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BOUNDARY BEHAVIOR OF SOLUTIONS OF THE HELMHOLTZ EQUATION

KENTARO HIRATA

ABSTRACT. This paper is concerned with the boundary behavior of solutions of the Helmholtz equation in $\mathbb{R}^n$. In particular, we give a Littlewood-type theorem to show that the approach region introduced by Korányi and Taylor (1983) is best possible.

1. INTRODUCTION

Let $n \geq 2$ and let us denote a typical point in $\mathbb{R}^n$ by $x = (x_1, \ldots, x_n)$. The usual inner product and norm are written respectively as $\langle x, y \rangle = x_1y_1 + \cdots + x_ny_n$ and $|x| = \sqrt{\langle x, x \rangle}$. The symbol $O(n)$ stands for the set of all orthogonal transformations on $\mathbb{R}^n$. Let $\lambda > 0$. We consider the Helmholtz equation

\[ \Delta u = \lambda^2 u \quad \text{in} \quad \mathbb{R}^n, \]

where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$. It is known that the Martin boundary for positive solutions of (1.1) can be identified with the unit sphere $S$ of $\mathbb{R}^n$, and that every positive solution $u$ of (1.1) can be represented as $u = K\mu$ for some Radon measure $\mu$ on $S$, where

\[ K\mu(x) = \int_S e^{\lambda \langle x, y \rangle} d\mu(y) \quad \text{for} \quad x \in \mathbb{R}^n. \]  

See [4, Corollary to Theorem 4] and [9]. Let $\sigma$ denote the surface measure on $S$. Since $K\sigma(x) \to +\infty$ as $x \to \infty$ (cf. Lemma 2.1), we investigate the behavior at infinity of the normalization $K\mu/K\sigma$. Let $e = (1, 0, \ldots, 0)$ and let $\Omega$ be an unbounded subset of $\mathbb{R}^n$ converging to $e$ at $\infty$ in the sense that $|x/|x| - e| \to 0$ as $x \to \infty$ within $\Omega$. We write $\Omega(y)$ for the image of $\Omega$ under an element of $O(n)$ mapping $e$ to $y$. Then $\{\Omega(y) : y \in S\}$ makes a collection of approach regions. By the notation $\Omega(y) \ni x \to \infty$, we mean that $x \to \infty$ within $\Omega(y)$. Korányi and Taylor [9] considered the following approach region. For $\alpha > 0$ and $y \in S$, define

\[ A_\alpha(y) = \left\{ x \in \mathbb{R}^n : |x - |x||y| \leq \alpha \sqrt{|x|} \right\}. \]

Theorem A. Let $\alpha > 0$ and let $\mu$ be a Radon measure on $S$. Then

\[ \lim_{A_\alpha(y) \ni x \to \infty} K\mu(x)/K\sigma(x) = \frac{d\mu}{d\sigma}(y) \quad \text{for} \ \sigma\text{-a.e.} \ y \in S. \]

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This result corresponds to Fatou’s theorem [5] for the boundary behavior of harmonic functions in the unit ball or the upper half space of $\mathbb{R}^n$ (see also [8, 12] for invariant harmonic functions in the unit ball of $\mathbb{C}^n$). The result corresponding to Nagel–Stein’s theorem [11] was established by Berman and Singman [3]. The potential theoretic extension was due to Gowrisankaran and Singman [6]. These results show that there exists an unbounded subset $\Omega$ of $\mathbb{R}^n$ converging to $e$ at $\infty$ such that
\[
\limsup_{\Omega \ni x \to \infty} \frac{|x - |x|e|}{\sqrt{|x|}} = +\infty
\]
and that
\[
\lim_{\Omega(y) \ni x \to \infty} \frac{K\mu}{K\sigma}(x) = \frac{d\mu}{d\sigma}(y) \quad \text{for } \sigma\text{-a.e. } y \in S,
\]
whenever $\mu$ is a Radon measure on $S$. Berman and Singman also showed the following (see [3, Theorem B and Remark 1.13(a)]).

