The characterizations of weighted Sobolev spaces by wavelets and scaling functions *

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Abstract

We prove that suitable wavelets and scaling functions give characterizations and unconditional bases of the weighted Sobolev space $L^{p,s}(w)$ with A_p or A_p^{loc} weights. In the case of $w \in A_p$, we use only wavelets with proper regularity. If we consider the case of $w \in A_p^{\mathrm{loc}}$, we obtain the results by applying wavelets and scaling functions in $C_{\mathrm{comp}}^{s+1}(\mathbb{R}^n)$. We also construct the greedy bases for $L^{p,s}(w)$ by normalizing the unconditional bases in both of two cases.
Keywords and Phrases. A_p weight, A_p^{loc} weight, wavelet, scaling function, weighted Sobolev space, unconditional basis, greedy basis.

1 Introduction

We can characterize the L^2-norm of $f \in L^2(\mathbb{R}^n)$ by the wavelet coefficients appeared in the wavelet expansion of f with the wavelet basis. In particular, if we use the wavelets with proper decay, proper smoothness or compact support, then they give characterizations and unconditional bases of various function spaces (cf. [1, 9, 10, 15, 18, 24]).

Now we would like to explain the study on weighted L^p spaces $L^p(w) := L^p(\mathbb{R}^n, w(x)dx)$ $(1 < p < \infty)$. Lemarié-Rieusset showed that the Daubechies wavelets give a characterization and an unconditional basis of $L^p(w)$ with $w \in A_p$. Here A_p means the Muckenhoupt A_p class. He also considered for the case of A_p^{loc}, which is an extension of A_p. As a result, he proved that a characterization and an unconditional basis of $L^p(w)$ with $w \in A_p^{\mathrm{loc}}$ were given by means of the Daubechies wavelets and the Daubechies scaling functions ([15]).

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After that, Aimar, Bernardis and Martín-Reyes showed that the similar result to [15] was valid for 1-regular wavelets in the case of A_p ([1]).

In this paper we study the weighted Sobolev spaces $L^{p,s}(w) := L^{p,s}(\mathbb{R}^n, w(x)dx)$ ($1 < p < \infty$, $s \in \mathbb{N}$) with $w \in A_p$ or $w \in A_p^{\mathrm{loc}}$. We shall need smoother wavelets and scaling functions in order to get the characterizations and the unconditional bases of $L^{p,s}(w)$. As a consequence, we have the similar results to the studies on $L^p(w)$ shown by [1] and [15].

Additionally we would like to comment on the construction of greedy bases. As noted in [10], the characterizations and the unconditional bases of $L^p(w)$ given by wavelets and scaling functions enable us to construct the greedy bases in $L^p(w)$. The same method is applicable to $L^{p,s}(w)$, that is, we can construct the greedy bases in $L^{p,s}(w)$ using wavelets and scaling functions.

Let us explain the outline of this article. In Section 2 we explain the fundamental theory on wavelets associated with an MRA. Next we define two classes of weights, namely, A_p and A_p^{loc} in Section 3. We introduce two kinds of bases in Section 4. One is an unconditional basis, and the other is a greedy basis defined by Konyagin and Temlyakov ([13]). In Section 5 we define $L^p(w)$ and $L^{p,s}(w)$. In Section 6 we state the density of $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ in $L^{p,s}(w)$, where $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ means the space of all infinitely differentiable functions with compact support. We describe some known results on $L^p(w)$ in Section 7. Our results are contained in Section 8, 9 and 10. We characterize $L^{p,s}(w)$ with $w \in A_p$ by wavelets in Section 8. On the other hand, we consider the characterization of $L^{p,s}(w)$ with $w \in A_p^{\mathrm{loc}}$ by wavelets and scaling functions in Section 9. Lastly, in Section 10, we construct the unconditional bases and the greedy bases in $L^{p,s}(w)$ by applying the results in Sections 8 and 9.

Throughout this paper, s means a positive integer, $1 < p < \infty$ and p' means the conjugate exponent of p, i.e., p' satisfies $1/p + 1/p' = 1$. We shall also note that the Fourier transform of a function f is defined by $\mathcal{F}[f](\xi) := \displaystyle\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx$.

2 Wavelets and scaling functions

First let us recall the definition of wavelet ([18, 24]).

Definition 2.1 *Let $\{\psi^e : 1 \le e \le 2^n - 1\}$ be a sequence of functions belong to $L^2(\mathbb{R}^n)$. We define*

$$\psi^e_{j,k}(x) := 2^{jn/2}\psi^e\left(2^j x - k\right) = 2^{jn/2}\psi^e\left(2^j x_1 - k_1, \cdots, 2^j x_n - k_n\right) \quad (x = (x_1, \cdots, x_n) \in \mathbb{R}^n)$$

for each $1 \le e \le 2^n - 1$, $j \in \mathbb{Z}$ and $k = (k_1, \cdots, k_n) \in \mathbb{Z}^n$. The sequence $\{\psi^e : 1 \le e \le 2^n - 1\}$ is called a wavelet set if $\{\psi^e_{j,k} : 1 \le e \le 2^n - 1,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$ forms an orthonormal basis in $L^2(\mathbb{R}^n)$. Then we say that $\{\psi^e_{j,k} : 1 \le e \le 2^n - 1,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$ is a wavelet basis in $L^2(\mathbb{R}^n)$ and that each ψ^e is a wavelet.

We shall point out that a sequence of closed subspaces of $L^2(\mathbb{R}^n)$ called MRA gives wavelets.

Definition 2.2 *An MRA (multiresolution analysis) is a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^n)$ such that*
(a) $V_j \subset V_{j+1}$ *for all* $j \in \mathbb{Z}$.
(b) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$.
(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.
(d) $f \in V_j$ *holds if and only if* $f(2^{-j}x) \in V_0$ *for all* $j \in \mathbb{Z}$.
(e) $f \in V_0$ *holds if and only if* $f(x-k) \in V_0$ *for every* $k \in \mathbb{Z}^n$.
(f) *There exists a function $\varphi \in V_0$ such that the system $\{\varphi(x-k)\}_{k\in\mathbb{Z}^n}$ is an orthonormal basis in V_0. We call φ a scaling function of $\{V_j\}_{j\in\mathbb{Z}}$.*

Given an MRA $\{V_j\}_{j\in\mathbb{Z}}$ with a scaling function φ, we can construct the associated wavelet set $\{\psi^e : 1 \le e \le 2^n - 1\}$ such that $\{\psi^e_{j,k} : 1 \le e \le 2^n - 1,\ k \in \mathbb{Z}^n\}$ forms an orthonormal basis in W_j for each $j \in \mathbb{Z}$. Here W_j is the orthogonal complement of V_j in V_{j+1}. Then the wavelet basis $\{\psi^e_{j,k} : 1 \le e \le 2^n - 1,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$ and the sequence $\{\varphi_{m,k}\}_{k\in\mathbb{Z}^n} \cup \{\psi^e_{j,k} : 1 \le e \le 2^n - 1,\ j \ge m,\ k \in \mathbb{Z}^n\}$ are orthonormal bases in $L^2(\mathbb{R}^n)$ for any fixed $m \in \mathbb{Z}$.

In the case of $n = 1$, we can write the wavelet explicitly using φ as follows. The function ψ defined by

$$\psi(x) := \sum_{l=-\infty}^{\infty} (-1)^l < \varphi(\cdot - l), \varphi(\cdot/2) > \varphi(2x + l + 1) \tag{1}$$

is a wavelet in $L^2(\mathbb{R})$ such that $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ forms an orthonormal basis in W_j for all $j \in \mathbb{Z}$ ([9, 18, 24]). Here $< \cdot, \cdot >$ means the L^2-inner product. If a scaling function φ has a certain smoothness or a compact support, then the wavelet ψ given by (1) has similar properties. It also clearly follows that $\psi(\cdot - l)$ is a wavelet and that $\{(\psi(\cdot - l))_{j,k}\}_{k\in\mathbb{Z}} = \{\psi_{j,k}\}_{k\in\mathbb{Z}}$ for any $l, j \in \mathbb{Z}$. Here we shall give remarkable examples of scaling functions and wavelets in $L^2(\mathbb{R})$.

Example 2.3
(a) Meyer constructed a real-valued scaling function φ such that $\varphi \in \mathcal{S}(\mathbb{R})$ and $\operatorname{supp}\mathcal{F}[\varphi] \subset \left[-\frac{4}{3}\pi, \frac{4}{3}\pi\right]$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz class. Then the wavelet ψ given by (1) satisfies that $\psi \in \mathcal{S}(\mathbb{R})$ and that $\operatorname{supp}\mathcal{F}[\psi] \subset \left\{\frac{2}{3}\pi \le |\xi| \le \frac{8}{3}\pi\right\}$. We say that φ is the Meyer scaling function and that ψ is the Meyer wavelet (cf. [18, 24]).
(b) For each positive integers $N \ge 2$, Daubechies constructed a real-valued scaling function such that

$$\varphi \in C^{r(N)}(\mathbb{R}) \quad \text{and} \quad \operatorname{supp}\varphi = [0, 2N-1], \tag{2}$$

where $r(N) > 0$ and $\lim_{N\to\infty} N^{-1}r(N) = 1 - \log 3 \cdot (2\log 2)^{-1} \simeq 0.2075$. $C^{\lambda}(\mathbb{R})$ is the set of all functions f such that $D^{[\lambda]}f$ are $(\lambda - [\lambda])$-Hölder continuous, and $[\lambda]$ means the maximal integer that is less than λ for $\lambda \in (0,\infty) - \mathbb{Z}$. Now define the function ψ by

$$\begin{aligned}\psi(x) &:= \sum_{l=-\infty}^{\infty} (-1)^l < \varphi(\cdot - l), \varphi(\cdot/2) > \varphi(2(x-N)+l+1) \\ &= \sum_{l=0}^{2N-1} (-1)^l < \varphi(\cdot - l), \varphi(\cdot/2) > \varphi(2x - 2N + l + 1). \qquad (3)\end{aligned}$$

Then ψ is a wavelet which satisfies that $\psi \in C^{r(N)}(\mathbb{R})$ and $\operatorname{supp}\psi = [0, 2N-1]$. We say that φ is the Daubechies scaling function and ψ is the Daubechies wavelet (cf. [6, 16]).

Next let us consider the case of several-variables. We can get wavelet sets directly from an MRA $\{V_j\}_{j\in\mathbb{Z}}$ in $L^2(\mathbb{R}^n)$ with a scaling function φ, however, it is difficult for us to describe the desired wavelets explicitly with φ in general ([18, 24]). We shall introduce the construction of wavelets in $L^2(\mathbb{R})$ by tensor products. Let φ^0 be the scaling function of an MRA in $L^2(\mathbb{R})$, φ^1 be the wavelet ψ in $L^2(\mathbb{R})$ given by (1) with φ^0, and $E := \{0,1\}^n - \{(0,\cdots,0)\}$. For $x = (x_1,\cdots,x_n) \in \mathbb{R}^n$ and $e = (e_1,\cdots,e_n) \in E$, we define

$$\varphi(x) := \prod_{\nu=1}^{n} \varphi^0(x_\nu), \quad \psi^e(x) := \prod_{\nu=1}^{n} \varphi^{e_\nu}(x_\nu) \qquad (4)$$

and $V_j := \overline{\operatorname{span}\{\varphi_{j,k}\}_{k\in\mathbb{Z}^n}}^{L^2(\mathbb{R}^n)}$ for $j \in \mathbb{Z}$. Here $\operatorname{span}\{\varphi_{j,k}\}_{k\in\mathbb{Z}^n}$ means the set of finite linear combinations of elements in $\{\varphi_{j,k}\}_{k\in\mathbb{Z}^n}$. Then $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA with the scaling function φ. Moreover $\{\psi^e\}_{e\in E}$ is a wavelet set such that $\{\psi^e_{j,k} : e \in E,\ k \in \mathbb{Z}^n\}$ forms an orthonormal basis in W_j for each $j \in \mathbb{Z}$.

3 A_p weights and A_p^{loc} weights

We consider the following two classes of weights in this paper.

Definition 3.1 *Let $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $w > 0$ a.e. and $w^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.*
(a) *We define the class of weights A_p which consists of all weights w satisfying*

$$A_p(w) := \sup_{Q:\text{cube}} \frac{1}{|Q|} w(Q) \left(\frac{1}{|Q|} \int_Q w(y)^{-1/(p-1)} dy \right)^{p-1} < \infty,$$

and say that $w \in A_p$ is an A_p weight. Here $w(Q) := \int_Q w(x)dx$ and $|Q|$ means the Lebesgue measure of Q.

