi|^n d\mu \} \) is a moment sequence of \( \nu_0 \).

Conversely if \( \int_{\mathcal{D}} \phi^n \bar{\phi}^m d\mu = 0 \) \( (n \neq m) \) and \( \{ \int_{\mathcal{D}} |\phi|^n d\mu \} \) is a moment sequence of \( \nu_0 \), then

\[
\int_{\mathcal{D}} \left| \sum_{n=0}^k a_n \phi^n \right|^2 d\mu = \sum_{n=0}^k |a_n|^2 \int_{\mathcal{D}} |\phi|^{2n} d\mu
\]

\[
= \sum_{n=0}^k |a_n|^2 \int_0^1 r^{2n} d\nu_0 = \int_{\mathcal{D}} \left| \sum_{n=0}^k a_n z^n \right|^2 d\nu.
\]
Theorem 2. If \( d\nu = d\nu_0(r)d\theta/2\pi \) then the following conditions are equivalent.

1. \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2(\mu) \).
2. \( \nu_0 = \mu_{|\phi|} \)
3. \( \int_0^1 F(r)d\nu_0 = \int_D F(|\phi|)d\mu \) for any Borel nonnegative function \( F \) on \([0,1]\).

Proof. (1) \(\Rightarrow\) (2) If \( G \) is a Borel set in \([0,1]\), then \( \nu_0(G) = \inf\{ \nu_0(V) : G \subset V, V \) is open in \([0,1]\}\) because \( \nu_0 \) is a Borel measure. Hence there exists a sequence of continuous functions \( \{ f_m \} \) such that \( f_m \to \chi_G \) a.e. \( \nu_0 \) on \([0,1]\) and \( \| f_m \|_\infty \leq \gamma < \infty \) \((m = 1, 2, \cdots)\).

By the Stone-Weierstrass theorem, \( \| f_m \| = \int_0^1 f_m(r)d\nu_0 = \int_D f_m(|\phi|)d\mu \) \((m = 1, 2, \cdots)\).

The following theorem shows that we can solve Problem 2 in the Introduction when \( C_\phi \) is onto.

Theorem 3. Suppose \( d\nu = d\nu_0(r)d\theta/2\pi \). If \( C_\phi \) is an isometric operator from \( H^2(\nu) \) onto \( H^2(\mu) \) then \( \phi^n \) is an inner function in \( H^2(\mu) \) for any \( n \geq 0 \).

Proof. Let \( F \in \mathcal{P} \) then there exists \( f \in H^2(\nu) \) such that \( F = f \circ \phi \). Let \( f = \sum_{j=0}^\infty a_j z^j \), since \( \sum_{j=0}^\infty |a_j|^2 \int_0^1 r^{2j}d\nu_0(r) < \infty \), \( F = \sum_{j=0}^\infty a_j \phi^j \) and \( \sum_{j=0}^\infty |a_j|^2 \int_D |\phi|^2 j d\mu < \infty \). By Theorem 1, for any \( \ell \geq 0 \)

\[
\int_D |\phi|^{2\ell} d\mu = a_0 \int_D |\phi|^{2\ell} d\mu = \int_D F d\mu \int_D |\phi|^{2\ell} d\mu
\]

because \( \int_D \phi d\mu = 0 \). This implies that \( \phi^\ell \) is an inner function in \( H^2(\mu) \) for any \( \ell \geq 0 \).

When \( d\nu = d\nu_0(r)d\theta/2\pi \), if \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2(\mu) \) then \( C_z \) is isometric from \( H^2(\nu) \) onto \( H^2(\mu) \).

Corollary 1. Suppose \( d\nu = d\nu_0(r)d\theta/2\pi \). If \( C_z \) is an isometric operator then \( z^n \) is an inner function in \( H^2(\mu) \) for any \( n \geq 0 \). Moreover \( d\mu = d\nu_1(r)d\theta/2\pi + d\delta_{r=0}d\mu_1(\theta) \), where \( \nu_1 \) is a Borel measure on \([0,1]\) and \( \mu_1 \) is a Borel measure on \( \partial D \). If \( \nu_0 \) does not have point mass on \( \{ r = 0 \} \) then \( \nu = \mu \).

