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<td>Author(s)</td>
<td>Hirata, Kentaro</td>
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<tr>
<td>Citation</td>
<td>Hokkaido University Preprint Series in Mathematics, 810, 1-8</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2006</td>
</tr>
<tr>
<td>DOI</td>
<td>10.14943/83960</td>
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<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/69618">http://hdl.handle.net/2115/69618</a></td>
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<td>pre810.pdf</td>
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BOUNDARY BEHAVIOR OF SOLUTIONS OF THE HELMHOLTZ EQUATION

KENTARO HIRATA

ABSTRACT. This paper is concerned with the boundary behavior of solutions of the Helmholtz equation in \( \mathbb{R}^n \). In particular, we give a Littlewood-type theorem to show that the approach region introduced by Korányi and Taylor (1983) is best possible.

1. INTRODUCTION

Let \( n \geq 2 \) and let us denote a typical point in \( \mathbb{R}^n \) by \( x = (x_1, \ldots, x_n) \). The usual inner product and norm are written respectively as \( \langle x, y \rangle = x_1y_1 + \cdots + x_ny_n \) and \( |x| = \sqrt{\langle x, x \rangle} \). The symbol \( O(n) \) stands for the set of all orthogonal transformations on \( \mathbb{R}^n \). Let \( \lambda > 0 \). We consider the Helmholtz equation

\[
\Delta u = \lambda^2 u \quad \text{in} \quad \mathbb{R}^n,
\]

where \( \Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2 \). It is known that the Martin boundary for positive solutions of (1.1) can be identified with the unit sphere \( S \) of \( \mathbb{R}^n \), and that every positive solution \( u \) of (1.1) can be represented as \( u = K\mu \) for some Radon measure \( \mu \) on \( S \), where

\[
K\mu(x) = \int_S e^{\lambda \langle x, y \rangle} \, d\mu(y) \quad \text{for} \quad x \in \mathbb{R}^n.
\]

See [4, Corollary to Theorem 4] and [9]. Let \( \sigma \) denote the surface measure on \( S \). Since \( K\sigma(x) \to +\infty \) as \( x \to \infty \) (cf. Lemma 2.1), we investigate the behavior at infinity of the normalization \( K\mu/K\sigma \). Let \( e = (1, 0, \ldots, 0) \) and let \( \Omega \) be an unbounded subset of \( \mathbb{R}^n \) converging to \( e \) at \( \infty \) in the sense that \( |x/|x| - e| \to 0 \) as \( x \to \infty \) within \( \Omega \). We write \( \Omega(y) \) for the image of \( \Omega \) under an element of \( O(n) \) mapping \( e \) to \( y \). Then \( \{\Omega(y) : y \in S\} \) makes a collection of approach regions. By the notation \( \Omega(y) \ni x \to \infty \), we mean that \( x \to \infty \) within \( \Omega(y) \). Korányi and Taylor [9] considered the following approach region. For \( \alpha > 0 \) and \( y \in S \), define

\[
\mathcal{A}_\alpha(y) = \left\{ x \in \mathbb{R}^n : |x - |x||y| \leq \alpha \sqrt{|x|} \right\}.
\]

Theorem A. Let \( \alpha > 0 \) and let \( \mu \) be a Radon measure on \( S \). Then

\[
\lim_{\mathcal{A}_\alpha(y) \ni x \to \infty} \frac{K\mu(x)}{K\sigma} = \frac{d\mu}{d\sigma}(y) \quad \text{for} \quad \sigma\text{-a.e.} \quad y \in S.
\]
This result corresponds to Fatou’s theorem [5] for the boundary behavior of harmonic functions in the unit ball or the upper half space of $\mathbb{R}^n$ (see also [8, 12] for invariant harmonic functions in the unit ball of $\mathbb{C}^n$). The result corresponding to Nagel–Stein’s theorem [11] was established by Berman and Singman [3]. The potential theoretic extension was due to Gowrisankaran and Singman [6]. These results show that there exists an unbounded subset $\Omega$ of $\mathbb{R}^n$ converging to $e$ at $\infty$ such that

