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<th>Title</th>
<th>Boundary behavior of solutions of the Helmholtz equation</th>
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</thead>
<tbody>
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BOUNDARY BEHAVIOR OF SOLUTIONS OF THE HELMHOLTZ EQUATION

KENTARO HIRATA

ABSTRACT. This paper is concerned with the boundary behavior of solutions of the Helmholtz equation in $\mathbb{R}^n$. In particular, we give a Littlewood-type theorem to show that the approach region introduced by Korányi and Taylor (1983) is best possible.

1. INTRODUCTION

Let $n \geq 2$ and let us denote a typical point in $\mathbb{R}^n$ by $x = (x_1, \ldots, x_n)$. The usual inner product and norm are written respectively as $\langle x, y \rangle = x_1y_1 + \cdots + x_ny_n$ and $|x| = \sqrt{\langle x, x \rangle}$. The symbol $O(n)$ stands for the set of all orthogonal transformations on $\mathbb{R}^n$. Let $\lambda > 0$. We consider the Helmholtz equation

$$\Delta u = \lambda^2 u \text{ in } \mathbb{R}^n,$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$. It is known that the Martin boundary for positive solutions of (1.1) can be identified with the unit sphere $S$ of $\mathbb{R}^n$, and that every positive solution $u$ of (1.1) can be represented as $u = K\mu$ for some Radon measure $\mu$ on $S$, where

$$K\mu(x) = \int_S e^{\lambda \langle x, y \rangle} d\mu(y) \quad \text{for } x \in \mathbb{R}^n.$$

See [4, Corollary to Theorem 4] and [9]. Let $\sigma$ denote the surface measure on $S$. Since $K\sigma(x) \to +\infty$ as $x \to \infty$ (cf. Lemma 2.1), we investigate the behavior at infinity of the normalization $K\mu/K\sigma$. Let $e = (1, 0, \ldots, 0)$ and let $\Omega$ be an unbounded subset of $\mathbb{R}^n$ converging to $e$ at $\infty$ in the sense that $|x| - |e| \to 0$ as $x \to \infty$ within $\Omega$. We write $\Omega(y)$ for the image of $\Omega$ under an element of $O(n)$ mapping $e$ to $y$. Then $\{\Omega(y) : y \in S\}$ makes a collection of approach regions. By the notation $\Omega(y) \ni x \to \infty$, we mean that $x \to \infty$ within $\Omega(y)$. Korányi and Taylor [9] considered the following approach region. For $\alpha > 0$ and $y \in S$, define

$$\mathcal{A}_\alpha(y) = \left\{ x \in \mathbb{R}^n : |x - |x||y| \leq \alpha \sqrt{|x|} \right\}.$$

**Theorem A.** Let $\alpha > 0$ and let $\mu$ be a Radon measure on $S$. Then

$$\lim_{\mathcal{A}_\alpha(y) \ni x \to \infty} \frac{K\mu(x)}{K\sigma} = \frac{d\mu}{d\sigma}(y) \quad \text{for } \sigma\text{-a.e. } y \in S.$$
This result corresponds to Fatou’s theorem [5] for the boundary behavior of harmonic functions in the unit ball or the upper half space of $\mathbb{R}^n$ (see also [8, 12] for invariant harmonic functions in the unit ball of $\mathbb{C}^n$). The result corresponding to Nagel–Stein’s theorem [11] was established by Berman and Singman [3]. The potential theoretic extension was due to Gowrisankaran and Singman [6]. These results show that there exists an unbounded subset $\Omega$ of $\mathbb{R}^n$ converging to $e$ at $\infty$ such that

$$\limsup_{\Omega(x) \ni x \to \infty} \frac{|x - |x|| e}{\sqrt{|x|}} = +\infty$$

and that

$$\lim_{\Omega(y) \ni x \to \infty} \frac{K\mu}{K\sigma}(x) = \frac{d\mu}{d\sigma}(y) \quad \text{for } \sigma\text{-a.e. } y \in S,$$

whenever $\mu$ is a Radon measure on $S$. Berman and Singman also showed the following (see [3, Theorem B and Remark 1.13(a)]).