Proof. By the remark above, \( C_z \) is isometric from \( H^2(\nu) \) onto \( H^2(\mu) \) because \( \mu_z = \mu \). By Theorem 3, \( z^n \) is inner in \( H^2(\mu) \) for any \( n \geq 0 \). Put \( C_0[0,1] = \{ u ; u \) is continuous on

\[
D \subset [0,1] \]

\[
\mu \]
\[ [0,1] \text{ and } u(0) = 0 \] and \( C_0(\partial D) = \{ f : f \text{ is continuous on } \partial D \text{ and } f(1) = 0 \}. \) Since \( r^nd\mu \) annihilates \( z^P + \overline{z}^P \) for any \( n \geq 0 \), for any \( j \neq 0 \), \( d\mu \perp \{ r^{2n+|j|}e^{ij\theta} ; n = 0, 1, 2, \ldots \}. \) By the Müntz-Szasz theorem \([6]\), \( d\mu \perp C_0[0,1)e^{ij\theta} \) for any \( j \neq 0 \) and so \( d\mu \perp C_0[0,1] \otimes C_0(\partial D). \) This implies that \( d\mu = d\nu_1(r)d\theta/2\pi + d\delta_{r=0}d\mu_1(\theta) \) where \( \nu_1 \) is a Borel measure on \([0,1]\) and \( \mu_1 \) is a Borel measure on \( T. \) If \( \nu_0 \) does not have point mass on \( \{ r = 0 \} \) then we may assume that \( \mu_1 = 0 \) and so \( d\mu = d\nu_1(r)d\theta/2\pi. \) By Theorem 2 \( \nu_0 = \mu_{|z|} \) and \( \mu_{|z|} = \nu_1 \) because \( d\mu = d\nu_1(r)d\theta/2\pi. \) \( \Box \)

§3. Radial measure

In this section we assume that \( \nu \) and \( \mu \) are radial measures, that is, \( d\nu = d\nu_0(r)d\theta/2\pi \) and \( d\mu = d\mu_0(r)d\theta/2\pi. \) Proposition 1 solves Problem 2 when \( \nu = \mu. \) By Theorem 2, if \( C_\phi \) is isometric from \( H^2(\nu) \) into \( H^2(\mu) \), then for some positive integer \( k \)

\[
\int_0^1 \log r d\nu_0 \leq k \int_0^1 \log r d\mu_0 + \int_0^{2\pi} \log |\phi(e^{i\theta})| d\theta
\]

as \( F(t) = \log t, \) using the inner outer factorization of \( \phi. \) Proposition 2 gives an exact formula for this.

**Proposition 1.** Suppose \( \nu \) is a radial measure. If \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2(\nu) \), then \( \phi^n \) is an inner function in \( H^2(\nu) \) for any \( n \geq 0. \)

Proof. By Theorem 1, \( \phi(0) = 0 \) because \( \nu \) is a radial measure and so by Schwarz’s lemma, \( |\phi(z)| \leq |z| \) \( (z \in D). \) Since \( \int_D |\phi(z)|^2 d\nu = \int_D |z|^2 d\nu, \) \( |\phi(z)| = |z| \) a.e. \( \nu. \) For \( f \in \mathcal{P}, \)

\[
\int_D f|\phi|^{2n} d\nu = \int_D f|z|^{2n} d\nu = f(0) \int_0^1 r^{2n} d\nu_0 = \int_\partial D f d\nu \int_D |\phi|^{2n} d\nu. \] \( \Box \)

**Proposition 2.** Suppose \( \nu \) and \( \mu \) are radial measures, that is, \( d\nu = d\nu_0(r)d\theta/2\pi \) and \( d\mu = d\mu_0(r)d\theta/2\pi. \) Let \( \phi = z^kBQ_h \) where \( k \) is a positive integer, \( B \) is a Blaschke product with \( B(0) \neq 0, \) \( Q(z) = \exp -\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\lambda(t) \) is a singular inner function and \( h \) is an outer function. If \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2(\mu) \), then

\[
\int_0^1 \log r d\nu_0 = k \int_0^1 \log r d\mu_0 + \int_0^1 d\mu_0 \int_0^r n(s, B) \frac{ds}{s} + \log |B(0)| - \mu_0([0, 1)) \lambda([0, 2\pi]) + \int_0^{2\pi} \log |\phi(e^{i\theta})| d\theta/2\pi
\]