$$\limsup_{\Omega(x) \ni x \to \infty} \frac{x - |x|e}{\sqrt{|x|}} = +\infty$$

and that

$$\lim_{\Omega(y) \ni x \to \infty} \frac{K_{\mu}}{K_{\sigma}}(x) = \frac{d\mu}{d\sigma}(y) \quad \text{for } \sigma \text{-a.e. } y \in S,$$

whenever $\mu$ is a Radon measure on $S$. Berman and Singman also showed the following (see [3, Theorem B and Remark 1.13(a)]).

**Theorem B.** Let $\Omega$ be an unbounded subset of $\mathbb{R}^n$ converging to $e$ at $\infty$ and satisfying

$$\limsup_{\Omega(y) \ni x \to \infty} \frac{x - |x|e}{\sqrt{|x|}} = +\infty.$$

Suppose in addition that $\Omega$ is invariant under all elements of $O(n)$ that preserve the point $e$. Then there exists a Radon measure $\mu$ on $S$ such that

$$\limsup_{\Omega(y) \ni x \to \infty} \frac{K_{\mu}}{K_{\sigma}}(x) = +\infty \quad \text{for every } y \in S.$$

Note that the second assumption on $\Omega$ can not be omitted from their construction even if “$\limsup$” in (1.3) is replaced by “$\lim$”.

The purpose of this paper is to show the following Littlewood-type theorem. See [10, 1, 2, 7] for harmonic or invariant harmonic functions.

**Theorem 1.1.** Let $\gamma$ be a curve in $\mathbb{R}^n$ converging to $e$ at $\infty$ and satisfying

$$\lim_{\gamma(x) \ni x \to \infty} \frac{|x - |x|e|}{\sqrt{|x|}} = +\infty.$$

Then there exists a solution $u$ of (1.1) such that $u/K_{\sigma}$ is bounded in $\mathbb{R}^n$ and that $u/K_{\sigma}$ admits no limits as $x \to \infty$ along $T\gamma$ for every $T \in O(n)$.

**Remark 1.2.** We indeed construct $u$ satisfying $-1 \leq u/K_{\sigma} \leq 1$ and

$$\liminf_{T \gamma(x) \ni x \to \infty} \frac{u}{K_{\sigma}}(x) = -1 \quad \text{and} \quad \limsup_{T \gamma(x) \ni x \to \infty} \frac{u}{K_{\sigma}}(x) = 1$$

for every $T \in O(n)$. Note that “$\lim$” in (1.4) can not be replaced by “$\limsup$” as mentioned above (cf. [3, 6]).

The proof of Theorem 1.1 is based on our previous work [7] for invariant harmonic functions in the unit ball of $\mathbb{C}^n$, which was a refinement of Aikawa’s method [1, 2] for harmonic functions in the unit disc or the upper half space of $\mathbb{R}^n$. In Section 4, we remark that our construction and estimates are applicable to show Theorem B.
2. Lemmas

The symbol $A$ denotes an absolute positive constant depending only on $\lambda$ and the dimension $n$, and may change from line to line. The following estimate is found in [3, Lemma 4.1].

**Lemma 2.1.** There exists a constant $A > 1$ such that

$$\frac{1}{A} e^{\lambda |x| |x|^{(1-n)/2}} \leq K \sigma(x) \leq A e^{\lambda |x| |x|^{(1-n)/2}}$$

whenever $|x| \geq 1$.

