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Invariant Subspaces Of Toeplitz Operators And Uniform Algebras

By

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Abstract Let $T_\phi$ be a Toeplitz operator on the one variable Hardy space $H^2$. We show that if $T_\phi$ has a nontrivial invariant subspace in the set of invariant subspaces of $T_z$ then $\phi$ belongs to $H^\infty$. In fact, we also study such a problem for the several variables Hardy space $H^2$. 
§1. Introduction

Let $X$ be a compact Hausdorff space, let $C(X)$ be the algebra of complex-valued continuous functions on $X$, and let $A$ be a uniform algebra on $X$. A probability measure $m$ (on $X$) denotes a representing measure for some nonzero complex homomorphism. The abstract Hardy space $H^p = H^p(m)$, $1 \leq p \leq \infty$, determined by $A$ is defined to be the closure of $A$ in $L^p = L^p(m)$ when $p$ is finite and to be the weak* closure of $A$ in $L^\infty = L^\infty(m)$ when $p = \infty$.

Let $P$ be the orthogonal projection from $L^2$ onto $H^2$. For $\phi$ in $L^\infty$, put

$$T_\phi f = P(\phi f) \quad (f \in H^2)$$

and then $T_\phi$ is called a Toeplitz operator. In this paper, we are interested in invariant subspaces of Toeplitz operators. Put $A = \{T_\phi ; \phi \in H^\infty\}$ and $A^* = \{T^*_\phi ; \phi \in H^\infty\}$. Lat $T_\phi$ denotes the set of all invariant subspaces of $T_\phi$, Lat $A = \cap \{\text{Lat } T_\phi ; \phi \in H^\infty\}$, and Lat $A^* = \cap \{\text{Lat } T^*_\phi ; \phi \in H^\infty\}$. We don’t know whether arbitrary $T_\phi$ has a nontrivial invariant subspace. When $\phi$ is in $H^\infty$ and $H^\infty$ has a nonconstant unimodular function $q$, $T_\phi$ has a nontrivial invariant subspace $M = qH^2$. Hence $\text{Lat } T_\phi \neq \{(0), H^2\}$.

Let $K$ be the orthogonal complement of $H^2$ in $L^2$. Then $L^2 = H^2 \oplus K$. $I(H^\infty)$ denotes the set of all unimodular functions in $H^\infty$. A function in $I(H^\infty)$ is called an inner function. For a subset $Y$ in $L^\infty, Y^\perp$ denotes $\{g \in L^1 : \int g\bar{f}dm = 0 \ (f \in Y)\}$.

In this paper we study the following four natural questions :

**Question 1.** If Lat $T_\phi \supseteq$ Lat $A$ then does $T_\phi$ belong to $A$?

**Question 2.** Suppose that $H^\infty$ is a weak* closed maximal algebra in $L^\infty$. If Lat $T_\phi \subsetneq$ Lat $A$ then is Lat $T_\phi = \{(0), H^2\}$?

**Question 3.** Is Lat $A^* \cap$ Lat $A = \{(0), H^2\}$?

**Question 4.** Can we describe Lat $T_\phi \cap$ Lat $A$ or equivalently Lat $T_\phi \cap$ Lat $A^*$?

In this paper, we will answer these four questions positively when $A$ is the disc algebra. In fact, for Question 1 we can do it for more general uniform algebras. However for Question 2 we could not answer even for simple uniform algebras. Question 3 can be answered for almost all uniform algebras.

In this paper $H^p(D^n)$ denotes the Hardy space on the polydisc $D^n$ and $H^p(\Omega)$ denotes the Hardy space on a finitely connected domain $\Omega$. $L^2(D)$ denotes the Bergman space on $D$ and put $N^2 = L^2(D) \ominus \{L^2(D) \oplus \bar{z}L^2_a(D)\}$. $H^p_0$ denotes the set of $\{f \in H^p ; \int f dm = 0\}$. $H^p(\Gamma)$ denotes the usual Hardy space on the dual group $\Gamma$ where $\Gamma$ is an ordered subgroup of the reals.
§2. Lat $A \subseteq \text{Lat } T_{\phi}$

In this section we study Question 1. Theorem 1 shows that Question 1 can be answered positively for very general uniform algebras.