(b) *We define the class of weights A_p^{loc} which consists of all weights w satisfying*

$$A_p^{\mathrm{loc}}(w) := \sup_{\substack{|Q|\le 1,\\ Q:\mathrm{cube}}} \frac{1}{|Q|}w(Q)\left(\frac{1}{|Q|}\int_Q w(y)^{-1/(p-1)}dy\right)^{p-1} < \infty, \tag{5}$$

and say that $w \in A_p^{\mathrm{loc}}$ is an A_p^{loc} weight.

Remark 3.2
(a) For example, $|x|^a \in A_p$ for $-n < a < n(p-1)$ (cf. [23, Section IX. 4]).
(b) The class of A_p^{loc} weights is independent of the upper bound for the cube size used in its definitions. Namely we can replace $|Q| \le 1$ by $|Q| \le r$ in (5) for any $0 < r < \infty$. In fact, if we define

$$A_p^{\mathrm{loc},r}(w) := \sup_{\substack{|Q|\le r,\\ Q:\mathrm{cube}}} \frac{1}{|Q|}w(Q)\left(\frac{1}{|Q|}\int_Q w(y)^{-1/(p-1)}dy\right)^{p-1}$$

for each $r > 0$, then it clearly follows that $A_p^{\mathrm{loc},r}(w) \le A_p^{\mathrm{loc},1}(w)$ if $0 < r \le 1$. On the other hand, Rychkov gave the estimation that $A_p^{\mathrm{loc},r}(w) \le r^{-p}e^{cr}A_p^{\mathrm{loc},1}(w)$ if $r > 1$, where $c > 0$ is a constant depended only on n, p and $A_p^{\mathrm{loc}}(w)$ (cf. [20]).
(c) We shall also remark that $A_p \subsetneq A_p^{\mathrm{loc}}$. In fact, $e^{r|x|} \in A_p^{\mathrm{loc}} - A_p$ for $r \in \mathbb{R} - \{0\}$.
(d) We have that $w \in A_p$ if and only if $w^{-1/(p-1)} \in A_{p'}$. In fact, it clearly follows that $A_p(w) = A_{p'}(w^{-1/(p-1)})^{p-1}$. The same result is true for the case of A_p^{loc}.

The next lemma is obtained from [20, Proof of Lemma 1.1], and states an useful relation between A_p and A_p^{loc}.

Lemma 3.3 *Let $a \in \mathbb{R}$, $r, t > 0$ and $w \in A_p^{\mathrm{loc}}$. We define*

$$\tau_m(u) := \begin{cases} u & \text{if} \quad u \in [t(m+a), t(m+a+r)) \\ 2t(m+a+r) - u & \text{if} \quad u \in [t(m+a+r), t(m+a+2r)) \end{cases}$$

for $m \in \mathbb{Z}$ and $u \in [t(m+a), t(m+a+2r))$. We also define $\{w_l\}_{l\in\mathbb{Z}^n}$ to fulfill that

$$w_l(x) = w(\tau_{l_1}(x_1), \cdots, \tau_{l_n}(x_n)) \quad \text{if} \quad x \in \prod_{\nu=1}^{n} [t(l_\nu + a), t(l_\nu + a + 2r)),$$

and that each w_l is a $2tr\mathbb{Z}^n$-periodic function on $\mathbb{R}^n$ for all $l \in \mathbb{Z}^n$. Then it follows that $\{w_l\}_{l\in\mathbb{Z}^n} \subset A_p$ with $A_p(w_l) \le 3^{np}A_p^{\mathrm{loc},t^n r^n}(w)$ for every $l \in \mathbb{Z}^n$.

4 Unconditional bases and greedy bases

Let us begin with introducing two kinds of bases. Let X be a Banach space, X^* be the dual space of X and A be a countable index set in this section.

4.1 Unconditional bases

It is known that there are several equivalent definitions of an unconditional basis in a Banach space ([11, 17]). We adopt the definition of an unconditional basis by [24, Chapter 7] in this paper.

Definition 4.1 *Let $\{x_m\}_{m\in A}$ be a sequence of elements in X and $\{\tilde{x}_k\}_{k\in A}$ be a sequence of elements in X^*.*

(a) *We say that the series $\sum_{m\in A} x_m$ is unconditionally convergent in X if the series $\sum_{i=1}^{\infty} x_{\sigma(i)}$ converges in X for all $\sigma : \mathbb{N} \to A$, a 1 to 1 and onto map.*

(b) *We call $\{x_m, \tilde{x}_m\}_{m\in A}$ an unconditional basis in X if the following three conditions are satisfied:*

(i) *$\{x_m, \tilde{x}_m\}_{m\in A}$ is a biorthogonal system, i.e., $\tilde{x}_k(x_m) = \delta_{m,k}$. Here $\delta_{m,k}$ means Kronecker's delta, that is, $\delta_{m,m} = 1$ and $\delta_{m,k} = 0$ if $m \neq k$.*

(ii) $\overline{\mathrm{span}\{x_m\}_{m\in A}}^{X} = X$.

(iii) *There exists a constant $0 < C < \infty$ such that $\left\| \sum_{m\in B} \tilde{x}_m(x) x_m \right\|_X \leq C\|x\|_X$ for every $x \in X$ and every finite subset $B \subset A$.*

Remark 4.2 Let $\{x_m, \tilde{x}_m\}_{m\in A}$ be an unconditional basis in X.

(a) ([24, Theorem 7.7 (i)]). The series $\sum_{m\in A} \tilde{x}_m(x) x_m$ converges unconditionally in X to x for every $x \in X$.

(b) ([24, Remark 7.2]). We see that the functionals $\{\tilde{x}_k\}_{k\in A} \subset X^*$ are determined by the vectors $\{x_m\}_{m\in A} \subset X$ from two conditions (i) and (ii) in Definition 4.1 (b). Thus we often say that $\{x_m\}_{m\in A}$ is an unconditional basis in X.

4.2 Greedy bases

We define a Schauder basis first.

Definition 4.3 *We say that $\{x_k\}_{k=1}^{\infty} \subset X$ is a Schauder basis if there exists an unique sequence $\{c_k(x)\}_{k=1}^{\infty} \subset \mathbb{C}$ such that $x = \sum_{k=1}^{\infty} c_k(x) x_k$ in X for all $x \in X$.*

We introduce two kinds of bases defined by Konyagin and Temlyakov.

Definition 4.4 *Let $\{x_k\}_{k=1}^{\infty}$ be a Schauder basis in X such that $\|x_k\|_X = 1$ for all $k \in \mathbb{N}$. We call $\{x_k\}_{k=1}^{\infty}$ a greedy basis for X if there exists a constant $0 < C < \infty$ such that for every $x \in X$ there exists a permutation ρ of $\mathbb{N}$ which satisfies*

$$\left|c_{\rho(1)}(x)\right| \geq \left|c_{\rho(2)}(x)\right| \geq \cdots \geq \left|c_{\rho(N)}(x)\right|$$

and

$$\left\| x - \sum_{k=1}^{N} c_{\rho(k)}(x) x_{\rho(k)} \right\|_X \leq C \inf_{y \in \Sigma_N} \|x - y\|_X ,$$

for every $N \in \mathbb{N}$*, where* $\Sigma_N := \left\{ \sum_{\nu \in \Lambda} \alpha_\nu x_\nu : \alpha_\nu \in \mathbb{C},\ \sharp\Lambda \leq N,\ \Lambda \subset \mathbb{N} \right\}$.

Definition 4.5 *Let* $\{x_k\}_{k=1}^{\infty}$ *be a Schauder basis in X such that* $\|x_k\|_X = 1$ *for all* $k \in \mathbb{N}$*. We say that* $\{x_k\}_{k=1}^{\infty}$ *is a democratic basis for X if there exists a constant* $0 < D < \infty$ *independent of P and Q such that*

$$\left\| \sum_{k \in P} x_k \right\|_X \leq D \left\| \sum_{k \in Q} x_k \right\|_X$$

for any finite subsets $P, Q \subset \mathbb{N}$ *with the same cardinality* $\sharp P = \sharp Q$*.*

Theorem 4.6 we describe next becomes the key in Section 10 later.

Theorem 4.6 ([13, Theorem 1]). *Let* $\{x_k\}_{k=1}^{\infty}$ *be a Schauder basis in X such that* $\|x_k\|_X = 1$ *for all* $k \in \mathbb{N}$*. Then* $\{x_k\}_{k=1}^{\infty}$ *is a greedy basis if and only if it is an unconditional and democratic basis.*

Remark 4.7 ([13, Section 3]). Konyagin and Temlyakov give some examples of bases, which are not democratic but unconditional, or which are not unconditional but democratic.

5 The weighted L^p spaces and the weighted Sobolev spaces

Definition 5.1 *Let* $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *with* $w > 0$ *a.e.. The weighted* L^p *space* $L^p(w) := L^p(\mathbb{R}^n, w(x)dx)$ *is the space of all measurable functions f with*

$$\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

Remark 5.2 Let $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $w > 0$ a.e..
(a) $(L^p(w), \|\cdot\|_{L^p(w)})$ is a Banach space.
(b) In addition, if w satisfies $w^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then $L^p(w) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Definition 5.3 *Let* $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *with* $w > 0$ *a.e. and* $w^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$*. The weighted Sobolev space* $L^{p,s}(w) := L^{p,s}(\mathbb{R}^n, w(x)dx)$ *is the space of all measurable functions f* *satisfying that* $f \in L^p(w)$ *and weak derivatives* $D^\alpha f \in L^p(w)$ *for every* $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_+^n$ *with* $|\alpha| \le s$. *Here*

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \quad \textit{and} \quad |\alpha| := \sum_{\nu=1}^{n} \alpha_\nu.$$

Let $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $w > 0$ a.e. and $w^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then $L^{p,s}(w)$ is a Banach space with the norm

$$\|f\|_{L^{p,s}(w)} := \sum_{|\alpha| \le s} \|D^\alpha f\|_{L^p(w)}.$$

Remark 5.4 (cf. [23, Section IX. 4]). For any $w \in A_p$, it follows that

$$\int_{\mathbb{R}^n} (1 + |x|)^{-np} w(x) dx < \infty.$$

Thus we see that $\mathcal{S}(\mathbb{R}^n) \subset L^{p,s}(w)$.

In the case of $w \in A_p$, we can replace $\| \cdot \|_{L^{p,s}(w)}$ as follows.

Theorem 5.5 *Let* $w \in A_p$. *Then there exists a constant* $C > 0$ *depended only on n, p,* $A_p(w)$ *and s such that*

$$C\|f\|_{L^{p,s}(w)} \le \|f\|_{L^p(w)} + \sum_{|\beta|=s} \|D^\beta f\|_{L^p(w)}$$

for all $f \in L^{p,s}(w)$, *i.e.,* $\| \cdot \|_{L^p(w)} + \sum_{|\beta|=s} \|D^\beta(\cdot)\|_{L^p(w)}$ *is equivalent to* $\| \cdot \|_{L^{p,s}(w)}$.

We can obtain Theorem 5.5 above by the same arguments as [9, Theorem 6.4 in Chapter 6] applying the next result given by Kurtz ([14, Theorem 4]).

Proposition 5.6 *Let* $w \in A_p$ *and* $m \in C^n(\mathbb{R}^n - \{(0, \cdots, 0)\})$. *Suppose that*

$$\sup_{R>0} R^{2|\alpha|-n} \int_{R \le |x| \le 2R} |D^\alpha m(x)|^2 dx < \infty$$

for all $|\alpha| \le n$. *Then the operator T defined by* $\mathcal{F}[Tf] = m\mathcal{F}[f]$ *is bounded on* $L^p(w)$.

We recall the definition of the Hardy-Littlewood maximal function.

Definition 5.7 *Let* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and* $B(0, r) := \{y \in \mathbb{R}^n : |y| < r\}$ *for* $r > 0$. *The Hardy-Littlewood maximal function of f is defined by*

$$Mf(x) := \sup_{r>0} \frac{1}{|B(0, r)|} \int_{B(0,r)} |f(x - y)| dy \quad (x \in \mathbb{R}^n).$$

Proposition 5.8 (cf. [2]). *Let $1 < q < \infty$ and $w \in A_p$. Then there exists a constant $C > 0$ depended only on n, p, q and $A_p(w)$ such that*

$$\left\| \|(Mf_\nu)_{\nu=1}^\infty\|_{l^q} \right\|_{L^p(w)} \le C \left\| \|(f_\nu)_{\nu=1}^\infty\|_{l^q} \right\|_{L^p(w)}$$

for all $(f_\nu)_{\nu=1}^\infty$ with

$$\left\| \|(f_\nu)_{\nu=1}^\infty\|_{l^q} \right\|_{L^p(w)} := \left(\int_{\mathbb{R}^n} \left(\sum_{\nu=1}^\infty |f_\nu(x)|^q \right)^{p/q} w(x)dx \right)^{1/p} < \infty.$$

6 Density of $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ in $L^{p,s}(w)$

We will need the following densities to obtain characterizations of $L^{p,s}(w)$.