The surface ball of center $y \in S$ and radius $r > 0$ is denoted by

$$Q(y, r) = \{x \in S : |x - y| < r\}.$$

Then we observe that

$$\lim_{r \to 0} \frac{\sigma(Q(y, r))}{r^{n-1}} = \nu_{n-1},$$

where $\nu_{n-1}$ is the volume of the unit ball of $\mathbb{R}^{n-1}$. Moreover, there exists a constant $A > 1$ such that

$$\frac{1}{A} r^{n-1} \leq \sigma(Q(y, r)) \leq Ar^{n-1} \quad \text{for } 0 < r \leq 2.$$

Let $\pi$ be the radial projection onto $S$, i.e., $\pi(x) = x/|x|$ for $x \in \mathbb{R}^n \setminus \{0\}$. For a Radon measure $\mu$ on $S$, we define the maximal function $M_{(c)} \mu$ with parameter $c \geq 1$ by

$$M_{(c)} \mu(x) = \sup \left\{ \frac{\mu(Q(\pi(x), r))}{r^{n-1}} : r \geq \frac{c}{\sqrt{|x|}} \right\}.$$

**Lemma 2.2.** Let $c \geq 1$ and let $\mu$ be a Radon measure on $S$. Then

$$\frac{K \mu}{K \sigma}(x) \leq A \left( |x|^{(n-1)/2} \mu(Q(\pi(x), c/\sqrt{|x|})) + \frac{1}{c} M_{(c)} \mu(x) \right)$$

whenever $|x| \geq 1$.

**Proof.** Let $|x| \geq 1$. Since $|x| - \langle x, y \rangle = |x| |\pi(x) - y|^2/2$ for $y \in S$, it follows from Lemma 2.1 that

$$\frac{K \mu}{K \sigma}(x) \leq A |x|^{(n-1)/2} \int_S e^{-(\lambda/2)|x||\pi(x) - y|^2} d\mu(y).$$

Let $Q_1 = Q(\pi(x), c/\sqrt{|x|})$ and $Q_j = Q(\pi(x), jc/\sqrt{|x|}) \setminus Q(\pi(x), (j - 1)c/\sqrt{|x|})$ for $j = 2, \ldots, N$, where $N$ is the smallest integer such that $Nc/\sqrt{|x|} > 2$. Then, for $j = 1, \ldots, N$,

$$\int_{Q_j} e^{-(\lambda/2)|x||\pi(x) - y|^2} d\mu(y) \leq e^{-(\lambda/2)((j-1)c)^2} \mu(Q(\pi(x), jc/\sqrt{|x|})).$$

Therefore the right hand side of (2.3) is bounded by

$$A \left( |x|^{(n-1)/2} \mu(Q(\pi(x), c/\sqrt{|x|})) + \sum_{j \geq 2} e^{-(\lambda/2)((j-1)c)^2} (jc)^{n-1} M_{(c)} \mu(x) \right).$$
Since \( \sum_{j \geq 2} e^{-(\lambda/2)(j-1)c^2} (je)^{n-1} \leq A/c \), we obtain the required estimate.

For an integrable function \( f \) on \( S \), we write \( Kf = K(f \, d\sigma) \) and \( M_c(f) = M_c(|f| \, d\sigma) \).

**Lemma 2.3.** The following statements hold.

(i) Let \( \mu \) be a Radon measure on \( S \). Then

\[
\frac{K\mu}{K\sigma}(x) \leq AM_{(1)}\mu(x)
\]

whenever \(|x| \geq 1\).

(ii) Let \( y \in S \), \( 0 < r < 1 \) and \( c \geq 1 \). Suppose that \( f \) is a Borel measurable function on \( S \) such that \( f = 1 \) on \( Q(y, cr) \) and \( |f| \leq 1 \) on \( S \). Then

\[
\frac{Kf}{K\sigma}(ty) \geq 1 - \frac{A}{c}
\]

whenever \( \sqrt{t} \geq 1/r \).