**Theorem B.** Let $\Omega$ be an unbounded subset of $\mathbb{R}^n$ converging to $e$ at $\infty$ and satisfying

$$\limsup_{\Omega(x) \ni x \to \infty} \frac{|x - |x|| e}{\sqrt{|x|}} = +\infty.$$  

Suppose in addition that $\Omega$ is invariant under all elements of $O(n)$ that preserve the point $e$. Then there exists a Radon measure $\mu$ on $S$ such that

$$\limsup_{\Omega(y) \ni x \to \infty} \frac{K\mu}{K\sigma}(x) = +\infty \quad \text{for every } y \in S.$$

Note that the second assumption on $\Omega$ can not be omitted from their construction even if “$\limsup$” in (1.3) is replaced by “$\lim$”.

The purpose of this paper is to show the following Littlewood-type theorem. See [10, 1, 2, 7] for harmonic or invariant harmonic functions.

**Theorem 1.1.** Let $\gamma$ be a curve in $\mathbb{R}^n$ converging to $e$ at $\infty$ and satisfying

$$\limsup_{\gamma(x) \ni x \to \infty} \frac{|x - |x|| e}{\sqrt{|x|}} = +\infty.$$  

Then there exists a solution $u$ of (1.1) such that $u/K\sigma$ is bounded in $\mathbb{R}^n$ and that $u/K\sigma$ admits no limits as $x \to \infty$ along $T\gamma$ for every $T \in O(n)$.

**Remark 1.2.** We indeed construct $u$ satisfying $-1 \leq u/K\sigma \leq 1$ and

$$\liminf_{T\gamma(x) \ni x \to \infty} \frac{u}{K\sigma}(x) = -1 \quad \text{and} \quad \limsup_{T\gamma(x) \ni x \to \infty} \frac{u}{K\sigma}(x) = 1$$

for every $T \in O(n)$. Note that “$\lim$” in (1.4) can not be replaced by “$\limsup$” as mentioned above (cf. [3, 6]).

The proof of Theorem 1.1 is based on our previous work [7] for invariant harmonic functions in the unit ball of $\mathbb{C}^n$, which was a refinement of Aikawa’s method [1, 2] for harmonic functions in the unit disc or the upper half space of $\mathbb{R}^n$. In Section 4, we remark that our construction and estimates are applicable to show Theorem B.
2. Lemmas

The symbol $A$ denotes an absolute positive constant depending only on $\lambda$ and the dimension $n$, and may change from line to line. The following estimate is found in [3, Lemma 4.1].

**Lemma 2.1.** There exists a constant $A > 1$ such that

\[
\frac{1}{A} e^{\lambda|x|} |x|^{(1-n)/2} \leq K \sigma(x) \leq A e^{\lambda|x|} |x|^{(1-n)/2}
\]

whenever $|x| \geq 1$.

The surface ball of center $y \in S$ and radius $r > 0$ is denoted by

\[ Q(y, r) = \{x \in S : |x - y| < r\} \]

Then we observe that

\[
\lim_{r \to 0} \frac{\sigma(Q(y, r))}{r^{n-1}} = \nu_{n-1},
\]

where $\nu_{n-1}$ is the volume of the unit ball of $\mathbb{R}^{n-1}$. Moreover, there exists a constant $A > 1$ such that

\[
\frac{1}{A} \nu_{n-1} \leq \sigma(Q(y, r)) \leq A \nu_{n-1} \quad \text{for } 0 < r \leq 2.
\]

Let $\pi$ be the radial projection onto $S$, i.e., $\pi(x) = x/|x|$ for $x \in \mathbb{R}^n \setminus \{0\}$. For a Radon measure $\mu$ on $S$, we define the maximal function $M(c)\mu$ with parameter $c \geq 1$ by

\[
M(c)\mu(x) = \sup \left\{ \frac{\mu(Q(\pi(x), r))}{r^{n-1}} : r \geq \frac{c}{\sqrt{|x|}} \right\}.
\]

**Lemma 2.2.** Let $c \geq 1$ and let $\mu$ be a Radon measure on $S$. Then

\[
\frac{K\mu}{K \sigma}(x) \leq A \left( |x|^{(n-1)/2} \mu(Q(\pi(x), c/\sqrt{|x|})) + \frac{c}{c} M(c)\mu(x) \right)
\]

whenever $|x| \geq 1$.

