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HOKKAIDO UNIVERSITY
Isometric Composition Operators Between Two Weighted Hardy Spaces

By

Takahiko Nakazi*

Abstract. We study isometric composition operators $C_\phi$ between two weighted Hardy spaces $H^2(\nu)$ and $H^2(\mu)$ when $\nu$ is a radial measure. The isometric $C_\phi$ is related to a moment sequence and such a $\phi$ is studied by the Nevanlinna counting function of $\phi$ when $\mu$ is the normalized Lebesgue measure on the unit circle.

§1. Introduction

Let $D$ be the open unit disc in the complex plane $\mathbb{C}$. We denote by $\mathcal{P}$ the set of all analytic polynomials and $H$ the set of all analytic functions on $D$. Let $\mu$ be a positive Borel measure on $D$ with $\mu(D) = 1$. $H^p(\mu)$ denotes the closure of all analytic polynomials in $L^p(d\mu)$ for $0 < p < \infty$. If $d\mu = d\theta/2\pi$, then $H^p(\mu) = \mathcal{H}^p$ is the classical Hardy space. If $d\mu = 2rdrd\theta/2\pi$, then $H^p(\mu) = L^p_a$ is the classical Bergman space. $\mathcal{H}^p$ and $L^p_a$ can be embedded in $H$. In this paper, we assume that $H^p(\mu)$ is embedded in $H$ for a general $\mu$. $H^\infty$ denotes the set of all bounded analytic functions on $D$. We also assume that $H^\infty = H \cap L^\infty(d\mu)$.

For an analytic self map $\phi$ of $D$, the composition operator $C_\phi$ is defined by $(C_\phi f)(z) = f(\phi(z))$ $(z \in D)$ for $f$ in $H$. Throughout this paper, we assume that $\nu$ and $\mu$ are positive Borel measures on $\overline{D}$ with $\nu(D) = \mu(D) = 1$. $\nu$ is called a radial measure if $d\nu = d\nu_0(r)d\theta/2\pi$ for a positive Borel measure $\nu_0$ on $[0,1]$. Since $d\theta/2\pi = d\delta_{r=1}d\theta/2\pi$, $d\theta/2\pi$ is a radial measure.

In this paper, we studied isometric composition operators from $H^2(\nu)$ into $H^2(\mu)$ when $\nu$ is a radial measure. As we show in the final section, our isometric composition operator $C_\phi$ is related to an isometric operator $T$ from $H^p(\nu)$ into $H^p(\mu)$ with $T1 = 1$ when $p \neq 2$. We have a long history for such isometric operators (see [8]). The onto isometries on $H^p$ or $L^p_a$ for $p \neq 2$ were described completely. Unfortunately into isometries have been known very little.

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Problem 1. For given measures $\nu$ and $\mu$, does there exist an isometric composition operator $C_\phi$ from $H^2(\nu)$ into $H^2(\mu)$? If there exists such a $C_\phi$, describe $\phi$.

A function $F$ in $H^2(\mu)$ is called an inner function in $H^2(\mu)$ if
\[ \int_D |F|^2 d\mu = \int_D |f|^2 d\mu \quad (f \in \mathcal{P}). \]

If $\phi^n$ is an inner function in $H^2(\mu)$ with $\int_D \phi d\mu = 0$ for any $n \geq 0$ then there exists a unique radial measure $\nu$ such that $C_\phi$ is isometric from $H^2(\nu)$ into $H^2(\mu)$ where $d\nu = d\nu_0(r)d\theta/2\pi$ and $1 \in \text{supp } \nu_0$. This is not difficult to prove. However we don’t know whether the converse is true.

Problem 2. If a composition operator $C_\phi$ is isometric from $H^2(\nu)$ into $H^2(\mu)$ then is $\phi^n$ an inner function in $H^2(\mu)$ with $\int_D \phi d\mu = 0$ for any $n \geq 0$?

A function $\phi$ in $H^\infty$ with $\|\phi\|_\infty = 1$ is called a Rudin’s orthogonal function in $H^2(\mu)$ if $\{\phi^n; n = 0, 1, 2, \ldots\}$ is a set of orthogonal functions in $H^2(\mu)$. If $\phi^n$ is an inner function in $H^2(\mu)$ with $\int_D \phi d\mu = 0$ for any $n \geq 0$ and $\|\phi\|_\infty = 1$ then $\phi$ is a Rudin’s orthogonal function in $H^2(\mu)$ because $\mathcal{P}$ is dense in $H^2(\mu)$ by its definition. We can ask whether the converse is true or not.

Problem 3. If $\phi$ is a Rudin’s orthogonal function in $H^2(\mu)$ then is $\phi^n$ an inner function in $H^2(\mu)$ with $\int_D \phi d\mu = 0$ for any $n \geq 0$?

