Faces of arrangements of hyperplanes and Arrow's impossibility theorem

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Abstract

In [T], Terao introduced an admissible map of chambers of a real central arrangement, and completely classified it. An admissible map is a generalization of a social welfare function and Terao's classification is that of Arrow's impossibility theorem in economics. In this article we consider an admissible map not of chambers but faces, and show that an admissible map of faces is a projection to a component if an arrangement is indecomposable and its cardinality is not less than three. From the view point of Arrow's theorem, our result corresponds to the impossibility theorem of a welfare function which permits the "tie" choice.

0 Main results

Let $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ be a nonempty real central arrangement of hyperplanes in $V := \mathbb{R}^l$, i.e., a collection of linear hyperplanes in V with cardinality n (we refer [OT] for elementary definitions and facts about arrangements of hyperplanes). We often use the term an "arrangement" instead of an "arrangement of hyperplanes". Let $\mathbf{Ch} = \mathbf{Ch}(\mathcal{A})$ be a set of chambers of a complement of $\mathcal{A}$, i.e., the connected components of $V \setminus \bigcup_{i=1}^n H_i$. Let us define the face of an arrangement $\mathcal{A}$, denoted by $\mathbf{Fc} = \mathbf{Fc}(\mathcal{A})$, by the following manner:

$$\mathbf{Fc}(\mathcal{A}) := \{\text{the connected components of } X \setminus \bigcup_{X \not\subset H \in \mathcal{A}} H \text{ for } X \in L(\mathcal{A})\},$$

where

$$L(\mathcal{A}) := \{\bigcap_{H \in \mathcal{B}} H |\ \mathcal{B} \subset \mathcal{A}\}$$

is called an **intersection lattice** of $\mathcal{A}$. It is clear that $\mathbf{Fc}$ contains $\mathbf{Ch}$ as a subset.

Before the main theorem, let us introduce some notations. For a vector $v \in V$ let v_i denote $x_i(v)$, where $\{x_1, \ldots, x_l\}$ is a fixed basis for V^*. For each hyperplane $H_j \in \mathcal{A}$, let us fix a linear form $\alpha_j \in V^*$ such that $H_j = \{\alpha_j = 0\}$. Define

$$H_j^+ := \{x \in V |\ \alpha_j(x) > 0\},\ H_j^- := \{x \in V |\ \alpha_j(x) < 0\}\ (j = 1, \ldots, n).$$

Throughout this article, let σ denote $+, -$ or 0. Let us put $B := \{+, -, 0\}$ and let $\mathbb{F}_2$ be a finite field of order 2. We sometimes identify $\mathbb{F}_2$ with the set $\{+, -\}$ with the usual multiplication. Let $1 \leq j \leq n$. The maps $\epsilon_j^\sigma : \mathbf{Fc} \to \mathbb{F}_2$ are defined, for $F \in \mathbf{Fc}$, by

$$\epsilon_j^\sigma(F) = \begin{cases} 1 & \text{if } F \subset H_j^\sigma, \\ 0 & \text{if } F \not\subset H_j^\sigma, \end{cases}$$

where $H_j^0 = H_j$. For a positive integer m, consider the m-time direct products $\mathbf{Fc}^m$ and $\mathbb{F}_2^m = \{0, 1\}^m$. Let $[m]$ denote the set $\{1, 2, \ldots, m\}$. Let $\mathbf{1}_{[m]} \in \mathbb{F}_2^m$ denote the vector such that $(\mathbf{1}_{[m]})_i = 1$ $(\forall i \in [m])$ and $\mathbf{0}_{[m]} \in \mathbb{F}_2^m$ such that $(\mathbf{0}_{[m]})_i = 0$ $(\forall i \in [m])$. For $S \subset [m]$, define the vector $\mathbf{1}_S \in \mathbb{F}_2^m$ by $(\mathbf{1}_S)_i = 1$ if $i \in S$ and $(\mathbf{1}_S)_i = 0$ if $i \notin S$. Any element $\mathcal{F} = (F_1, \ldots, F_m) \in \mathbf{Fc}^m$ can be expressed, by taking the suitable partition $\{S_j\}_{j \in J}$ of $[m]$ and set of faces $\{\overline{F}_j\}_{j \in J}$, as $F_i = \overline{F}_j$ if and only if $i \in S_j$. We write this $\mathcal{F}$ as

