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Isometric Composition Operators Between Two Weighted Hardy Spaces

By

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Abstract. We study isometric composition operators $C_\phi$ between two weighted Hardy spaces $H^2(\nu)$ and $H^2(\mu)$ when $\nu$ is a radial measure. The isometric $C_\phi$ is related to a moment sequence and such a $\phi$ is studied by the Nevanlinna counting function of $\phi$ when $\mu$ is the normalized Lebesgue measure on the unit circle.

§1. Introduction

Let $D$ be the open unit disc in the complex plane $\mathbb{C}$. We denote by $\mathcal{P}$ the set of all analytic polynomials and $H$ the set of all analytic functions on $D$. Let $\mu$ be a positive Borel measure on $\overline{D}$ with $\mu(\overline{D}) = 1$. $H^p(\mu)$ denotes the closure of all analytic polynomials in $L^p(d\mu)$ for $0 < p < \infty$. If $d\mu = d\theta/2\pi$, then $H^p(\mu) = H^p$ is the classical Hardy space. If $d\mu = 2rdrd\theta/2\pi$, then $H^p(\mu) = L^p_a$ is the classical Bergman space. $H^p$ and $L^p_a$ can be embedded in $H$. In this paper, we assume that $H^p(\mu)$ is embedded in $H$ for a general $\mu$. $H^\infty$ denotes the set of all bounded analytic functions on $D$. We also assume that $H^\infty = H \cap L^\infty(d\mu)$.

For an analytic self map $\phi$ of $D$, the composition operator $C_\phi$ is defined by $(C_\phi f)(z) = f(\phi(z)) \ (z \in D)$ for $f$ in $H$. Throughout this paper, we assume that $\nu$ and $\mu$ are positive Borel measures on $\overline{D}$ with $\nu(\overline{D}) = \mu(\overline{D}) = 1$. $\nu$ is called a radial measure if $d\nu = dv_0(r)d\theta/2\pi$ for a positive Borel measure $v_0$ on $[0, 1]$. Since $d\theta/2\pi = d\delta_{r=1}d\theta/2\pi$, $d\theta/2\pi$ is a radial measure.

In this paper, we studied isometric composition operators from $H^2(\nu)$ into $H^2(\mu)$ when $\nu$ is a radial measure. As we show in the final section, our isometric composition operator $C_\phi$ is related to an isometric operator $T$ from $H^p(\nu)$ into $H^p(\mu)$ with $T1 = 1$ when $p \neq 2$. We have a long history for such isometric operators (see [8]). The onto isometries on $H^p$ or $L^p_a$ for $p \neq 2$ were described completely. Unfortunately into isometries have been known very little.

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Problem 1. For given measures $\nu$ and $\mu$, does there exist an isometric composition operator $C_\phi$ from $H^2(\nu)$ into $H^2(\mu)$? If there exists such a $C_\phi$, describe $\phi$.

A function $F$ in $H^2(\mu)$ is called an inner function in $H^2(\mu)$ if

$$ \int_D |F|^2 d\mu = \int_D f d\mu \int_D |F|^2 d\mu \quad (f \in \mathcal{P}). $$

If $\phi^n$ is an inner function in $H^2(\mu)$ with $\int_D \phi d\mu = 0$ for any $n \geq 0$ then there exists a unique radial measure $\nu$ such that $C_\phi$ is isometric from $H^2(\nu)$ into $H^2(\mu)$ where $d\nu = dv_0(r) d\theta/2\pi$ and $1 \in \text{supp } \nu_0$. This is not difficult to prove. However we don’t know whether the converse is true.

Problem 2. If a composition operator $C_\phi$ is isometric from $H^2(\nu)$ into $H^2(\mu)$ then is $\phi^n$ an inner function in $H^2(\mu)$ with $\int_D \phi d\mu = 0$ for any $n \geq 0$?

A function $\phi$ in $H^\infty$ with $\|\phi\|_\infty = 1$ is called a Rudin’s orthogonal function in $H^2(\mu)$ if $\{\phi^n; n = 0, 1, 2, \ldots\}$ is a set of orthogonal functions in $H^2(\mu)$. If $\phi^n$ is an inner function in $H^2(\mu)$ with $\int_D \phi d\mu = 0$ for any $n \geq 0$ and $\|\phi\|_\infty = 1$ then $\phi$ is a Rudin’s orthogonal function in $H^2(\mu)$ because $\mathcal{P}$ is dense in $H^2(\mu)$ by its definition. We can ask whether the converse is true or not.

Problem 3. If $\phi$ is a Rudin’s orthogonal function in $H^2(\mu)$ then is $\phi^n$ an inner function in $H^2(\mu)$ with $\int_D \phi d\mu = 0$ for any $n \geq 0$?