**Theorem B.** Let $\Omega$ be an unbounded subset of $\mathbb{R}^n$ converging to $e$ at $\infty$ and satisfying
\[
(1.3) \quad \limsup_{\Omega \ni x \to \infty} \frac{|x - |x|e|}{\sqrt{|x|}} = +\infty.
\]
Suppose in addition that $\Omega$ is invariant under all elements of $O(n)$ that preserve the point $e$. Then there exists a Radon measure $\mu$ on $S$ such that
\[
\limsup_{\Omega(y) \ni x \to \infty} \frac{K\mu}{K\sigma}(x) = +\infty \quad \text{for every } y \in S.
\]

Note that the second assumption on $\Omega$ can not be omitted from their construction even if “$\limsup$” in (1.3) is replaced by “$\lim$”.

The purpose of this paper is to show the following Littlewood-type theorem. See [10, 1, 2, 7] for harmonic or invariant harmonic functions.

**Theorem 1.1.** Let $\gamma$ be a curve in $\mathbb{R}^n$ converging to $e$ at $\infty$ and satisfying
\[
(1.4) \quad \lim_{\gamma \ni x \to \infty} \frac{|x - |x|e|}{\sqrt{|x|}} = +\infty.
\]
Then there exists a solution $u$ of (1.1) such that $u/K\sigma$ is bounded in $\mathbb{R}^n$ and that $u/K\sigma$ admits no limits as $x \to \infty$ along $T\gamma$ for every $T \in O(n)$.

**Remark 1.2.** We indeed construct $u$ satisfying $-1 \leq u/K\sigma \leq 1$ and
\[
\liminf_{T \ni x \to \infty} \frac{u}{K\sigma}(x) = -1 \quad \text{and} \quad \limsup_{T \ni x \to \infty} \frac{u}{K\sigma}(x) = 1
\]
for every $T \in O(n)$. Note that “$\lim$” in (1.4) can not be replaced by “$\limsup$” as mentioned above (cf. [3, 6]).

The proof of Theorem 1.1 is based on our previous work [7] for invariant harmonic functions in the unit ball of $\mathbb{C}^n$, which was a refinement of Aikawa’s method [1, 2] for harmonic functions in the unit disc or the upper half space of $\mathbb{R}^n$. In Section 4, we remark that our construction and estimates are applicable to show Theorem B.
2. LEMMAS

The symbol $A$ denotes an absolute positive constant depending only on $\lambda$ and the dimension $n$, and may change from line to line. The following estimate is found in [3, Lemma 4.1].

**Lemma 2.1.** There exists a constant $A > 1$ such that

$$\frac{1}{A}e^{\lambda|x|} |x|^{(1-n)/2} \leq K \sigma(x) \leq A e^{\lambda|x|} |x|^{(1-n)/2}$$

whenever $|x| \geq 1$.

The surface ball of center $y \in S$ and radius $r > 0$ is denoted by

$$Q(y, r) = \{ x \in S : |x - y| < r \}.$$ 

Then we observe that

(2.1) \[ \lim_{r \to 0} \frac{\sigma(Q(y, r))}{r^n} = \nu_{n-1}, \]

where $\nu_{n-1}$ is the volume of the unit ball of $\mathbb{R}^{n-1}$. Moreover, there exists a constant $A > 1$ such that

(2.2) \[ \frac{1}{A} r^{n-1} \leq \sigma(Q(y, r)) \leq A r^{n-1} \quad \text{for } 0 < r \leq 2. \]

Let $\pi$ be the radial projection onto $S$, i.e., $\pi(x) = x/|x|$ for $x \in \mathbb{R}^n \setminus \{0\}$. For a Radon measure $\mu$ on $S$, we define the maximal function $M_{(c)} \mu$ with parameter $c \geq 1$ by

$$M_{(c)} \mu(x) = \sup \left\{ \frac{\mu(Q(\pi(x), r))}{r^n} : r \geq \frac{c}{\sqrt{|x|}} \right\}.$$ 