Theorem 6.1 ([19, Theorem 1.1]). *Let $w \in A_p$. Then $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $L^{p,s}(w)$.*

Theorem 6.2 *Let $w \in A_p^{\mathrm{loc}}$. Then $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $L^{p,s}(w)$.*

We can easily prove Theorem 6.2 by the same arguments as the proof of [19, Theorem 1.1] with the following uniformly boundedness stated in Lemma 6.3.

Lemma 6.3 *Let $w \in A_p^{\mathrm{loc}}$ and $\eta \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ with non-negative, radial and decreasing as a function on $(0, \infty)$. Define $\eta_t(x) := t^{-n}\eta(x/t)$ for $t > 0$. Then there exists a constant $C > 0$ depended on n, p, $A_p^{\mathrm{loc}}(w)$ and η such that $\|\eta_t * f\|_{L^p(w)} \le C \|f\|_{L^p(w)}$ for all $0 < t \le 1$ and $f \in L^p(w)$.*

Here we say that a non-negative and bounded function η is radial and decreasing as a function on $(0, \infty)$ if η satisfies (a) $\eta(x) = \eta(y)$ if $|x| = |y|$, and (b) $\eta(x) \le \eta(y)$ if $|x| \ge |y|$. The next lemma describes a relation between such η and the Hardy-Littlewood maximal function M (cf. [7, Proposition 2.7], [21, p.63]).

Lemma 6.4 *Let η be a function in $L^1(\mathbb{R}^n)$ which is non-negative, bounded, radial and decreasing as a function on $(0, \infty)$. Then $|\eta_t * f(x)| \le \|\eta\|_{L^1(\mathbb{R}^n)} Mf(x)$ for all $t > 0$ and a.e. $x \in \mathbb{R}^n$.*

Proof of Lemma 6.3 Let us take $J \in \mathbb{N}$ so that $\operatorname{supp} \eta \subset [-J, J]^n$ and denote $H_{t,l} := \prod_{\nu=1}^n [tl_\nu, t(l_\nu + 1))$ for $t > 0$ and $l \in \mathbb{Z}^n$. For all $0 < t \le 1$ and $f \in L^p(w)$, we get that

$$\eta_t * f(x) = \sum_{l \in \mathbb{Z}^n} \eta_t * \left(f \cdot \chi_{H_{t,l}} \right)(x) = \sum_{l \in \mathbb{Z}^n} \int_{H_{t,l}} t^{-n} \eta \left(\frac{x-y}{t} \right) f(y)dy.$$

Here remark that $\operatorname{supp}\eta((\cdot - y)/t) \subset \prod_{\nu=1}^{n}[t(-J+l_\nu), t(J+l_\nu+1)] =: B_{t,l}$ for each $l \in \mathbb{Z}^n$ and $y \in H_{t,l}$, i.e., $\operatorname{supp}\eta_t * \left(f \cdot \chi_{H_{t,l}}\right) \subset B_{t,l}$. On the other hand, for all $x \in \mathbb{R}^n$ and $0 < t \le 1$, there exists an unique $L = L(x,t) \in \mathbb{Z}^n$ such that $x \in H_{t,L}$. Additionally write $\mathcal{K}(L) := \{l \in \mathbb{Z}^n : L_\nu - J \le l_\nu \le L_\nu + J \text{ for all } 1 \le \nu \le n\}$. Then we obtain that $\eta_t * f(x) = \sum_{l\in\mathcal{K}(L)} \eta_t * \left(f \cdot \chi_{H_{t,l}}\right)(x)$. By Hölder's inequality, it follows that

$$|\eta_t * f(x)|^p \le \sum_{l\in\mathcal{K}(L)} \left|\eta_t * \left(f \cdot \chi_{H_{t,l}}\right)(x)\right|^p \cdot \sharp\mathcal{K}(L)^{p-1} \le (2J+1)^{n(p-1)} \sum_{l\in\mathbb{Z}^n} \left|\eta_t * \left(f \cdot \chi_{H_{t,l}}\right)(x)\right|^p .$$

Thus we obtain that

$$\begin{aligned} \|\eta_t * f\|_{L^p(w)}^p &\le (2J+1)^{n(p-1)} \int_{\mathbb{R}^n} \sum_{l\in\mathbb{Z}^n} \left|\eta_t * \left(f \cdot \chi_{H_{t,l}}\right)(x)\right|^p w(x)dx \\ &= (2J+1)^{n(p-1)} \sum_{l\in\mathbb{Z}^n} \int_{B_{t,l}} \left|\eta_t * \left(f \cdot \chi_{H_{t,l}}\right)(x)\right|^p w(x)dx. \end{aligned}$$

Following Lemma 3.3, we obtain $\{w_l\}_{l\in\mathbb{Z}^n} \subset A_p$ such that $w_l = w$ on $B_{t,l}$ and

$$A_p(w_l) \le 3^{np} A_p^{\mathrm{loc},t^n(2J+1)^n}(w) \le 3^{np} A_p^{\mathrm{loc},(2J+1)^n}(w)$$

for every $l \in \mathbb{Z}^n$ and $0 < t \le 1$. On the other hand, by Lemma 6.4, we see that $\left|\eta_t * \left(f \cdot \chi_{H_{t,l}}\right)(x)\right| \le \|\eta\|_{L^1(\mathbb{R}^n)} M\left(f \cdot \chi_{H_{t,l}}\right)(x)$. Hence we have that

$$\|\eta_t * f\|_{L^p(w)}^p \le (2J+1)^{n(p-1)} \|\eta\|_{L^1(\mathbb{R}^n)}^p \sum_{l\in\mathbb{Z}^n} \int_{B_{t,l}} M\left(f \cdot \chi_{H_{t,l}}\right)(x)^p w_l(x)dx.$$

In addition, by Proposition 5.8, there exists a constant $C > 0$ depended only on n, p, $A_p^{\mathrm{loc}}(w)$ and J such that $\left\|M\left(f \cdot \chi_{H_{t,l}}\right)\right\|_{L^p(w_l)} \le C \left\|f \cdot \chi_{H_{t,l}}\right\|_{L^p(w_l)} = C \left\|f \cdot \chi_{H_{t,l}}\right\|_{L^p(w)}$. Thus we have

$$\begin{aligned} \|\eta_t * f\|_{L^p(w)}^p &\le C^p (2J+1)^{n(p-1)} \|\eta\|_{L^1(\mathbb{R}^n)}^p \sum_{l\in\mathbb{Z}^n} \left\|f \cdot \chi_{H_{t,l}}\right\|_{L^p(w)}^p \\ &= C^p (2J+1)^{n(p-1)} \|\eta\|_{L^1(\mathbb{R}^n)}^p \|f\|_{L^p(w)}^p . \qquad \square \end{aligned}$$

7 Wavelets, scaling functions and $L^p(w)$

In this section we introduce known results about the characterizations and the constructions of bases of $L^p(w)$.

Notation 7.1

(a) We define a dyadic cube $Q_{j,k} := \prod_{\nu=1}^{n} \left[2^{-j}k_\nu, 2^{-j}(k_\nu + 1)\right)$ for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$.
(b) χ_E means the characteristic function of E for a measurable set $E \subset \mathbb{R}^n$.
(c) $\chi := \chi_{[0,1)^n}$, that is, $\chi_{j,k} = 2^{jn/2}\chi_{Q_{j,k}}$ for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$.

Definition 7.2 *Let $r \in \mathbb{N}$. A function f on $\mathbb{R}^n$ is r-regular if for every $m \in \mathbb{N}$ there exists a constant $0 < C_m < \infty$ such that $|D^\alpha f(x)| \leq C_m(1 + |x|)^{-m}$ for all $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq r$.*

For example, the Meyer wavelet is r-regular (see Example 2.3 (a)). Moreover if we take a large $N \in \mathbb{N}$ sufficiently, the Daubechies wavelet described in Example 2.3 (b) becomes r-regular.

Lemarié-Rieusset gave a characterization and an unconditional basis of $L^p(w)$ with $w \in A_p$ by the Daubechies wavelets in the case of one-variable. His proof is due to the boundedness of Calderón-Zygmund operators on $L^p(w)$. Following the same method, Aimar, Bernardis and Martín-Reyes showed that the result given by Lemarié-Rieusset was valid for 1-regular wavelets. More precisely, they obtained the next theorem.

Theorem 7.3 (cf. [15, 1]). *Let $w \in A_p$ and $\{\psi^e : 1 \leq e \leq 2^n - 1\}$ be a wavelet set constructed by an MRA such that each ψ^e is 1-regular. Then there exist two constants $0 < c \leq C < \infty$ depended only on n, p, $A_p(w)$ and $\{\psi^e\}_e$ such that for every $f \in L^p(w)$,*

$$c \, \|f\|_{L^p(w)} \leq \left\| \left(\sum_{e=1}^{2^n-1} \sum_{j=-\infty}^{\infty} \sum_{k \in \mathbb{Z}^n} \left| \left\langle f, \psi^e_{j,k} \right\rangle \chi_{j,k} \right|^2 \right)^{1/2} \right\|_{L^p(w)} \leq C \, \|f\|_{L^p(w)} .$$

Additionally the wavelet basis $\{\psi^e_{j,k} : 1 \leq e \leq 2^n - 1,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$ forms an unconditional basis in $L^p(w)$.

On the other hand, Lemarié-Rieusset gave the next result. The result shows that we need not only wavelets but also scaling functions which construct wavelets if we consider $L^p(w)$ with $w \in A_p^{\mathrm{loc}}$. Although he proved it in the case of one-variable, it is true in the case of several-variables with obvious modifications applying tensor products. We call that φ is the Daubechies scaling function in $L^2(\mathbb{R}^n)$ if φ is given by (4) with the Daubechies scaling function φ^0 in $L^2(\mathbb{R})$. At the same time we say that $\{\psi^e\}_{e \in E}$ is the Daubechies wavelet set associated with φ if each wavelet ψ^e is given by (4) with φ^0 and φ^1, where $\varphi^1 := \psi$ is the Daubechies wavelet in $L^2(\mathbb{R})$ given by (3) with φ^0.

Theorem 7.4 (cf. [15, Proposition 2 (ii)]). *Let $w \in A_p^{\mathrm{loc}}$, $m \in \mathbb{Z}$, φ be the Daubechies scaling function in $L^2(\mathbb{R}^n)$ and $\{\psi^e\}_{e \in E}$ be the Daubechies wavelet set associated with φ.*

Define

$$\mathcal{M}_{p,w,m}(f) := \left(\sum_{k\in\mathbb{Z}^n} \left| \langle f, \varphi_{m,k} \rangle \, \|\varphi_{m,k}\|_{L^p(w)} \right|^p \right)^{1/p} + \left\| \left(\sum_{e\in E} \sum_{j=m}^{\infty} \sum_{k\in\mathbb{Z}^n} \left| \langle f, \psi^e_{j,k} \rangle \chi_{j,k} \right|^2 \right)^{1/2} \right\|_{L^p(w)}.$$

Then there exist two constants $0 < c \le C < \infty$ *depended only on n, p,* $A_p^{\mathrm{loc}}(w)$*, m and* φ *such that for all* $f \in L^p(w)$*,*

$$c\|f\|_{L^p(w)} \le \mathcal{M}_{p,w,m}(f) \le C\|f\|_{L^p(w)}.$$

Additionally the sequence $\{\varphi_{m,k}\}_{k\in\mathbb{Z}^n} \cup \{\psi^e_{j,k} : e \in E,\ j \ge m,\ k \in \mathbb{Z}^n\}$ *forms an unconditional basis in* $L^p(w)$*.*

Applying the characterizations and the constructions of the unconditional bases above, we can construct the greedy bases for $L^p(w)$. Namely the next theorem follows (cf. [10, Section 6]).