Lemma 1. Let $M$ be a closed subspace of $H^2$. $M \in \text{Lat } T_{\phi}$ if and only if $\phi M \subseteq M \oplus K$.

Proof. By definition of a Toeplitz operator, this is clear.

Lemma 2. If $\phi$ is a function in $L^\infty$ and Lat $A \subseteq \text{Lat } T_{\phi}$ then $\phi = \phi_0 + \bar{k}_0$ where $\phi_0 \in H^2$ and $\bar{k}_0 \in \cap \{q\bar{K} ; q \in I(H^\infty)\}$.

Proof. Since $L^2 = H^2 \oplus \bar{K}$, there exist $h \in H^2$ and $k \in \bar{K}$ such that $\phi = h + \bar{k}$.

If $q \in I(H^\infty)$ then $qH^2 \in \text{Lat } A$ and so by Lemma 1 $\phi q = qh + \bar{q}k \in qH^2 + \bar{K}$. Since $T_{\phi q} \subseteq qH^2$ and $\phi h \subseteq qH^2$, $P(q\bar{K}) \subseteq qH^2$. Hence $q\bar{k} = q\ell + \bar{q}t$ where $\ell \in H^2$ and $t \in \bar{K}$. Therefore $\bar{k} = \ell + \bar{q}t$ and $\ell = \bar{k} - q\bar{t} \in H^2 \cap \bar{K} = \langle 0 \rangle$. Hence $\ell = 0$ and $\bar{k} = q\bar{t}$. This implies that $k$ belongs to $q\bar{K}$ for any $q \in I(H^\infty)$.

Theorem 1. Suppose that $\cap \{q\bar{K} ; q \in I(H^\infty)\} = \{0\}$. If $\phi$ is a function in $L^\infty$ and Lat $A \subseteq \text{Lat } T_{\phi}$ then $\phi$ belongs to $H^\infty$.

Proof. Lemmas 1 and 2 imply the theorem trivially.

Corollary 1. Suppose that $H^2 = H^2(T^N)$. If $\phi$ is a function in $L^\infty$ and Lat $A \subseteq \text{Lat } T_{\phi}$ then $\phi$ belongs to $H^\infty$.

Proof. $\bar{K}$ is an invariant subspace under multiplications by the coordinates functions $z_1, \ldots, z_n$. $\cap \{z_1^{\ell_1} \cdots z_n^{\ell_n} K ; (\ell_1, \ldots, \ell_n) \geq (0, \ldots, 0)\}$ is a reducing subspace and so $\cap z_1^{\ell_1} \cdots z_n^{\ell_n} K = \chi E L^2$ for some characteristic function $\chi E$. Since $\chi E L^2$ is orthogonal to $\bar{H}^2$, $\chi E = 0$ and so $\langle 0 \rangle = \cap z_1^{\ell_1} \cdots z_n^{\ell_n} \bar{K} = \cap \{q\bar{K} ; q \in I(H^\infty)\}$.

Corollary 2. Suppose that $H^2 = H^2(\Omega)$. If $\phi$ is a function in $L^\infty$ and Lat $A \subseteq \text{Lat } T_{\phi}$ then $\phi$ belongs to $H^\infty$.

Proof. Let $Z$ be the Ahlfors function for $\Omega$ then $|Z| = 1$ on $\partial \Omega = X$ (see [3]). $\cap_{n=0}^{\infty} \bar{Z}^n \bar{K}$ is invariant under the multiplications by $Z$ and $\bar{H}^\infty$. Since $H^\infty$ is a weak* maximal subalgebra of $L^\infty$, $\cap_{n=0}^{\infty} \bar{Z}^n \bar{K} = \chi E L^2$. Since $\chi E L^2$ is orthogonal to $H^2$, $\chi E = 0$ and so $\cap \{q\bar{K} ; q \in I(H^\infty)\} = \{0\}$.