When $d\mu = d\theta/2\pi$, Problem 3 was studied by several people, for example, [2],[3],[5],[6] and [10]. C. Bishop [2] and C. Sundberg [10] gave counter examples. Hence there exists a Rudin’s orthogonal function which is not an inner function in $H^2(d\theta/2\pi)$.

Problem 3 has a strong connection with Problem 2. In fact, if $C_\phi$ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$ then by Theorem 1 $\phi$ is a Rudin’s orthogonal function in $H^2(\mu)$. Conversely if $\phi$ is a Rudin’s orthogonal function in $H^2(\mu)$ then by Proposition 8 there exists a unique radial measure $\nu$ such that $C_\phi$ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$.

For each $\phi$, we will use two Borel measures $\mu_\phi$ on $\overline{D}$ and $\mu_{|\phi|}$ on $[0,1]$. For a Borel set $E$ in $\overline{D}$ $\mu_\phi(E) = \mu(\{z \in \overline{D}; \phi(z) \in E\})$ and for a Borel set $G$ in $[0,1]$ $\mu_{|\phi|}(G) = \mu(\{z \in \overline{D}; |\phi(z)| \in G\})$. 

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§2. General case

In this section we assume that $\nu$ is a radial measure, $\mu$ is an arbitrary measure and $\phi$ is an analytic selfmap with $\|\phi\|_{\infty} = 1$. We say that $\{a_n\}$ is a moment sequence of $\nu_0$, a positive Borel measure on $[0,1]$, if $a_n = \int_0^1 r^n d\nu_0$ \((n = 0, 1, 2, \ldots)\).

**Theorem 1.** Suppose $d\nu = d\nu_0(r)d\theta/2\pi$. Then $C_{\phi}$ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$ if and only if \(\int_{\mathbb{T}} \phi^n \bar{\phi}^m d\mu = 0 \ (n \neq m)\) and $\{\int_{\mathbb{T}} |\phi|^n d\mu\}$ is a moment sequence of $\nu_0$.

Proof. If $C_{\phi}$ is isometric, by the polarization formula

$$\delta_{nm} \int_0^1 r^n r^m d\nu_0(r) = \int_{\mathbb{T}} z^n \bar{z}^m d\nu = \int_{\mathbb{T}} \phi^n \bar{\phi}^m d\mu$$

because $\nu$ is a radial measure. Hence

$$\int_{\mathbb{T}} |\phi|^n d\mu = \int_0^1 r^{2n} d\nu_0 \ (n = 0, 1, 2, \ldots).$$

It is elementary to see that $x = \sqrt{1 - (1 - x^2)} = \sum_{n=0}^{\infty} a_n(1 - x^2)^n$ and $\sum_{n=0}^{\infty} |a_n|(1 - x^2)^n < \infty \ (0 \leq x \leq 1)$. Hence by Lebesgue’s dominated convergence theorem

$$\int_{\mathbb{T}} |\phi| d\mu = \int_{\mathbb{T}} \sum_{n=0}^{\infty} a_n(1 - |\phi|^2)^n d\mu = \sum_{n=0}^{\infty} a_n \int_{\mathbb{T}} (1 - |\phi|^2)^n d\mu$$

$$= \sum_{n=0}^{\infty} a_n \int_0^1 (1 - r^2)^n d\nu_0 = \int_0^1 \sum_{n=0}^{\infty} a_n (1 - r^2)^n d\nu_0 = \int_0^1 r d\nu_0$$

because $\sum_{n=0}^{k} a_n(1 - |\phi|^2)^n \leq \sum_{n=0}^{\infty} |a_n|$ and $\sum_{n=0}^{k} a_n(1 - r^2)^2 \leq \sum_{n=0}^{\infty} |a_n| < \infty$. Similarly, as $x^{2\ell+1} = \sqrt{1 - (1 - x^{2\ell+2})}$ we can show that $\int_{\mathbb{T}} |\phi|^{2n+1} d\mu = \int_0^1 r^{2n+1} d\nu_0 \ (n = 0, 1, 2, \ldots)$.

Thus $\{\int_{\mathbb{T}} |\phi|^n d\mu\}$ is a moment sequence of $\nu_0$.

Conversely if $\int_{\mathbb{T}} \phi^n \bar{\phi}^m d\mu = 0 \ (n \neq m)$ and $\{\int_{\mathbb{T}} |\phi|^n d\mu\}$ is a moment sequence of $\nu_0$, then

$$\int_{\mathbb{T}} \left| \sum_{n=0}^{k} a_n \phi^n \right|^2 d\mu = \sum_{n=0}^{k} |a_n|^2 \int_{\mathbb{T}} |\phi|^2 d\mu$$

$$= \sum_{n=0}^{k} |a_n|^2 \int_0^1 r^{2n} d\nu_0 = \int_{\mathbb{T}} \left| \sum_{n=0}^{k} a_n z^n \right|^2 d\nu.$$
Hence $C_\phi$ is isometric. □

**Theorem 2.** If $d\nu = dv_0(r)d\theta/2\pi$ then the following conditions are equivalent.
(1) $C_\phi$ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$.
(2) $\nu_0 = \mu|_\phi$
(3) $\int_0^1 F(r)d\nu_0 = \int_{\partial D} F(|\phi|)d\mu$ for any Borel nonnegative function $F$ on $[0,1]$.