where \( n(s, B) \) is the number of zeros of \( B \) on the closed disc \( \{ z \in \mathbb{C} : |z| \leq r \}. \)
Proof. Let \( n(s,B) = n(s,BQh) \) be the number of zeros of \( BQh \) on the closed disc \( \{ z \in \mathbb{C}; |z| \leq r \} \). Then, by Theorem 2 and [1, §2 of Chapter 5]

\[
\int_{0}^{1} \log r \, d\nu_0 = \int_{0}^{1} d\mu_0 \int_{0}^{2\pi} \log |\phi(e^{i\theta})| d\theta / 2\pi + \mu_0(\{1\}) \int_{0}^{2\pi} \log |\phi(e^{i\theta})| d\theta / 2\pi \\
= \int_{0}^{1} d\mu_0 \left\{ \log r^k + \int_{0}^{r} n(s,B) \frac{ds}{s} \right\} + \mu_0([0,1)) \log |B(0)Q(0)h(0)| \\
+ \mu_0(\{1\}) \int_{0}^{2\pi} \log |\phi(e^{i\theta})| d\theta / 2\pi \\
= k \int_{0}^{1} \log r \, d\mu_0 + \int_{0}^{1} d\mu_0 \int_{0}^{r} n(s,B) \frac{ds}{s} + \log |B(0)| \\
- \mu_0([0,1)) \lambda([0,2\pi]) + \int_{0}^{2\pi} \log |\phi(e^{i\theta})| d\theta / 2\pi
\]

because \( \mu_0(\{1\}) \int_{0}^{1} n(s,B) \frac{ds}{s} = -\mu_0(\{1\}) \log |B(0)|. \) □

§4. Special cases

In this section we assume that \( \nu \) or \( \mu \) is the normalized Lebesgue measure or the normalized area measure. Proposition 3 solves Problems 1 and 2 when \( \nu \) is the normalized Lebesgue measure on the circle and \( \mu \) is a radial measure. Proposition 5 solves Problem 2 when \( \nu \) is a radial measure or the Lebesgue measure on the circle. Corollary 3 solves Problem 2 negatively when \( d\nu = 2rdrd\theta/2\pi \) and \( d\mu = d\theta/2\pi \).

**Proposition 3.** Let \( \mu \) be a radial measure. \( C_\phi \) is an isometric operator from \( H^2 \) into \( H^2(\mu) \) if and only if \( \phi^n \) is an inner function with \( \int_{\partial D} \phi \, d\mu = 0 \) in \( H^2(\mu) \) for any \( n \geq 1 \) and \( H^2(d\mu) = H^2. \)

Proof. If \( C_\phi \) is isometric, by Theorem 1 \( \int_{\partial D} \phi^n \overline{\phi}^m \, d\mu = 0 \) \((n \neq m)\) and we have

\[
1 = \int_{0}^{2\pi} |z|^2 d\theta / 2\pi = \int_{\partial D} |\phi|^2 \, d\mu \leq 1.
\]

Hence \( |\phi(z)| = 1 \) a.e. \( \mu \) and so supp \( \mu \subset \partial D \). This implies that \( d\mu = d\delta_{r=1} d\theta / 2\pi \) because \( \mu \) is a radial measure. The converse is clear. □

**Proposition 4.** Suppose \( d\nu = d\nu_0(r) d\theta / 2\pi \) and \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2. \)

(1) \( \nu_0(\{a\}) > 0 \) for \( 0 \leq a \leq 1 \) if and only if \( d\theta / 2\pi(\{e^{i\theta}; |\phi(e^{i\theta})| = a\}) > 0. \)
\( d\nu_0 = d\delta_{r=1} \) if and only if \( \phi \) is an inner function in \( H^2 \).

(3) \( \nu_0 \) is a discrete measure if and only if \(|\phi| = \sum_{n=1}^{\infty} a_n \chi_{E_n} \) where \( 0 \leq a_n \leq 1 \), and \( d\theta/2\pi(E_n) = \nu_0 \{a_n\} \) \((n = 1, 2, \cdots)\).

Proof. Since \( \nu_0(G) = d\theta/2\pi\{e^{i\theta}; |\phi(e^{i\theta})| \in G\} \) for a Borel set \( G \) in \([0,1]\) by Theorem 2, it is easy to see. \( \square \)

Proof. This is just (2) of Proposition 4.