Corollary 3. Let $A$ be a Dirichlet algebra (see [4]). If $\phi$ is a function in $L^\infty$ and Lat $A \subseteq \text{Lat } T_{\phi}$ then $\phi$ belongs to $H^\infty$. 

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Proof. Since $H^\infty$ is a uniform algebra which has the annulus property ([2],[6]) on a totally disconnected space, by [2, Theorem 1] the set of quotients of inner functions is norm dense in the set of unimodular functions in $L^\infty$. In this situation, $\bar{K} = H^2_0$ and $Y = \cap\{\bar{q}K; q \in I(H^\infty)\} \subset H^2$. $\bar{q}Y = Y$ for any $q$ in $I(H^\infty)$ and so $\bar{q}_1\bar{q}_2Y \subseteq Y$ for any $q_1, q_2$ in $I(H^\infty)$. Hence $\phi Y \subseteq Y$ for any unimodular function $\phi$ in $L^\infty$. Hence $Y = \chi_E L^2$ for the characteristic function $\chi_E$ for some set $E$. Since $Y \subset H^2$, $Y$ must be $\{0\}$.

**Proposition 1.** Suppose that $H^2 = L^2_a(D)$, $\phi$ is a function in $L^\infty$ and $\text{Lat } A \subseteq \text{Lat } T_\phi$. Then the following are valid.

1. $\phi$ belongs to $L^2_a(D) + N^2$.
2. If $\phi = f + \ell$ where $f \in H^\infty$ and $\ell \in N^2$ then $\text{Lat } T_\ell \supseteq \text{Lat } A$.

Proof. (1) Since $zL^2_a \subseteq \text{Lat } T_\phi$ by hypothesis, $\mathcal{C} \subseteq \text{Lat } T_\phi^* = \text{Lat } \bar{T}_\phi$ and so $\bar{\phi} = \bar{c} + \bar{\ell}$ where $c \in \mathcal{C}$ and $k \in zL^2_a(D) + N^2$. Hence $\phi \in L^2_a(D) + N^2$. (2) If $\phi = f + \ell$ and $M \in \text{Lat } A$ then $\phi M \subseteq M + \bar{K}$. Hence $(f + \ell)g = fg + \ell g \in M + \bar{K}$ for any $g \in M$. Since $\bar{g} \in M$, $\ell g \in M + \bar{K}$ for any $g \in M$ and so $\ell M \subseteq M + \bar{K}$. Thus $M \subseteq \text{Lat } T_\ell$.

A bounded operator $B$ is called reflexive if whenever $C$ is a bounded operator and $\text{Lat } B \subseteq \text{Lat } C$ then $C$ belongs to the closed algebra (in weak operator topology) generated by $B$. When $B$ is subnormal, it is known that $B$ is reflexive [7]. Hence if $f$ is a nonzero function in $H^\infty$ and $\text{Lat } T_\phi \supseteq \text{Lat } T_f$ then $T_\phi$ belongs to the closed algebra generated by $T_f$. Hence $T_\phi$ belongs to $A$. Usually $A \not\subseteq \text{Lat } T_f$ and so this does not answer Question 1. However if there exists a function $f$ in $H^\infty$ such that $\text{Lat } T_f = \text{Lat } A$ then the above result about subnormal operators answers Question 1. Hence when $H^2 = H^2(\mathbb{T})$, if $\text{Lat } T_\phi \supseteq \text{Lat } A$ then $T_\phi$ belongs to $A$ because $\text{Lat } T_2 = \text{Lat } A$. Therefore Corollary 1 is not new for $N = 1$. Similarly Question 1 can be answered for $H^2 = L^2_a(D)$. Hence Proposition 1 is a very weak result.

§3. $\text{Lat } T_\phi \not\subseteq \text{Lat } A$

In this section we study Question 2. Theorem 2 shows that Question 2 can be answered positively for the disc algebra. In fact, it gives a few results for more general uniform algebras about Question 2.