Proof. (1) $\Rightarrow$ (2) If $G$ is a Borel set in $[0,1]$, then $\nu_0(G) = \inf\{\nu_0(V) : G \subset V, \text{ } V \text{ open in } [0,1]\}$ because $\nu_0$ is a Borel measure. Hence there exists a sequence of continuous functions $\{f_m\}$ such that $f_m \to \chi_G$ a.e. $\nu_0$ on $[0,1]$ and $\|f_m\| \leq \gamma < \infty (m = 1, 2, \cdots)$. By the Stone-Weierstrass theorem,
$$\int_0^1 f_m(r)d\nu_0 = \int_{\partial D} f_m(|\phi|)d\mu \ (m = 1, 2, \cdots)$$
because $\int_0^1 r^n d\nu_0 = \int_{\partial D} |\phi|^n d\mu \ (n = 0, 1, 2, \cdots)$. Thus $\nu_0(G) = \mu(\{z \in \partial D : |\phi(z)| \in G\})$.
(2) $\Rightarrow$ (3) is clear. (3) $\Rightarrow$ (1) is a result of Theorem 1. □

The following theorem shows that we can solve Problem 2 in the Introduction when $C_\phi$ is onto.

**Theorem 3.** Suppose $d\nu = dv_0(r)d\theta/2\pi$. If $C_\phi$ is an isometric operator from $H^2(\nu)$ onto $H^2(\mu)$ then $\phi^n$ is an inner function in $H^2(\mu)$ for any $n \geq 0$.

Proof. Let $F \in \mathcal{P}$ then there exists $f \in H^2(\nu)$ such that $F = f \circ \phi$. Let $f = \sum_{j=0}^{\infty} a_j z^j$, since $\sum_{j=0}^{\infty} |a_j|^2 \int_0^1 r^{2j} d\nu_0(r) < \infty$, $F = \sum_{j=0}^{\infty} a_j \phi^j$ and $\sum_{j=0}^{\infty} |a_j|^2 \int_{\partial D} |\phi|^2 d\mu < \infty$. By Theorem 1, for any $\ell \geq 0$
$$\int_{\partial D} F|\phi|^{2\ell} d\mu = a_0 \int_{\partial D} |\phi|^{2\ell} d\mu = \int_{\partial D} F d\mu \int_{\partial D} |\phi|^{2\ell} d\mu$$
because $\int_{\partial D} \phi d\mu = 0$. This implies that $\phi^\ell$ is an inner function in $H^2(\mu)$ for any $\ell \geq 0$.

When $d\nu = dv_0(r)d\theta/2\pi$, if $C_\phi$ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$ then $C_z$ is isometric from $H^2(\nu)$ onto $H^2(\mu_\phi)$.

**Corollary 1.** Suppose $d\nu = dv_0(r)d\theta/2\pi$. If $C_z$ is an isometric operator then $z^n$ is an inner function in $H^2(\mu)$ for any $n \geq 0$. Moreover $d\mu = dv_1(r)d\theta/2\pi + d\delta_{r=0}d\mu_1(\theta)$ where $\nu_1$ is a Borel measure on $[0,1]$ and $\mu_1$ is a Borel measure on $\partial D$. If $\nu_0$ does not have point mass on $\{r = 0\}$ then $\nu = \mu$.

Proof. By the remark above, $C_z$ is isometric from $H^2(\nu)$ onto $H^2(\mu)$ because $\mu_z = \mu$. By Theorem 3, $z^n$ is inner in $H^2(\mu)$ for any $n \geq 0$. Put $C_0[0,1] = \{u : u$ is continuous on
[0,1] and \( u(0) = 0 \) and \( C_0(\partial D) = \{ f : f \text{ is continuous on } \partial D \text{ and } f(1) = 0 \} \). Since \( r^n d\mu \) annihilates \( zP + \overline{zP} \) for any \( n \geq 0 \), for any \( j \neq 0 \), \( d\mu \perp \{ r^{2n+|j|}e^{ij\theta} ; n = 0, 1, 2, \ldots \} \). By the Müntz-Szasz theorem [6], \( d\mu \perp C_0[0,1]e^{ij\theta} \) for any \( j \neq 0 \) and so \( d\mu \perp C_0[0,1] \otimes C_0(\partial D) \). This implies that \( d\mu = d\nu_1(r) d\theta/2\pi + d\mu_1 d\mu_1(\theta) \) where \( \nu_1 \) is a Borel measure on \([0,1]\) and \( \mu_1 \) is a Borel measure on \( T \). If \( \nu_0 \) does not have point mass on \( \{ r = 0 \} \) then we may assume that \( \mu_1 = 0 \) and so \( d\mu = d\nu_1(r) d\theta/2\pi \). By Theorem 2 \( \nu_0 = \mu_{|z|} \) and \( \mu_{|z|} = \nu_1 \) because \( d\mu = d\nu_1(r) d\theta/2\pi \). □