**Lemma 2.2.** Let $c \geq 1$ and let $\mu$ be a Radon measure on $S$. Then

$$\frac{K \mu}{K \sigma} (x) \leq A \left( |x|^{(n-1)/2} \mu(Q(\pi(x), c/\sqrt{|x|})) + \frac{1}{c} M_{(c)} \mu(x) \right)$$

whenever $|x| \geq 1$.

**Proof.** Let $|x| \geq 1$. Since $|x| - \langle x, y \rangle = |x| |\pi(x) - y|^2/2$ for $y \in S$, it follows from Lemma 2.1 that

(2.3) \[ \frac{K \mu}{K \sigma} (x) \leq A |x|^{(n-1)/2} \int_S e^{-\lambda/2 |x|^2} \mu(\pi(x), y) d\mu(y). \]

Let $Q_1 = Q(\pi(x), c/\sqrt{|x|})$ and $Q_j = Q(\pi(x), j c/\sqrt{|x|}) \setminus Q(\pi(x), (j - 1)c/\sqrt{|x|})$ for $j = 2, \ldots, N$, where $N$ is the smallest integer such that $N c/\sqrt{|x|} > 2$. Then, for $j = 1, \ldots, N$,

$$\int_{Q_j} e^{-\lambda/2 |x|^2} \mu(\pi(x), y) d\mu(y) \leq e^{-\lambda/2 ((j - 1)c)^2} \mu(Q(\pi(x), j c/\sqrt{|x|})).$$

Therefore the right hand side of (2.3) is bounded by

$$A \left( |x|^{(n-1)/2} \mu(Q(\pi(x), c/\sqrt{|x|})) + \sum_{j \geq 2} e^{-\lambda/2 (j - 1)c^2} (jc)^{n-1} M_{(c)} \mu(x) \right).$$
Since $\sum_{j \geq 2} e^{-(\lambda/2)((j-1)c)^2} (j\epsilon)^{n-1} \leq A/c$, we obtain the required estimate. \hfill \Box

For an integrable function $f$ on $S$, we write $Kf = K(f \, d\sigma)$ and $M(c)f = M(c)(|f| \, d\sigma)$.

**Lemma 2.3.** The following statements hold.

(i) Let $\mu$ be a Radon measure on $S$. Then

$$\frac{K\mu}{K\sigma}(x) \leq AM(\mu)\mu(x)$$

whenever $|x| \geq 1$.

(ii) Let $y \in S$, $0 < r < 1$ and $c \geq 1$. Suppose that $f$ is a Borel measurable function on $S$ such that $f = 1$ on $Q(y, cr)$ and $|f| \leq 1$ on $S$. Then

$$\frac{Kf}{K\sigma}(ty) \geq 1 - \frac{A}{c}$$

whenever $\sqrt{1} \geq 1/r$.

**Proof.** Lemma 2.2 with $c = 1$ gives (i). To show (ii), let $g = (1 - f)/2$. Then $g = 0$ on $Q(y, cr)$ and $|g| \leq 1$ on $S$. Observe from Lemma 2.2 and (2.2) that if $\sqrt{1} \geq 1/r$, then

$$\frac{Kg}{K\sigma}(ty) \leq \frac{A}{c} \sup \left\{ \frac{\sigma(Q(y, \rho))}{\rho^{n-1}} : \rho \geq \frac{c}{\sqrt{t}} \right\} \leq \frac{A}{c}. $$

Since $Kf = K\sigma - 2Kg$, we obtain (ii). \hfill \Box

For a set $E$, let $\text{diam } E = \sup\{|x - y| : x, y \in E\}$.