$$\mathcal{F} = \sum_{j \in J} \overline{F}_j \mathbf{1}_{S_j}.$$

We let the same notation ϵ_j^σ also denote the map $\mathbf{Fc}^m \to \mathbb{F}_2^m$ induced from $\mathbf{Fc} \to \mathbb{F}_2$ by the following manner:

$$\epsilon_j^\sigma(F_1, \ldots, F_m) := (\epsilon_j^\sigma(F_1), \ldots, \epsilon_j^\sigma(F_m)) \in B^m$$

for $(F_1, \ldots, F_n) \in \mathbf{Fc}^m$.

Definition 0.1
A map $\Phi : \mathbf{Fc}^m \to \mathbf{Fc}$ is called an **admissible map of faces** *if there exists a family of maps φ_j^σ $(1 \leq j \leq n,\ \sigma \in \{+, -, 0\})$ which satisfies the following two conditions:*

(1) $\varphi_j^\sigma(\mathbf{1}_{[m]}) = 1$ *for each j, $1 \leq j \leq n$ and $\sigma \in \{+, -, 0\}$.*

(2) *the diagram*

$$\begin{tikzcd} \mathbf{Fc}^m \arrow[r, "\Phi"] \arrow[d, "\epsilon_j^\sigma"'] & \mathbf{Fc} \arrow[d, "\epsilon_j^\sigma"] \\ \mathbb{F}_2^m \arrow[r, "\phi_j^\sigma"] & \mathbb{F}_2 \end{tikzcd}$$

commutes for each $j,\ 1 \le j \le n$ *and* $\sigma \in \{+, -, 0\}$.

In [T], an admissible map of chambers is defined and Definition 0.1 is a generalization of it. In this article, from now on, we assume that all admissible maps are those of faces.

Definition 0.2
A central arrangement $\mathcal{A}$ *is said to be* **decomposable** *if there exist nonempty arrangements* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *such that* $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ *(disjoint) and* $\mathrm{rank}(\mathcal{A}) = \mathrm{rank}(\mathcal{A}_1) + \mathrm{rank}(\mathcal{A}_2)$ *(where* $\mathrm{rank}(\mathcal{A}) := \mathrm{codim}_V(\cap_{H\in\mathcal{A}} H)$*). A central arrangement* $\mathcal{A}$ *is said to be* **indecomposable** *if it is not decomposable.*

In [T], Terao completely classified admissible maps of chambers of real central arrangements. Then it is natural to extend his result to that of faces of arrangements. Our main theorem is the following characterization of an admissible map of faces of indecomposable arrangements.

Theorem 0.3
If $\mathcal{A}$ *is a central real indecomposable arrangement and* $|\mathcal{A}| \ge 3$*, then any admissible map of faces is a projection to a component.*

Theorem 0.3 is a generalization of Theorem 1.5 (2) in [T] which plays an important role in the classification of admissible maps of chambers, and also of Arrow's impossibility theorem in [A]. Roughly speaking, Theorem 0.3 is an impossibility theorem for a welfare function which permits the “tie” choice. To explain the Arrow's impossibility theorem and its relation with Theorem 0.3, let us quote a remark of [T] in section one.

In the impossibility theorem, we assume that a society of m people have l policy options and that every individual has his/her own order of preferences on the l policy options. A social welfare function can be interpreted as a voting system by which the individual preferences are aggregated into a single societal preference. We require the following two requirements for a reasonable social welfare function:

(A) the society prefers the option i to the option j if every individual prefers the option i to the option j (Pareto property), and (B) whether the society prefers the option i to the option j only depends which individuals prefer the option i to the option j (pairwise independence).

The conclusion of Arrow's impossibility theorem is, for $l \ge 3$, the only social welfare function satisfying the two requirements (A) and (B) is a dictatorship, that is, the social preference has to be equal to the preference of one particular individual.