**Lemma 3.** Let $Q$ be a function in $I(H^\infty)$. Then $\bar{K} = \bigoplus_{n=0}^{\infty} (\bar{K} \ominus Q\bar{K})Q^n \ominus \bigcap_{n=0}^{\infty} Q^n \bar{K}$.

Proof. Since $|Q| = 1$ a.e. and $\bar{Q}\bar{K} \subseteq \bar{K}$, $\bar{Q}$ is an isometry on $\bar{K}$. Hence this is well known and called a Wold decomposition.
Theorem 2. Suppose that \( \text{Lat } T_\phi \subset \text{Lat } A \). If \( M \in \text{Lat } T_\phi \) and \( \{ \bar{Q}^n \bar{K} : Q \in \mathcal{I} \} = \{ 0 \} \) for some subset \( \mathcal{I} \) in \( I(H^\infty) \) then there exists a nonconstant \( Q \) in \( \mathcal{I} \) such that \( M \cap (H^2 \ominus QH^2) \neq \{ 0 \} \) or \( \phi M \subseteq M \).

Proof. If \( M \in \text{Lat } T_\phi \) then by Lemma 1 there exist \( f \in M, g \in M \) and \( k \in K \) such that \( \phi f = g + \bar{k} \). If \( \phi M \not\subseteq M \) then we may assume that \( k \neq 0 \). For any fixed \( Q \in \mathcal{I} \), by Lemma 3 \( \bar{K} = \left\{ \sum_{n=0}^{\infty} (\bar{K} \ominus \bar{Q} \bar{K})\bar{Q}^n \right\} \cap \{ \bar{Q}^n \bar{K} \} \) and so

\[
\bar{k} = \sum_{n=0}^{\infty} k_n \bar{Q}^n + k_\infty
\]

where \( k_n \in \bar{K} \ominus \bar{Q} \bar{K} \) \((n = 0, 1, 2, \cdots)\) and \( k_\infty \in \bigcap_{n=0}^{\infty} \bar{Q}^n \bar{K} \). Then \( Q\bar{k} = Qk_0 + \sum_{n=1}^{\infty} k_n \bar{Q}^{n-1} + Qk_\infty \) and by Lemma 1 \( Q\bar{k} \) belongs to \( M + \bar{K} \) because \( \phi f = g + \bar{k} \) and \( QM \subseteq M \).

Suppose that there does not exist a nonconstant function \( Q \) in \( \mathcal{I} \) such that \( M \cap (H^2 \ominus QH^2) \neq \{ 0 \} \). Then we will get a contradiction. By what was proved above, \( Qk_0 \) belongs to \( M \cap (H^2 \ominus QH^2) \) \( \neq \{ 0 \} \). Hence \( k_0 \equiv 0 \). Next we consider \( Q^2 \bar{k} \) and then \( k_1 \equiv 0 \) follows. Proceeding similarly we can show that \( \bar{k} = k_\infty \). By hypothesis, this implies that \( k \equiv 0 \) because \( Q \) is arbitrary in \( \mathcal{I} \). This contradiction implies that there exists \( Q \in I \) such that \( M \cap (H^2 \ominus QH^2) \neq \{ 0 \} \).

Corollary 4. Suppose that \( H^2 = H^2(T^N) \), \( \phi \) is a function in \( L^\infty \) and \( \text{Lat } T_\phi \subset \text{Lat } A \). If \( M \in \text{Lat } T_\phi \) and \( M \neq \{ 0 \} \) then \( M \) contains a nonzero function which is \((N-1)\)-variable. Hence if \( N = 1 \) then \( M = H^2 \).

Proof. It is known that if \( \phi M \subseteq M \) then \( \phi \in H^\infty \). Hence we may assume that \( \phi M \not\subseteq M \). Put \( \mathcal{I} = \{ z_1, \cdots, z_N \} \) then \( \mathcal{I} \) satisfies the condition of Theorem 2. By Theorem 2, there exists \( z_j \) such that \( 1 \leq j \leq N \) and \( (H^2 \ominus z_j H^2) \cap M \neq \{ 0 \} \). Since \( H^2 \ominus z_j H^2 = H^2(z_j', T^{N-1}) \) where \( z = (z_j, z'_j) \), \( M \) contains a nonzero \((N-1)\)-variable function.