**Lemma 2.4.** Let $\gamma$ be a curve in $\mathbb{R}^n$ converging to $e$ at $\infty$ and satisfying (1.4). Then there exist sequences of numbers $\{a_j\}_{j \geq 1}$, $\{b_j\}_{j \geq 1}$ and subarcs $\{\gamma_j\}_{j \geq 1}$ of $\gamma$ with the following properties:

(i) $1 < a_1 < b_1 < \cdots < a_j < b_j < a_{j+1} < b_{j+1} < \cdots \to +\infty$,  
(ii) $a_j \leq \sqrt{|x|} \leq b_j$ for $x \in \gamma_j$,  
(iii) $b_{j-1} \text{ diam } \pi(\gamma_j) \leq 1$ if $j \geq 2$,  
(iv) $\lim_{j \to +\infty} a_j \text{ diam } \pi(\gamma_j) = +\infty$.

**Proof.** Let $\{a_j\}$ be a sequence such that $a_j \to +\infty$ as $j \to +\infty$, and let us choose $\{a_j\}$, $\{b_j\}$ and $\{\gamma_j\}$ inductively. By (1.4), we find $a_1 > \max\{1, \inf_{x \in \gamma} \sqrt{|x|}\}$ with

$$\sqrt{|x|} - \pi(x) - e \geq \alpha_1 \quad \text{for } x \in \gamma \cap \{\sqrt{|x|} \geq a_1\}. $$

Let $\gamma'$ be the connected component of $\gamma \cap \{\sqrt{|x|} \geq a_1\}$ which converges to $\infty$, and let $x_1 \in \gamma' \cap \{\sqrt{|x|} = a_1\}$. Then

$$\text{diam } \pi(\gamma') \geq |\pi(x_1) - e| \geq \frac{\alpha_1}{a_1}. $$

Let $\gamma''$ be a subarc of $\gamma'$ starting from $x_1$ toward $\infty$ such that

$$\sup_{x \in \gamma''} \sqrt{|x|} < +\infty \quad \text{and} \quad \text{diam } \pi(\gamma'') \geq \frac{1}{2} \text{ diam } \pi(\gamma').$$
We take $b_1 > \sup_{x \in \gamma''} \sqrt{|x|}$. Let $\gamma_1$ be the connected component of $\gamma \cap \{a_1 \leq \sqrt{|x|} \leq b_1\}$ containing $\gamma''$. Then
\[
\text{diam } \pi(\gamma_1) \geq \frac{\alpha_1}{2a_1}.
\]
We next choose $a_2$, $b_2$ and $\gamma_2$ as follows. By (1.4) and the fact that $|\pi(x) - e| \to 0$ as $x \to \infty$ along $\gamma$, we find $a_2 > b_1$ such that
\[
(2.4) \quad \frac{1}{2b_1} \geq |\pi(x) - e| \geq \frac{\alpha_2}{\sqrt{|x|}} \quad \text{for } x \in \gamma \cap \{\sqrt{|x|} \geq a_2\}.
\]
Repeat the above process to get $b_2 > a_2$ and $\gamma_2$ such that $a_2 \leq \sqrt{|x|} \leq b_2$ for $x \in \gamma_2$ and diam $\pi(\gamma_2) \geq \alpha_2/2a_2$. Then (2.4) also yields that
\[
\text{diam } \pi(\gamma_2) \leq 2 \sup_{x \in \gamma_2} |\pi(x) - e| \leq \frac{1}{b_1}.
\]
Continue this process to obtain the required sequences. 