**Proof.** Lemma 2.2 with \( c = 1 \) gives (i). To show (ii), let \( g = (1 - f)/2 \). Then \( g = 0 \) on \( Q(y, cr) \) and \( |g| \leq 1 \) on \( S \). Observe from Lemma 2.2 and (2.2) that if \( \sqrt{t} \geq 1/r \), then

\[
\frac{Kg}{K\sigma}(ty) \leq \frac{A}{c}M_{(c)}g(ty) \leq \frac{A}{c} \sup \left\{ \frac{\sigma(Q(y, \rho))}{\rho^{n-1}} : \rho \geq \sqrt{t} \right\} \leq \frac{A}{c}.
\]

Since \( Kf = K\sigma - 2Kg \), we obtain (ii). \( \square \)

For a set \( E \), let \( \text{diam} \, E = \sup\{|x - y| : x, y \in E\} \).

**Lemma 2.4.** Let \( \gamma \) be a curve in \( \mathbb{R}^n \) converging to \( e \) at \( \infty \) and satisfying (1.4). Then there exist sequences of numbers \( \{a_j\}_{j \geq 1}, \{b_j\}_{j \geq 1} \) and subarcs \( \{\gamma_j\}_{j \geq 1} \) of \( \gamma \) with the following properties:

(i) \( 1 < a_1 < b_1 < \cdots < a_j < b_j < a_{j+1} < b_{j+1} < \cdots \to +\infty \),

(ii) \( a_j \leq \sqrt{|x|} \leq b_j \) for \( x \in \gamma_j \),

(iii) \( b_{j-1} \, \text{diam} \, \pi(\gamma_j) \leq 1 \) if \( j \geq 2 \),

(iv) \( \lim_{j \to \infty} a_j \, \text{diam} \, \pi(\gamma_j) = +\infty \).

**Proof.** Let \( \{a_j\} \) be a sequence such that \( a_j \to +\infty \) as \( j \to +\infty \), and let us choose \( \{a_j\} \), \( \{b_j\} \) and \( \{\gamma_j\} \) inductively. By (1.4), we find \( a_1 > \max\{1, \inf_{x \in \gamma} \sqrt{|x|} \} \) with

\[
\sqrt{|x|} \pi(x) - e \geq \alpha_1 \quad \text{for} \quad x \in \gamma \cap \{\sqrt{|x|} \geq a_1\}.
\]

Let \( \gamma' \) be the connected component of \( \gamma \cap \{\sqrt{|x|} \geq a_1\} \) which converges to \( \infty \), and let \( x_1 \in \gamma' \cap \{\sqrt{|x|} = a_1\} \). Then

\[
\text{diam} \, \pi(\gamma') \geq |\pi(x_1) - e| \geq \frac{\alpha_1}{a_1}.
\]

Let \( \gamma'' \) be a subarc of \( \gamma' \) starting from \( x_1 \) toward \( \infty \) such that

\[
\sup_{x \in \gamma''} \sqrt{|x|} < +\infty \quad \text{and} \quad \text{diam} \, \pi(\gamma'') \geq \frac{1}{2} \text{diam} \, \pi(\gamma').
\]
We take $b_1 > \sup_{x \in \gamma^n} \sqrt{|x|}$. Let $\gamma_1$ be the connected component of $\gamma \cap \{a_1 \leq \sqrt{|x|} \leq b_1\}$ containing $\gamma''$. Then
$$\text{diam } \pi(\gamma_1) \geq \frac{\alpha_1}{2a_1}. $$
We next choose $a_2, b_2$ and $\gamma_2$ as follows. By (1.4) and the fact that $|\pi(x) - e| \to 0$ as $x \to \infty$ along $\gamma$, we find $a_2 > b_1$ such that
$$\frac{1}{2b_1} \geq |\pi(x) - e| \geq \frac{\alpha_2}{\sqrt{|x|}} \text{ for } x \in \gamma \cap \{|\sqrt{|x|}| \geq a_2\}. $$
Repeat the above process to get $b_2 > a_2$ and $\gamma_2$ such that $a_2 \leq \sqrt{|x|} \leq b_2$ for $x \in \gamma_2$ and $\text{diam } \pi(\gamma_2) \geq \alpha_2/2a_2$. Then (2.4) also yields that
$$\text{diam } \pi(\gamma_2) \leq 2 \sup_{x \in \gamma_2} |\pi(x) - e| \leq \frac{1}{b_1}. $$
Continue this process to obtain the required sequences. □