Now we consider when \( d\nu = rdrd\theta/\pi \) or \( d\mu = rdrd\theta/\pi \).

**Proposition 5.** If \( C_\phi \) is an isometric operator from \( L^2_a \) into \( H^2(\mu) \), then \( \mu(\{z \in D; |\phi| = b\}) = 0 \) and \( \int_{D}(b - |\phi|)^{-1}d\mu = \infty \) for any \( 0 \leq b \leq 1 \).

Proof. It is clear by Theorem 2. \( \square \)

**Corollary 2.** If \( C_\phi \) is an isometric operator from \( L^2_a \) into \( H^2 \), then \( \phi \) is not inner in \( H^2 \).

**Proposition 6.** Suppose \( d\nu = d\nu_0(r)d\theta/2\pi \). If \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( L^2_a \), then \( \int_0^1 \log rd\nu_0 = -\frac{k}{4} + \int_0^1 2rdr \int_0^r n(s,B) \frac{ds}{s} + \log |B(0)| - \lambda([0,2\pi]) + \int_0^{2\pi} \log |\phi(e^{i\theta})|d\theta/2\pi \), where the inner part of \( \phi \) is \( z^kBQ \), \( B \) is a Blaschke product with \( B(0) \neq 0 \), \( Q(z) = \exp -\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\lambda \) is a singular inner function. Hence if \( \phi \) is a shricht function, then \( \int_0^1 \log rd\nu_0 = -\frac{1}{4} + \int_0^{2\pi} \log |\phi(e^{i\theta})|d\theta/2\pi \).

Proof. It is clear by Proposition 2. \( \square \)

§5. Nevanlinna counting function

Suppose \( \nu \) or \( \mu \) is the normalized Lebesgue measure or the normalized area measure. We assume that \( \phi \) is a non-constant function in \( H^\infty \) with \( ||\phi||_\infty = 1 \). The Nevanlinna counting function of \( \phi \), \( N_\phi \), is defined on \( D\{\phi(0)\} \) by

\[
N_\phi(w) = \sum_{\phi(z) = w} \frac{1}{|z|},
\]

where multiplicities are counted and \( N_\phi(w) \) is taken to be zero if \( w \) is not in the range of \( \phi \). Corollary 4 seems to be interesting in spite of Corollary 3.
Theorem 4. Suppose \( d\nu = d\nu_0(r)d\theta/2\pi \). Then, \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2 \) if and only if

\[
N_\phi(z) = \int_{|z|}^{1} \log \frac{r}{|z|} d\nu_0(r)
\]

for nearly all \( z \) in \( D \).

Proof. The ‘only if’ part was proved in [6, Lemma 3]. If \( N_\phi(z) = \int_{|z|}^{1} \log \frac{r}{|z|} d\nu_0(r) \) for nearly all \( z \) in \( D \), by the Littlewood-Paley theorem (see [3]),

\[
\int_{0}^{2\pi} \phi^n(e^{i\theta})\bar{\phi}^m(e^{i\theta})d\theta/2\pi = 2nm\int_{D} z^{n-1}z^{m-1}N_\phi(|z|)dA(z)
= 4nm\delta_{nm}\int_{0}^{1} r^{n+m-1} \left( \int_{r}^{1} \log \frac{s}{r} d\nu_0(s) \right) dr
= 4nm\delta_{nm}\int_{0}^{1} d\nu_0(s) \int_{0}^{s} r^{m+n-1} \left( \log \frac{s}{r} \right) dr
= \frac{4nm}{(n+m)^2}\delta_{nm}\int_{0}^{1} s^{n+m}d\nu_0(s).
\]

When \( n = m \), \( \int_{0}^{2\pi} |\phi(e^{i\theta})|^{2n}d\theta/2\pi = \int_{0}^{1} s^{2n}d\nu_0(s) \) for \( n = 0, 1, 2, \ldots \). Hence by Theorem 1 and its proof, \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2 \).

Lemma. \( D \setminus \{ z \in D : \phi'(z) = 0 \} \) can be decomposed into an at most countable disjoint collection \( \{ R_n \} \) of “semi-closed” polar rectangles, on each of which \( \phi \) is schricht.