Let us consider a braid arrangement $\mathcal{A}$ in $V = \mathbb{R}^l$, i.e.,

$$\mathcal{A} := \{H_{ij} |\ 1 \leq i < j \leq l, \text{ where } H_{ij} := \ker(x_i - x_j)\}.$$

This arrangement is known to be indecomposable (see [T]). Then it is easy to see that an order of preferences on l policy options is equivalent to a choice of chambers of a braid arrangement. Similarly, social welfare function is an admissible map of chambers of a braid arrangement and the dictatorship by the h-th individual is the projection to the h-th component. Moreover, the condition (A) and (B) correspond to (1) and (2) in Definition 0.1. Hence Theorem 1.5 in [T] is a generalization of Arrow's impossibility theorem. In other words, Arrow's impossibility theorem can be formulated, by using Theorem 1.5 in [T], as:

If $\mathcal{A}$ is a braid arrangement with $l \geq 3$, then every admissible map of chambers is projective.

In Theorem 0.3, we consider faces of arrangements and its admissible maps. Let us apply Theorem 0.3 to a braid arrangement by using the interpretation above. If an individual has a face F as his/her order of preferences and $F \subset H_{ij}$, then the individual considers that the option i and j are matched, in other words, they are "tie". So the admissible map of faces of a braid arrangement corresponds to the welfare function in which the "tie" choice of options is permitted. Then Theorem 0.3 asserts that, if $l \geq 3$, the only social welfare function satisfying the two requirements (A) and (B) is, even if we permit the tie choice, a dictatorship.

1 Proof of Theorem 0.3

We introduce some results in [T]. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central real arrangement in $V = \mathbb{R}^l$. Let $\mathcal{B}$ be a subarrangement of $\mathcal{A}$. We say $\mathcal{B}$ is **dependent** if

$$\operatorname{rank}(\mathcal{B}) := \operatorname{codim}_V(\bigcap_{H \in \mathcal{B}} H) < |\mathcal{B}|.$$

A subarrangement $\mathcal{B}$ of $\mathcal{A}$ is said to be **independent** if it is not dependent. If $\mathcal{B}$ is a minimally dependent subset, then $\mathcal{B}$ is called a **circuit**. Note that any indecomposable arrangement contains at least one circuit. We introduce a graph $\Gamma(\mathcal{A})$ associated with $\mathcal{A}$. The set of vertices of $\Gamma(\mathcal{A})$ consists of hyperplanes in $\mathcal{A}$. Two vertices H_i, H_j of $\Gamma(\mathcal{A})$ are connected by an edge if and only if there exists a circuit in $\mathcal{A}$ containing $\{H_i, H_j\}$.

Lemma 1.1 ([T], Lemma 2.1)
A non-empty real central arrangement $\mathcal{A}$ is indecomposable if and only if the associated graph $\Gamma(\mathcal{A})$ is connected.

Lemma 1.1 plays one of the key roles in the proof of Theorem 0.3. Next we check the following fact.

Proposition 1.2 (cf. [T], Proposition 2.4)
Assume that $\sigma \in B$ and $1 \leq j \leq n$. Then the map $\epsilon_j^\sigma : \mathbf{Fc}^m \to \mathbb{F}_2^m$ is surjective.

Proof. If $\sigma = +$ or $-$, then this is just the Proposition 2.4 in [T]. So we show the case when $\sigma = 0$. Take any $S \subset [m]$ and faces F_0 and F such that $F_0 \subset H_j$ and $F \not\subset H_j$. Define $\mathcal{F} := F_0 \mathbf{1}_S + F \mathbf{1}_{S^c} \in \mathbf{Fc}^m$. Then we have $\epsilon_j^0(\mathcal{F}) = \mathbf{1}_S$ and this completes the proof. $\square$