3. CONSTRUCTION

Throughout this section, we suppose that $\{a_j\}_{j \geq 1}$, $\{b_j\}_{j \geq 1}$ and $\{\gamma_j\}_{j \geq 1}$ are as in Lemma 2.4. Let
\[
(3.1) \quad \ell_j = \frac{\text{diam } \pi(\gamma_j)}{3}, \quad c_j = \sqrt{a_j \text{ diam } \pi(\gamma_j)} \quad \text{and} \quad \rho_j = \frac{c_j}{a_j}.
\]
Then, by Lemma 2.4,
\[
(3.2) \quad \lim_{j \to +\infty} \ell_j = 0, \quad \lim_{j \to +\infty} \frac{\rho_j}{\ell_j} = 0 \quad \text{and} \quad \lim_{j \to +\infty} c_j = +\infty.
\]
Therefore, in the construction below, we may assume that $\rho_j < \ell_j$ for every $j \in \mathbb{N}$. For each $j \in \mathbb{N}$, we choose finitely many points $\{y_j^\mu\}_\mu$ in $S$ such that

(I) $S = \bigcup_{\mu} Q(y_j^\mu, \ell_j)$,

(II) $Q(y_j^\mu, \ell_j/2) \cap Q(y_j^\nu, \ell_j/2) = \emptyset$ if $\mu \neq \nu$.

We define
\[
(3.3) \quad M_j = \bigcup_{\nu} \{y \in S : |y - y_j^\nu| = \ell_j\},
\]
\[
(3.4) \quad G_j = \left\{x \in \mathbb{R}^n : a_j \leq \sqrt{|x|} \leq b_j \text{ and } \pi(x) \in M_j\right\}.
\]
Then we have the following.

**Lemma 3.1.** $T\gamma_j \cap G_j \neq \emptyset$ for any $T \in O(n)$ and $j \in \mathbb{N}$.

**Proof.** By (I), we find $\nu$ with $\pi(T\gamma_j) \cap Q(y_j^\nu, \ell_j) \neq \emptyset$. Since diam $\pi(T\gamma_j) = \text{diam } \pi(\gamma_j) = 3\ell_j$, we see that $\pi(T\gamma_j) \cap M_j \neq \emptyset$. Therefore it follows from $T\gamma_j \subset \{a_j \leq \sqrt{|x|} \leq b_j\}$ that $T\gamma_j \cap G_j \neq \emptyset$. 

\[\square\]
Let \( R_j^\nu = \{ y \in S : \ell_j - \rho_j < |y - y_j^\nu| < \ell_j + \rho_j \} \) and define

\[
(3.5) \quad E_j = \bigcup_\nu R_j^\nu.
\]

Note that \( Q(y, \rho_j) \subset E_j \) if \( y \in M_j \). By \( \mathcal{X}_E \) we denote the characteristic function of \( E \).

**Lemma 3.2.** The following properties for the above \( \{E_j\}_{j \geq 1} \) hold.

\[(i) \quad \lim_{j \to +\infty} \left( \sup \left\{ \frac{K \mathcal{X}_{E_j}}{K\sigma}(x) : \sqrt{|x|} \leq b_{j-1} \right\} \right) = 0.
\]

\[(ii) \quad \lim_{j \to +\infty} \sigma(E_j) = 0.
\]

**Proof.** Since the value \( \sigma(R_j^\nu) \) is independent of \( \nu \), we write \( \sigma_j = \sigma(R_j^\nu) \). For a moment, we fix \( j \) and let \( \sqrt{|x|} \leq b_{j-1} \). By Lemma 2.3(i),

\[
\frac{K \mathcal{X}_{E_j}}{K\sigma}(x) \leq AM_1 \mathcal{X}_{E_j}(x)
\]

\[
\leq A \sup \left\{ \sum_\nu \frac{\sigma(R_j^\nu \cap Q(\pi(x), r))}{r^{n-1}} : r \geq \frac{1}{\sqrt{|x|}} \right\}
\]

\[
\leq A \sup \left\{ \frac{\sigma_j N_j}{r^{n-1}} : r \geq \frac{1}{\sqrt{|x|}} \right\},
\]