Proof. It is known in [9, p186]. □

Corollary 3. Suppose \( \phi \) is a finite-to-one map. Then \( C_\phi \) is not an isometric operator from \( L^2_a \) into \( H^2 \).

Proof. By Lemma, there exists the inverse \( \psi_n \) of the restriction of \( \phi \) to \( R_n \). Let \( w \in \phi(R_{j_1}) \). If \( \phi \) is an \( \ell \) to 1 map, then there exist \( j_2, \ldots, j_\ell \) such that \( \psi_{j_1}(z) = \psi_{j_2}(z) = \cdots = \psi_{j_\ell}(z) = w \). Hence there exists a small disc \( \Delta \) in \( \phi(R_{j_1}) \) such that

\[
N_\phi(z) = \sum_{z=\phi(w)} \log \frac{1}{|w|} = \sum_{i=1}^{\ell} \log \frac{1}{|\psi_{j_i}(z)|}
\]

for all \( w \in \Delta \). Therefore there exists a subdisc \( \Delta_0 \) in \( \Delta \) such that \( N_\phi(z) \) is harmonic on \( \Delta_0 \). On the other hand, by Proposition 5

\[
N_\phi(z) = 2 \int_{|z|}^{1} \left( \log \frac{r}{|z|} \right) rdr = \frac{|z|^2 - 1}{2} + \log \frac{1}{|z|}.
\]

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This contradicts that \( N_\phi(z) \) is harmonic on \( \Delta_0 \). □

**Theorem 5.** Suppose \( \phi \) is a contractive function in \( H^\infty \) such that \( \phi \) is a finite-to-one map and \( |\phi| = \sum_{j=1}^{\ell} a_j \chi\{E_j \} \), where \( 0 < a_j < a_{j+1} \), \( \sum_{j=1}^{\ell} \chi\{E_j \} = 1 \) and \( E_j \) is a measurable set in \( \partial D \) where \( 1 \leq \ell \leq \infty \). If the inner part of \( z - \phi \) is a Blaschke product for each \( z \in D \), then \( C_\phi \) is not an isometric operator from \( H^2(\nu) \) into \( H^2 \) for any \( d\nu = d\nu_0(r) d\theta/2\pi \) if \( \ell \neq 1 \).

Proof. Suppose \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2 \) for some \( d\nu = d\nu_0(r) d\theta/2\pi \). By Proposition 4, \( \nu_0 \) is a discrete measure and \( d\theta/2\pi(E_j) = \nu_0(\{a_j\}) \) (\( j = 1, 2, \cdots \)). Since \( \phi(0) = 0 \), by Lemma 2 in [6] and Proposition 7

\[
N_\phi(z) = \int_0^{2\pi} \log|z - \phi(e^{i\theta})| d\theta/2\pi + \log \frac{1}{|z|} = \int_{|z|}^1 \log \frac{r}{|z|} d\nu_0(r)
\]

for \( z \in D \setminus \{0\} \). If \( |z| \leq a_1 \), then

\[
\int_{|z|}^1 \log \frac{r}{|z|} d\nu_0(r) = \sum_{j=1}^\infty \left( \log \frac{a_j}{|z|} \right) \nu_0(\{a_j\}) = \sum_{j=1}^\infty \nu_0(\{a_j\}) \log \frac{1}{|z|} + \sum_{j=1}^\infty \nu_0(\{a_j\}) \log a_j = \log \frac{1}{|z|} + \sum_{j=1}^\infty \nu_0(\{a_j\}) \log a_j.
\]

Hence if \( |z| \leq a_1 \) then

\[
\int_0^{2\pi} \log |z - \phi(e^{i\theta})| d\theta/2\pi = \sum_{j=1}^\infty \nu_0(\{a_j\}) \log a_j = \alpha.
\]

If \( a_1 < |z| \leq a_2 \), then

\[
\int_{|z|}^1 \log \frac{r}{|z|} d\nu_0(r) = \sum_{j=2}^\infty \left( \log \frac{a_j}{|z|} \right) \nu_0(\{a_j\}) = \sum_{j=2}^\infty \nu_0(\{a_j\}) \log \frac{1}{|z|} + \sum_{j=2}^\infty \nu_0(\{a_j\}) \log a_j
\]

\[
\int_0^{2\pi} \log |z - \phi(e^{i\theta})| d\theta/2\pi = -\nu_0(\{a_1\}) \log \frac{1}{|z|} + \sum_{j=2}^\infty \nu_0(\{a_j\}) \log a_j = \beta \log \frac{1}{|z|} + \gamma.
\]
where \( \beta \neq 0 \)