Lemma 1.3
Assume that $1 \leq j \leq n$, $S \subset [m]$ and $\sigma \in \{+,-\}$. Then the followings hold:

$$\begin{aligned} \text{If } \varphi_j^0(\mathbf{1}_S) &= 1, \text{ then } \varphi_j^\sigma(\mathbf{1}_{S^c}) = 0. \\ \text{If } \varphi_j^0(\mathbf{1}_S) &= 0, \text{ then } \varphi_j^+(\mathbf{1}_{S^c}) = 1 \text{ or } \varphi_j^-(\mathbf{1}_{S^c}) = 1. \end{aligned}$$

Proof. First assume that $\varphi_j^0(\mathbf{1}_S) = 1$. By Proposition 1.2 we may choose, for each $\sigma \in \{+,-\}$, a face $\mathcal{F}^\sigma \in \mathbf{Fc}^m$ such that $\epsilon_j^0(\mathcal{F}^\sigma) = \mathbf{1}_S$, $\epsilon_j^\sigma(\mathcal{F}^\sigma) = \mathbf{1}_{S^c}$. Then the commutativity of an admissible map shows that $\epsilon_j^0(\Phi(\mathcal{F}^\sigma)) = 1$. This means $\Phi(\mathcal{F}^\sigma) \subset H_j$, hence $\epsilon_j^\sigma(\Phi(\mathcal{F}^\sigma)) = 0$. Again the commutativity shows $\varphi_j^\sigma(\epsilon_j^\sigma(\mathcal{F}^\sigma)) = \varphi_j^\sigma(\mathbf{1}_{S^c}) = 0$. Next assume that $\varphi_j^0(\mathbf{1}_S) = 0$. In this case, by using the same argument as the above, we have $\Phi(\mathcal{F}^\sigma) \subset H_j^+ \cup H_j^-$. So it holds that $\varphi_j^+(\mathbf{1}_{S^c}) = 1$ or $\varphi_j^-(\mathbf{1}_{S^c}) = 1$. $\square$

Corollary 1.4
Assume that $1 \leq j \leq n$ and $\sigma \in B$. Then it holds that $\varphi_j^\sigma(\mathbf{0}_{[m]}) = 0$.

Proof. Immediately from Lemma 1.3. $\square$

Next we introduce two results from [T].

Lemma 1.5 ([T], Lemma 3.3)
Let $\mathcal{B} = \{H_1, \ldots, H_\nu\} \subset \mathcal{A}$ be a circuit with $3 \leq \nu \leq n$. Define a map

$$\delta_{\mathcal{B}} : \mathbf{Ch}(\mathcal{B}) \to \{+,-\}^\nu$$

by $\delta_{\mathcal{B}}(C) = (\sigma_i)_{i=1}^{\nu} \in \{+,-\}^{\nu}$ $(C \in \mathbf{Ch}(\mathcal{B}))$, *where*

$$\sigma_i = \begin{cases} + & \text{if } C \subset H_i^+, \\ - & \text{if } C \subset H_i^-. \end{cases}$$

Then the followings hold:

(1) $|\mathbf{Ch}(\mathcal{B})| = 2^{\nu} - 2$.

(2) *there exists* $\tau \in \{+,-\}^{\nu}$ *such that* $Im(\delta_{\mathcal{B}}) = \{+,-\}^{\nu} \setminus \{\tau, -\tau\}$.

From now on, we assume that any arrangement is indecomposable and its cardinality is not less than three.

Lemma 1.6 ([T], Lemma 3.4)
Assume that $\sigma = +$ *or* $-$. *Then The maps* φ_j^{σ} *does not depend on* j *and* σ.

Proof. The same proof as that of Lemma 3.4 in [T] can be applied if we only use chambers. $\square$

By Lemma 1.6, we can define $\varphi := \varphi_j^{\sigma}$ for $1 \leq j \leq n$ and $\sigma \in \{+,-\}$. Moreover, let us put

$$K := \{v \in \mathbb{F}_2^m |\ \varphi(v) = 1\}.$$

To show Theorem 0.3 we need the similar result to Lemma 1.6 for φ_j^0 and $K_j^0 := \{v \in \mathbb{F}_2^m |\ \varphi_j^0(v) = 1\}$. First we show the following.