**Proof.** Let $|x| \geq 1$. Since $|x| - \langle x, y \rangle = |x| |\pi(x) - y|^2/2$ for $y \in S$, it follows from Lemma 2.1 that

\[
\frac{K\mu}{K \sigma}(x) \leq A |x|^{(n-1)/2} \int_S e^{-(\lambda/2)|x||\pi(x) - y|^2} d\mu(y).
\]

Let $Q_1 = Q(\pi(x), c/\sqrt{|x|})$ and $Q_j = Q(\pi(x), j c/\sqrt{|x|}) \setminus Q(\pi(x), (j - 1)c/\sqrt{|x|})$ for $j = 2, \ldots, N$, where $N$ is the smallest integer such that $N c/\sqrt{|x|} > 2$. Then, for $j = 1, \ldots, N$,

\[
\int_{Q_j} e^{-(\lambda/2)|x||\pi(x) - y|^2} d\mu(y) \leq e^{-(\lambda/2)((j-1)c)^2} \mu(Q(\pi(x), j c/\sqrt{|x|})).
\]

Therefore the right hand side of (2.3) is bounded by

\[
A \left( |x|^{(n-1)/2} \mu(Q(\pi(x), c/\sqrt{|x|})) \right) + \sum_{j \geq 2} e^{-(\lambda/2)((j-1)c)^2} (jc)^{n-1} M(c)\mu(x).
\]
Since \( \sum_{j \geq 2} e^{-\lambda j/2} (j \varepsilon)^{n-1} \leq A/c \), we obtain the required estimate. \( \square \)

For an integrable function \( f \) on \( S \), we write \( Kf = K(f \, d\sigma) \) and \( M(\varepsilon) f = M(\varepsilon)(|f| \, d\sigma) \).

**Lemma 2.3.** The following statements hold.

(i) Let \( \mu \) be a Radon measure on \( S \). Then
\[
\frac{K \mu}{K \sigma}(x) \leq AM(1) \mu(x)
\]
whenever \( |x| \geq 1 \).

(ii) Let \( y \in S \), \( 0 < r < 1 \) and \( c \geq 1 \). Suppose that \( f \) is a Borel measurable function on \( S \) such that \( f = 1 \) on \( Q(y, cr) \) and \( |f| \leq 1 \) on \( S \). Then
\[
\frac{Kf}{K \sigma}(ty) \geq 1 - \frac{A}{c}
\]
whenever \( \sqrt{t} \geq 1/r \).

**Proof.** Lemma 2.2 with \( c = 1 \) gives (i). To show (ii), let \( g = (1 - f)/2 \). Then \( g = 0 \) on \( Q(y, cr) \) and \( |g| \leq 1 \) on \( S \). Observe from Lemma 2.2 and (2.2) that if \( \sqrt{t} \geq 1/r \), then
\[
\frac{Kg}{K \sigma}(ty) \leq \frac{A}{c} M(\varepsilon) g(ty) \leq \frac{A}{c} \sup \left\{ \frac{\sigma(Q(y, \rho))}{\rho^{n-1}} : \rho \geq \frac{c}{\sqrt{t}} \right\} \leq \frac{A}{c}.
\]
Since \( Kf = K \sigma - 2Kg \), we obtain (ii). \( \square \)

For a set \( E \), let \( \text{diam} \, E = \sup \{|x - y| : x, y \in E\} \).

**Lemma 2.4.** Let \( \gamma \) be a curve in \( \mathbb{R}^n \) converging to \( e \) at \( \infty \) and satisfying (1.4). Then there exist sequences of numbers \( \{a_j\}_{j \geq 1}, \{b_j\}_{j \geq 1} \) and subarcs \( \{\gamma_j\}_{j \geq 1} \) of \( \gamma \) with the following properties:

(i) \( 1 < a_1 < b_1 < \cdots < a_j < b_j < a_{j+1} < b_{j+1} < \cdots \to +\infty \),

(ii) \( a_j \leq \sqrt{|x|} \leq b_j \) for \( x \in \gamma_j \),

(iii) \( b_{j-1} \text{ diam } \pi(\gamma_j) \leq 1 \) if \( j \geq 2 \),

(iv) \( \lim_{j \to +\infty} a_j \text{ diam } \pi(\gamma_j) = +\infty \).