3. CONSTRUCTION

Throughout this section, we suppose that $\{a_j\}_{j \geq 1}$, $\{b_j\}_{j \geq 1}$ and $\{\gamma_j\}_{j \geq 1}$ are as in Lemma 2.4. Let
$$\ell_j = \frac{\text{diam } \pi(\gamma_j)}{3}, \quad c_j = \sqrt{a_j \text{ diam } \pi(\gamma_j)} \quad \text{and} \quad \rho_j = \frac{c_j}{a_j}. $$
Then, by Lemma 2.4,
$$\lim_{j \to +\infty} \ell_j = 0, \quad \lim_{j \to +\infty} \frac{\rho_j}{\ell_j} = 0 \quad \text{and} \quad \lim_{j \to +\infty} c_j = +\infty. $$

Therefore, in the construction below, we may assume that $\rho_j < \ell_j$ for every $j \in \mathbb{N}$. For each $j \in \mathbb{N}$, we choose finitely many points $\{y_j^\nu\}_\nu$ in $S$ such that

(I) $S = \bigcup_{\nu} Q(y_j^\nu, \ell_j)$,

(II) $Q(y_j^\mu, \ell_j/2) \cap Q(y_j^\nu, \ell_j/2) = \emptyset$ if $\mu \neq \nu$.

We define
$$M_j = \bigcup_{\nu} \{y \in S : |y - y_j^\nu| = \ell_j\}, $$
$$G_j = \left\{x \in \mathbb{R}^n : a_j \leq \sqrt{|x|} \leq b_j \text{ and } \pi(x) \in M_j\right\}. $$

Then we have the following.

**Lemma 3.1.** $T\gamma_j \cap G_j \neq \emptyset$ for any $T \in O(n)$ and $j \in \mathbb{N}$.

**Proof.** By (I), we find $\nu$ with $\pi(T\gamma_j) \cap Q(y_j^\nu, \ell_j) \neq \emptyset$. Since $\text{diam } \pi(T\gamma_j) = \text{diam } \pi(\gamma_j) = 3\ell_j$, we see that $\pi(T\gamma_j) \cap M_j \neq \emptyset$. Therefore it follows from $T\gamma_j \subset \{a_j \leq \sqrt{|x|} \leq b_j\}$ that $T\gamma_j \cap G_j \neq \emptyset$. □
Let $R'_j = \{ y \in S : \ell_j - \rho_j < |y - y'_j| < \ell_j + \rho_j \}$ and define

(3.5) \[ E_j = \bigcup_{\nu} R'_j. \]

Note that $Q(y, \rho_j) \subset E_j$ if $y \in M_j$. By $X_E$ we denote the characteristic function of $E$.

**Lemma 3.2.** The following properties for the above $\{E_j\}_{j \geq 1}$ hold.

(i) \[ \lim_{j \to +\infty} \left( \sup_{x} \left\{ \frac{K X_{E_i}}{K \sigma}(x) : \sqrt{|x|} \leq b_{j-1} \right\} \right) = 0. \]

(ii) \[ \lim_{j \to +\infty} \sigma(E_j) = 0. \]