§3. Radial measure

In this section we assume that \( \nu \) and \( \mu \) are radial measures, that is, \( d\nu = d\nu_0(r) d\theta/2\pi \) and \( d\mu = d\mu_0(r) d\theta/2\pi \). Proposition 1 solves Problem 2 when \( \nu = \mu \). By Theorem 2, if \( C_\phi \) is isometric from \( H^2(\nu) \) into \( H^2(\nu) \), then for some positive integer \( k \)

\[
\int_0^1 \log r d\nu_0 \leq k \int_0^1 \log r d\mu_0 + \int_0^{2\pi} \log |\phi(e^{i\theta})| d\theta
\]

as \( F(t) = \log t \), using the inner outer factorization of \( \phi \). Proposition 2 gives an exact formula for this.

**Proposition 1.** Suppose \( \nu \) is a radial measure. If \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2(\nu) \), then \( \phi^n \) is an inner function in \( H^2(\nu) \) for any \( n \geq 0 \).

Proof. By Theorem 1, \( \phi(0) = 0 \) because \( \nu \) is a radial measure and so by Schwarz’s lemma, \( |\phi(z)| \leq |z| \) \( (z \in D) \). Since \( \int_D |\phi(z)|^2 d\nu = \int_D |z|^2 d\nu, |\phi(z)| = |z| \) a.e. \( \nu \). For \( f \in \mathcal{P}, \)

\[
\int_D f|\phi|^{2n} d\nu = \int_D f|z|^{2n} d\nu = f(0) \int_0^1 r^{2n} d\nu_0 = \int_D f d\nu \int_D |\phi|^{2n} d\nu. \quad \square
\]

**Proposition 2.** Suppose \( \nu \) and \( \mu \) are radial measures, that is, \( d\nu = d\nu_0(r) d\theta/2\pi \) and \( d\mu = d\mu_0(r) d\theta/2\pi \). Let \( \phi = z^k B Q h \) where \( k \) is a positive integer, \( B \) is a Blaschke product with \( B(0) \neq 0 \), \( Q(z) = \exp -\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\lambda(t) \) is a singular inner function and \( h \) is an outer function. If \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2(\mu) \), then

\[
\int_0^1 \log r d\nu_1 = k \int_0^1 \log r d\mu_0 + \int_0^1 d\mu_0 \int_0^r n(s, B) \frac{ds}{s} + \\
\log |B(0)| - \frac{\mu_0([0,1]) \lambda([0,2\pi])}{\pi} + \int_0^{2\pi} \log |\phi(e^{i\theta})| d\theta/2\pi
\]

where \( n(s, B) \) is the number of zeros of \( B \) on the closed disc \( \{ z \in \mathbb{C} ; |z| \leq r \} \).
Proof. Let \( n(s, B) = n(s, BQh) \) be the number of zeros of \( BQh \) on the closed disc \( \{ z \in \mathbb{C} ; |z| \leq r \} \). Then, by Theorem 2 and [1, §2 of Chapter 5]

\[
\int_{0}^{1} \log r \, d\nu_0 = \int_{0}^{1} d\mu_0 \int_{0}^{2\pi} \log |\phi(\alpha)| \, d\theta + \mu_0(\{1\}) \int_{0}^{2\pi} \log |\phi(\alpha)| \, d\theta + \mu_0(\{0, 1\}) \lambda([0, 2\pi]) + \int_{0}^{2\pi} \log |\phi(\alpha)| \, d\theta
\]

because \( \mu_0(\{1\}) \int_{0}^{1} n(s, B) \frac{ds}{s} = -\mu_0(\{1\}) \log |B(0)| \).

\[\square\]

§4. Special cases

In this section we assume that \( \nu \) or \( \mu \) is the normalized Lebesgue measure or the normalized area measure. Proposition 3 solves Problems 1 and 2 when \( \nu \) is the normalized Lebesgue measure on the circle and \( \mu \) is a radial measure. Proposition 5 solves Problem 2 when \( \nu \) is a radial measure or the Lebesgue measure on the circle. Corollary 3 solves Problem 2 negatively when \( d\nu = 2rdrd\theta/2\pi \) and \( d\mu = d\theta/2\pi \).