Corollary 5. Suppose that \( H^2 = H^2(\Omega) \), \( \text{Lat } T_\phi \subset \text{Lat } A \) and \( Z \) is the Alfors function for \( \Omega \) (see [3]). If \( M \in \text{Lat } T_\phi \) and \( M \neq \{ 0 \} \) then \( M \cap (H^2 \ominus ZH^2) \neq \{ 0 \} \).

Proof. Put \( \mathcal{I} = \{ Z \} \) then \( \mathcal{I} \) satisfies the condition of Theorem 2. It is known that if \( \phi M \subseteq M \) then \( \phi \in H^\infty \). Hence we may assume that \( \phi M \not\subseteq M \).

Proposition 2. If \( T_\phi \) is subnormal and \( \text{Lat } T_\phi \subseteq \text{Lat } A \) then \( T_\phi \) commutes with \( A \) and so \( T_\phi f = P(\phi_0 f) \) \((f \in H^\infty)\) for some \( \phi_0 \) in \( H^2 \). If \( A \) is a uniform algebra which approximates in modulus on \( X \) then \( \phi \) belongs to \( H^2 \cap L^\infty \).

Proof. If \( T_\phi \) is subnormal and \( \text{Lat } T_\phi \subseteq \text{Lat } A \) then it is known [7] that \( A \) is contained in the closed algebra generated by \( T_\phi \). Hence \( T_\phi \) commutes with \( A \). Let \( \phi_0 = \)
$T_\phi f = T_\phi T_f 1 = T_f T_\phi 1 = P(\phi_0 f)$ for $f \in H^\infty$. Since $\|\phi_0 f\|_2 \leq \|T_\phi\| \|f\|_2$ ($f \in H^\infty$),

$$\left| \int_X \phi_0 f g dm \right| \leq \|\phi\|_\infty \|f\|_2 \|g\|_2 \quad (f, g \in H^\infty).$$

Hence

$$\left| \int_X \phi_0 | f |^2 dm \right| \leq \|\phi\|_\infty \|f^2\|_1.$$

Since $A$ approximates in modulus on $X$, $\phi_0$ belongs to $H^2 \cap L^\infty$. It is easy to see that $\phi = \phi_0$.

**Corollary 6.** Suppose that $H^2 = H^2(T^N)$ or $H^2 = H^2(\Omega)$. If $T_\phi$ is subnormal then $\text{Lat } T_\phi \subset \text{Lat } A$ or $\phi$ belongs to $H^\infty$.

Proof. A uniform algebra $A$ approximates in modulus on $X$, that is, for every positive continuous function $g$ on $X$ and $\varepsilon > 0$, there is an $f$ in $A$ with $|g - |f|| < \varepsilon$ if the set of unimodular elements of $A$ separates points of $X$ (see [6, Lemma 4.12]). Since the coordinate functions $z_1, \cdots, z_n$ separate $T^N$, the polydisc algebra approximates in modulus on $T^N$. If $T_\phi$ is subnormal on $H^2(T^N)$ and $\text{Lat } T_\phi \subset \text{Lat } A$ then by Proposition 2 $\phi$ belongs to $H^2(T^N) \cap L^\infty = H^\infty(T^N)$. If $A = H^\infty(\Omega)$ then by [3, Lemma 4.8] $I(H^\infty(\Omega))$ separates $X = \text{the maximal ideal space of } L^\infty(\partial D)$. Hence Corollary 6 for $H^2 = H^2(\Omega)$ follows from Proposition 2.

**Proposition 3.** If $\text{Lat } T_\phi \subset \text{Lat } A$, then $\text{Lat } T^*_\phi \cap \text{Lat } T_\phi \subset \text{Lat } A^* \cap \text{Lat } A$.