For each \( z \in D \), put

\[
z - \phi(\zeta) = q_z(\zeta)h_z(\zeta) \quad (\zeta \in D)
\]

where \( q_z(\zeta) \) is inner and \( h_z(\zeta) \) is outer. Since \( \phi \) is a finite-to-one map, \( q_z \) is a finite Blaschke product by hypothesis and so

\[
q_{\phi(t)}(\zeta) = \prod_{j=1}^{n} \frac{\zeta - b_j(t)}{1 - b_j(t)\zeta} \quad (t \in D).
\]

Then, since \( \phi(0) = 0 \),

\[
\phi(t) = (-1)^n \left( \prod_{j=1}^{n} b_j(t) \right) h_{\phi(t)}(0) \quad (t \in D).
\]

Put \( D_r = \{ t \in \mathbb{C}; |t| \leq r \} \) for \( 0 < r < 1 \). If both \( \phi \) and \( \phi' \) have no zeros on \( \partial D_r \) then there is a division \( \{ D_r^n \}_{1 \leq j \leq n} \) of \( D_r \) such that \( \phi \) is one-to-one on \( D_r^n \) for \( 1 \leq j \leq n \). For, \( \phi \) is conformal in a neighborhood of each point on \( \partial D_r \) and so \( \arg \phi \) is increasing on, \( \partial D_r \). Put \( \phi_j = \phi \mid D_r^n \) and \( b_j(t) = \phi_j^{-1}(\phi(t)) \) for \( 1 \leq j \leq n \). Then \( b_j(t) \) is analytic except \( \phi'(t) = 0 \) when \( \phi(t) \) in \( \phi(D_r) \). Hence \( h_{\phi(t)}(0) \) is analytic except \( \phi'(t) = 0 \) and \( \bigcup_{j=1}^{n} \{ t \in D; b_j(t) = 0 \} \) when \( \phi(t) \) in \( \phi(D_r) \). Since \( \phi(0) = 0 \), \( \{ t \in D; |\phi(t)| < a_1 \} \) is a nonempty open set. We can choose \( r \) such that \( \{ t \in D; |\phi(t)| < a_1 \} \cap \phi(D_r) \neq \emptyset \). If \( |\phi(t)| \leq a_1 \), by what was proved above,

\[
\alpha = \int_{0}^{2\pi} \log |\phi(t) - \phi(e^{i\theta})|d\theta/2\pi
= \int_{0}^{2\pi} \log |h_{\phi(t)}(e^{i\theta})|d\theta/2\pi = \log |h_{\phi(t)}(0)|.
\]

Hence \( |h_{\phi(t)}(0)| = e^\alpha \) and so \( h_{\phi(t)}(0) \) is constant on \( D_r \). If \( a_1 < |\phi(t)| \leq a_2 \), by what was proved above,

\[
\beta \log \frac{1}{|\phi(t)|} + \gamma = \int_{0}^{2\pi} \log |\phi(t) - \phi(e^{i\theta})|d\theta/2\pi
= \int_{0}^{2\pi} \log |h_{\phi(t)}(e^{i\theta})|d\theta/2\pi = \log |h_{\phi(t)}(0)|
\]

and so \( |h_{\phi(t)}(0)| = e^\gamma |\phi(t)|^\beta \). Since there exists \( 0 < r < 1 \) such that \( \{ t \in D; a_1 < |\phi(t)| < a_2 \} \cap \phi(D_r) \neq \emptyset \), this implies that \( |\phi(t)| \) is constant there and so \( \phi \) is constant on \( D \). This contradicts that \( \phi \) is a finite-to-one map. Therefore \( C_\phi \) is not isometric. \( \square \)

If \( \phi \) is a one-to-one map then it is known [4, Theorem 3.17] that the inner part of \( z - \phi \) is a Blaschke product for each \( z \in D \). Hence we need not such a hypothesis in
Theorem 5. Unfortunately we could not prove it in general, that is, for a finite-to-one map.