**Proposition 3.** Let \( \mu \) be a radial measure. \( C_\phi \) is an isometric operator from \( H^2 \) into \( H^2(\mu) \) if and only if \( \phi^n \) is an inner function with \( \int_D \phi^n d\mu = 0 \) in \( H^2(\mu) \) for any \( n \geq 1 \) and \( H^2(d\mu) = H^2 \).

Proof. If \( C_\phi \) is isometric, by Theorem 1 \( \int_D \phi^n \bar{\phi}^m d\mu = 0 \) \( (n \neq m) \) and we have

\[
1 = \int_{0}^{2\pi} |z|^2 \, d\theta/2\pi = \int_D |\phi|^2 \, d\mu \leq 1.
\]

Hence \( |\phi(z)| = 1 \) a.e. \( \mu \) and so supp \( \mu \subset \partial D \). This implies that \( d\mu = d\delta_{r=1} d\theta/2\pi \) because \( \mu \) is a radial measure. The converse is clear. \( \square \)

**Proposition 4.** Suppose \( d\nu = d\nu_0(r)dr/2\pi \) and \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2 \).

(1) \( \nu_0(\{a\}) > 0 \) for \( 0 \leq a \leq 1 \) if and only if \( d\theta/2\pi(\{e^{i\theta}; |\phi(e^{i\theta})| = a\}) > 0 \).
(2) $d\nu_0 = d\delta_{r=1}$ if and only if $\phi$ is an inner function in $H^2$.

(3) $\nu_0$ is a discrete measure if and only if $|\phi| = \sum_{n=1}^{\infty} a_n \chi_{E_n}$ where $0 \leq a_n \leq 1$, and $d\theta/2\pi(E_n) = \nu_0(\{a_n\})$ $(n = 1, 2, \ldots)$.

Proof. Since $\nu_0(G) = d\theta/2\pi \{e^{i\theta} ; |\phi(e^{i\theta})| \in G\}$ for a Borel set $G$ in $[0,1]$ by Theorem 2, it is easy to see. □

Proof. This is just (2) of Proposition 4.

Now we consider when $d\nu = r dr d\theta/\pi$ or $d\mu = r dr d\theta/\pi$.

**Proposition 5.** If $C\phi$ is an isometric operator from $L^2_a$ into $H^2(\mu)$, then $\mu(\{z \in D ; |\phi| = b\}) = 0$ and $\int_D (b - |\phi|)^{-1} d\mu = \infty$ for any $0 \leq b \leq 1$.

Proof. It is clear by Theorem 2. □

**Corollary 2.** If $C\phi$ is an isometric operator from $L^2_a$ into $H^2$, then $\phi$ is not inner in $H^2$.

**Proposition 6.** Suppose $d\nu = d\nu_0(r) d\theta/2\pi$. If $C\phi$ is an isometric operator from $H^2(\nu)$ into $L^2_a$, then $\int_0^1 \log r d\nu_0 = -\frac{k}{4} + \int_0^1 2rd\theta \int_0^r n(s, B) \frac{ds}{s} + \log |B(0)| - \lambda([0,2\pi])$ + $\int_0^{2\pi} \log |\phi(e^{i\theta})| d\theta/2\pi$, where the inner part of $\phi$ is $z^k BQ$, $B$ is a Blaschke product with $B(0) \neq 0$, $Q(z) = \exp -\int_0^{2\pi} e^{it} + \frac{z}{e^{it} - z} d\lambda$ is a singular inner function. Hence if $\phi$ is a shricht function, then $\int_0^1 \log r d\nu_0 = -\frac{1}{4} + \int_0^{2\pi} \log |\phi(e^{i\theta})| d\theta/2\pi$.

Proof. It is clear by Proposition 2. □

§5. Nevanlinna counting function

Suppose $\nu$ or $\mu$ is the normalized Lebesgue measure or the normalized area measure. We assume that $\phi$ is a non-constant function in $H^\infty$ with $||\phi||_\infty = 1$. The Nevanlinna counting function of $\phi$, $N_\phi$, is defined on $D \setminus \{\phi(0)\}$ by

$$N_\phi(w) = \sum_{\phi(z) = w} \log \frac{1}{|z|},$$

where multiplicities are counted and $N_\phi(w)$ is taken to be zero if $w$ is not in the range of $\phi$. Corollary 4 seems to be interesting in spite of Corollary 3.
Theorem 4. Suppose $d\nu = d\nu_0(r)d\theta/2\pi$. Then, $C_\phi$ is an isometric operator from $H^2(\nu)$ into $H^2$ if and only if

$$N_\phi(z) = \int_{|z|}^{1} \frac{r}{|z|} d\nu_0(r)$$

for nearly all $z$ in $D$.