Theorem 7.5
(a) *Let* $w \in A_p$ *and* $\{\psi^e : 1 \le e \le 2^n - 1\}$ *be a wavelet set constructed by an MRA such that each* ψ^e *is* 1*-regular. Define*

$$\widetilde{\psi^e}_{j,k} := \frac{\psi^e_{j,k}}{\|\psi^e_{j,k}\|_{L^p(w)}}$$

for $1 \le e \le 2^n-1$, $j \in \mathbb{Z}$ *and* $k \in \mathbb{Z}^n$*. Then the sequence* $\left\{\widetilde{\psi^e}_{j,k} : 1 \le e \le 2^n - 1,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\right\}$ *forms a greedy basis for* $L^p(w)$*.*
(b) *Let* $w \in A_p^{\mathrm{loc}}$*,* φ *be the Daubechies scaling function in* $L^2(\mathbb{R}^n)$ *and* $\{\psi^e\}_{e\in E}$ *be the Daubechies wavelet set associated with* φ*. Define*

$$\tilde{\varphi}_{m,k} := \frac{\varphi_{m,k}}{\|\varphi_{m,k}\|_{L^p(w)}} \quad \textit{and} \quad \widetilde{\psi^e}_{j,k} := \frac{\psi^e_{j,k}}{\|\psi^e_{j,k}\|_{L^p(w)}}$$

for $e \in E$*,* $j \ge m$ *and* $k \in \mathbb{Z}^n$*. Then the sequence* $\{\tilde{\varphi}_{m,k}\}_{k\in\mathbb{Z}^n} \cup \{\widetilde{\psi^e}_{j,k} : e \in E,\ j \ge m,\ k \in \mathbb{Z}^n\}$ *forms a greedy basis for* $L^p(w)$*.*

In Section 10, we will construct the greedy bases for $L^{p,s}(w)$ by means of wavelets and scaling functions following the similar method.

8 The characterization of $L^{p,s}(w)$ with $w \in A_p$ by wavelets

8.1 Statement of the result

Following statements in [9, Chapter 6], we can obtain the next characterization of $L^{p,s}(w)$ with $w \in A_p$ by wavelets.

Theorem 8.1 *Let $w \in A_p$ and $\{\psi^e : 1 \leq e \leq 2^n - 1\}$ be a wavelet set constructed by an MRA such that each wavelet ψ^e is $(s+1)$-regular. Then there exist two constants $0 < c \leq C < \infty$ depended only on n, p, $A_p(w)$, s and $\{\psi^e\}_e$ such that for all $f \in L^{p,s}(w)$,*

$$c\|f\|_{L^{p,s}(w)} \leq \left\| \left(\sum_{e=1}^{2^n-1} \sum_{j=-\infty}^{\infty} \sum_{k \in \mathbb{Z}^n} (1 + 2^{2js}) \left| < f, \psi^e_{j,k} > \chi_{j,k} \right|^2 \right)^{1/2} \right\|_{L^p(w)} \leq C\|f\|_{L^{p,s}(w)}.$$

Remark that we need some improvements on [9] to obtain Theorem 8.1. We use Theorem 5.5, Theorem 6.1 and Theorem 7.3 described already, in addition, Lemma 8.3 and Proposition 8.4 as follows. We shall introduce the class of functions $\mathcal{R}^r(\mathbb{R}^n)$ in order to state them.

Definition 8.2 *Let $r \in \mathbb{Z}_+$. We define the class of functions $\mathcal{R}^r(\mathbb{R}^n)$ which consists of all functions f satisfying that there exist constants $0 < \varepsilon, \gamma < \infty$ and $0 < C_\alpha < \infty$ for each $\alpha \in Z^n_+$ with $|\alpha| \leq r + 1$ such that*

(i) $\displaystyle\int_{\mathbb{R}^n} x^\alpha f(x) dx = 0$ *for every* $|\alpha| \leq r + 1$,

(ii) $|f(x)| \leq C_{(0,\cdots,0)}(1 + |x|)^{-(2+s+\gamma)n}$,

(iii) $|D^\alpha f(x)| \leq C_\alpha (1 + |x|)^{-(1+\varepsilon)n}$ *for every* $1 \leq |\alpha| \leq r + 1$.

For example, if ψ^e is an $(r + 1)$-regular wavelet constructed by an MRA for some $r \in \mathbb{Z}_+$, then $\psi^e \in \mathcal{R}^r(\mathbb{R}^n)$ (cf. [18]).

Lemma 8.3 *Let $r \in \mathbb{Z}_+$, $\{\Phi^e : 1 \leq e \leq 2^n - 1\}$, $\{\psi^e : 1 \leq e \leq 2^n - 1\} \subset \mathcal{R}^r(\mathbb{R}^n)$ and $w \in A_p$. Define*

$$\mathcal{W}[r, \{\Phi^e\}_e](f) := \left(\sum_{e=1}^{2^n-1} \sum_{j=-\infty}^{\infty} \sum_{k \in \mathbb{Z}^n} \left| 2^{jr} < f, \Phi^e_{j,k} > \chi_{j,k} \right|^2 \right)^{1/2}.$$

If $\{\psi^e\}_e$ is a wavelet set, then there exists a constant $C > 0$ depended only on n, p, $A_p(w)$, r, $\{\Phi^e\}_e$ and $\{\psi^e\}_e$ such that for all $f \in L^p(w)$,

$$\|\mathcal{W}[r, \{\Phi^e\}_e](f)\|_{L^p(w)} \leq C \left\| \mathcal{W}[r, \{\psi^e\}_e](f) \right\|_{L^p(w)}.$$

Hernández and Weiss proved Lemma 8.3 for the non-weighted case using the non-weighted version of Proposition 5.8 ([9, Theorem 4.9 and Theorem 6.21 in Chapter 6]). By the same arguments as [9] with Proposition 5.8, we can get Lemma 8.3.

We also have the next proposition.

Proposition 8.4 *Let $w \in A_p$ and $\{\Phi^e : 1 \leq e \leq 2^n - 1\} \subset \mathcal{R}^s(\mathbb{R}^n)$. Then there exists a constant $C > 0$ depended only on n, p, $A_p(w)$, s and $\{\Phi^e\}_e$ such that for all $f \in L^{p,s}(w)$,*

$$\|\mathcal{W}[s, \{\Phi^e\}_e](f)\|_{L^p(w)} \leq C\|f\|_{L^{p,s}(w)}.$$

8.2 Proof of Proposition 8.4

In this subsection we prove Proposition 8.4. The next proposition and Lemma 8.3 will be important.

Proposition 8.5 ([4, Theorem 1.2], cf. [3, 12]). *Let $w \in A_p$, $\lambda > n$ and $\{\phi_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^n)$. Define $\phi^{**}_{j,\lambda}(f)(x) := \sup_{y\in\mathbb{R}^n}\left\{|\phi_j * f(x-y)|(1+2^j|y|)^{-\lambda}\right\}$ and assume the following:*
(i) *There exists a constant $a > 0$ independent of j such that* $\operatorname{supp}\mathcal{F}[\phi_j] \subset \left\{2^{j-a} \le |\xi| \le 2^{j+a}\right\}$ *for all $j \in \mathbb{Z}$.*
(ii) *For each $\alpha \in \mathbb{Z}^n_+$, there exists a constant $C_\alpha > 0$ such that $\left|D^\alpha\mathcal{F}[\phi_j](\xi)\right| \le C_\alpha 2^{-j|\alpha|}$ for all $\xi \in \mathbb{R}^n$ and $j \in \mathbb{Z}$.*
Then there exists a constant $C > 0$ depended only on n, p, $A_p(w)$, s, λ and $\{\phi_j\}_{j\in\mathbb{Z}}$ such that for every $f \in L^{p,s}(w)$,

$$\left\|\left\{\sum_{j\in\mathbb{Z}}\left(2^{js}\phi^{**}_{j,\lambda}(f)\right)^2\right\}^{1/2}\right\|_{L^p(w)} \le C\,\|f\|_{L^{p,s}(w)}.$$

Let $\{\psi^e\}_{e\in E}$ be the Meyer wavelet set constructed by tensor products (4) with the Meyer scaling function φ^0 and the Meyer wavelet φ^1 in $L^2(\mathbb{R})$. By Lemma 8.3, there exists a constant $C_0 > 0$ depended only on n, p, $A_p(w)$, s, $\{\Phi^e\}_e$ and $\{\psi^e\}_e$ such that for every $f \in L^{p,s}(w)$,

$$\|\mathcal{W}[s,\{\Phi^e\}_e](f)\|_{L^p(w)} \le C_0\left\|\mathcal{W}[s,\{\psi^e\}_e](f)\right\|_{L^p(w)}.$$

Denote $\phi^e_j(y) := 2^{jn}\psi^e(-2^jy)$ for $j \in \mathbb{Z}$ and $e \in E$. Take $\lambda > n$ arbitrarily. Following the same calculations as [9, Proof of Theorem 4.2 in Chapter 6], we have that

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^n}\left|< f,\psi^e_{j,k} > \chi_{j,k}(x)\right|^2 &= \sum_{k\in\mathbb{Z}^n}\left|\int_{\mathbb{R}^n} f(z)2^{jn/2}\psi^e(2^jz-k)dz\right|^2\chi_{j,k}(x)^2\\
&= \sum_{k\in\mathbb{Z}^n}\left|\int_{\mathbb{R}^n} f(z)2^{-jn/2}\phi^e_j(2^{-j}k-z)dz\right|^2\chi_{j,k}(x)^2\\
&= \sum_{k\in\mathbb{Z}^n}\left|\phi^e_j * f(2^{-j}k)\right|^2\chi_{Q_{j,k}}(x)\\
&\le \sum_{k\in\mathbb{Z}^n}\left\{\sup_{y\in Q_{j,k}}\left|\phi^e_j * f(y)\right|^2\right\}\chi_{Q_{j,k}}(x)\\
&\le \left\{\sup_{|z|\le 2^{-j}\sqrt{n}}\left|\phi^e_j * f(x-z)\right|\right\}^2\\
&\le \left\{\sup_{|z|\le 2^{-j}\sqrt{n}}\frac{\left|\phi^e_j * f(x-z)\right|}{(1+2^j|z|)^\lambda}\right\}^2\cdot\sup_{|z|\le 2^{-j}\sqrt{n}}(1+2^j|z|)^{2\lambda}
\end{aligned}$$

$$= \left(1+\sqrt{n}\right)^{2\lambda}\phi^{e**}_{j,\lambda}(f)(x)^2.$$

Namely we obtain that

$$\mathcal{W}[s,\{\psi^e\}_e](f) \le \left(1+\sqrt{n}\right)^{\lambda}\left[\sum_{e\in E}\left\{\sum_{j=-\infty}^{\infty}\left(2^{js}\phi^{e**}_{j,\lambda}(f)\right)^2\right\}\right]^{1/2}.$$

Now remark that $\{\phi_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^n)$ satisfies the assumptions of Proposition 8.5. Hence there exists a constant $C_1 > 0$ depended only on n, p, $A_p(w)$, s, λ and $\{\psi^e\}_e$ such that

$$\left\|\mathcal{W}[s,\{\psi^e\}_e](f)\right\|_{L^p(w)} \le \left(1+\sqrt{n}\right)^{\lambda}(2^n-1)C_1\|f\|_{L^{p,s}(w)}.$$

Consequently we get

$$\|\mathcal{W}[s,\{\Phi^e\}_e](f)\|_{L^p(w)} \le \left(1+\sqrt{n}\right)^{\lambda}(2^n-1)C_0C_1\|f\|_{L^{p,s}(w)}. \qquad \square$$

8.3 Proof of Theorem 8.1

Theorem 7.3 and Proposition 8.4 shows the right-hand side inequality. We will prove the left-hand side inequality. By Theorem 5.5 and Theorem 7.3, we have only to estimate $\left\|D^\beta f\right\|_{L^p(w)}$ for all $\beta \in \mathbb{Z}^n_+$ with $|\beta| = s$ and $f \in L^{p,s}(w)$. By the duality, it follows that

$$\left\|D^\beta f\right\|_{L^p(w)} = \sup_g\left\{\left|\int_{\mathbb{R}^n} D^\beta f(x)\cdot g(x)dx\right| \;:\; \|g\|_{L^{p'}(v)} \le 1\right\},$$

where $v := w^{-1/(p-1)}$. Following Theorem 6.1 and the right-hand side inequality, it suffices to prove that

$$\left|\int_{\mathbb{R}^n} D^\beta f(x)\cdot g(x)dx\right| \le C\left\|\mathcal{W}[s,\{\psi^e\}_e](f)\right\|_{L^p(w)}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ with $\|g\|_{L^{p'}(v)} \le 1$, where $C > 0$ is a constant independent of β, f and g. Because $\{\psi^e_{j,k} : 1 \le e \le 2^n-1,\ j\in\mathbb{Z},\ k\in\mathbb{Z}^n\}$ forms an orthonormal basis in $L^2(\mathbb{R}^n)$, we obtain that