**Proof.** Since the value $\sigma(R'_j)$ is independent of $\nu$, we write $\sigma_j = \sigma(R'_j)$. For a moment, we fix $j$ and let $\sqrt{|x|} \leq b_{j-1}$. By Lemma 2.3(i),

\[
\frac{K X_{E_i}}{K \sigma}(x) \leq AM(1) X_{E_j}(x) \leq A \sup_{\nu} \left\{ \frac{\sigma(R'_j \cap Q(\pi(x), r))}{r^{n-1}} : r \geq \frac{1}{\sqrt{|x|}} \right\} \leq A \sup_{\nu} \left\{ \frac{\sigma_j}{r^{n-1} N_j : r \geq \frac{1}{\sqrt{|x|}}} \right\},
\]

where $N_j$ is the number of $\nu$ such that $R'_j \cap Q(\pi(x), r) \neq \emptyset$. If $r \geq 1/\sqrt{|x|}$, then $r \geq 1/b_{j-1} \geq \text{diam} \pi(\gamma_j) = 3\ell_j$ by Lemma 2.4. Therefore $R'_j \cap Q(\pi(x), r) \neq \emptyset$ implies $Q(y'_j, \ell_j/2) \subset Q(\pi(x), 2r)$. It follows from (II) that $N_j \leq A(r/\ell_j)^{n-1}$. Hence we obtain

(3.6) \[ \sup_{\nu} \left\{ \frac{K X_{E_i}}{K \sigma}(x) : \sqrt{|x|} \leq b_{j-1} \right\} \leq A \frac{\sigma_j}{\ell_j^{n-1}}. \]

Observe from (2.1) and (3.2) that

\[
\frac{\sigma_j}{\ell_j^{n-1}} = \left( \frac{\ell_j + \rho_j}{\ell_j} \right)^{n-1} \frac{\sigma(Q(y, \ell_j + \rho_j))}{(\ell_j + \rho_j)^{n-1}} - \left( \frac{\ell_j - \rho_j}{\ell_j} \right)^{n-1} \frac{\sigma(Q(y, \ell_j - \rho_j))}{(\ell_j - \rho_j)^{n-1}} \rightarrow 0 \quad \text{as} \quad j \rightarrow +\infty.
\]

This together with (3.6) concludes (i).

Taking $x = 0$ in (i), we obtain

\[ \sigma(E_j) = \sigma(S) \frac{K X_{E_i}}{K \sigma}(0) \rightarrow 0 \quad \text{as} \quad j \rightarrow +\infty. \]

Thus (ii) follows. \qed

**Proof of Theorem 1.1.** In view of Lemma 3.2, taking a subsequence of $j$ if necessary, we may assume that

(3.7) \[ \frac{K X_{E_i}}{K \sigma}(x) \leq 2^{-j} \quad \text{for} \quad \sqrt{|x|} \leq b_{j-1}, \]
and \( \sigma(E_j) \leq 2^{-j} \). Then \( \sigma(\bigcap_k \bigcup_{i \geq k} E_i) = 0 \). For \( j \in \mathbb{N} \), let

\[
 f_j(y) = \begin{cases} 
(-1)^{I_j(y)} & \text{if } y \in \bigcup_{1 \leq i \leq j} E_i, \\
0 & \text{if } y \not\in \bigcup_{1 \leq i \leq j} E_i,
\end{cases}
\]

where \( I_j(y) = \max \{ i : y \in E_i, 1 \leq i \leq j \} \). Then we see that \( f_j \) converges \( \sigma \)-a.e. on \( S \) to

\[
f(y) = \begin{cases} 
(-1)^{I(y)} & \text{if } y \in \bigcup_{i \geq 1} E_i \setminus \bigcap_k \bigcup_{i \geq k} E_i, \\
0 & \text{if } y \not\in \bigcup_{i \geq 1} E_i \text{ or } y \in \bigcap_k \bigcup_{i \geq k} E_i,
\end{cases}
\]

where \( I(y) = \max \{ i : y \in E_i \} \) for \( y \in \bigcup_{i \geq 1} E_i \setminus \bigcap_k \bigcup_{i \geq k} E_i \). Also, we have the following:

\[
|f_j| \leq 1, \quad |f_{j+1} - f_j| \leq 2\chi_{E_{j+1}} \text{ on } S; \quad f_j = (-1)^j \text{ on } E_j; \quad Kf_j \rightarrow Kf \text{ on } \mathbb{R}^n.
\]