Proof. The ‘only if’ part was proved in [6, Lemma 3]. If $N_\phi(z) = \int_{|z|}^{1} \frac{r}{|z|} d\nu_0(r)$ for nearly all $z$ in $D$, by the Littlewood-Paley theorem (see [3]),

$$\int_{0}^{2\pi} \phi^n(e^{i\theta})\bar{\phi}^m(e^{i\theta}) d\theta/2\pi = 2nm\int_{D} z^{n-1}z^{m-1}N_\phi(|z|)dA(z) = 4nm\delta_{nm}\int_{0}^{1} r^{n+m-1} \left( \int_{r}^{1} \log \frac{s}{r} d\nu_0(s) \right) dr = \frac{4nm}{(n + m)^2}\delta_{nm}\int_{0}^{1} s^{n+m}d\nu_0(s).$$

When $n = m$, $\int_{0}^{2\pi} |\phi(e^{i\theta})|^{2n} d\theta/2\pi = \int_{0}^{1} s^{2n}d\nu_0(s)$ for $n = 0, 1, 2, \cdots$. Hence by Theorem 1 and its proof, $C_\phi$ is an isometric operator from $H^2(\nu)$ into $H^2$.

Lemma. $D \setminus \{ z \in D : \phi'(z) = 0 \}$ can be decomposed into an at most countable disjoint collection $\{ R_n \}$ of “semi-closed” polar rectangles, on each of which $\phi$ is schricht.

Proof. It is known in [9, p186]. □

Corollary 3. Suppose $\phi$ is a finite-to-one map. Then $C_\phi$ is not an isometric operator from $L^2_a$ into $H^2$.

Proof. By Lemma, there exists the inverse $\psi_n$ of the restriction of $\phi$ to $R_n$. Let $w \in \phi(R_{j_1})$. If $\phi$ is an $\ell$ to 1 map, then there exist $j_2, \cdots, j_\ell$ such that $\psi_{j_1}(z) = \psi_{j_2}(z) = \cdots = \psi_{j_\ell}(z) = w$. Hence there exists a small disc $\Delta$ in $\phi(R_{j_1})$ such that

$$N_\phi(z) = \sum_{z = \phi(w)} \log \frac{1}{|w|} = \sum_{t=1}^{\ell} \log \frac{1}{|\psi_{j_t}(z)|}$$

for all $w \in \Delta$. Therefore there exists a subdisc $\Delta_0$ in $\Delta$ such that $N_\phi(z)$ is harmonic on $\Delta_0$. On the other hand, by Proposition 5

$$N_\phi(z) = 2 \int_{|z|}^{1} \left( \log \frac{r}{|z|} \right) r dr = \frac{|z|^2 - 1}{2} + \log \frac{1}{|z|}. $$
This contradicts that \( N_\phi(z) \) is harmonic on \( \Delta_0 \). \( \square \)

**Theorem 5.** Suppose \( \phi \) is a contractive function in \( H^\infty \) such that \( \phi \) is a finite-to-one map and \(|\phi| = \sum_{j=1}^{\ell} a_j \chi_{E_j} \) where \( 0 < a_j < a_{j+1} \), \( \sum_{j=1}^{\ell} \chi_{E_j} = 1 \) and \( E_j \) is a measurable set in \( \partial D \) where \( 1 \leq \ell \leq \infty \). If the inner part of \( z - \phi \) is a Blaschke product for each \( z \in D \), then \( C_\phi \) is not an isometric operator from \( H^2(\nu) \) into \( H^2 \) for any \( d\nu = d\nu_0(r) d\theta/2\pi \) if \( \ell \neq 1 \).

Proof. Suppose \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2 \) for some \( d\nu = d\nu_0(r) d\theta/2\pi \). By Proposition 4, \( \nu_0 \) is a discrete measure and \( d\theta/2\pi(E_j) = \nu_0(\{a_j\}) \) (\( j = 1, 2, \cdots \)). Since \( \phi(0) = 0 \), by Lemma 2 in [6] and Proposition 7

\[
N_\phi(z) = \int_0^{2\pi} \log |z - \phi(e^{i\theta})| d\theta/2\pi + \log \frac{1}{|z|} = \int_{|z|}^{1} \log \frac{r}{|z|} d\nu_0(r)
\]

for \( z \in D \{0\} \). If \( |z| \leq a_1 \), then

\[
\int_{|z|}^{1} \log \frac{r}{|z|} d\nu_0(r) = \sum_{j=1}^{\infty} \left( \log \frac{a_j}{|z|} \right) \nu_0(\{a_j\}) = \sum_{j=1}^{\infty} \nu_0(\{a_j\}) \log \frac{1}{|z|} + \sum_{j=1}^{\infty} \nu_0(\{a_j\}) \log a_j
\]

\[
= \log \frac{1}{|z|} + \sum_{j=1}^{\infty} \nu_0(\{a_j\}) \log a_j.
\]