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} D^\beta f(x)\cdot g(x)dx\right| &= \left|\int_{\mathbb{R}^n} f(x)\cdot D^\beta g(x)dx\right| \\
&= \left|\int_{\mathbb{R}^n}\left\{\sum_{e=1}^{2^n-1}\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n} < f,\psi^e_{j,k} > \psi^e_{j,k}(x)\right\}\cdot\left\{\sum_{e=1}^{2^n-1}\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n} < D^\beta g,\overline{\psi^e_{j,k}} > \overline{\psi^e_{j,k}(x)}\right\}dx\right| \\
&= \left|\sum_{e=1}^{2^n-1}\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n} < f,\psi^e_{j,k} >< D^\beta g,\overline{\psi^e_{j,k}} >\right|
\end{aligned}$$

$$\leq \sum_{e=1}^{2^n-1}\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n}\left|< f, \psi^e_{j,k} >< g, \overline{2^{js}(D^\beta\psi^e)_{j,k}} >\right| \cdot \int_{\mathbb{R}^n}\chi_{j,k}(x)^2 dx$$
$$= \int_{\mathbb{R}^n}\sum_{e=1}^{2^n-1}\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n}\left|2^{js} < f, \psi_{j,k} > \chi_{j,k}(x) \cdot < g, \overline{(D^\beta\psi^e)_{j,k}} > \chi_{j,k}(x)\right| dx.$$

Therefore by the Cauchy-Schwartz inequality and Hölder's inequality, we have that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} D^\beta f(x)\cdot g(x)dx\right| &\leq \int_{\mathbb{R}^n}\mathcal{W}[s,\{\psi^e\}_e](f)(x)\cdot\mathcal{W}\left[0,\left\{\overline{D^\beta\psi^e}\right\}_e\right](g)(x)dx \\ &\leq \left\|\mathcal{W}[s,\{\psi^e\}_e](f)\right\|_{L^p(w)}\left\|\mathcal{W}\left[0,\left\{\overline{D^\beta\psi^e}\right\}_e\right](g)\right\|_{L^{p'}(v)}.\end{aligned}$$

Now let $\{\Psi^e : 1 \leq e \leq 2^n - 1\}$ be a wavelet set constructed by an MRA such that each Ψ^e is 1-regular. Following Lemma 8.3 and Theorem 7.3, we get

$$\left\|\mathcal{W}\left[0,\left\{\overline{D^\beta\psi^e}\right\}_e\right](g)\right\|_{L^{p'}(v)} \leq C_0\left\|\mathcal{W}[0,\{\Psi^e\}_e](g)\right\|_{L^{p'}(v)} \leq C_1\|g\|_{L^{p'}(v)} \leq C_1,$$

where C_0 and C_1 are constants depended only on n, p, $A_p(w)$, s, $\{\psi^e\}_e$ and $\{\Psi^e\}_e$. □

9 The characterization of $L^{p,s}(w)$ with $w \in A_p^{\mathrm{loc}}$ by wavelets and scaling functions

In this section, we characterize $L^{p,s}(w)$ with $w \in A_p^{\mathrm{loc}}$ by wavelets and scaling functions with proper smoothness and compact support.

9.1 Statement of the result

We have the next main result.

Theorem 9.1 *Let $w \in A_p^{\mathrm{loc}}$, φ be the Daubechies scaling function in $L^2(\mathbb{R}^n)$ and $\{\psi^e\}_{e\in E}$ be the Daubechies wavelet set associated with φ. Suppose that φ, $\psi^e \in C^{s+1}_{\mathrm{comp}}(\mathbb{R}^n)$ for all $e \in E$. Define*

$$\mathcal{V}[s,\{\psi^e\}_e](f) := \left(\sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}\left|2^{js} < f, \psi^e_{j,k} > \chi_{j,k}\right|^2\right)^{1/2}$$

and

$$\mathcal{N}^s_{p,w}(f) := \left(\sum_{k\in\mathbb{Z}^n}\left|< f, \varphi_{0,k} > \|\varphi_{0,k}\|_{L^p(w)}\right|^p\right)^{1/p} + \left\|\mathcal{V}[s,\{\psi^e\}_e](f)\right\|_{L^p(w)}.$$

Then there exist two constants $0 < c \le C < \infty$ *depended only on n, p,* $A_p^{\mathrm{loc}}(w)$*, s and* φ *such that for all* $f \in L^{p,s}(w)$*,*

$$c\|f\|_{L^{p,s}(w)} \le \mathcal{N}^s_{p,w}(f) \le C\|f\|_{L^{p,s}(w)}.$$

We need the following proposition in order to prove the characterization above.

Proposition 9.2 *Let* $w \in A_p^{\mathrm{loc}}$ *and* $\{\Psi^e\}_{e\in E}$ *be a set of functions in* $\mathcal{R}^s(\mathbb{R}^n)$ *with compact support. Then there exists a constant* $C > 0$ *depended only on n, p,* $A_p^{\mathrm{loc}}(w)$*, s and* $\{\Psi^e\}_{e\in E}$ *such that for all* $f \in L^{p,s}(w)$*,*

$$\|\mathcal{V}[s, \{\Psi^e\}_e](f)\|_{L^p(w)} \le C\|f\|_{L^{p,s}(w)}.$$

9.2 Proof of Theorem 9.1

First we show the right-hand side inequality. The estimation of $\left\|\mathcal{V}[s, \{\psi^e\}_e](f)\right\|_{L^p(w)}$ is shown by Proposition 9.2. We will estimate the first term of $\mathcal{N}^s_{p,w}(f)$. Let $N \ge 2$ be the positive integer such that $\operatorname{supp}\varphi = \operatorname{supp}\psi^e = [0, 2N-1]^n$ for every $e \in E$. Denote $G_k := \prod_{\nu=1}^n [k_\nu, k_\nu + 2N - 1] = \operatorname{supp}\varphi_{0,k}$ and $v := w^{-1/(p-1)}$. By Hölder's inequality, we obtain that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^n} \left|< f, \varphi_{0,k} > \|\varphi_{0,k}\|_{L^p(w)}\right|^p \\
= &\sum_{k\in\mathbb{Z}^n} \left|< f, \varphi_{0,k} >\right|^p \cdot \|\varphi_{0,k}\|^p_{L^p(w)} \\
\le &\sum_{k\in\mathbb{Z}^n} \int_{G_k} |f(x)|^p\, w(x)dx \cdot \left(\int_{G_k} |\varphi_{0,k}(x)|^{p'} v(x)dx\right)^{p-1} \cdot \|\varphi\|^p_{L^\infty(\mathbb{R}^n)} w(G_k) \\
\le &\|\varphi\|^{2p}_{L^\infty(\mathbb{R}^n)} \sum_{k\in\mathbb{Z}^n} \int_{G_k} |f(x)|^p\, w(x)dx \cdot w(G_k) v(G_k)^{p-1} \\
\le &\|\varphi\|^{2p}_{L^\infty(\mathbb{R}^n)} (2N-1)^{np} A_p^{\mathrm{loc},(2N-1)^n}(w) \sum_{k\in\mathbb{Z}^n} \int_{G_k} |f(x)|^p\, w(x)dx \\
\le &\|\varphi\|^{2p}_{L^\infty(\mathbb{R}^n)} (2N-1)^{n(p+1)} A_p^{\mathrm{loc},(2N-1)^n}(w) \|f\|^p_{L^p(w)}.
\end{aligned}$$

Next we prove the left-hand side inequality. By the duality, we see that

$$\|D^\alpha f\|_{L^p(w)} = \sup_g \left\{ \left| \int_{\mathbb{R}^n} D^\alpha f(x) \cdot g(x)dx \right| \ :\ \|g\|_{L^{p'}(v)} \le 1 \right\},$$

for all $f \in L^{p,s}(w)$ and $|\alpha| \le s$. Thus, following Theorem 6.2 and the right-hand side inequality, it suffices to show that

$$\left| \int_{\mathbb{R}^n} D^\alpha f(x) \cdot g(x)dx \right| \le C\mathcal{N}^s_{p,w}(f)$$

for all $f, g \in C^{\infty}_{\mathrm{comp}}(\mathbb{R}^n)$ with $\|g\|_{L^{p'}(v)} \le 1$, where $C > 0$ is a constant independent of α, f and g. Because $\{\varphi_{0,k}\}_{k\in\mathbb{Z}^n} \cup \{\psi^e_{j,k} : e \in E,\ j \ge 0,\ k \in \mathbb{Z}^n\}$ forms an orthonormal basis in $L^2(\mathbb{R}^n)$, we obtain that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} D^\alpha f(x)\cdot g(x)dx\right| = \left|\int_{\mathbb{R}^n} f(x)\cdot D^\alpha g(x)dx\right| \\
&\quad= \Bigg|\int_{\mathbb{R}^n}\left\{\sum_{k\in\mathbb{Z}^n} < f,\varphi_{0,k} > \varphi_{0,k}(x) + \sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} < f,\psi^e_{j,k} > \psi^e_{j,k}(x)\right\} \\
&\qquad\times\left\{\sum_{k\in\mathbb{Z}^n} < D^\alpha g,\varphi_{0,k} > \varphi_{0,k}(x) + \sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} < D^\alpha g,\psi^e_{j,k} > \psi^e_{j,k}(x)\right\}dx\Bigg| \\
&\quad= \left|\sum_{k\in\mathbb{Z}^n} < f,\varphi_{0,k} >< D^\alpha g,\varphi_{0,k} > + \sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} < f,\psi^e_{j,k} >< D^\alpha g,\psi^e_{j,k} >\right|.
\end{aligned}$$

We estimate $\left|\sum_{k\in\mathbb{Z}^n} < f,\varphi_{0,k} >< D^\alpha g,\varphi_{0,k} >\right|$ first. By Hölder's inequality, we see that

$$1 = \int_{\mathbb{R}^n} |\varphi_{0,k}(x)|^2 dx \le \|\varphi_{0,k}\|_{L^p(w)}\|\varphi_{0,k}\|_{L^{p'}(v)}$$

for every $k \in \mathbb{Z}^n$. We shall also remark that $\left|< D^\alpha g,\varphi_{0,k} >\right| = \left|< g,(D^\alpha\varphi)_{0,k} >\right|$. Using Hölder's inequality again, it follows that

$$\begin{aligned}
&\left|\sum_{k\in\mathbb{Z}^n} < f,\varphi_{0,k} >< D^\alpha g,\varphi_{0,k} >\right| \\
&\quad\le \sum_{k\in\mathbb{Z}^n}\left|< f,\varphi_{0,k} > \|\varphi_{0,k}\|_{L^p(w)}\cdot < g,(D^\alpha\varphi)_{0,k} > \|\varphi_{0,k}\|_{L^{p'}(v)}\right| \\
&\quad\le \left(\sum_{k\in\mathbb{Z}^n}\left|< f,\varphi_{0,k} > \|\varphi_{0,k}\|_{L^p(w)}\right|^p\right)^{1/p}\cdot\left(\sum_{l\in\mathbb{Z}^n}\left|< g,(D^\alpha\varphi)_{0,l} > \|\varphi_{0,l}\|_{L^{p'}(v)}\right|^{p'}\right)^{1/p'}.
\end{aligned}$$

Remark that $\operatorname{supp}\varphi_{0,l}$, $\operatorname{supp}(D^\alpha\varphi)_{0,l} \subset G_l$ for each $l \in \mathbb{Z}^n$. Using Hölder's inequality once more, we have that

$$\begin{aligned}
&\sum_{l\in\mathbb{Z}^n}\left|< g,(D^\alpha\varphi)_{0,l} > \|\varphi_{0,l}\|_{L^{p'}(v)}\right|^{p'} \\
&\quad= \sum_{l\in\mathbb{Z}^n}\left|\int_{G_l} g(x)\cdot(D^\alpha\varphi)_{0,l}(x)dx\right|^{p'}\cdot\int_{G_l}|\varphi_{0,l}(x)|^{p'}v(x)dx \\
&\quad\le \sum_{l\in\mathbb{Z}^n}\int_{G_l}|g(x)|^{p'}v(x)dx\cdot\left(\int_{G_l}\left|(D^\alpha\varphi)_{0,l}(x)\right|^p v(x)^{-p/p'}dx\right)^{p'/p}\cdot\int_{G_l}|\varphi_{0,l}(x)|^{p'}v(x)dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \max_{|\alpha|\leq s} \|D^\alpha\varphi\|^{p'}_{L^\infty(\mathbb{R}^n)}\|\varphi\|^{p'}_{L^\infty(\mathbb{R}^n)} \cdot \sum_{l\in\mathbb{Z}^n}\int_{G_l}|g(x)|^{p'}v(x)dx\cdot\left(\int_{G_l}v(x)^{-p/p'}dx\right)^{p'/p}v(G_l)\\
&\leq \max_{|\alpha|\leq s} \|D^\alpha\varphi\|^{p'}_{L^\infty(\mathbb{R}^n)}\|\varphi\|^{p'}_{L^\infty(\mathbb{R}^n)}(2N-1)^{np'}A_{p'}^{\mathrm{loc},(2N-1)^n}(v)\cdot\sum_{l\in\mathbb{Z}^n}\int_{G_l}|g(x)|^{p'}v(x)dx\\
&= \max_{|\alpha|\leq s} \|D^\alpha\varphi\|^{p'}_{L^\infty(\mathbb{R}^n)}\|\varphi\|^{p'}_{L^\infty(\mathbb{R}^n)}(2N-1)^{n(p'+1)}A_{p'}^{\mathrm{loc},(2N-1)^n}(v)\|g\|^{p'}_{L^{p'}(v)}.
\end{aligned}
$$

