Abstract. It is known that the weight enumerator of a self-dual doubly-even code in genus \( g = 1 \) can be uniquely written as an isobaric polynomial in certain homogeneous polynomials with integral coefficients. We settle the case where \( g = 2 \) and prove the non-existence of such polynomials under some conditions.

1. Introduction. In this paper we deal with binary self-dual doubly-even codes only. We refer to [3] for the general facts on coding theory. We shall first recall our problem in the case where \( g = 1 \), which explains what this paper concerns about. It is known that the weight enumerator of any self-dual doubly-even code can be uniquely written as an isobaric polynomial in \( \varphi_8 = x^8 + 14x^4y^4 + y^8 \) and \( \varphi_{24} = x^4y^4(x^4 - y^4)^4 \) with integral coefficients ([5]). We note that \( \varphi_{24} \) itself is not the weight enumerator of a code but a linear combination of the weight enumerators with rational coefficients.

We shall add a few words on this basis. We consider the elements in \( \mathbb{Z}[x, y] \) for simplicity. The choice of \( \varphi_8 \) is unique (up to \( \pm 1 \)) since there exists a unique self-dual doubly-even code \( e_8 \) of length 8. Next we assume that another homogeneous polynomial \( \xi \) of degree 24 has the property in question, i.e., the weight enumerator of any self-dual doubly-even code can be written as an isobaric polynomial in \( \varphi_8 \) and \( \xi \) with integral coefficients. We put \( \xi = ax^{24} + bx^{20}y^4 + \cdots, \) \( a, b \in \mathbb{Z}, \) in which the unwritten part consists of terms of degree less than 20 in \( x \). There are 85 classes self-dual doubly-even codes of length 32 ([1], [2]) and the weight enumerator of these classes should be written as \( m\varphi_8^4 + n\varphi_8\xi \), in which \( m, n \) are integers. Examining these conditions for all classes, we know that \( -42a + b \) must be a divisor of 1. We have that \( \xi = a\varphi_8 \pm \varphi_{24} \) and conversely, such \( \xi \) has the said property.

In the rest of this paper we restrict ourselves to the case where \( g = 2 \) when considering the weight enumerators. Let \( C \) be a binary self-dual doubly-even code and \( W_C = W_C(x, y, z, w) \) the weight enumerator of \( C \) in genus 2 (cf. [6], [4], [7]). We remark that \( W_C \) is symmetric in \( x, y, z, w \). We shall denote by \( \mathfrak{M} \) the graded ring over the field \( \mathbb{C} \) of complex numbers generated by \( W_C \) of all self-dual doubly-even codes. The degree \( d \)-part \( \mathfrak{M}_d \) of \( \mathfrak{M} \) is a finite dimensional vector space over \( \mathbb{C} \). The four elements
$W_{e_8}, W_{d_{24}}, W_{g_{24}}, W_{d_{40}}$ are algebraically independent over $\mathbb{C}$ and the graded ring $\mathfrak{W}$ is a free $\mathbb{C}[W_{e_8}, W_{d_{24}}, W_{g_{24}}, W_{d_{40}}]$-module with a basis $1, W_{d_{2}}, W_{g_{24}}, W_{d_{40}}$, where $g_{24}$ is the extended Golay code of length 24. The dimension formula of this ring is

$$\sum_{d \geq 0} (\dim \mathfrak{W}_d) t^d = \frac{1 + t^{32}}{(1 - t^8)(1 - t^{24})^2(1 - t^{40})}$$

$$= 1 + t^8 + t^{16} + 3t^{24} + 4t^{32} + 5t^{40} + 8t^{48} + 10t^{56} + \cdots$$

We always keep this formula in mind through the next section.

2. Result. For the proof of our Theorem, we shall construct homogeneous polynomials $X_{8}, X_{24}, Y_{24}, X_{32}, X_{40}$ of degrees 8, 24, 32, 40, respectively. This is done by analyzing the vector spaces $\mathfrak{W}_d, d = 8, 24, 32, 40$.