Proof. If $M \in \text{Lat } T^*_\phi$ then $M^\perp \in \text{Lat } T_\phi$ and so $M^\perp \in \text{Lat } A$ because $\text{Lat } T_\phi \subset \text{Lat } A$. Hence $M \in \text{Lat } A^*$ and so $\text{Lat } T^*_\phi \subset \text{Lat } A^*$.

By Proposition 3, when $\text{Lat } A^* \cap \text{Lat } A = \{(0), H^2\}$, if $\text{Lat } T_\phi \subset \text{Lat } A$ then $T_\phi$ does not have a nontrivial reducing subspace. Hence if $T_\phi$ is normal then $\text{Lat } T_\phi \notin \text{Lat } A$. Therefore it is important to know that $\text{Lat } A^* \cap \text{Lat } A = \{(0), H^2\}$, that is, $A$ is irreducible.

§4. $A^* \cap \text{Lat } A$

In this section we study Question 3. Theorem 3 shows that Question 3 can be answered positively for usual uniform algebras. Recall $A^* = \{T^*_\phi ; \phi \in H^\infty\}$.

**Theorem 3.** If $M \in \text{Lat } A^* \cap \text{Lat } A$ then $M \subset \chi_E L^2 \subset M + \hat{K}$ where $E = \cup \{\text{supp } f ; f \in M\}$. Hence if $E = X$ then $M = H^2$.

Proof. If $\phi \in L^\infty$ then by the Stone-Weierstrass theorem for any $\varepsilon > 0$ there exist $f_1, \cdots, f_n$ and $g_1, \cdots, g_n$ in $H^\infty$ such that $\| \phi - \sum_{j=1}^{n} f_j \bar{g}_j \|_\infty < \varepsilon$. Since $T_{f_j \bar{g}_j} M \subset M$ for
\[ j = 1, \ldots, n, \ T_{\phi} M \subset M. \] By Lemma 1 \( \phi M \subset M \oplus \bar{K}. \) Thus \( \chi_E L^2 \subset M \oplus \bar{K}. \) If \( E = X \) then \( L^2 = M \oplus \bar{K} \) and so \( M = H^2. \)

**Corollary 7.** Suppose that there does not exist a nonzero function in \( H^2 \) such that \( m(\{x \in X \ ; \ f(x) = 0\}) \neq 0. \) If \( M \in \text{Lat} \ A \cap \text{Lat} \ A^* \) then \( M = \langle 0 \rangle \) or \( H^2. \)

**§5. Lat \( T_{\phi} \cap \text{Lat} \ A \)**

In this section we study Question 4. We don’t know whether \( \text{Lat} \ T_{\phi} \neq \{\langle 0 \rangle, H^2\}. \) However we show that \( \text{Lat} \ T_{\phi} \cap \text{Lat} \ A = \{\langle 0 \rangle, H^2\} \) if \( \phi \notin H^\infty \) and \( H^2 = H^2(T). \) For any \( M \) in \( \text{Lat} \ T_{\phi}, \) put

\[ K_M = \{k \in K \ ; \ \bar{k} = \phi f - g \text{ for some } f \text{ and } g \in M\} \]

then \( K_M \subset K \) and \( \phi M \subset M + \bar{K}_M \) (see Lemma 1).

**Theorem 4.** If \( M \in \text{Lat} \ T_{\phi} \cap \text{Lat} \ A \) then \( K_M \times (H^2 \ominus M) \subseteq (H^\infty)^\perp \) and \( T_k^* (H^\infty) \subseteq M \) for any \( k \) in \( K_M. \)

Proof. By the remark above, if \( M \in \text{Lat} \ T_{\phi} \cap \text{Lat} \ A \) then \( \phi M \subset M + \bar{K}_M. \)

If \( k \in K_M \) then by its definition there exist \( f \) and \( g \) such that \( \phi f = g + \bar{k}. \) For any \( \ell \in H^\infty, \phi f \ell = g \ell + \bar{k} \ell \in M + \bar{K}_M \) and so \( P(\bar{k} \ell) \in M. \) Since

\[ \bar{k} \ell = P(\bar{k} \ell) + (I - P)(\bar{k} \ell) \in M + \bar{K}_M, \]

if \( s \in H^2 \ominus M \) then \( \langle \bar{k} \ell, s \rangle = \langle P(\bar{k} \ell), s \rangle = 0. \) Hence \( ks \) belongs to \( (H^\infty)^\perp \) and so \( K_M \times (H^2 \ominus M) \subseteq (H^\infty)^\perp. \) The above proof implies that \( T_k^* (H^\infty) \subseteq M. \)