When $d\mu = d\theta/2\pi$, Problem 3 was studied by several people, for example, [2],[3],[5],[6] and [10]. C. Bishop [2] and C. Sundberg [10] gave counter examples. Hence there exists a Rudin’s orthogonal function which is not an inner function in $H^2(d\theta/2\pi)$.

Problem 3 has a strong connection with Problem 2. In fact, if $C_\phi$ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$ then by Theorem 1 $\phi$ is a Rudin’s orthogonal function in $H^2(\mu)$. Conversely if $\phi$ is a Rudin’s orthogonal function in $H^2(\mu)$ then by Proposition 8 there exists a unique radial measure $\nu$ such that $C_\phi$ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$.

For each $\phi$, we will use two Borel measures $\mu_\phi$ on $\overline{D}$ and $\mu_{|\phi|}$ on $[0,1]$. For a Borel set $E$ in $\overline{D}$ $\mu_\phi(E) = \mu(\{z \in \mathcal{D}; \phi(z) \in E\})$ and for a Borel set $G$ in $[0,1]$ $\mu_{|\phi|}(G) = \mu(\{z \in \overline{D}; |\phi(z)| \in G\})$.  

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§2. General case

In this section we assume that $\nu$ is a radial measure, $\mu$ is an arbitrary measure and $\phi$ is an analytic selfmap with $\|\phi\|_\infty = 1$. We say that $\{a_n\}$ is a moment sequence of $\nu_0$, a positive Borel measure on $[0,1]$, if $a_n = \int_0^1 r^n d\nu_0 \quad (n = 0, 1, 2, \ldots)$.

**Theorem 1.** Suppose $d\nu = d\nu_0(r)d\theta/2\pi$. Then $C_\phi$ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$ if and only if $\int_D \phi^n \bar{\phi}_m d\mu = 0 \ (n \neq m)$ and $\left\{\int_D |\phi|^n d\mu\right\}$ is a moment sequence of $\nu_0$.

Proof. If $C_\phi$ is isometric, by the polarization formula

$$\delta_{nm} \int_0^1 r^n r^m d\nu_0(r) = \int_D z^n \bar{z}^m d\nu = \int_D \phi^n \bar{\phi}_m d\mu$$

because $\nu$ is a radial measure. Hence

$$\int_D |\phi|^{2n} d\mu = \int_0^1 r^{2n} d\nu_0 \quad (n = 0, 1, 2, \ldots).$$

It is elementary to see that $x = \sqrt{1 - (1 - x^2)} = \sum_{n=0}^\infty a_n(1 - x^2)^n$ and $\sum_{n=0}^\infty |a_n|(1 - x^2)^n < \infty \ (0 \leq x \leq 1)$. Hence by Lebesgue’s dominated convergence theorem

$$\int_D |\phi| d\mu = \int_D \sum_{n=0}^\infty a_n(1 - |\phi|^2)^n d\mu = \sum_{n=0}^\infty a_n \int_D (1 - |\phi|^2)^n d\mu$$

$$= \sum_{n=0}^\infty a_n \int_0^1 (1 - r^2)^n d\nu_0 = \int_0^1 \sum_{n=0}^\infty a_n (1 - r^2)^n d\nu_0 = \int_0^1 r d\nu_0$$

because $\sum_{n=0}^k a_n(1 - |\phi|^2)^n \leq \sum_{n=0}^\infty |a_n|$ and $\sum_{n=0}^k a_n(1 - r^2)^2 \leq \sum_{n=0}^\infty |a_n| < \infty$. Similarly, as $x^{2n+1} = \sqrt{1 - (1 - x^{4n+2})}$ we can show that $\int_D |\phi|^{2n+1} d\mu = \int_0^1 r^{2n+1} d\nu_0 \ (n = 0, 1, 2, \ldots)$. Thus $\left\{\int_D |\phi|^n d\mu\right\}$ is a moment sequence of $\nu_0$.

Conversely if $\int_D \phi^n \bar{\phi}_m d\mu = 0 \ (n \neq m)$ and $\left\{\int_D |\phi|^n d\mu\right\}$ is a moment sequence of $\nu_0$, then

$$\int_D \left|\sum_{n=0}^k a_n \phi^n\right|^2 d\mu = \sum_{n=0}^k |a_n|^2 \int_D |\phi|^2 d\mu$$

$$= \sum_{n=0}^k |a_n|^2 \int_0^1 r^{2n} d\nu_0 = \int_D \left|\sum_{n=0}^k a_n z^n\right|^2 d\nu.$$
Hence $C_\phi$ is isometric. □

**Theorem 2.** If $d\nu = dv_0(r)d\theta/2\pi$ then the following conditions are equivalent.

1. $C_\phi$ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$.
2. $\nu_0 = \mu_{|\phi|}$
3. $\int_0^1 F(r)d\nu_0 = \int_D F(|\phi|)d\mu$ for any Borel nonnegative function $F$ on $[0,1]$.