where \( N_j \) is the number of \( \nu \) such that \( R_j^\nu \cap Q(\pi(x), r) \neq \emptyset \). If \( r \geq 1/\sqrt{|x|} \), then \( r \geq 1/b_{j-1} \geq \text{diam } \pi(\gamma_j) = 3\ell_j \) by Lemma 2.4. Therefore \( R_j^\nu \cap Q(\pi(x), r) \neq \emptyset \) implies \( Q(y_j^\nu, \ell_j/2) \subset Q(\pi(x), 2r) \). It follows from (II) that \( N_j \leq A(r/\ell_j)^{n-1} \). Hence we obtain

\[
(3.6) \quad \sup \left\{ \frac{K \mathcal{X}_{E_j}}{K\sigma}(x) : \sqrt{|x|} \leq b_{j-1} \right\} \leq A \frac{\sigma_j}{\ell_j^{n-1}}.
\]

Observe from (2.1) and (3.2) that

\[
\frac{\sigma_j}{\ell_j^{n-1}} = \left( \frac{\ell_j + \rho_j}{\ell_j} \right)^{n-1} \frac{\sigma(Q(y, \ell_j + \rho_j))}{(\ell_j + \rho_j)^{n-1}} - \left( \frac{\ell_j - \rho_j}{\ell_j} \right)^{n-1} \frac{\sigma(Q(y, \ell_j - \rho_j))}{(\ell_j - \rho_j)^{n-1}}
\]

\[
\to 0 \quad \text{as } j \to +\infty.
\]

This together with (3.6) concludes (i).

Taking \( x = 0 \) in (i), we obtain

\[
\sigma(E_j) = \sigma(S) \frac{K \mathcal{X}_{E_j}}{K\sigma}(0) \to 0 \quad \text{as } j \to +\infty.
\]

Thus (ii) follows. \( \square \)

**Proof of Theorem 1.1.** In view of Lemma 3.2, taking a subsequence of \( j \) if necessary, we may assume that

\[
(3.7) \quad \frac{K \mathcal{X}_{E_j}}{K\sigma}(x) \leq 2^{-j} \quad \text{for } \sqrt{|x|} \leq b_{j-1},
\]

\[
\lim_{j \to +\infty} \left( \sup \left\{ \frac{K \mathcal{X}_{E_j}}{K\sigma}(x) : \sqrt{|x|} \leq b_{j-1} \right\} \right) = 0.
\]
and \( \sigma(E_j) \leq 2^{-j} \). Then \( \sigma(\bigcap_k \bigcup_{i \geq k} E_i) = 0 \). For \( j \in \mathbb{N} \), let

\[
f_j(y) = \begin{cases} 
(-1)^{I_j(y)} & \text{if } y \in \bigcup_{1 \leq i \leq j} E_i, \\
0 & \text{if } y \notin \bigcup_{1 \leq i \leq j} E_i,
\end{cases}
\]

where \( I_j(y) = \max\{i : y \in E_i, 1 \leq i \leq j\} \). Then we see that \( f_j \) converges \( \sigma \)-a.e. on \( S \) to

\[
f(y) = \begin{cases} 
(-1)^{I(y)} & \text{if } y \in \bigcup_{i \geq 1} E_i \setminus \bigcap_k \bigcup_{i \geq k} E_i, \\
0 & \text{if } y \notin \bigcup_{i \geq 1} E_i \text{ or } y \in \bigcap_k \bigcup_{i \geq k} E_i,
\end{cases}
\]

where \( I(y) = \max\{i : y \in E_i\} \) for \( y \in \bigcup_{i \geq 1} E_i \setminus \bigcap_k \bigcup_{i \geq k} E_i \). Also, we have the following:

\[
|f_j| \leq 1, \quad |f_{j+1} - f_j| \leq 2\mathcal{X}_{E_{j+1}} \text{ on } S; \quad f_j = (-1)^j \text{ on } E_j; \quad Kf_j \rightarrow Kf \text{ on } \mathbb{R}^n.
\]