Hence denoting $C_1 := \max_{|\alpha|\leq s}\|D^\alpha\varphi\|_{L^\infty(\mathbb{R}^n)}\|\varphi\|_{L^\infty(\mathbb{R}^n)}(2N-1)^{n(1+1/p')}A_{p'}^{\mathrm{loc},(2N-1)^n}(v)^{1/p'}$, we get

$$
\left|\sum_{k\in\mathbb{Z}^n} < f,\varphi_{0,k} >< D^\alpha g,\varphi_{0,k} >\right| \leq C_1\left(\sum_{k\in\mathbb{Z}^n}\left|< f,\varphi_{0,k} > \|\varphi_{0,k}\|_{L^p(w)}\right|^p\right)^{1/p}\|g\|_{L^{p'}(v)}.
$$

Next we estimate $\left|\sum_{e\in E}\sum_{j=0}^\infty\sum_{k\in\mathbb{Z}^n} < f,\psi^e_{j,k} >< D^\alpha g,\psi^e_{j,k} >\right|$. By straightforward calculations and Hölder's inequality, we obtain that

$$
\begin{aligned}
&\left|\sum_{e\in E}\sum_{j=0}^\infty\sum_{k\in\mathbb{Z}^n} < f,\psi^e_{j,k} >< D^\alpha g,\psi^e_{j,k} >\right|\\
&\quad\leq \sum_{e\in E}\sum_{j=0}^\infty\sum_{k\in\mathbb{Z}^n}\left|< f,\psi^e_{j,k} >< g, 2^{j|\alpha|}(D^\alpha\psi^e)_{j,k} >\right|\cdot\int_{\mathbb{R}^n}\chi_{j,k}(x)^2dx\\
&\quad= \int_{\mathbb{R}^n}\sum_{e\in E}\sum_{j=0}^\infty\sum_{k\in\mathbb{Z}^n}\left|2^{js} < f,\psi^e_{j,k} > \chi_{j,k}(x)\cdot 2^{j(|\alpha|-s)} < g,(D^\alpha\psi^e)_{j,k} > \chi_{j,k}(x)\right|dx\\
&\quad= \int_{\mathbb{R}^n}\sum_{e\in E}\sum_{j=0}^\infty\sum_{k\in\mathbb{Z}^n}\left|2^{js} < f,\psi^e_{j,k} > \chi_{j,k}(x)\cdot < g,(D^\alpha\psi^e)_{j,k} > \chi_{j,k}(x)\right|dx\\
&\quad\leq \int_{\mathbb{R}^n}\mathcal{V}\left[s,\{\psi^e\}_e\right](f)(x)\cdot\mathcal{V}\left[0,\{D^\alpha\psi^e\}_e\right](g)(x)dx\\
&\quad\leq \left\|\mathcal{V}\left[s,\{\psi^e\}_e\right](f)\right\|_{L^p(w)}\left\|\mathcal{V}\left[0,\{D^\alpha\psi^e\}_e\right](g)\right\|_{L^{p'}(v)}.
\end{aligned}
$$

Now remark that $\operatorname{supp}\mathcal{V}\left[0,\{D^\alpha\psi^e\}_e\right](g\cdot\chi_{Q_{0,l}}) \subset \prod_{\nu=1}^n[l_\nu-2N+1, l_\nu+2] =: E_l$ for each $l\in\mathbb{Z}^n$. On the other hand, for all $x\in\mathbb{R}^n$, there exists an unique $L = L(x)\in\mathbb{Z}^n$ such that $x\in Q_{0,L}$. Denoting $\mathcal{B}(L) := \{l\in\mathbb{Z}^n : L_\nu-1\leq l_\nu\leq L_\nu+2N-1 \text{ for all } 1\leq\nu\leq n\}$, we get

$$
\left|\mathcal{V}\left[0,\{D^\alpha\psi^e\}_e\right](g)(x)\right|^{p'} \leq \left|\sum_{l\in\mathbb{Z}^n}\mathcal{V}\left[0,\{D^\alpha\psi^e\}_e\right](g\cdot\chi_{Q_{0,l}})(x)\right|^{p'}
$$

$$
\begin{aligned}
&= \left|\sum_{l\in\mathcal{B}(L)} \mathcal{V}\left[0,\{D^\alpha\psi^e\}_e\right](g\cdot\chi_{Q_{0,l}})(x)\right|^{p'} \\
&\le \sum_{l\in\mathcal{B}(L)} \left|\mathcal{V}\left[0,\{D^\alpha\psi^e\}_e\right](g\cdot\chi_{Q_{0,l}})(x)\right|^{p'}\cdot\sharp\mathcal{B}(L)^{p'/p} \\
&\le (2N+1)^{n(p'-1)}\sum_{l\in\mathbb{Z}^n} \left|\mathcal{V}\left[0,\{D^\alpha\psi^e\}_e\right](g\cdot\chi_{Q_{0,l}})(x)\right|^{p'}.
\end{aligned}
$$

Therefore it follows that

$$
\begin{aligned}
&\left\|\mathcal{V}\left[0,\{D^\alpha\psi^e\}_e\right](g)\right\|_{L^{p'}(v)}^{p'} \\
&\quad\le (2N+1)^{n(p'-1)}\int_{\mathbb{R}^n}\sum_{l\in\mathbb{Z}^n} \left|\mathcal{V}\left[0,\{D^\alpha\psi^e\}_e\right](g\cdot\chi_{Q_{0,l}})(x)\right|^{p'} v(x)dx \\
&\quad= (2N+1)^{n(p'-1)}\sum_{k\in\mathbb{Z}^n}\int_{Q_{0,k}}\sum_{l\in\mathbb{Z}^n} \left|\mathcal{V}\left[0,\{D^\alpha\psi^e\}_e\right](g\cdot\chi_{Q_{0,l}})(x)\right|^{p'} v(x)dx \\
&\quad= (2N+1)^{n(p'-1)}\sum_{k\in\mathbb{Z}^n}\int_{Q_{0,k}}\sum_{l\in\mathcal{B}(k)} \left|\mathcal{V}\left[0,\{D^\alpha\psi^e\}_e\right](g\cdot\chi_{Q_{0,l}})(x)\right|^{p'} v(x)dx \\
&\quad\le (2N+1)^{n(p'-1)}\sum_{k\in\mathbb{Z}^n}\sum_{l\in\mathcal{B}(k)}\int_{E_l} \left|\mathcal{V}\left[0,\{D^\alpha\psi^e\}_e\right](g\cdot\chi_{Q_{0,l}})(x)\right|^{p'} v(x)dx.
\end{aligned}
$$

By Lemma 3.3, we can construct $\{v_l\}_{l\in\mathbb{Z}^n}\subset A_{p'}$ such that $v_l = v$ on E_l and $A_{p'}(v_l)\le 3^{np'}A_{p'}^{\mathrm{loc},(2N+1)^n}(v)$ for every $l\in\mathbb{Z}^n$. In addition, let $\{\Psi^e : 1\le e\le 2^n-1\}$ be a wavelet set constructed by an MRA such that each Ψ^e is 1-regular. Following Lemma 8.3 and Theorem 7.3, we have that

$$
\begin{aligned}
\left\|\mathcal{V}\left[0,\{D^\alpha\psi^e\}_e\right](g\cdot\chi_{Q_{0,l}})\right\|_{L^{p'}(v_l)}^{p'} &\le \left\|\mathcal{W}\left[0,\{D^\alpha\psi^e\}_e\right](g\cdot\chi_{Q_{0,l}})\right\|_{L^{p'}(v_l)}^{p'} \\
&\le C_2\left\|\mathcal{W}\left[0,\{\Psi^e\}_e\right](g\cdot\chi_{Q_{0,l}})\right\|_{L^{p'}(v_l)}^{p'} \\
&\le C_3\left\|g\cdot\chi_{Q_{0,l}}\right\|_{L^{p'}(v_l)}^{p'} = C_3\left\|g\cdot\chi_{Q_{0,l}}\right\|_{L^{p'}(v)}^{p'},
\end{aligned}
$$

where $0 < C_2, C_3 < \infty$ are constants depended only on n, p, $A_p^{\mathrm{loc}}(w)$, s, $\{\psi^e\}_e$ and $\{\Psi^e\}_e$. Now write $C_4 := (2N+1)^{n/p}C_3^{1/p'}$. Then it follows that

$$
\left\|\mathcal{V}\left[0,\{D^\alpha\psi^e\}_e\right](g)\right\|_{L^{p'}(v)}^{p'} \le (2N+1)^{n(p'-1)}\sum_{k\in\mathbb{Z}^n}\sum_{l\in\mathcal{B}(k)} C_3\left\|g\cdot\chi_{Q_{0,l}}\right\|_{L^{p'}(v)}^{p'} = C_4^{p'}\|g\|_{L^{p'}(v)}^{p'}.
$$

Namely we get

$$
\left|\sum_{e\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} < f,\psi_{j,k}^e >< D^\alpha g,\psi_{j,k}^e >\right| \le C_4\left\|\mathcal{V}\left[s,\{\psi^e\}_e\right](f)\right\|_{L^p(w)}\|g\|_{L^{p'}(v)}.
$$

Consequently we have that

$$\left|\int_{\mathbb{R}^n} D^\alpha f(x)\cdot g(x)dx\right| \le \max\{C_1, C_4\}\mathcal{N}^s_{p,w}(f)\cdot \|g\|_{L^{p'}(v)} \le \max\{C_1, C_4\}\mathcal{N}^s_{p,w}(f). \qquad \square$$

9.3 Proof of Proposition 9.2

Let φ be the Daubechies scaling function in $C^s_{\mathrm{comp}}(\mathbb{R}^n)$ with $\operatorname{supp}\varphi = [0, 2N-1]^n$ for some positive integer $N \ge 2$ and write $f_l(x) := f(x)\varphi(x-l)$ for each $l \in \mathbb{Z}^n$. Then we have $\sum_{l\in\mathbb{Z}^n} \varphi(x-l) = 1$ by [9, Proposition 3.14 in Chapter 5] or [24, Proposition 2.17]. Thus we get the decomposition that $f = \sum_{l\in\mathbb{Z}^n} f_l$. Here we can take $m \in \mathbb{N}$ so that $\operatorname{supp}\Psi \subset [-m, m]^n$. Then we see that $\operatorname{supp}\mathcal{V}[s, \{\Psi^e\}_e](f_l) \subset \prod_{\nu=1}^n [l_\nu - m, l_\nu + 2N + m] =: D_l$ for each $l \in \mathbb{Z}^n$. On the other hand, for all $x \in \mathbb{R}^n$, there exists an unique $L = L(x) \in \mathbb{Z}^n$ such that $x \in Q_{0,L}$. Denoting $\mathcal{A}(L) := \{l \in \mathbb{Z}^n : L_\nu + 1 - 2N - m \le l_\nu \le L_\nu + m \text{ for all } 1 \le \nu \le n\}$, we get that

$$\begin{aligned}
|\mathcal{V}[s, \{\Psi^e\}_e](f)(x)|^p &\le \left|\sum_{l\in\mathbb{Z}^n} \mathcal{V}[s, \{\Psi^e\}_e](f_l)(x)\right|^p = \left|\sum_{l\in\mathcal{A}(L)} \mathcal{V}[s, \{\Psi^e\}_e](f_l)(x)\right|^p \\
&\le \sum_{l\in\mathcal{A}(L)} |\mathcal{V}[s, \{\Psi^e\}_e](f_l)(x)|^p \cdot \sharp\mathcal{A}(L)^{p/p'} \\
&\le (2m + 2N)^{n(p-1)} \sum_{l\in\mathbb{Z}^n} |\mathcal{V}[s, \{\Psi^e\}_e](f_l)(x)|^p .
\end{aligned}$$