Proof. (1) $\Rightarrow$ (2) If $G$ is a Borel set in $[0,1]$, then $\nu_0(G) = \inf\{\nu_0(V) : G \subset V, V$ is open in $[0,1]\}$ because $\nu_0$ is a Borel measure. Hence there exists a sequence of continuous functions $\{f_m\}$ such that $f_m \rightarrow \chi_G$ a.e. $\nu_0$ on $[0,1]$ and $\|f_m\|_\infty \leq \gamma < \infty (m = 1, 2, \ldots)$. By the Stone-Weierstrass theorem,

$$\int_0^1 f_m(r)d\nu_0 = \int_D f_m(|\phi|)d\mu \quad (m = 1, 2, \ldots)$$

because $\int_0^1 r^n d\nu_0 = \int_D |\phi|^n d\mu \quad (n = 0, 1, 2, \ldots)$. Thus $\nu_0(G) = \mu(\{z \in \partial D : |\phi(z)| \in G\})$.

(2) $\Rightarrow$ (3) is clear. (3) $\Rightarrow$ (1) is a result of Theorem 1. □

The following theorem shows that we can solve Problem 2 in the Introduction when $C_\phi$ is onto.

**Theorem 3.** Suppose $d\nu = dv_0(r)d\theta/2\pi$. If $C_\phi$ is an isometric operator from $H^2(\nu)$ onto $H^2(\mu)$ then $\phi^n$ is an inner function in $H^2(\mu)$ for any $n \geq 0$.

Proof. Let $F \in \mathcal{P}$ then there exists $f \in H^2(\nu)$ such that $F = f \circ \phi$. Let $f = \sum_{j=0}^\infty a_j z^j$, since $\sum_{j=0}^\infty |a_j|^2 \int_0^1 r^{2j}d\nu_0(r) < \infty$, $F = \sum_{j=0}^\infty a_j \phi^j$ and $\sum_{j=0}^\infty |a_j|^2 \int_D |\phi|^{2j}d\mu < \infty$. By Theorem 1, for any $\ell \geq 0$

$$\int_D F|\phi|^{2\ell}d\mu = a_0 \int_D |\phi|^{2\ell}d\mu = \int_D Fd\mu \int_D |\phi|^{2\ell}d\mu$$

because $\int_D |\phi|d\mu = 0$. This implies that $\phi^\ell$ is an inner function in $H^2(\mu)$ for any $\ell \geq 0$.

When $d\nu = dv_0(r)d\theta/2\pi$, if $C_\phi$ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$ then $C_z$ is isometric from $H^2(\nu)$ onto $H^2(\mu_0)$.

**Corollary 1.** Suppose $d\nu = dv_0(r)d\theta/2\pi$. If $C_z$ is an isometric operator then $z^n$ is an inner function in $H^2(\mu)$ for any $n \geq 0$. Moreover $d\mu = dv_1(r)d\theta/2\pi + d\delta_{r=0}d\mu_1(\theta)$.

where $v_1$ is a Borel measure on $[0,1]$ and $\mu_1$ is a Borel measure on $\partial D$. If $\nu_0$ does not have point mass on $\{r = 0\}$ then $\nu = \mu$.

Proof. By the remark above, $C_z$ is isometric from $H^2(\nu)$ onto $H^2(\mu)$ because $\mu_z = \mu$. By Theorem 3, $z^n$ is inner in $H^2(\mu)$ for any $n \geq 0$. Put $C_0[0,1] = \{u; u$ is continuous on
[0,1] and \( u(0) = 0 \} and \( C_0(\partial D) = \{ f : f \text{ is continuous on } \partial D \text{ and } f(1) = 0 \}. \) Since \( r^n d\mu \) annihilates \( zP + \bar{z}P \) for any \( n \geq 0, \) for any \( j \neq 0, \) \( d\mu \perp \{ r^{2n+|j|} e^{ij\theta} ; n = 0, 1, 2, \ldots \} \). By the Müntz-Szasz theorem \([6], \) \( d\mu \perp C_0[0,1] e^{ij\theta} \) for any \( j \neq 0 \) and so \( d\mu \perp C_0[0,1] \otimes C_0(\partial D). \) This implies that \( d\mu = dv_1(r) d\theta/2\pi + d\nu_1 = d\mu_1(\theta) \) where \( \nu_1 \) is a Borel measure on \([0,1] \) and \( \mu_1 \) is a Borel measure on \( T. \) If \( \nu_0 \) does not have point mass on \( \{ r = 0 \} \) then we may assume that \( \mu_1 = 0 \) and so \( d\mu = dv_1(r) d\theta/2\pi. \) By Theorem 2 \( \nu_0 = \mu_{|z|} \) and \( \mu_{|z|} = \nu_1 \) because \( d\mu = dv_1(r) d\theta/2\pi. \) □