**Proof.** Let \( \{\alpha_j\} \) be a sequence such that \( \alpha_j \to +\infty \) as \( j \to +\infty \), and let us choose \( \{a_j\}, \{b_j\} \) and \( \{\gamma_j\} \) inductively. By (1.4), we find \( a_1 > \max\{1, \inf_{x \in \gamma} \sqrt{|x|}\} \) with
\[
\sqrt{|x|} \pi(x) - e \geq \alpha_1 \quad \text{for } x \in \gamma \cap \{\sqrt{|x|} \geq a_1\}.
\]
Let \( \gamma' \) be the connected component of \( \gamma \cap \{\sqrt{|x|} \geq a_1\} \) which converges to \( \infty \), and let \( x_1 \in \gamma' \cap \{\sqrt{|x|} = a_1\} \). Then
\[
\text{diam } \pi(\gamma') \geq |\pi(x_1) - e| \geq \frac{\alpha_1}{a_1}.
\]
Let \( \gamma'' \) be a subarc of \( \gamma' \) starting from \( x_1 \) toward \( \infty \) such that
\[
\sup_{x \in \gamma''} \sqrt{|x|} < +\infty \quad \text{and} \quad \text{diam } \pi(\gamma'') \geq \frac{1}{2} \text{ diam } \pi(\gamma').
\]
We take \( b_1 > \sup_{x \in \gamma'} \sqrt{|x|} \). Let \( \gamma_1 \) be the connected component of \( \gamma \cap \{ a_1 \leq \sqrt{|x|} \leq b_1 \} \) containing \( \gamma'' \). Then
\[
\text{diam } \pi(\gamma_1) \geq \frac{\alpha_1}{2a_1}.
\]
We next choose \( a_2, b_2 \) and \( \gamma_2 \) as follows. By (1.4) and the fact that \( |\pi(x) - e| \to 0 \) as \( x \to \infty \) along \( \gamma \), we find \( a_2 > b_1 \) such that
\[
\frac{1}{2b_1} \geq |\pi(x) - e| \geq \frac{\alpha_2}{\sqrt{|x|}} \quad \text{for } x \in \gamma \cap \{ \sqrt{|x|} \geq a_2 \}.
\]
Repeat the above process to get \( b_2 > a_2 \) and \( \gamma_2 \) such that \( a_2 \leq \sqrt{|x|} \leq b_2 \) for \( x \in \gamma_2 \) and \( \text{diam } \pi(\gamma_2) \geq \alpha_2/2a_2 \). Then (2.4) also yields that
\[
\text{diam } \pi(\gamma_2) \leq 2 \sup_{x \in \gamma_2} |\pi(x) - e| \leq \frac{1}{b_1}.
\]
Continue this process to obtain the required sequences. \( \square \)

### 3. Construction

Throughout this section, we suppose that \( \{ a_j \}_{j \geq 1}, \{ b_j \}_{j \geq 1} \) and \( \{ \gamma_j \}_{j \geq 1} \) are as in Lemma 2.4. Let
\[
\ell_j = \frac{3}{\text{diam } \pi(\gamma_j)}, \quad c_j = \sqrt{a_j \text{diam } \pi(\gamma_j)} \quad \text{and} \quad \rho_j = \frac{c_j}{a_j}.
\]
Then, by Lemma 2.4,
\[
\lim_{j \to +\infty} \ell_j = 0, \quad \lim_{j \to +\infty} \frac{\rho_j}{\ell_j} = 0 \quad \text{and} \quad \lim_{j \to +\infty} c_j = +\infty.
\]
Therefore, in the construction below, we may assume that \( \rho_j < \ell_j \) for every \( j \in \mathbb{N} \). For each \( j \in \mathbb{N} \), we choose finitely many points \( \{ y_j^\nu \}_\nu \) in \( S \) such that
\[
\begin{align*}
\text{(I)} & \quad S = \bigcup_{\nu} Q(y_j^\nu, \ell_j), \\
\text{(II)} & \quad Q(y_j^\mu, \ell_j/2) \cap Q(y_j^\nu, \ell_j/2) = \emptyset \text{ if } \mu \neq \nu.
\end{align*}
\]
We define
\[
M_j = \bigcup_{\nu} \{ y \in S : |y - y_j^\nu| = \ell_j \},
\]
\[
G_j = \left\{ x \in \mathbb{R}^n : a_j \leq \sqrt{|x|} \leq b_j \text{ and } \pi(x) \in M_j \right\}.
\]
Then we have the following.

**Lemma 3.1.** \( T\gamma_j \cap G_j \neq \emptyset \) for any \( T \in O(n) \) and \( j \in \mathbb{N} \).

**Proof.** By (I), we find \( \nu \) with \( \pi(T\gamma_j) \cap Q(y_j^\nu, \ell_j) \neq \emptyset \). Since \( \text{diam } \pi(T\gamma_j) = \text{diam } \pi(\gamma_j) = 3\ell_j \), we see that \( \pi(T\gamma_j) \cap M_j \neq \emptyset \). Therefore it follows from \( T\gamma_j \subset \{ a_j \leq \sqrt{|x|} \leq b_j \} \) that \( T\gamma_j \cap G_j \neq \emptyset \). \( \square \)
Let \( R^\nu_j = \{ y \in S : \ell_j - \rho_j < |y - y^\nu_j| < \ell_j + \rho_j \} \) and define

\[
E_j = \bigcup \nu R^\nu_j.
\]

Note that \( Q(y, \rho_j) \subset E_j \) if \( y \in M_j \). By \( X_E \) we denote the characteristic function of \( E \).