Let \( T \in O(n) \). By Lemma 3.1, we find \( x_j \in T\gamma \cap G_j \) for each \( j \in \mathbb{N} \). Then \( a_j \leq \sqrt{|x_j|} \leq b_j \) and \( Q(\pi(x_j), c_j/a_j) \subset E_j \). If \( j \) is even, then Lemma 2.3(ii) and (3.7) give

\[
\frac{Kf}{K\sigma}(x_j) = \frac{Kf_j}{K\sigma}(x_j) + \sum_{k \geq j} \frac{K(f_{k+1} - f_k)}{K\sigma}(x_j)
\geq \frac{Kf_j}{K\sigma}(x_j) - 2 \sum_{k \geq j} \frac{K\chi_{E_{k+1}}}{K\sigma}(x_j)
\geq 1 - \frac{A}{c_j} - 2^{1-j}.
\]

Similarly, if \( j \) is odd, then

\[
\frac{Kf}{K\sigma}(x_j) \leq -1 + \frac{A}{c_j} + 2^{1-j}.
\]

Hence we conclude from (3.2) that

\[
\liminf_{T\gamma \ni x \to \infty} \frac{Kf}{K\sigma}(x) = -1 < 1 = \limsup_{T\gamma \ni x \to \infty} \frac{Kf}{K\sigma}(x).
\]

Obviously, \( u = Kf \) is a solution of (1.1) such that \(-1 \leq u/K\sigma \leq 1 \) on \( \mathbb{R}^n \). Thus the proof of Theorem 1.1 is complete.

\[\square\]

4. Remark

Our construction and estimates in Sections 2 and 3 are applicable to show Theorem B. Suppose that an unbounded subset \( \Omega \) of \( \mathbb{R}^n \) satisfies the assumption in Theorem B. Then we find a sequence \( \{x_j\} \) in \( \Omega \) converging to \( e \) at \( \infty \) such that

\[
\lim_{j \to +\infty} \frac{|x_j - |x_j| e|}{|x_j|} = +\infty.
\]

Taking a subsequence of \( j \) if necessary, we may assume that \( \sqrt{|x_j - |x_j| e|} - e| \leq 1 \). Let \( \omega_j = \{T_e(x_j) : T_e \in O(n) \text{ preserves } e\} \) and let \( \omega = \bigcup_j \omega_j \). Note that \( \omega \) is a subset of \( \Omega \).
converging to $e$ at $\infty$. Let $a_j = b_j = \sqrt{|x_j|}$ and define
\[
\ell_j = \frac{|\pi(x_j) - e|}{3}, \quad c_j = \sqrt{a_j|\pi(x_j) - e|} \quad \text{and} \quad \rho_j = \frac{c_j}{a_j},
\]
in place of (3.1). Then these satisfy (3.2) and $3\ell_j \leq 1/b_{j-1}$. Let $M_j$, $G_j$ and $E_j$ be as in (3.3), (3.4) and (3.5) respectively. Then the conclusions in Lemma 3.2 hold in this setting as well. Note that $\omega_j$ and $G_j$ lie on the sphere of center at the origin and radius $|x_j|$. Let $T \in O(n)$. Since $\{y \in S : |y - Te| = 3\ell_j\} \subset \pi(T\omega_j)$, we see that $\pi(T\omega_j) \cap M_j \neq \emptyset$, and so $T\omega_j \cap G_j \neq \emptyset$. Hence we observe the existence of $f$ such that
\[
\liminf_{T\omega \ni x \to \infty} \frac{Kf}{K\sigma}(x) \neq \limsup_{T\omega \ni x \to \infty} \frac{Kf}{K\sigma}(x) \quad \text{for every } T \in O(n).
\]
Thus $Kf/K\sigma$ admits no limits as $x \to \infty$ along $\Omega(y)$ for every $y \in S$.

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