Hence if \( |z| \leq a_1 \) then

\[
\int_0^{2\pi} \log |z - \phi(e^{i\theta})| d\theta/2\pi = \sum_{j=1}^{\infty} \nu_0(\{a_j\}) \log a_j = \alpha.
\]

If \( a_1 < |z| \leq a_2 \), then

\[
\int_{|z|}^{1} \log \frac{r}{|z|} d\nu_0(r) = \sum_{j=2}^{\infty} \left( \log \frac{a_j}{|z|} \right) \nu_0(\{a_j\}) = \sum_{j=2}^{\infty} \nu_0(\{a_j\}) \log \frac{1}{|z|} + \sum_{j=2}^{\infty} \nu_0(\{a_j\}) \log a_j
\]

\[
\int_0^{2\pi} \log |z - \phi(e^{i\theta})| d\theta/2\pi = -\nu_0(\{a_1\}) \log \frac{1}{|z|} + \sum_{j=2}^{\infty} \nu_0(\{a_j\}) \log a_j
\]

\[
= \beta \log \frac{1}{|z|} + \gamma.
\]

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where \( \beta \neq 0 \)

For each \( z \in D \), put

\[
z - \phi(\zeta) = q_z(\zeta)h_z(\zeta) \quad (\zeta \in D)
\]

where \( q_z(\zeta) \) is inner and \( h_z(\zeta) \) is outer. Since \( \phi \) is a finite-to-one map, \( q_z \) is a finite Blaschke product by hypothesis and so

\[
q_{\phi(t)}(\zeta) = \prod_{j=1}^{n} \frac{\zeta - b_j(t)}{1 - b_j(t)\zeta} \quad (t \in D).
\]

Then, since \( \phi(0) = 0 \),

\[
\phi(t) = (-1)^n \left( \prod_{j=1}^{n} b_j(t) \right) h_{\phi(t)}(0) \quad (t \in D).
\]

Put \( D_r = \{ t \in \mathbb{C}; |t| \leq r \} \) for \( 0 < r < 1 \). If both \( \phi \) and \( \phi' \) have no zeros on \( \partial D_r \) then there is a division \( \{ D_r^j \}_{1 \leq j \leq n} \) of \( D_r \) such that \( \phi \) is one-to-one on \( D_r^j \) for \( 1 \leq j \leq n \). For, \( \phi \) is conformal in a neighborhood of each point on \( \partial D_r \) and so arg \( \phi \) is increasing on, \( \partial D_r \). Put \( \phi_j = \phi \mid D_r^j \) and \( b_j(t) = \phi_j^{-1}(\phi(t)) \) for \( 1 \leq j \leq n \). Then \( b_j(t) \) is analytic except \( \phi'(t) = 0 \) when \( \phi(t) \) in \( \phi(D_r) \). Hence \( h_{\phi(t)}(0) \) is analytic except \( \phi'(t) = 0 \) and \( \bigcup_{j=1}^{n} \{ t \in D; b_j(t) = 0 \} \) when \( \phi(t) \) in \( \phi(D_r) \). Since \( \phi(0) = 0 \), \( \{ t \in D; |\phi(t)| < a_1 \} \) is a nonempty open set. We can choose \( r \) such that \( \{ t \in D; |\phi(t)| < a_1 \} \cap \phi(D_r) \neq \emptyset \). If \( |\phi(t)| \leq a_1 \), by what was proved above,

\[
\alpha = \int_{0}^{2\pi} \log |\phi(t) - \phi(e^{i\theta})| d\theta / 2\pi
\]

\[
= \int_{0}^{2\pi} \log |h_{\phi(t)}(e^{i\theta})| d\theta / 2\pi = \log |h_{\phi(t)}(0)|.
\]

Hence \( |h_{\phi(t)}(0)| = e^\alpha \), and so \( h_{\phi(t)}(0) \) is constant on \( D_r \). If \( a_1 < |\phi(t)| \leq a_2 \), by what was proved above,

\[
\beta \log \frac{1}{|\phi(t)|} + \gamma = \int_{0}^{2\pi} \log |\phi(t) - \phi(e^{i\theta})| d\theta / 2\pi
\]

\[
= \int_{0}^{2\pi} \log |h_{\phi(t)}(e^{i\theta})| d\theta / 2\pi = \log |h_{\phi(t)}(0)|
\]

and so \( |h_{\phi(t)}(0)| = e^{\gamma} |\phi(t)|^\beta \). Since there exists \( 0 < r < 1 \) such that \( \{ t \in D; a_1 < |\phi(t)| < a_2 \} \cap \phi(D_r) \neq \emptyset \), this implies that \( |\phi(t)| \) is constant there and so \( \phi \) is constant on \( D \). This contradicts that \( \phi \) is a finite-to-one map. Therefore \( C_\phi \) is not isometric. \( \square \)

If \( \phi \) is a one-to-one map then it is known \([4, \text{Theorem 3.17}]\) that the inner part of \( z - \phi \) is a Blaschke product for each \( z \in D \). Hence we need not such a hypothesis in
Theorem 5. Unfortunately we could not prove it in general, that is, for a finite-to-one map.