§6. Rudin’s orthogonal function

In this section, we study Rudin’s orthogonal functions. By Theorem 1, if \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2(\mu) \) then \( \phi \) is a Rudin’s orthogonal function. Proposition 7 implies the converse. This was proved by the author [6] when \( d\mu = d\theta/2\pi \). The proof is valid for an arbitrary \( \mu \). However we give a new proof due to K. Izuchi.

**Proposition 7.** If \( \phi \) is a Rudin’s orthogonal function in \( H^2(\mu) \) then there exists a unique radial measure \( \nu \) such that \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2(\mu) \) where \( d\nu = d\nu_0(r)d\theta/2\pi \) and \( 1 \in \text{supp } \nu_0 \).

Proof. Put \( \nu_0 = \mu|\phi| \) and \( d\nu = d\nu_0d\theta/2\pi \), then Theorems 1 and 2 imply the proposition. □

**Corollary 4.** Suppose \( \phi \) is a finite-to-one map and \( \phi \) is a Rudin’s orthogonal function. If the inner part of \( z - \phi \) is a Blaschke product for each \( z \in D \) and \( |\phi| = \sum_{j=1}^\ell a_j \chi_{E_j} \),

where \( 0 \leq a_j < a_{j+1}, \sum_{j=1}^\ell \chi_{E_j} = 1 \) and \( E_j \) is a measurable set in \( \partial D \) where \( 1 \leq \ell \leq \infty \), then \(|\phi| = 1 \) and so \( \phi \) is a finite Blaschke product.

Proof. If \( \phi \) is a Rudin’s orthogonal function, then by Proposition 7 and Theorem 5, \( \ell = 1 \) and so \( \phi \) is a finite Blaschke product. □

In Corollary 4, if \( \phi \) is one-to-one map then the inner part of \( z - \phi \) is a Blaschke product (see [4. Theorem 3.17]). Hence we can take off such a condition. However in such a case Corollary 4 is not new. In fact, P. S. Bourdon [3] showed that if \( \phi \) is univalent and a Rudin’s orthogonal function then \( \phi \) is just the coordinate function \( z \).

§7. Final remark

The research in this paper gives more general one. Suppose \( 0 < p < \infty \) and \( p \neq 2 \). \( T \) is an isometric operator from \( H^p(\nu) \) into \( H^p(\mu) \) with \( T1 = 1 \) if and only if \( T = C_\phi \) for some \( \phi \) in \( H^\infty \) with \( \|\phi\|_\infty = 1 \) and \( C_\phi \) is an isometric operator from \( H^p(\nu) \) into \( H^p(\mu) \). For the ‘if’ part is trivial. For the ‘only if’ part, if \( T \) is isometric and \( T1 = 1 \), then by [5, Theorem 7.5.3] \( T(fg) = Tf \cdot Tg \) a.e. \( \mu \) and \( \|Tf\|_\infty = \|f\|_\infty \) for
all \( f \in \mathcal{P}, \ g \in \mathcal{P} \). Hence if \( \phi = Tz \) then \( \phi \) belongs to \( H^\infty \) and \( \| \phi \|_\infty = 1 \). Therefore \( Tf = C_\phi f \) \((f \in \mathcal{P})\) and so \( Tf = C_\phi f \) \((f \in H^p(\nu))\). When \( p \neq 2 \), if \( C_\phi \) is an isometric operator from \( H^p(\nu) \) into \( H^p(\mu) \), then \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2(\mu) \). For by [5, Theorem 8.5.3], for all \( f \in \mathcal{P} \) and \( g \in \mathcal{P} \)

\[
\int_D C_\phi f \cdot \overline{C_\phi g} \, d\mu = \int_D f \overline{g} \, d\mu
\]

and \( \| C_\phi f \|_\infty = \| f \|_\infty \). This implies that \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2(\mu) \).

We give two open problems:

1. Are there any isometric \( C_\phi \) from \( L^2_a \) into \( H^2 \) ?

2. When \( \nu_0 \) is a discrete measure and not a dirac measure, are there any isometric \( C_\phi \) from \( H^2(\nu) \) to \( H^2 \) where \( dv = d\nu_0(r) d\theta/2\pi \) ?

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