Therefore it follows that

$$\begin{aligned}
\|\mathcal{V}[s, \{\Psi^e\}_e](f)\|^p_{L^p(w)} &\le (2m + 2N)^{n(p-1)} \int_{\mathbb{R}^n} \sum_{l\in\mathbb{Z}^n} |\mathcal{V}[s, \{\Psi^e\}_e](f_l)(x)|^p\, w(x)dx \\
&= (2m + 2N)^{n(p-1)} \sum_{k\in\mathbb{Z}^n} \int_{Q_{0,k}} \sum_{l\in\mathbb{Z}^n} |\mathcal{V}[s, \{\Psi^e\}_e](f_l)(x)|^p\, w(x)dx \\
&= (2m + 2N)^{n(p-1)} \sum_{k\in\mathbb{Z}^n} \int_{Q_{0,k}} \sum_{l\in\mathcal{A}(k)} |\mathcal{V}[s, \{\Psi^e\}_e](f_l)(x)|^p\, w(x)dx \\
&\le (2m + 2N)^{n(p-1)} \sum_{k\in\mathbb{Z}^n} \sum_{l\in\mathcal{A}(k)} \int_{D_l} |\mathcal{V}[s, \{\Psi^e\}_e](f_l)(x)|^p\, w(x)dx.
\end{aligned}$$

By Lemma 3.3, we get $\{w_l\}_{l\in\mathbb{Z}^n} \subset A_p$ such that $A_p(w_l) \le 3^{np} A_p^{\mathrm{loc}, 2^n(m+N)^n}(w)$ and $w_l = w$ on D_l for all $l \in \mathbb{Z}^n$. Thus we obtain that

$$\|\mathcal{V}[s, \{\Psi^e\}_e](f)\|^p_{L^p(w)} \le (2m + 2N)^{n(p-1)} \sum_{k\in\mathbb{Z}^n} \sum_{l\in\mathcal{A}(k)} \|\mathcal{V}[s, \{\Psi^e\}_e](f_l)\|^p_{L^p(w_l)} .$$

Denote $G_l := \prod_{\nu=1}^{n} [l_\nu, l_\nu + 2N - 1] = \operatorname{supp} \varphi(x - l)$. Then we see that $w_l = w$ on G_l for every $l \in \mathbb{Z}^n$. Additionally by Proposition 8.4, there exists a constant $C_0 > 0$ depended only on n, p, $A_p^{\mathrm{loc}}(w)$, s and $\{\Psi^e\}_e$ such that for each $k \in \mathbb{Z}^n$ and $l \in \mathcal{A}(k)$,

$$\begin{aligned}
\|\mathcal{V}[s, \{\Psi^e\}_e](f_l)\|_{L^p(w_l)}^p &\le \|\mathcal{W}[s, \{\Psi^e\}_e](f_l)\|_{L^p(w_l)}^p \\
&\le C_0 \|f_l\|_{L^{p,s}(w_l)}^p = C_0 \left(\sum_{|\alpha| \le s} \|D^\alpha(f_l)\|_{L^p(w_l)} \right)^p \\
&= C_0 \left(\sum_{|\alpha| \le s} \left\| \sum_{\beta \le \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta f \cdot D^{\alpha - \beta} (\varphi(\cdot - l)) \right\|_{L^p(w_l)} \right)^p \\
&\le C_1 \sum_{|\alpha| \le s} \int_{G_l} |D^\alpha f(x)|^p \, w_l(x) dx = C_1 \sum_{|\alpha| \le s} \int_{G_l} |D^\alpha f(x)|^p \, w(x) dx,
\end{aligned}$$

where $C_1 > 0$ is a constant depended only on n, p, s, C_0 and φ. Thus we obtain that

$$\begin{aligned}
&\|\mathcal{V}[s, \{\Psi^e\}_e](f)\|_{L^p(w)} \\
&\quad \le \left\{ (2m + 2N)^{n(p-1)} \cdot C_1 \cdot (2N - 1)^n (2m + 2N)^n \sum_{|\alpha| \le s} \|D^\alpha f\|_{L^p(w)}^p \right\}^{1/p} \\
&\quad \le C_2 \|f\|_{L^{p,s}(w)},
\end{aligned}$$

where $C_2 > 0$ is a constant depended only on n, p, $A_p^{\mathrm{loc}}(w)$, s, $\{\Psi^e\}_e$ and φ. □

10 The unconditional bases and the greedy bases of $L^{p,s}(w)$

As mentioned in [10], in the case of weighted L^p spaces $L^p(w)$, if we have the characterization and the unconditional basis of $L^p(w)$ by wavelets and scaling functions, then we can establish the greedy basis given by them following the same statements as [5]. In this section, we show the similar arguments are applicable to $L^{p,s}(w)$.

Theorem 10.1
(a) *Let* $w \in A_p$ *and* $\{\psi^e : 1 \le e \le 2^n - 1\}$ *be a wavelet set constructed by an MRA such that each* ψ^e *is* $(s+1)$*-regular. Then the following* (a1) *and* (a2) *hold:*
(a1) *The wavelet basis* $\left\{ \psi_{j,k}^e : 1 \le e \le 2^n - 1,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n \right\}$ *forms an unconditional basis for* $L^{p,s}(w)$.
(a2) *Define*

$$\widetilde{\psi^e}_{j,k} := \frac{\psi_{j,k}^e}{\|\psi_{j,k}^e\|_{L^{p,s}(w)}}$$

for $1 \le e \le 2^n-1$, $j \in \mathbb{Z}$ *and* $k \in \mathbb{Z}^n$. *Then the sequence* $\left\{\widetilde{\psi^e}_{j,k} : 1 \le e \le 2^n-1,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\right\}$ *forms a greedy basis for* $L^{p,s}(w)$.

(b) *Let* $w \in A_p^{\rm loc}$, φ *be the Daubechies scaling function in* $L^2(\mathbb{R}^n)$ *and* $\{\psi^e\}_{e\in E}$ *be the Daubechies wavelet set associated with* φ *such that* $\varphi,\ \psi^e \in C^{s+1}_{\rm comp}(\mathbb{R}^n)$ *for all* $e \in E$. *Then the following* (b1) *and* (b2) *hold:*

(b1) *The sequence* $\{\varphi_{0,k}\}_{k\in\mathbb{Z}^n} \cup \{\psi^e_{j,k} : e \in E,\ j \ge 0,\ k \in \mathbb{Z}^n\}$ *forms an unconditional basis for* $L^{p,s}(w)$.

(b2) *Define*

$$\tilde{\varphi}_{0,k} := \frac{\varphi_{0,k}}{\|\varphi_{0,k}\|_{L^{p,s}(w)}} \quad and \quad \widetilde{\psi^e}_{j,k} := \frac{\psi^e_{j,k}}{\|\psi^e_{j,k}\|_{L^{p,s}(w)}}$$

for $e \in E$, $j \ge m$ *and* $k \in \mathbb{Z}^n$. *Then the sequence* $\{\tilde{\varphi}_{0,k}\}_{k\in\mathbb{Z}^n} \cup \{\widetilde{\psi^e}_{j,k} : e \in E,\ j \ge 0,\ k \in \mathbb{Z}^n\}$ *forms a greedy basis for* $L^{p,s}(w)$.

Proof of Theorem 10.1 We have only to show (b). (a) is proved by the same arguments as the proof of (b) below.

To begin with, we shall prove (b1). It suffices to check the following two conditions: (I) There exists a constant $0 < C < \infty$ independent of f, A and B such that $\left\|T_{A,B}f\right\|_{L^{p,s}(w)} \le C\|f\|_{L^{p,s}(w)}$ for all $f \in L^{p,s}(w)$ and all finite subsets $A \subset \mathbb{Z}^n$ and $B \subset E \times \mathbb{Z}_+ \times \mathbb{Z}^n$, where $T_{A,B}f := \sum_{k\in A} \langle f, \varphi_{0,k}\rangle \varphi_{0,k} + \sum_{(e,j,k)\in B} \left\langle f, \psi^e_{j,k}\right\rangle \psi^e_{j,k}$.

(II) $L^{p,s}(w) = \overline{\operatorname{span}\{\varphi_{0,k}\}_{k\in\mathbb{Z}^n} \cup \operatorname{span}\{\psi^e_{j,k} : e \in E,\ j \ge 0,\ k \in \mathbb{Z}^n\}}^{L^{p,s}(w)}$.

We show the condition (I) first. By the orthonormality and Theorem 9.1, we obtain

$$c\left\|T_{A,B}f\right\|_{L^{p,s}(w)} \le \mathcal{N}^s_{p,w}(T_{A,B}f) \le \mathcal{N}^s_{p,w}(f) \le C\|f\|_{L^{p,s}(w)},$$

where $0 < c \le C < \infty$ are constants appeared in Theorem 9.1.

Next we check (II). We shall prove that $\lim_{A\nearrow\mathbb{Z}^n, B\nearrow E\times\mathbb{Z}_+\times\mathbb{Z}^n} \mathcal{N}^s_{p,w}(f - T_{A,B}f) = 0$, since $c\left\|f - T_{A,B}f\right\|_{L^{p,s}(w)} \le \mathcal{N}^s_{p,w}(f - T_{A,B}f)$ for all $f \in L^{p,s}(w)$ by Theorem 9.1. Now define

$$\mathcal{N}_1(f) := \left(\sum_{k\in\mathbb{Z}^n} \left|\langle f, \varphi_{0,k}\rangle \left\|\varphi_{0,k}\right\|_{L^p(w)}\right|^p\right)^{1/p} \quad \text{and} \quad \mathcal{N}_2(f) := \left\|\mathcal{V}\left[s, \{\psi^e\}_e\right](f)\right\|_{L^p(w)}.$$

Then we have $\mathcal{N}^s_{p,w}(f - T_{A,B}f) = \mathcal{N}_1(f - T_{A,B}f) + \mathcal{N}_2(f - T_{A,B}f)$. We omit the detail because it is obvious that the orthonormality of the system $\{\varphi_{0,k}\}_{k\in\mathbb{Z}^n} \cup \{\psi^e_{j,k} : e \in E,\ j \ge 0,\ k \in \mathbb{Z}^n\}$ with regard to the L^2-inner product, the boundedness of $\mathcal{N}_\nu(f - T_{A,B}f)$ for each $\nu = 1, 2$ and Lebesgue's dominated convergence theorem give us the desired result.

Secondarily we prove (b2). The proof we give here is essentially same as [10, Section 5.3], and based on [5, the proof of Lemma 4.1]. We prepare the following two lemmas in order to get (b2).

Lemma 10.2 *Let $w \in A_p^{\mathrm{loc}}$. Then w satisfies the dyadic reverse doubling condition, i.e., there exists a constant $1 < d < \infty$ independent of I and I' such that $dw(I') \le w(I)$ for all dyadic cubes I, I' with $I' \subsetneqq I$.*

The proof of Lemma 10.2 is found in [8, p.141] or [22, Proof of Corollary 1.1]. We also have the next estimate.

Lemma 10.3 *Let $w \in A_p^{\mathrm{loc}}$, φ be the Daubechies scaling function in $L^2(\mathbb{R}^n)$ and $\{\psi^e\}_{e\in E}$ be the Daubechies wavelet set associated with φ such that φ, $\psi^e \in C^{s+1}_{\mathrm{comp}}(\mathbb{R}^n)$ for all $e \in E$. Define*

$$\widetilde{\mathcal{V}}\left[s, \{\psi^e\}_e\right](f) := \left(\sum_{e\in E} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \left| w(Q_{j,k})^{-1/p} \left\| \psi^e_{j,k} \right\|_{L^{p,s}(w)} < f, \psi^e{}_{j,k} > \chi_{Q_{j,k}} \right|^2 \right)^{1/2}.$$

Then it follows that

$$c \left\| \widetilde{\mathcal{V}}\left[s, \{\psi^e\}_e\right](f) \right\|_{L^p(w)} \le \left\| \mathcal{V}\left[s, \{\psi^e\}_e\right](f) \right\|_{L^p(w)} \le C \left\| \widetilde{\mathcal{V}}\left[s, \{\psi^e\}_e\right](f) \right\|_{L^p(w)}$$

for all $f \in L^{p,s}(w)$, where $0 < c \le C < \infty$ are the constants appeared in Theorem 9.1.