§3. Radial measure

In this section we assume that \( \nu \) and \( \mu \) are radial measures, that is, \( d\nu = dv_0(r) d\theta/2\pi \) and \( d\mu = d\mu_0(r) d\theta/2\pi. \) Proposition 1 solves Problem 2 when \( \nu = \mu. \) By Theorem 2, if \( C_\phi \) is isometric from \( H^2(\nu) \) into \( H^2(\mu) \), then for some positive integer \( k \)

\[
\int_0^1 \log r d\nu_0 \leq k \int_0^1 \log r d\mu_0 + \int_0^{2\pi} \log |\phi(e^{i\theta})| d\theta
\]

as \( F(t) = \log t \), using the inner outer factorization of \( \phi. \) Proposition 2 gives an exact formula for this.

**Proposition 1.** Suppose \( \nu \) is a radial measure. If \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2(\nu) \), then \( \phi^n \) is an inner function in \( H^2(\nu) \) for any \( n \geq 0. \)

Proof. By Theorem 1, \( \phi(0) = 0 \) because \( \nu \) is a radial measure and so by Schwarz’s lemma, \( |\phi(z)| \leq |z| \) (\( z \in D \)). Since \( \int_D |\phi(z)|^2 d\nu = \int_D |z|^2 d\nu, \) \( |\phi(z)| = |z| \) a.e. \( \nu. \) For \( f \in \mathcal{P}, \)

\[
\int_D f |\phi|^{2n} d\nu = \int_D f |z|^{2n} d\nu = f(0) \int_0^1 r^{2n} d\nu_0 = \int_D f d\nu \int_D |\phi|^{2n} d\nu. \]

**Proposition 2.** Suppose \( \nu \) and \( \mu \) are radial measures, that is, \( d\nu = dv_0(r) d\theta/2\pi \) and \( d\mu = d\mu_0(r) d\theta/2\pi. \) Let \( \phi = z^k BQh \) where \( k \) is a positive integer, \( B \) is a Blaschke product with \( B(0) \neq 0, \) \( Q(z) = \exp - \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\lambda(t) \) is a singular inner function and \( h \) is an outer function. If \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2(\mu) \), then

\[
\int_0^1 \log r d\nu_1 = k \int_0^1 \log r d\mu_0 + \int_0^1 d\mu_0 \int_0^r n(s,B) \frac{ds}{s} + \int_0^{2\pi} \log |B(\theta)| - \mu_0([0,1]) \lambda([0,2\pi]) + \int_0^{2\pi} \log |\phi(e^{i\theta})| d\theta/2\pi
\]

where \( n(s,B) \) is the number of zeros of \( B \) on the closed disc \( \{ z \in \mathbb{C} : |z| \leq r \}. \)
Proof. Let \( n(s, B) = n(s, BQh) \) is the number of zeros of \( BQh \) on the closed disc \( \{ z \in \mathbb{C} ; |z| \leq r \} \). Then, by Theorem 2 and [1, §2 of Chapter 5]

\[
\int_0^1 \log r \, d\nu_0 = \int_0^1 d\mu_0 \int_0^{2\pi} \log |\phi(re^{i\theta})| \, d\theta/2\pi + \mu_0(\{1\}) \int_0^{2\pi} \log |\phi(e^{i\theta})| \, d\theta/2\pi
\]

\[
= \int_0^1 d\mu_0 \left\{ \log r^k + \int_0^r n(s, B) \frac{ds}{s} \right\} + \mu_0([0, 1)) \log |B(0)Q(0)h(0)| + \mu_0(\{1\}) \int_0^{2\pi} \log |\phi(e^{i\theta})| \, d\theta/2\pi
\]

\[
= k \int_0^1 \log r \, d\mu_0 + \int_0^1 d\mu_0 \int_0^r n(s, B) \frac{ds}{s} + \log |B(0)|
\]

\[
- \mu_0([0, 1)) \lambda([0, 2\pi]) + \int_0^{2\pi} \log |\phi(e^{i\theta})| \, d\theta/2\pi
\]

because \( \mu_0(\{1\}) \int_0^1 n(s, B) \frac{ds}{s} = -\mu_0(\{1\}) \log |B(0)| \). □

§4. Special cases

In this section we assume that \( \nu \) or \( \mu \) is the normalized Lebesgue measure or the normalized area measure. Proposition 3 solves Problems 1 and 2 when \( \nu \) is the normalized Lebesgue measure on the circle and \( \mu \) is a radial measure. Proposition 5 solves Problem 2 when \( \nu \) is a radial measure or the Lebesgue measure on the circle. Corollary 3 solves Problem 2 negatively when \( d\nu = 2rdrd\theta/2\pi \) and \( d\mu = d\theta/2\pi \).