**Corollary 8.** Suppose that \( H^2 = H^2(\Omega), \ \mathcal{C}\backslash \Omega \) has \( n \) components and \( \phi \notin H^\infty. \) If \( M \in \text{Lat} \ T_{\phi} \cap \text{Lat} \ A \) then \( \dim(H^2 \ominus M) \leq n. \)

Proof. By Theorem 4

\[ K_M \times (H^2 \ominus M) \subseteq (H^\infty)^\perp \cap (H^\infty)^\perp = (H^\infty + \bar{H}^\infty)^\perp \cap L^1 \]

and \( \dim(H^\infty + \bar{H}^\infty)^\perp \cap L^1 = n \) because \( \mathcal{C}\backslash \Omega \) has \( n \) components. If \( K_M = \langle 0 \rangle \) then \( \phi M \subset M. \) It is known [4] that \( L^\infty \) is generated by \( \phi \) and \( H^\infty \) in the weak* topology. Hence \( M \in \text{Lat} \ A \cap \text{Lat} \ A^* = \{\langle 0 \rangle, H^2\} \) by Corollary 7 and so \( M = H^2. \) It is clear that if \( K_M \neq \langle 0 \rangle \) then \( \dim(H^2 \ominus M) \leq n. \)

**Corollary 9.** If \( H^2 = H^2(T) \) and \( \phi \notin H^\infty \) then \( \text{Lat} \ T_{\phi} \cap \text{Lat} \ A = \{\langle 0 \rangle, H^2\}. \)

Proof. When \( \Omega \) is the open unit disc, \( H^2(\Omega) = H^2(T) \) and so by Corollary 8 \( \text{Lat} \ T_{\phi} \cap \text{Lat} \ A = \{\langle 0 \rangle, H^2\}. \)
Corollary 10. Let $A$ be a Dirichlet algebra. If $\phi \notin H^\infty$ then $\text{Lat}_T \cap \text{Lat}_A = \{\langle 0 \rangle, H^2\}$.

Proof. It is known that $(\overline{H^\infty})^\perp \cap (H^\infty)^\perp = \langle 0 \rangle$. The corollary is a result of Theorem 4.

In general, it seems to be difficult to describe $\text{Lat}_T \cap \text{Lat}_A$. When $H^2 = H^2(\Omega)$ and $\bar{\phi} \in H^\infty$, $\text{Lat}_T \cap \text{Lat}_A = \{\langle 0 \rangle, H^2\}$ by Corollary 8. In fact, if $M \in \text{Lat}_T \cap \text{Lat}_A$ then $\bar{\phi}(H^2 \ominus M) \subseteq H^2 \ominus M$. Since $\dim(H^2 \ominus M) < \infty$ by Corollary 8, $M$ must be equal to $H^2$. When $H^2 = H^2(T^2)$ and $\phi = \bar{z}$, $\text{Lat}_T \cap \text{Lat}_A = \{\langle 0 \rangle, qH^2(w, T); q = q(w) \text{ is a one variable inner function}\}$ where $z$ and $w$ are the independent variables on $T^2$. In fact, if $M \in \text{Lat}_T \cap \text{Lat}_A$ then $T^{*}_z M_1$ is orthogonal to $M$ where $M_1 = M \ominus zM$. Since $T^{*}_z M_1 \subset M$, $T^{*}_z M_1 = \langle 0 \rangle$ and so $M_1 \subset H^2(w, T)$. Corollary 10 shows that $\text{Lat}_A^{*} \cap \text{Lat}_A = \{\langle 0 \rangle, H^2\}$ if $A$ is a Dirichlet algebra.

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