Proposition 1.7
The set K_j^0 *does not depend on* j.

Proof. Choose a circuit $\mathcal{B} \subset \mathcal{A}$. We may assume that $\mathcal{B} = \{H_1, \ldots, H_{\nu}\}$ $(3 \leq \nu \leq n)$. Take a face F_0 satisfying $F_0 \subset \bigcap_{i=1}^{\nu} H_i \neq \emptyset$, F_1 such that $F_1 \subset (\bigcap_{j \in [\nu] \setminus \{1,\nu\}} H_j) \setminus (H_1 \cup H_{\nu})$, and faces F_i $(2 \leq i \leq \nu)$ such that $F_i \subset (\bigcap_{j \in [\nu] \setminus \{i-1,i\}} H_j) \setminus (H_{i-1} \cup H_i)$. It is obvious that we can choose these faces by the definition of a circuit. For $S_i \in K_i^0$ $(1 \leq i \leq \nu)$, put

$$\begin{aligned} \mathcal{F}_i : &= F_0 \mathbf{1}_{S_i} + F_{i+1} \mathbf{1}_{S_i^c} \ (1 \leq i \leq \nu - 1), \\ \mathcal{F}_{\nu} : &= F_0 \mathbf{1}_{S_{\nu}} + F_1 \mathbf{1}_{S_{\nu}^c}. \end{aligned}$$

Then for $i \neq \nu$ it holds that $\epsilon_i^0(\mathcal{F}_i) = \epsilon_{i+1}^0(\mathcal{F}_i) = \mathbf{1}_{S_i}$ and $\epsilon_j^0(\mathcal{F}_i) = \mathbf{1}_{[m]}$ $(j \neq i, i+1)$. For $i = \nu$ it holds that $\epsilon_{\nu}^0(\mathcal{F}_{\nu}) = \epsilon_1^0(\mathcal{F}_{\nu}) = \mathbf{1}_{S_{\nu}}$ and $\epsilon_j^0(\mathcal{F}_{\nu}) = \mathbf{1}_{[m]}$ $(j \neq 1, \nu)$. By the commutativity of Φ and the assumption $\varphi_j^0(\mathbf{1}_{[m]}) = 1$, it holds that

$$\begin{aligned} \Phi(\mathcal{F}_i) &\subset H_1 \cap H_2 \cap \cdots \cap H_i \cap H_{i+2} \cap \cdots \cap H_{\nu} \subset H_{i+1} \ (1 \leq i \leq \nu - 1), \\ \Phi(\mathcal{F}_{\nu}) &\subset H_2 \cap H_3 \cap \cdots \cap H_{\nu} \subset H_1. \end{aligned}$$

This implies $S_i \in K^0_{i+1}$ and $S_\nu \in K^0_1$, hence $K^0_i \subset K^0_{i+1}$ for $1 \le i \le \nu - 1$ and $K^0_\nu \subset K^0_1$. Therefore it holds that $K^0_i = K^0_j$ for any $i,\ j$ with $\mathbf{1} \le i,\ j \le \nu$. Since $\mathcal{A}$ is indecomposable, Lemma 1.1 completes the proof. $\square$

By Proposition 1.7 we can define $K^0 := K^0_i$ and $\varphi^0 := \varphi^0_i$ by the same manner as K and φ.

Lemma 1.8
It holds that $K^0 \subset K$.

Proof. Choose a circuit $\mathcal{B} = \{H_1, \dots, H_\nu\} \subset \mathcal{A}$. By Lemma 1.5, there exists $\tau = (\tau_1, \dots, \tau_\nu) \in \{+, -\}^\nu$ such that

$$\{+, -\}^\nu = \mathrm{Im}(\delta_{\mathcal{B}}) \cup \{\tau, -\tau\} \text{ (disjoint).} \tag{1}$$