Proposition 3. Let \( \mu \) be a radial measure. \( C_\phi \) is an isometric operator from \( H^2 \) into \( H^2(\mu) \) if and only if \( \phi^n \) is an inner function with \( \int_D \phi \, d\mu = 0 \) in \( H^2(\mu) \) for any \( n \geq 1 \) and \( H^2(d\mu) = H^2 \).

Proof. If \( C_\phi \) is isometric, by Theorem 1 \( \int_D \phi^n \overline{\phi}^m \, d\mu = 0 \) \( (n \neq m) \) and we have

\[
1 = \int_0^{2\pi} |z|^2 d\theta/2\pi = \int_D |\phi|^2 \, d\mu \leq 1.
\]

Hence \( |\phi(z)| = 1 \) a.e. \( \mu \) and so \( \text{supp} \, \mu \subset \partial D \). This implies that \( d\mu = d\delta_{r=1}d\theta/2\pi \) because \( \mu \) is a radial measure. The converse is clear. □

Proposition 4. Suppose \( d\nu = d\nu_0(r)d\theta/2\pi \) and \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2 \).

1) \( \nu_0([a]) > 0 \) for \( 0 \leq a \leq 1 \) if and only if \( d\theta/2\pi(\{e^{i\theta} ; |\phi(e^{i\theta})| = a\}) > 0 \).
\[ d\nu_0 = d\delta_{r=1} \text{ if and only if } \phi \text{ is an inner function in } H^2. \]

(3) \( \nu_0 \) is a discrete measure if and only if \(|\phi| = \sum_{n=1}^{\infty} a_n \chi_{E_n} \) where \( 0 \leq a_n \leq 1 \), and \( d\theta/2\pi(E_n) = \nu_0(\{a_n\}) \) \((n = 1, 2, \cdots)\).

Proof. Since \( \nu_0(G) = d\theta/2\pi \{ e^{i\theta}; |\phi(e^{i\theta})| \in G \} \) for a Borel set \( G \) in \([0,1]\) by Theorem 2, it is easy to see. □

Proof. This is just (2) of Proposition 4.

Now we consider when \( d\nu = rdrd\theta/\pi \) or \( d\mu = rdrd\theta/\pi \).

**Proposition 5.** If \( C_\phi \) is an isometric operator from \( L^2_a \) into \( H^2(\mu) \), then \( \mu(\{z \in D; |\phi| = b\}) = 0 \) and \( \int_D (b - |\phi|)^{-1}d\mu = \infty \) for any \( 0 \leq b \leq 1 \).

Proof. It is clear by Theorem 2. □

**Corollary 2.** If \( C_\phi \) is an isometric operator from \( L^2_a \) into \( H^2 \), then \( \phi \) is not inner in \( H^2 \).

**Proposition 6.** Suppose \( d\nu = dv_0(r)d\theta/2\pi \). If \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( L^2_a \), then \( \int_0^1 \log rdv_0 = -\frac{k}{4} + \int_0^1 2rdr \int_0^r n(s,B) \frac{ds}{s} + \log |B(0)| - \lambda([0,2\pi]) \)

\[ + \int_0^{2\pi} \log |\phi(e^{i\theta})|d\theta/2\pi, \text{ where the inner part of } \phi \text{ is } z^kBQ, \text{ } B \text{ is a Blaschke product with } B(0) \neq 0, \text{ } Q(z) = \exp -\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}d\lambda \text{ is a singular inner function. Hence if } \phi \text{ is a shricht function, then } \int_0^1 \log rdv_0 = -\frac{1}{4} + \int_0^{2\pi} \log |\phi(e^{i\theta})|d\theta/2\pi. \]

Proof. It is clear by Proposition 2. □

§5. Nevanlinna counting function

Suppose \( \nu \) or \( \mu \) is the normalized Lebesgue measure or the normalized area measure. We assume that \( \phi \) is a non-constant function in \( H^\infty \) with \( \|\phi\|_\infty = 1 \). The Nevanlinna counting function of \( \phi \), \( N_\phi \), is defined on \( D \backslash \{\phi(0)\} \) by

\[ N_\phi(w) = \sum_{\phi(z) = w} \log \frac{1}{|z|}, \]

where multiplicities are counted and \( N_\phi(w) \) is taken to be zero if \( w \) is not in the range of \( \phi \). Corollary 4 seems to be interesting in spite of Corollary 3.
**Theorem 4.** Suppose \( d\nu = d\nu_0(r)d\theta/2\pi \). Then, \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2 \) if and only if

\[
N_\phi(z) = \int_{|z|}^1 \log \frac{r}{|z|} d\nu_0(r)
\]

for nearly all \( z \) in \( D \).