§6. Rudin's orthogonal function

In this section, we study Rudin’s orthogonal functions. By Theorem 1, if $C_\phi$ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$ then $\phi$ is a Rudin’s orthogonal function. Proposition 7 implies the converse. This was proved by the author [6] when $d\mu = d\theta/2\pi$. The proof is valid for an arbitrary $\mu$. However we give a new proof due to K. Izuchi.

Proposition 7. If $\phi$ is a Rudin’s orthogonal function in $H^2(\mu)$ then there exists a unique radial measure $\nu$ such that $C_\phi$ is an isometric operator from $H^2(\nu)$ into $H^2(\mu)$, where $d\nu = d\nu_0(r)d\theta/2\pi$ and $1 \in \text{supp } \nu_0$.

Proof. Put $\nu_0 = \mu|\phi|$ and $d\nu = d\nu_0d\theta/2\pi$, then Theorems 1 and 2 imply the proposition. □

Corollary 4. Suppose $\phi$ is a finite-to-one map and $\phi$ is a Rudin’s orthogonal function. If the inner part of $z-\phi$ is a Blaschke product for each $z \in D$ and $|\phi| = \sum_{j=1}^\ell a_j \chi_{E_j}$, where $0 \leq a_j < a_{j+1}$, $\sum_{j=1}^\ell \chi_{E_j} = 1$ and $E_j$ is a measurable set in $\partial D$ where $1 \leq \ell \leq \infty$, then $|\phi| = 1$ and so $\phi$ is a finite Blaschke product.

Proof. If $\phi$ is a Rudin’s orthogonal function, then by Proposition 7 and Theorem 5, $\ell = 1$ and so $\phi$ is a finite Blaschke product. □

In Corollary 4, if $\phi$ is one-to-one map then the inner part of $z-\phi$ is a Blaschke product (see [4,Theorem 3.17]). Hence we can take off such a condition. However in such a case Corollary 4 is not new. In fact, P. S. Bourdon [3] showed that if $\phi$ is univalent and a Rudin’s orthogonal function then $\phi$ is just the coordinate function $z$.

§7. Final remark

The research in this paper gives more general one. Suppose $0 < p < \infty$ and $p \neq 2$, $T$ is an isometric operator from $H^p(\nu)$ into $H^p(\mu)$ with $T1 = 1$ if and only if $T = C_\phi$ for some $\phi$ in $H^\infty$ with $||\phi||_\infty = 1$ and $C_\phi$ is an isometric operator from $H^p(\nu)$ into $H^p(\mu)$. For the ‘if’ part is trivial. For the ‘only if’ part, if $T$ is isometric and $T1 = 1$, then by [5, Theorem 7.5.3] $T(fg) = Tf \cdot Tg$ a.e. $\mu$ and $||Tf||_\infty = ||f||_\infty$ for
all \( f \in \mathcal{P}, \ g \in \mathcal{P} \). Hence if \( \phi = Tz \) then \( \phi \) belongs to \( H^\infty \) and \( \|\phi\|_\infty = 1 \). Therefore \( Tf = C_\phi f \) \((f \in \mathcal{P})\) and so \( Tf = C_\phi f \) \((f \in H^p(\nu))\). When \( p \neq 2 \), if \( C_\phi \) is an isometric operator from \( H^p(\nu) \) into \( H^p(\mu) \), then \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2(\mu) \). For by [5, Theorem 8.5.3], for all \( f \in \mathcal{P} \) and \( g \in \mathcal{P} \)

\[
\int_{\mathcal{D}} C_\phi f \cdot \overline{C_\phi g} d\mu = \int_{\mathcal{D}} f \overline{g} d\mu
\]

and \( \|C_\phi f\|_\infty = \|f\|_\infty \). This implies that \( C_\phi \) is an isometric operator from \( H^2(\nu) \) into \( H^2(\mu) \).

We give two open problems:

(1) Are there any isometric \( C_\phi \) from \( L^2_a \) into \( H^2 \)?

(2) When \( \nu_0 \) is a discrete measure and not a dirac measure, are there any isometric \( C_\phi \) from \( H^2(\nu) \) to \( H^2 \) where \( dv = d\nu_0(r) d\theta / 2\pi \) ?

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