From (1) we may choose a chamber $C' \in \mathbf{Ch}(\mathcal{B})$ with

$$\delta_{\mathcal{B}}(C') = (\tau_1, \tau_2, \dots, \tau_{\nu-1}, -\tau_\nu).$$

When $\nu \ge 4$, an arrangement $\mathcal{B}' := \{H_1 \cap H_\nu, H_2 \cap H_\nu, \dots, H_{\nu-1} \cap H_\nu\}$ is also a circuit and it holds that

$$\{+, -\}^{\nu-1} = \mathrm{Im}(\delta_{\mathcal{B}'}) \cup \{(\tau_1, \dots, \tau_{\nu-1}), (-\tau_1, \dots, -\tau_{\nu-1})\}. \tag{2}$$

When $\nu = 3$, it is easy to see that (2) holds too. Hence Lemma 1.5 allows us to choose a face $F' \subset H_\nu$ with

$$\delta_{\mathcal{B}'}(F') = (-\tau_1, \tau_2, \dots, \tau_{\nu-1}).$$

Let us choose a chamber C and a face F of $\mathcal{A}$ such that $C \subset C'$ and $F \subset F'$. Now take an element $T \in K^0_\nu$ and define $\mathcal{F} := F\mathbf{1}_T + C\mathbf{1}_{T^c}$. Then it holds that

$$\begin{aligned} \epsilon^0_\nu(\mathcal{F}) &= \mathbf{1}_T, \\ \epsilon^{-\tau_1}_1(\mathcal{F}) &= \mathbf{1}_T, \\ \epsilon^{\tau_j}_j(\mathcal{F}) &= \mathbf{1}_{[m]} \ (2 \le j \le \nu - 1). \end{aligned}$$

This and the equation (2) show that

$$\Phi(\mathcal{F}) \subset H^{\tau_2}_2 \cap \cdots \cap H^{\tau_{\nu-1}}_{\nu-1} \cap H_\nu \subset H^{-\tau_1}_1.$$

So we have $T \in K^{-\tau_1}_1$, thus $K^0_\nu = K^0 \subset K^{-\tau_1}_1 = K$. $\square$

Lemma 1.9
(1) $\mathbf{1}_{[m]} \in K^0 \subset K$.

(2) *For* $S_1, S_2 \in K^\sigma$ $(\sigma \in B)$, *it holds that* $S_1 \cap S_2 \in K^\sigma$, *where* $K^+ := K =: K^-$.

Proof. (1) is obvious. Show (2). If $\sigma \in \{+, -\}$ then this is Lemma 3.5 (3) in [T]. So we assume $\sigma = 0$. Choose a circuit $\mathcal{B} = \{H_1, \dots, H_\nu\} \subset \mathcal{A}$ for $3 \leq \nu \leq n$ and take faces $F'_0,\ F'_1,\ F'_2$ and F'_3 of $\mathcal{B}$ by the following manner:

$$\begin{array}{rcl} F'_0 & \subset & H_1 \cap H_2 \cap \cdots \cap H_\nu, \\ F'_1 & \subset & (\bigcap_{j \in [\nu] \setminus \{2,3\}} H_j) \setminus (H_2 \cup H_3), \\ F'_2 & \subset & (\bigcap_{j \in [\nu] \setminus \{1,3\}} H_j) \setminus (H_1 \cup H_3), \\ F'_3 & \subset & (\bigcap_{j \in [\nu] \setminus \{1,2,3\}} H_j) \setminus (H_1 \cup H_2 \cup H_3). \end{array}$$

Choose faces F_i $(0 \leq i \leq 3)$ of $\mathcal{A}$ such that $F_i \subset F'_i$, and let us define the face $\mathcal{F} \in \mathbf{Fc}^m$ by

$$\mathcal{F} := F_0 \mathbf{1}_{S_1 \cap S_2} + F_1 \mathbf{1}_{S_1 \setminus S_2} + F_2 \mathbf{1}_{S_2 \setminus S_1} + F_3 \mathbf{1}_{(S_1 \cup S_2)^c}.$$