(degree 8) The extended Hamming code $e_8$ of length 8 is a unique self-dual doubly-even code of this length. We put $X_8 = W_{e_8}$. $X_8$ is also characterized by

$$0x^{24} + x^{20}(y^4 + \cdots) + 0x^{18}(y^2 z^2 w^2) + \cdots, 0x^{24} + 0x^{20}(y^4 + \cdots) + x^{18}(y^2 z^2 w^2) + \cdots.$$

As we remarked, the weight enumerator in this paper is symmetric and $x^{20}(y^4 + \cdots)$ stands for $x^{20}(y^4 + z^4 + w^4)$. We note that 0 as a coefficient of $x^{18}(y^2 z^2 w^2)$ in the first formula is not much of importance. The general form of the elements in $\mathfrak{W}_{24}$ is

$$a_0 x^{24} + a_1 x^{20}(y^4 + \cdots) + a_2 x^{18}(y^2 z^2 w^2) + \cdots$$

and is uniquely written as

$$a_0 X_8^3 + (-42a_0 + a_1)X_{24} + (-504a_0 + a_2)Y_{24}.$$  

(degree 24) Two polynomials $X_{24}, Y_{24}$ are characterized by

$$0x^{24} + x^{20}(y^4 + \cdots) + 0x^{18}(y^2 z^2 w^2) + \cdots, 0x^{24} + 0x^{20}(y^4 + \cdots) + x^{18}(y^2 z^2 w^2) + \cdots.$$  

We remark that 0 as a coefficient of $x^{24}(y^8 + \cdots)$ stands for $x^{24}(y^8 + z^8 + w^8)$. We note that 0 as a coefficient of $x^{18}(y^2 z^2 w^2)$ in the first formula is not much of importance. The similar remark also holds in the following (degree 40). The general form of the elements in $\mathfrak{W}_{32}$ is

$$a_0 x^{32} + a_1 x^{28}(y^4 + \cdots) + a_2 x^{26}(y^2 z^2 w^2) + x^{24}(a_3(y^8 + \cdots) + a_4(y^4 z^4 + \cdots) + \cdots$$
and is uniquely written as
\[ a_0X^4_8 + (-56a_0 + a_1)X_8X_{24} + (-672a_0 + a_2)X_8Y_{24} + (784a_0 - 33a_1 - 2a_2 + a_4)X_{32}, \]

where \( a_3 = 620a_0 + 10a_1. \)

(degree 40) The polynomial \( X_{40} \) is characterized by
\[ 0x^{40} + 0x^{36}(y^4 + \cdots) + 0x^{34}(y^2z^2w^2) + 0x^{32}(y^4z^4 + \cdots) + x^{28}(y^4z^4w^4) + \cdots. \]

The general form of the elements in \( \mathfrak{W}_{40} \) is
\[ a_0x^{40} + a_1x^{36}(y^4 + \cdots) + a_2x^{34}(y^2z^2w^2) + x^{32}(a_3(y^8 + \cdots) + a_4(y^4z^4 + \cdots)) \]
\[ + a_5x^{30}(y^6z^2w^2 + \cdots) + x^{28}(a_6(y^{12} + \cdots) + a_7(y^8z^4 + \cdots) + a_8(y^4z^4w^4)) + \cdots \]

and is uniquely written as
\[ a_0X^5_8 + (-70a_0 + a_1)X^2_8X_{24} + (-840a_0 + a_2)X^3_8Y_{24} + (1960a_0 - 61a_1 - 2a_2 + a_4)X_8X_{32} \]
\[ + (196560a_0 - 7350a_1 - 880a_2 + 150a_4 + a_8)X_{40}, \]

where we have the relations \( a_3 = 285a_0 + 24a_1, a_5 = 84a_1 - 8a_2 + 12a_4, a_6 = 21280a_0 + 92a_1, a_7 = 225a_1 + 32a_2 + 11a_4. \)