Proof. The ‘only if’ part was proved in [6, Lemma 3]. If \( N_\phi(z) = \int_{|z|}^1 \log \frac{r}{|z|} d\nu_0(r) \) for nearly all \( z \) in \( D \), by the Littlewood-Paley theorem (see [3]),

\[
\int_0^{2\pi} \phi^n(e^{i\theta})\overline{\phi}^m(e^{i\theta})d\theta/2\pi = 2nm \int_B z^{n-1}z^{m-1}N_\phi(|z|)dA(z)
= 4nm\delta_{nm} \int_0^1 r^{n+m-1} \left( \int_{r}^1 \log \frac{s}{r} d\nu_0(s) \right) dr
= 4nm\delta_{nm} \int_0^1 d\nu_0(s) \int_0^s r^{m+n-1} \left( \log \frac{s}{r} \right) dr
= \frac{4nm}{(n+m)^2} \delta_{nm} \int_0^1 s^{n+m}d\nu_0(s).
\]

When \( n = m \), \( \int_0^{2\pi} |\phi(e^{i\theta})|^{2n}d\theta/2\pi = \int_0^1 s^{2n}d\nu_0(s) \) for \( n = 0, 1, 2, \cdots \). Hence by Theorem 1 and its proof, \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2 \).

**Lemma.** \( D \setminus \{ z \in D : \phi'(z) = 0 \} \) can be decomposed into an at most countable disjoint collection \( \{ R_n \} \) of “semi-closed” polar rectangles, on each of which \( \phi \) is schricht.

Proof. It is known in [9, p186]. □

**Corollary 3.** Suppose \( \phi \) is a finite-to-one map. Then \( C_\phi \) is not an isometric operator from \( L^2(\nu) \) into \( H^2 \).

Proof. By Lemma, there exists the inverse \( \psi_n \) of the restriction of \( \phi \) to \( R_n \). Let \( w \in \phi(R_{j_1}) \). If \( \phi \) is an \( \ell \) to 1 map, then there exist \( j_2, \cdots, j_\ell \) such that \( \psi_{j_1}(z) = \psi_{j_2}(z) = \cdots = \psi_{j_\ell}(z) = w \). Hence there exists a small disc \( \Delta \) in \( \phi(R_{j_1}) \) such that

\[
N_\phi(z) = \sum_{z=\phi(w)} \log \frac{1}{|w|} = \sum_{t=1}^\ell \log \frac{1}{|\psi_{j_t}(z)|}
\]

for all \( w \in \Delta \). Therefore there exists a subdisc \( \Delta_0 \) in \( \Delta \) such that \( N_\phi(z) \) is harmonic on \( \Delta_0 \). On the other hand, by Proposition 5

\[
N_\phi(z) = 2 \int_{|z|}^1 \left( \log \frac{r}{|z|} \right) r dr = \frac{|z|^2 - 1}{2} + \log \frac{1}{|z|}.
\]
This contradicts that $N_\phi(z)$ is harmonic on $\Delta_0$. □

**Theorem 5.** Suppose $\phi$ is a contractive function in $H^\infty$ such that $\phi$ is a finite-to-one map and $|\phi| = \sum_{j=1}^{\ell} a_j \chi_{E_j}$ where $0 < a_j < a_{j+1}$, $\sum_{j=1}^{\ell} \chi_{E_j} = 1$ and $E_j$ is a measurable set in $\partial D$ where $1 \leq \ell \leq \infty$. If the inner part of $z - \phi$ is a Blaschke product for each $z \in D$, then $C_\phi$ is not an isometric operator from $H^2(\nu)$ into $H^2$ for any $d\nu = d\nu_0(r)d\theta/2\pi$ if $\ell \neq 1$.

Proof. Suppose $C_\phi$ is an isometric operator from $H^2(\nu)$ into $H^2$ for some $d\nu = d\nu_0(r)d\theta/2\pi$. By Proposition 4, $\nu_0$ is a discrete measure and $d\theta/2\pi(E_j) = \nu_0(\{a_j\})$ $(j = 1, 2, \ldots)$. Since $\phi(0) = 0$, by Lemma 2 in [6] and Proposition 7

\[
N_\phi(z) = \int_0^{2\pi} \log |z - \phi(e^{i\theta})|d\theta/2\pi + \log \frac{1}{|z|} = \int_{|z|}^{1} \log \frac{r}{|z|}d\nu_0(r)
\]

for $z \in D\setminus\{0\}$. If $|z| \leq a_1$, then

\[
\int_{|z|}^{1} \log \frac{r}{|z|}d\nu_0(r) = \sum_{j=1}^\infty \left( \log \frac{a_j}{|z|} \right) \nu_0(\{a_j\}) = \sum_{j=1}^\infty \nu_0(\{a_j\}) \log \frac{1}{|z|} + \sum_{j=1}^\infty \nu_0(\{a_j\}) \log a_j = \log \frac{1}{|z|} + \sum_{j=1}^\infty \nu_0(\{a_j\}) \log a_j.
\]