Let \( T \in O(n) \). By Lemma 3.1, we find \( x_j \in T\gamma \cap G_j \) for each \( j \in \mathbb{N} \). Then \( a_j \leq \sqrt{|x_j|} \leq b_j \) and \( Q(\pi(x_j), c_j/a_j) \subset E_j \). If \( j \) is even, then Lemma 2.3(ii) and (3.7) give

\[
\frac{Kf}{K\sigma}(x_j) = \frac{Kf_j}{K\sigma}(x_j) + \sum_{k \geq j} \frac{K(f_{k+1} - f_k)}{K\sigma}(x_j)
\]

\[
\geq \frac{Kf_j}{K\sigma}(x_j) - 2 \sum_{k \geq j} \frac{\mathcal{X}_{E_{k+1}}}{K\sigma}(x_j)
\]

\[
\geq 1 - \frac{A}{c_j} - 2^{1-j}.
\]

Similarly, if \( j \) is odd, then

\[
\frac{Kf}{K\sigma}(x_j) \leq -1 + \frac{A}{c_j} + 2^{1-j}.
\]

Hence we conclude from (3.2) that

\[
\liminf_{T\gamma \ni x \to \infty} \frac{Kf}{K\sigma}(x) = -1 < 1 = \limsup_{T\gamma \ni x \to \infty} \frac{Kf}{K\sigma}(x).
\]

Obviously, \( u = Kf \) is a solution of (1.1) such that \(-1 \leq u/K\sigma \leq 1\) on \( \mathbb{R}^n \). Thus the proof of Theorem 1.1 is complete. \( \square \)

4. Remark

Our construction and estimates in Sections 2 and 3 are applicable to show Theorem B. Suppose that an unbounded subset \( \Omega \) of \( \mathbb{R}^n \) satisfies the assumption in Theorem B. Then we find a sequence \( \{x_j\} \) in \( \Omega \) converging to \( e \) at \( \infty \) such that

\[
\lim_{j \to +\infty} \frac{|x_j - |x_j||}{\sqrt{|x_j|}} = +\infty.
\]

Taking a subsequence of \( j \) if necessary, we may assume that \( \sqrt{|x_{j-1}||\pi(x_j) - e|} \leq 1 \). Let \( \omega_j = \{T_e(x_j) : T_e \in O(n) \text{ preserves } e\} \) and let \( \omega = \bigcup_j \omega_j \). Note that \( \omega \) is a subset of \( \Omega \).
converging to $e$ at $\infty$. Let $a_j = b_j = \sqrt{|x_j|}$ and define

$$\ell_j = \frac{|\pi(x_j) - e|}{3}, \quad c_j = \sqrt{a_j|\pi(x_j) - e|} \quad \text{and} \quad \rho_j = \frac{c_j}{a_j},$$

in place of (3.1). Then these satisfy (3.2) and $3\ell_j \leq 1/b_{j-1}$. Let $M_j$, $G_j$ and $E_j$ be as in (3.3), (3.4) and (3.5) respectively. Then the conclusions in Lemma 3.2 hold in this setting as well. Note that $\omega_j$ and $G_j$ lie on the sphere of center at the origin and radius $|x_j|$. Let $T \in O(n)$. Since $\{y \in S : |y - Te| = 3\ell_j\} \subset \pi(T\omega_j)$, we see that $\pi(T\omega_j) \cap M_j \neq \emptyset$, and so $T\omega_j \cap G_j \neq \emptyset$. Hence we observe the existence of $f$ such that

$$\liminf_{T\omega \ni x \to \infty} \frac{Kf}{K\sigma}(x) \neq \limsup_{T\omega \ni x \to \infty} \frac{Kf}{K\sigma}(x) \quad \text{for every} \ T \in O(n).$$

Thus $Kf/K\sigma$ admits no limits as $x \to \infty$ along $\Omega(y)$ for every $y \in S$.

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