**Lemma 3.2.** The following properties for the above \( \{E_j\}_{j \geq 1} \) hold.

(i) \[ \lim_{j \to +\infty} \left( \sup \left\{ \frac{KX_{E_j}}{K\sigma}(x) : \sqrt{|x|} \leq b_j - 1 \right\} \right) = 0. \]

(ii) \[ \lim_{j \to +\infty} \sigma(E_j) = 0. \]

**Proof.** Since the value \( \sigma(R^\nu_j) \) is independent of \( \nu \), we write \( \sigma_j = \sigma(R^\nu_j) \). For a moment, we fix \( j \) and let \( \sqrt{|x|} \leq b_j - 1 \). By Lemma 2.3(i),

\[
\frac{KX_{E_j}}{K\sigma}(x) \leq AM_1 \frac{X_{E_j}}{\sigma}(x) \leq A \sup \left\{ \sigma_j \left( \frac{r}{\ell_j} \right)^{n-1} N_j : r \geq \frac{1}{\sqrt{|x|}} \right\},
\]

where \( N_j \) is the number of \( \nu \) such that \( R^\nu_j \cap Q(\pi(x), r) \neq \emptyset \). If \( r \geq 1/\sqrt{|x|} \), then \( r \geq 1/b_j - 1 \geq \text{diam} \pi(\gamma_j) = 3\ell_j \) by Lemma 2.4. Therefore \( R^\nu_j \cap Q(\pi(x), r) \neq \emptyset \) implies \( Q(y^\nu_j, \ell_j/2) \subset Q(\pi(x), 2r) \). It follows from (II) that \( N_j \leq A(r/\ell_j)^{n-1} \). Hence we obtain

\[
\sup \left\{ \frac{KX_{E_j}}{K\sigma}(x) : \sqrt{|x|} \leq b_j - 1 \right\} \leq A \frac{\sigma_j}{\ell_j^{n-1}}.
\]

Observe from (2.1) and (3.2) that

\[
\frac{\sigma_j}{\ell_j^{n-1}} = \left( \frac{\ell_j + \rho_j}{\ell_j} \right)^{n-1} \frac{\sigma(Q(y, \ell_j + \rho_j))}{(\ell_j + \rho_j)^{n-1}} - \left( \frac{\ell_j - \rho_j}{\ell_j} \right)^{n-1} \frac{\sigma(Q(y, \ell_j - \rho_j))}{(\ell_j - \rho_j)^{n-1}} \to 0 \quad \text{as} \quad j \to +\infty.
\]

This together with (3.6) concludes (i).

Taking \( x = 0 \) in (i), we obtain

\[
\sigma(E_j) = \sigma(S) \frac{KX_{E_j}}{K\sigma}(0) \to 0 \quad \text{as} \quad j \to +\infty.
\]

Thus (ii) follows.

**Proof of Theorem 1.1.** In view of Lemma 3.2, taking a subsequence of \( j \) if necessary, we may assume that

\[
\frac{KX_{E_j}}{K\sigma}(x) \leq 2^{-j} \quad \text{for} \quad \sqrt{|x|} \leq b_j - 1,
\]

(3.7)
and \( \sigma(E_j) \leq 2^{-j} \). Then \( \sigma(\bigcap_k \bigcup_{i \geq k} E_i) = 0 \). For \( j \in \mathbb{N} \), let

\[
f_j(y) = \begin{cases} 
(-1)^{I_j(y)} & \text{if } y \in \bigcup_{1 \leq i \leq j} E_i, \\
0 & \text{if } y \notin \bigcup_{1 \leq i \leq j} E_i,
\end{cases}
\]

where \( I_j(y) = \max\{i : y \in E_i, 1 \leq i \leq j\} \). Then we see that \( f_j \) converges \( \sigma \)-a.e. on \( S \) to

\[
f(y) = \begin{cases} 
(-1)^{I(y)} & \text{if } y \in \bigcup_{i \geq 1} E_i \setminus \bigcap_k \bigcup_{i \geq k} E_i, \\
0 & \text{if } y \notin \bigcup_{i \geq 1} E_i \text{ or } y \in \bigcap_k \bigcup_{i \geq k} E_i,
\end{cases}
\]

where \( I(y) = \max\{i : y \in E_i\} \) for \( y \in \bigcup_{i \geq 1} E_i \setminus \bigcap_k \bigcup_{i \geq k} E_i \). Also, we have the following:

\[
|f_j| \leq 1, \quad |f_{j+1} - f_j| \leq 2\mathcal{X}_{E_{j+1}} \text{ on } S; \quad f_j = (-1)^j \text{ on } E_j; \quad Kf_j \to Kf \text{ on } \mathbb{R}^n.
\]