Hence if $|z| \leq a_1$ then

\[
\int_0^{2\pi} \log |z - \phi(e^{i\theta})|d\theta/2\pi = \sum_{j=1}^\infty \nu_0(\{a_j\}) \log a_j = \alpha.
\]

If $a_1 < |z| \leq a_2$, then

\[
\int_{|z|}^{1} \log \frac{r}{|z|}d\nu_0(r) = \sum_{j=2}^\infty \left( \log \frac{a_j}{|z|} \right) \nu_0(\{a_j\}) = \sum_{j=2}^\infty \nu_0(\{a_j\}) \log \frac{1}{|z|} + \sum_{j=2}^\infty \nu_0(\{a_j\}) \log a_j
\]

\[
\int_0^{2\pi} \log |z - \phi(e^{i\theta})|d\theta/2\pi = -\nu_0(\{a_1\}) \log \frac{1}{|z|} + \sum_{j=2}^\infty \nu_0(\{a_j\}) \log a_j = \beta \log \frac{1}{|z|} + \gamma.
\]

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where $\beta \neq 0$

For each $z \in D$, put

$$z - \phi(\zeta) = q_z(\zeta)h_z(\zeta) \quad (\zeta \in D)$$

where $q_z(\zeta)$ is inner and $h_z(\zeta)$ is outer. Since $\phi$ is a finite-to-one map, $q_z$ is a finite Blaschke product by hypothesis and so

$$q_{\phi(t)}(\zeta) = \prod_{j=1}^{n} \frac{\zeta - b_j(t)}{1 - b_j(t)\zeta} \quad (t \in D).$$

Then, since $\phi(0) = 0$,

$$\phi(t) = (-1)^n \left( \prod_{j=1}^{n} b_j(t) \right) h_{\phi(t)}(0) \quad (t \in D).$$

Put $D_r = \{ t \in \mathbb{C}; |t| \leq r \}$ for $0 < r < 1$. If both $\phi$ and $\phi'$ have no zeros on $\partial D_r$ then there is a division $\{ D_r^j \}_{1 \leq j \leq n}$ of $D_r$ such that $\phi$ is one-to-one on $D_r^j$ for $1 \leq j \leq n$. For, $\phi$ is conformal in a neighborhood of each point on $\partial D_r$ and so arg $\phi$ is increasing on, $\partial D_r$. Put $\phi_j = \phi | D_r^j$ and $b_j(t) = \phi_j^{-1}(\phi(t))$ for $1 \leq j \leq n$. Then $b_j(t)$ is analytic except $\phi'(t) = 0$ when $\phi(t)$ in $\phi(D_r)$. Hence $h_{\phi(t)}(0)$ is analytic except $\phi'(t) = 0$ and $\bigcup_{j=1}^{n} \{ t \in D; b_j(t) = 0 \}$ when $\phi(t)$ in $\phi(D_r)$. Since $\phi(0) = 0$, $\{ t \in D; |\phi(t)| < a_1 \}$ is a nonempty open set. We can choose $r$ such that $\{ t \in D; |\phi(t)| < a_1 \} \cap \phi(D_r) \neq \emptyset$. If $|\phi(t)| \leq a_1$, by what was proved above,

$$\alpha = \int_{0}^{2\pi} \log |\phi(t) - \phi(e^{i\theta})|d\theta/2\pi$$

$$= \int_{0}^{2\pi} \log |h_{\phi(t)}(e^{i\theta})|d\theta/2\pi = \log |h_{\phi(t)}(0)|.$$ 

Hence $|h_{\phi(t)}(0)| = e^\alpha$. and so $h_{\phi(t)}(0)$ is constant on $D_r$. If $a_1 < |\phi(t)| \leq a_2$, by what was proved above,

$$\beta \log \frac{1}{|\phi(t)|} + \gamma = \int_{0}^{2\pi} \log |\phi(t) - \phi(e^{i\theta})|d\theta/2\pi$$

$$= \int_{0}^{2\pi} \log |h_{\phi(t)}(e^{i\theta})|d\theta/2\pi = \log |h_{\phi(t)}(0)|$$

and so $|h_{\phi(t)}(0)| = e^\gamma|\phi(t)|^{\beta}$. Since there exists $0 < r < 1$ such that $\{ t \in D; a_1 < |\phi(t)| < a_2 \} \cap \phi(D_r) \neq \emptyset$, this implies that $|\phi(t)|$ is constant there and so $\phi$ is constant on $D$. This contradicts that $\phi$ is a finite-to-one map. Therefore $C_{\phi}$ is not isometric. □

If $\phi$ is a one-to-one map then it is known [4, Theorem 3.17] that the inner part of $z - \phi$ is a Blaschke product for each $z \in D$. Hence we need not such a hypothesis in
Theorem 5. Unfortunately we could not prove it in general, that is, for a finite-to-one map.