Proof of Lemma 10.3 For each $e \in E$, $j \ge 0$ and $k \in \mathbb{Z}^n$, we have that

$$\mathcal{N}^s_{p,w}(\psi^e_{j,k}) = \left\| 2^{js} \chi_{j,k} \right\|_{L^p(w)} = 2^{js+jn/2} w(Q_{j,k})^{1/p}.$$

On the other hand, by Theorem 9.1, we obtain

$$C^{-1} \mathcal{N}^s_{p,w}(\psi^e_{j,k}) \le \left\| \psi^e_{j,k} \right\|_{L^{p,s}(w)} \le c^{-1} \mathcal{N}^s_{p,w}(\psi^e_{j,k}).$$

Namely it follows that

$$C^{-1} 2^{js} \le w(Q_{j,k})^{-1/p} \left\| \psi^e_{j,k} \right\|_{L^{p,s}(w)} 2^{-jn/2} \le c^{-1} 2^{js}.$$

This estimation shows the desired result. □

Now let us return to the proof of Theorem 10.1 (b2). Following Theorem 10.1 (b1) and Theorem 4.6, it is enough to prove that $\{\tilde{\varphi}_{0,k}\}_{k\in\mathbb{Z}^n} \cup \{\widetilde{\psi^e}_{j,k} : e \in E,\ j \ge 0,\ k \in \mathbb{Z}^n\}$ is democratic. We see that $\{\varphi_{0,k}\}_{k\in\mathbb{Z}^n} \cup \{\psi^e_{j,k} : e \in E,\ j \ge 0,\ k \in \mathbb{Z}^n\}$ forms an unconditional basis for $L^{p,s}(w)$ by (b1). Thus for all $f \in L^{p,s}(w)$ we can write

$$f = \sum_{k\in\mathbb{Z}^n} a_k(f) \tilde{\varphi}_{0,k} + \sum_{e\in E} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} b^e_{j,k}(f) \widetilde{\psi^e}_{j,k},$$

where $a_k(f) :=< f, \varphi_{0,k} > \|\varphi_{0,k}\|_{L^{p,s}(w)}$ and $b^e_{j,k}(f) :=< f, \psi^e_{j,k} > \|\psi^e_{j,k}\|_{L^{p,s}(w)}$. Now define

$$\widetilde{\mathcal{N}}^s_{p,w}(f) := \left(\sum_{k\in\mathbb{Z}^n} \left|< f, \varphi_{0,k} > \|\varphi_{0,k}\|_{L^{p,s}(w)}\right|^p \right)^{1/p} + \left\| \widetilde{\mathcal{V}}\left[s, \{\psi^e\}_e\right](f) \right\|_{L^p(w)}.$$

By Theorem 9.1, we see that

$$c\|\varphi_{0,k}\|_{L^{p,s}(w)} \le \mathcal{N}^s_{p,w}(\varphi_{0,k}) = \|\varphi_{0,k}\|_{L^p(w)} \le \|\varphi_{0,k}\|_{L^{p,s}(w)}.$$

Thus Lemma 10.3 gives the estimate that

$$c'\|f\|_{L^{p,s}(w)} \le \widetilde{\mathcal{N}}^s_{p,w}(f) \le C'\|f\|_{L^{p,s}(w)},$$

where $0 < c' \le C' < \infty$ are constants depended only on n, p, $A^{\mathrm{loc}}_p(w)$, s and φ. Then we have

$$\begin{aligned} c'\|f\|_{L^{p,s}(w)} &\le \left(\sum_{k\in\mathbb{Z}^n} |a_k(f)|^p \right)^{1/p} + \left\| \left(\sum_{e\in E} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \left| w(Q_{j,k})^{-1/p} b^e_{j,k}(f) \chi_{Q_{j,k}} \right|^2 \right)^{1/2} \right\|_{L^p(w)} \\ &\le C'\|f\|_{L^{p,s}(w)}. \end{aligned} \tag{6}$$

Let us denote $\tilde{\varphi}_Q := \tilde{\varphi}_{j,k}$ and $\widetilde{\psi^e}_Q := \widetilde{\psi^e}_{j,k}$ for a dyadic cube $Q = Q_{j,k}$. Now we take finite subsets $A_\nu \subset \{Q_{0,k} : k \in \mathbb{Z}^n\}$, $E_\nu \subset E$ and $B_\nu \subset \{Q_{j,k} : j \ge 0,\ k \in \mathbb{Z}^n\}$ satisfying $\sharp A_1 + \sharp(E_1 \times B_1) = \sharp A_2 + \sharp(E_2 \times B_2)$ arbitrarily, and write $g_\nu := \sum_{I\in A_\nu} \tilde{\varphi}_I + \sum_{e\in E_\nu} \sum_{J\in B_\nu} \widetilde{\psi^e}_J$ for $\nu = 1, 2$. Using (6) and $1 \le \sharp E_1 \le \sharp E = 2^n - 1$, we obtain

$$\begin{aligned} &c'\|g_1\|_{L^{p,s}(w)} \\ &\le (\sharp A_1)^{1/p} + \left\| \left(\sum_{e\in E_1} \sum_{J\in B_1} \left| w(J)^{-1/p} \chi_J \right|^2 \right)^{1/2} \right\|_{L^p(w)} \\ &= (\sharp A_1)^{1/p} + (\sharp E_1)^{1/2} \cdot \left\{ \int_{\bigcup_{J'\in B_1} J'} \left(\sum_{J\in B_1} w(J)^{-2/p} \chi_J(x) \right)^{p/2} w(x)dx \right\}^{1/p} \\ &\le (\sharp A_1)^{1/p} + (2^n - 1)^{1/2} (\sharp E_1)^{1/p} \cdot \left\{ \int_{\bigcup_{J'\in B_1} J'} \left(\sum_{J\in B_1} w(J)^{-2/p} \chi_J(x) \right)^{p/2} w(x)dx \right\}^{1/p}. \end{aligned} \tag{7}$$

For each $x \in \bigcup_{J\in B_1} J$, $J_1(x)$ denotes the minimal dyadic cube in B_1 with regard to the inclusion relation that contains x. Then we get

$$\sum_{J\in B_1} w(J)^{-2/p} \chi_J(x) \le \sum_{r=0}^{\infty} w(J_r)^{-2/p}, \tag{8}$$

where $J_0 := J_1(x)$, J_r is a dyadic cube satisfying $J_{r-1} \subset J_r$ and $2^n |J_{r-1}| = |J_r|$ for every $r \in \mathbb{N}$. By Lemma 10.2, we obtain

$$w(J_r) \geq dw(J_{r-1}) \geq \cdots \geq d^r w(J_0) = d^r w(J_1(x))$$

for all $r \in \mathbb{N}$. Thus we have

$$\sum_{r=0}^{\infty} w(J_r)^{-2/p} \leq \sum_{r=0}^{\infty} (d^r w(J_1(x)))^{-2/p} = C_0 w(J_1(x))^{-2/p}, \tag{9}$$

where $C_0 := (1 - d^{-2/p})^{-1}$. Following (8) and (9), we obtain

$$\begin{aligned} \int_{\bigcup_{J' \in B_1} J'} \left(\sum_{J \in B_1} w(J)^{-2/p} \chi_J(x) \right)^{p/2} w(x)dx &\leq \int_{\bigcup_{J \in B_1} J} \left(C_0 w(J_1(x))^{-2/p} \right)^{p/2} w(x)dx \\ &= C_0^{p/2} \int_{\bigcup_{J \in B_1} J} w(J_1(x))^{-1} w(x)dx. \end{aligned} \tag{10}$$

Now we set $\tilde{J} := \left\{ x \in \bigcup_{J \in B_1} J : J_1(x) = J \right\}$ for each $J \in B_1$. Then, since $\tilde{J} \subset J$ and $\bigcup_{J \in B_1} J = \bigcup_{J \in B_1} \tilde{J}$, it follows that

$$\begin{aligned} \int_{\bigcup_{J \in B_1} J} w(J_1(x))^{-1} w(x)dx &= \int_{\bigcup_{J \in B_1} \tilde{J}} w(J_1(x))^{-1} w(x)dx \\ &= \int_{\bigcup_{J \in B_1} \tilde{J}} w(J)^{-1} w(x)dx \\ &\leq \sum_{J \in B_1} \int_J w(J)^{-1} w(x)dx \\ &= \sharp B_1. \end{aligned} \tag{11}$$

Following (7)-(11), we have $c\|g_1\|_{L^{p,s}(w)} \leq (\sharp A_1)^{1/p} + C_0^{1/2}(2^n - 1)^{1/2} (\sharp E_1)^{1/p} (\sharp B_1)^{1/p}$. Hence there exists a constant $0 < C_1 < \infty$ independent of g_1, A_1, E_1 and B_1 such that

$$\|g_1\|_{L^{p,s}(w)} \leq C_1 \left\{ \sharp A_1 + \sharp(E_1 \times B_1) \right\}^{1/p}. \tag{12}$$

On the other hand, applying (6) to $f = g_2$ and using $1 \leq \sharp E_2 \leq \sharp E = 2^n - 1$, we have

$$C'\|g_2\|_{L^{p,s}(w)} \geq (\sharp A_2)^{1/p} + \left\| \left(\sum_{e \in E_2} \sum_{J \in B_2} \left| w(J)^{-1/p} \chi_J \right|^2 \right)^{1/2} \right\|_{L^p(w)}$$

$$
\begin{aligned}
&= (\sharp A_2)^{1/p} + (\sharp E_2)^{1/2} \cdot \left\{ \int_{\bigcup\limits_{J' \in B_2} J'} \left(\sum_{J \in B_2} w(J)^{-2/p} \chi_J(y) \right)^{p/2} w(y)dy \right\}^{1/p} \\
&\geq (\sharp A_2)^{1/p} + (2^n - 1)^{-1/p} (\sharp E_2)^{1/p} \\
&\qquad \times \left\{ \int_{\bigcup\limits_{J' \in B_2} J'} \left(\sum_{J \in B_2} w(J)^{-2/p} \chi_J(y) \right)^{p/2} w(y)dy \right\}^{1/p}. \qquad (13)
\end{aligned}
$$

For each $y \in \bigcup\limits_{J \in B_2} J$, $J_2(y)$ denotes the minimal dyadic cube in B_2 with regard to the inclusion relation that contains y. Then we have

$$
\left(\sum_{J \in B_2} w(J)^{-2/p} \chi_J(y) \right)^{p/2} \geq w(J_2(y))^{-1}. \qquad (14)
$$

Now using the same argument as (8)-(9), replacing " B_1, $-2/p$ and $J_1(x)$ " by " B_2, -1 and $J_2(y)$ " respectively, we get

$$
\sum_{J \in B_2} w(J)^{-1} \chi_J(y) \leq C_0' w\left(J_2(y)\right)^{-1}, \qquad (15)
$$

where C_0' is a constant depended only on p and d. Following (13)-(15), we obtain

$$
\begin{aligned}
&C' \|g_2\|_{L^{p,s}(w)} \\
&\geq (\sharp A_2)^{1/p} + (2^n - 1)^{-1/p} (\sharp E_2)^{1/p} \cdot \left(\int_{\bigcup\limits_{J' \in B_2} J'} C_0'^{-1} \sum_{J \in B_2} w(J)^{-1} \chi_J(y) w(y)dy \right)^{1/p} \\
&= (\sharp A_2)^{1/p} + (2^n - 1)^{-1/p} (\sharp E_2)^{1/p} \cdot \left(C_0'^{-1} \sum_{J \in B_2} w(J)^{-1} \int_J w(y)dy \right)^{1/p} \\
&= (\sharp A_2)^{1/p} + C_0'^{-1/p} (2^n - 1)^{-1/p} (\sharp E_2)^{1/p} (\sharp B_2)^{1/p}.
\end{aligned}
$$

Namely there exists a constant $0 < C_2 < \infty$ independent of g_2, A_2, E_2 and B_2 such that

$$
C_2 \|g_2\|_{L^{p,s}(w)} \geq \{\sharp A_2 + \sharp(E_2 \times B_2)\}^{1/p}. \qquad (16)
$$

Following $\sharp A_1 + \sharp(E_1 \times B_1) = \sharp A_2 + \sharp(E_2 \times B_2)$, (12) and (16), we get

$$
\|g_1\|_{L^{p,s}(w)} \leq C_1 C_2 \|g_2\|_{L^{p,s}(w)}.
$$

Consequently we have proved that the sequence $\{\tilde{\varphi}_{0,k}\}_{k\in\mathbb{Z}^n} \cup \{\widetilde{\psi^e}_{j,k} : e \in E,\ j \geq 0,\ k \in \mathbb{Z}^n\}$ is democratic. □

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