Let \( T \in O(n) \). By Lemma 3.1, we find \( x_j \in T\gamma \cap G_j \) for each \( j \in \mathbb{N} \). Then \( a_j \leq \sqrt{|x_j|} \leq b_j \) and \( Q(\pi(x_j), c_j/a_j) \subset E_j \). If \( j \) is even, then Lemma 2.3(ii) and (3.7) give

\[
\frac{Kf}{K\sigma}(x_j) = \frac{Kf_j}{K\sigma}(x_j) + \sum_{k \geq j} \frac{K(f_{k+1} - f_k)}{K\sigma}(x_j)
\]

\[
\geq \frac{Kf_j}{K\sigma}(x_j) - 2 \sum_{k \geq j} \frac{\mathcal{X}_{E_{k+1}}}{K\sigma}(x_j)
\]

\[
\geq 1 - \frac{A}{c_j} - 2^{1-j}.
\]

Similarly, if \( j \) is odd, then

\[
\frac{Kf}{K\sigma}(x_j) \leq -1 + \frac{A}{c_j} + 2^{1-j}.
\]

Hence we conclude from (3.2) that

\[
\liminf_{T\gamma \ni x \to \infty} \frac{Kf}{K\sigma}(x) = -1 < 1 = \limsup_{T\gamma \ni x \to \infty} \frac{Kf}{K\sigma}(x).
\]

Obviously, \( u = Kf \) is a solution of (1.1) such that \( -1 \leq u/K\sigma \leq 1 \) on \( \mathbb{R}^n \). Thus the proof of Theorem 1.1 is complete. \( \square \)

4. Remark

Our construction and estimates in Sections 2 and 3 are applicable to show Theorem B. Suppose that an unbounded subset \( \Omega \) of \( \mathbb{R}^n \) satisfies the assumption in Theorem B. Then we find a sequence \( \{x_j\} \) in \( \Omega \) converging to \( e \) at \( \infty \) such that

\[
\lim_{j \to +\infty} \frac{|x_j - |x_j||}{\sqrt{|x_j|}} = +\infty.
\]

Taking a subsequence of \( j \) if necessary, we may assume that \( \sqrt{|x_{j-1}|} |\pi(x_j) - e| \leq 1 \). Let \( \omega_j = \{T_e(x_j) : T_e \in O(n) \text{ preserves } e\} \) and let \( \omega = \bigcup_j \omega_j \). Note that \( \omega \) is a subset of \( \Omega \)
converging to $e$ at $\infty$. Let $a_j = b_j = \sqrt{|x_j|}$ and define

$$
\ell_j = \frac{|\pi(x_j) - e|}{3}, \quad c_j = \sqrt{a_j|\pi(x_j) - e|} \quad \text{and} \quad \rho_j = \frac{c_j}{a_j},
$$

in place of (3.1). Then these satisfy (3.2) and $3\ell_j \leq 1/b_{j-1}$. Let $M_j$, $G_j$ and $E_j$ be as in (3.3), (3.4) and (3.5) respectively. Then the conclusions in Lemma 3.2 hold in this setting as well. Note that $\omega_j$ and $G_j$ lie on the sphere of center at the origin and radius $|x_j|$. Let $T \in O(n)$. Since $\{y \in S : |y - Te| = 3\ell_j\} \subset \pi(T\omega_j)$, we see that $\pi(T\omega_j) \cap M_j \neq \emptyset$, and so $T\omega_j \cap G_j \neq \emptyset$. Hence we observe the existence of $f$ such that

$$
\liminf_{T\omega \ni x \to \infty} \frac{Kf}{K\sigma}(x) \neq \limsup_{T\omega \ni x \to \infty} \frac{Kf}{K\sigma}(x) \quad \text{for every} \ T \in O(n).
$$

Thus $Kf/K\sigma$ admits no limits as $x \to \infty$ along $\Omega(y)$ for every $y \in S$.

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