The homogeneous polynomials we have obtained can be written as
\[ X_8 = W_8, \]
\[ X_{24} = 5 \cdot 2^{-2}3^{-1}7^{-1}W_8^3 - 2^{-3}11^{-1}W_{d_{24}} - 17 \cdot 2^{-1}3^{-1}7^{-1}11^{-1}W_{g_{24}}, \]
\[ Y_{24} = -2^{-3}3^{-1}7^{-1}W_8^3 + 2^{-4}3^{-1}11^{-1}W_{d_{24}} + 2^{-3}3^{-1}7^{-1}11^{-1}W_{g_{24}}, \]
\[ X_{32} = 67 \cdot 2^{-10}3^{-1}7^{-1}W_8^3 - 5 \cdot 2^{-7}11^{-1}W_8W_{d_{24}} - 2^{-3}3^{-1}7^{-1}11^{-1}W_{e_8}W_{g_{24}} + 2^{-10}W_{d_{24}^2}, \]
\[ X_{40} = -461 \cdot 2^{-13}3^{-1}5^{-1}7^{-1}11^{-1}W_8^5 + 13 \cdot 2^{-9}3^{-1}11^{-1}14^{-1}W_{e_8}^2 W_{d_{24}} \]
\[ + 2^{-6}3^{-1}7^{-1}11^{-1}41^{-1}W_{e_8}^3 W_{g_{24}} - 3 \cdot 2^{-13}41^{-1}W_{e_8}W_{d_{24}} + 2^{-10}3^{-1}5^{-1}11^{-1}W_{d_{24}^2}. \]

We note that \( X_8, X_{24}, Y_{24}, X_{32}, X_{40} \) are in \( \mathbf{Z}[x, y, z, w] \) and that they generate the ring \( \mathfrak{W} \).

These being prepared, we prove

**Theorem.** There exist no five homogeneous polynomials of degrees 8, 24, 24, 32, 40 in \( \mathfrak{W} \cap \mathbf{Z}[x, y, z, w] \) such that the weight enumerator of any self-dual doubly-even code can be written as an isobaric polynomial in these five elements with integral coefficients.

**Proof.** Suppose that there exist such homogeneous polynomials of degrees 8, 24, 24, 32, 40 satisfying the property in Theorem. As we discussed in this
section, any element in \( \mathfrak{M} \cap \mathbb{Z}[x, y, z, w] \) of degree at most equal to 40 can be uniquely written as an isobaric polynomial in \( X_8, X_{24}, Y_{24}, X_{32}, X_{40} \) with integral coefficients and the five assumed polynomials are hence integral polynomials in \( X_8, \ldots, X_{40} \). Therefore \( X_8, \ldots, X_{40} \) also enjoy the property in Theorem, i.e., the weight enumerator of any self-dual doubly-even code can be written as

\[
\sum_{i,j,k,l,m \in \mathbb{Z}_{\geq 0}} a_{ijklm} X_i^8 X_j^{24} Y_k^{24} X_l^{32} X_m^{40},
\]

in which all \( a_{ijklm} \) are integers. The weight enumerator of the code \( d_{50}^+ \) is, however, written as

\[
X_8^7 + 2^{15} \cdot 5 \cdot 7 X_8^4 X_{24} + 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11 X_8^4 Y_{24} + 2^5 \cdot 7 \cdot 23 X_8^3 X_{32} \\
+ 2^{10} X_8 \cdot 139 \cdot 3 \cdot 2 X_8^2 X_{40} + 2^8 \cdot 7 \cdot 7 X_8 X_{24}^{24} + 2^{10} \cdot 3 \cdot 7 \cdot 11 X_8 X_{24} Y_{24} \\
+ 2^{10} \cdot 7 \cdot 6521 \cdot 3 \cdot 2 X_8 Y_{24} X_{32} + 2^{11} \cdot 5 \cdot 7 X_{24} X_{32} + 2^{12} \cdot 7 \cdot 227 \cdot 3 \cdot 1 Y_{24} X_{32}.
\]

This expression is unique and we get a contradiction. This completes the proof of Theorem.

If we take a self-dual doubly-even code \( C \) of length 48, and write \( W_C \) as an isobaric polynomial in \( X_8, X_{24}, Y_{24}, X_{32}, X_{40} \), then we can show that the coefficients in this expression are in \( \mathbb{Z}[\frac{1}{3}] \). It was, therefore, expected to find a counter example to our assumption in the proof of Theorem at this length, but it did not work out that way.

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**References**