Then it holds that

$$\begin{array}{rcl} \epsilon^0_1(\mathcal{F}) & = & \mathbf{1}_{S_1}, \\ \epsilon^0_2(\mathcal{F}) & = & \mathbf{1}_{S_2}, \\ \epsilon^0_3(\mathcal{F}) & = & \mathbf{1}_{S_1 \cap S_2}, \\ \epsilon^0_j(\mathcal{F}) & = & \mathbf{1}_{[m]}\ (4 \leq j \leq \nu). \end{array}$$

These imply that

$$\Phi(\mathcal{F}) \subset H_1 \cap H_2 \cap H_4 \cap \cdots \cap H_\nu \subset H_3.$$

Hence it holds that $\varphi^0(\mathbf{1}_{S_1 \cap S_2}) = 1$, and this completes the proof. □

Proof of Theorem 0.3. By the assumption of Φ and Corollary 1.4, we can see that $[m] \in K^0$ and $\emptyset \notin K^0$. Hence Lemma 1.9 (2) implies

$$\emptyset \neq S_0 := \bigcap_{S \in K^0} S \in K^0 \subset K.$$

Take any $h_0 \in S_0$. Then Lemma 1.9 (2) shows $S_0 \setminus h_0 \notin K^0$. Hence Lemma 1.3 implies $\varphi(1_{S_0^c \cup h_0}) = 1$, thus $S_0^c \cup h_0 \in K$. So Lemma 1.9 (2) implies $(S_0^c \cup h_0) \cap S_0 = h_0 \in K$. Let us show $h_0 \in K^0$. If not, then $[m] \setminus h_0 \in K$.

Hence Lemma 1.9 (2) implies $([m] \setminus h_0) \cap h_0 = \emptyset \in K$, which contradicts Corollary 1.4. Hence $h_0 \in K^0$. It is easy to show that $S \in K^0$ if and only if $h_0 \in S$, and $K^0 \subset K$ implies $S \in K$ if and only if $h_0 \in S$. Hence we can see that φ^0, φ and Φ are all projections to the h_0-th component. $\square$

Remark 1.1
Theorem 0.3 is not true for a decomposable arrangement. For example, let us consider the Boolean arrangement $\mathcal{A}$ defined by

$$\mathcal{A} := \{H_i |\ 1 \le i \le l, \text{where } H_i := \ker(x_i)\}.$$

Then we can see that the majority decision rule defines a non-projection admissible map. i.e., for a face $\mathcal{F} = (F_1, \ldots, F_m) \in \mathbf{Fc}^m$ and for $j,\ 1 \le j \le n$, let us define

$$\begin{aligned} N_j^+(\mathcal{F}) : &= |\{i \in [m] \mid \epsilon_j^+(F_i) = 1\}|, \\ N_j^-(\mathcal{F}) : &= |\{i \in [m] \mid \epsilon_j^-(F_i) = 1\}|, \\ N_j^0(\mathcal{F}) : &= |\{i \in [m] \mid \epsilon_j^0(F_i) = 1\}|. \end{aligned}$$

Since $\mathcal{A}$ is Boolean, any face $F \in \mathbf{Fc}$ can be determined uniquely by the image of the map $\delta_{\mathcal{A}}(F) \in B^l$. Note that $\mathrm{Im}(\delta_{\mathcal{A}}) = B^l$. Let us define a map $\Phi : \mathbf{Fc}^m \to \mathbf{Fc}$ by $\delta_{\mathcal{A}}(\Phi(\mathcal{F})) = (\sigma_i)_{i=1}^l \in B^l$, where

$$\sigma_i = \begin{cases} + & \text{if } N_i^+(\mathcal{F}) \ge \max\{N_i^-(\mathcal{F}), N_i^0(\mathcal{F})\}, \\ 0 & \text{if } N_i^0(\mathcal{F}) > \max\{N_i^+(\mathcal{F}), N_i^-(\mathcal{F})\} \text{ or } N_i^0(\mathcal{F}) = N_i^-(\mathcal{F}) > N_i^+(\mathcal{F}), \\ - & \text{if } N_i^-(\mathcal{F}) > \max\{N_i^+(\mathcal{F}), N_i^0(\mathcal{F})\}. \end{cases}$$

Then it is easy to see that this Φ is an admissible map and not a projection.

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