§6. Rudin’s orthogonal function

In this section, we study Rudin’s orthogonal functions. By Theorem 1, if $C_\phi$ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$ then $\phi$ is a Rudin’s orthogonal function. Proposition 7 implies the converse. This was proved by the author [6] when $d\mu = d\theta/2\pi$. The proof is valid for an arbitrary $\mu$. However we give a new proof due to K. Izuchi.

Proposition 7. If $\phi$ is a Rudin’s orthogonal function in $H^2(\mu)$ then there exists a unique radial measure $\nu$ such that $C_\phi$ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$ where $d\nu = d\nu_0(r)d\theta/2\pi$ and $1 \in \text{supp } \nu_0$.

Proof. Put $\nu_0 = \mu|\phi|$ and $d\nu = d\nu_0d\theta/2\pi$, then Theorems 1 and 2 imply the proposition. □

Corollary 4. Suppose $\phi$ is a finite-to-one map and $\phi$ is a Rudin’s orthogonal function. If the inner part of $z - \phi$ is a Blaschke product for each $z \in D$ and $|\phi| = \sum_{j=1}^{\ell} a_j \chi_{E_j}$, where $0 \leq a_j < a_{j+1}$, $\sum_{j=1}^{\ell} \chi_{E_j} = 1$ and $E_j$ is a measurable set in $\partial D$ where $1 \leq \ell \leq \infty$, then $|\phi| = 1$ and so $\phi$ is a finite Blaschke product.

Proof. If $\phi$ is a Rudin’s orthogonal function, then by Proposition 7 and Theorem 5, $\ell = 1$ and so $\phi$ is a finite Blaschke product. □

In Corollary 4, if $\phi$ is one-to-one map then the inner part of $z - \phi$ is a Blaschke product (see [4, Theorem 3.17]). Hence we can take off such a condition. However in such a case Corollary 4 is not new. In fact, P. S. Bourdon [3] showed that if $\phi$ is univalent and a Rudin’s orthogonal function then $\phi$ is just the coordinate function $z$.

§7. Final remark

The research in this paper gives more general one. Suppose $0 < p < \infty$ and $p \neq 2$. $T$ is an isometric operator from $H^p(\nu)$ into $H^p(\mu)$ with $T1 = 1$ if and only if $T = C_\phi$ for some $\phi$ in $H^\infty$ with $\|\phi\|_\infty = 1$ and $C_\phi$ is an isometric operator from $H^p(\nu)$ into $H^p(\mu)$. For the ‘if’ part is trivial. For the ‘only if’ part, if $T$ is isometric and $T1 = 1$, then by [5, Theorem 7.5.3] $T(fg) = Tf \cdot Tg$ a.e. $\mu$ and $\|Tf\|_\infty = \|f\|_\infty$ for
all \( f \in \mathcal{P}, \ g \in \mathcal{P} \). Hence if \( \phi = Tz \) then \( \phi \) belongs to \( H^\infty \) and \( \|\phi\|_\infty = 1 \). Therefore \( Tf = C_\phi f \ (f \in \mathcal{P}) \) and so \( Tf = C_\phi f \ (f \in H^p(\nu)) \). When \( p \neq 2 \), if \( C_\phi \) is an isometric operator from \( H^p(\nu) \) into \( H^p(\mu) \), then \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2(\mu) \).

For by [5, Theorem 8.5.3], for all \( f \in \mathcal{P} \) and \( g \in \mathcal{P} \)

\[
\int_D C_\phi f \cdot \overline{C_\phi g} d\mu = \int_D f \overline{g} d\mu
\]

and \( \|C_\phi f\|_\infty = \|f\|_\infty \). This implies that \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2(\mu) \).

We give two open problems:

(1) Are there any isometric \( C_\phi \) from \( L^2_a \) into \( H^2 \) ?

(2) When \( \nu_0 \) is a discrete measure and not a dirac measure, are there any isometric \( C_\phi \) from \( H^2(\nu) \) to \( H^2 \) where \( dv = d\nu_0(r) d\theta/2\pi \) ?

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