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Invariant Subspaces Of Toeplitz Operators And Uniform Algebras

By

Takahiko Nakazi

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Abstract Let $T_\phi$ be a Toeplitz operator on the one variable Hardy space $H^2$. We show that if $T_\phi$ has a nontrivial invariant subspace in the set of invariant subspaces of $T_z$ then $\phi$ belongs to $H^\infty$. In fact, we also study such a problem for the several variables Hardy space $H^2$. 


§1. Introduction

Let $X$ be a compact Hausdorff space, let $C(X)$ be the algebra of complex-valued continuous functions on $X$, and let $A$ be a uniform algebra on $X$. A probability measure $m$ (on $X$) denotes a representing measure for some nonzero complex homomorphism. The abstract Hardy space $H^p = H^p(m)$, $1 \leq p \leq \infty$, determined by $A$ is defined to be the closure of $A$ in $L^p = L^p(m)$ when $p$ is finite and to be the weak$^*$ closure of $A$ in $L^\infty = L^\infty(m)$ when $p = \infty$.

Let $P$ be the orthogonal projection from $L^2$ onto $H^2$. For $\phi$ in $L^\infty$, put

$$T_\phi f = P(\phi f) \quad (f \in H^2)$$

and then $T_\phi$ is called a Toeplitz operator. In this paper, we are interested in invariant subspaces of Toeplitz operators. Put $A = \{T_\phi \; ; \; \phi \in H^\infty\}$ and $A^* = \{T_\phi^* \; ; \; \phi \in H^\infty\}$. Let $\text{Lat} T_\phi$ denote the set of all invariant subspaces of $T_\phi$. Let $A = \cap \{\text{Lat} T_\phi \; ; \; \phi \in H^\infty\}$ and $A^* = \cap \{\text{Lat} T_\phi^* ; \phi \in H^\infty\}$. We don't know whether arbitrary $T_\phi$ has a nontrivial invariant subspace. When $\phi$ is in $H^\infty$ and $H^\infty$ has a nonconstant unimodular function $q$, $T_\phi$ has a nontrivial invariant subspace $M = qH^2$. Hence $\text{Lat} T_\phi \neq \{(0), H^2\}$.

Let $K$ be the orthogonal complement of $H^2$ in $L^2$. Then $L^2 = H^2 \oplus K$. $I(H^\infty)$ denotes the set of all unimodular functions in $H^\infty$. A function in $I(H^\infty)$ is called an inner function. For a subset $Y$ in $L^\infty$, $Y^\perp$ denotes $\{g \in L^1 : \int g\bar{f}dm = 0 \quad (f \in Y)\}$.

In this paper we study the following four natural questions:

**Question 1.** If $\text{Lat} T_\phi \supseteq \text{Lat} A$ then does $T_\phi$ belong to $A$?

**Question 2.** Suppose that $H^\infty$ is a weak$^*$ closed maximal algebra in $L^\infty$. If $\text{Lat} T_\phi \subset \text{Lat} A$ then is $\text{Lat} T_\phi = \{(0), H^2\}$?

**Question 3.** Is $\text{Lat} A^* \cap \text{Lat} A = \{(0), H^2\}$?

**Question 4.** Can we describe $\text{Lat} T_\phi \cap \text{Lat} A$ or equivalently $\text{Lat} T_\phi \cap \text{Lat} A^*$?

In this paper, we will answer these four questions positively when $A$ is the disc algebra. In fact, for Question 1 we can do it for more general uniform algebras. However for Question 2 we could not answer even for simple uniform algebras. Question 3 can be answered for almost all uniform algebras.

In this paper $H^p(D^n)$ denotes the Hardy space on the polydisc $D^n$ and $H^p(\Omega)$ denotes the Hardy space on a finitely connected domain $\Omega$. $L^2_\alpha(D)$ denotes the Bergman space on $D$ and put $N^2 = L^2(D) \ominus \{L^2_\alpha(D) \oplus \bar{z}L^2_\alpha(D)\}$. $H^p_\alpha$ denotes the set of $\{f \in H^p ; \int f dm = 0\}$. $H^p(\Gamma)$ denotes the usual Hardy space on the dual group $\hat{\Gamma}$ where $\Gamma$ is an ordered subgroup of the reals.
§ 2. \textit{Lat }$\mathcal{A} \subseteq \text{Lat } T_\phi$

In this section we study Question 1. Theorem 1 shows that Question 1 can be answered positively for very general uniform algebras.

\textbf{Lemma 1.} Let $M$ be a closed subspace of $H^2$. $M \in \text{Lat } T_\phi$ if and only if $\phi M \subseteq M \oplus \bar{K}$.

Proof. By definition of a Toeplitz operator, this is clear.

\textbf{Lemma 2.} If $\phi$ is a function in $L^\infty$ and Lat $\mathcal{A} \subseteq \text{Lat } T_\phi$ then $\phi = \phi_0 + \bar{k}_0$ where $\phi_0 \in H^2$ and $\bar{k}_0 \in \mathcal{N}\{\bar{q}K \ ; \ q \in I(H^\infty)\}$.

Proof. Since $L^2 = H^2 \oplus \bar{K}$, there exist $h \in H^2$ and $k \in \bar{K}$ such that $\phi = h + \bar{k}$. If $q \in I(H^\infty)$ then $qH^2 \in \text{Lat } \mathcal{A}$ and so by Lemma 1 $\phi = qh + \bar{q}k \in qH^2 + \bar{K}$. Since $T_\phi \bar{q} \in qH^2$ and $qh \in qH^2$, $P(qk) \in qH^2$. Hence $qk = q\ell + t$ where $\ell \in H^2$ and $t \in \bar{K}$. Therefore $\bar{k} = \ell + \bar{q}t$ and $\ell = k - \bar{q}t \in H^2 \cap \bar{K} = \langle 0 \rangle$. Hence $\ell = 0$ and $\bar{k} = \bar{q}t$. This implies that $k$ belongs to $\bar{q}K$ for any $q \in I(H^\infty)$.

\textbf{Theorem 1.} Suppose that $\cap\{\bar{q}K \ ; \ q \in I(H^\infty)\} = \langle 0 \rangle$. If $\phi$ is a function in $L^\infty$ and Lat $\mathcal{A} \subseteq \text{Lat } T_\phi$ then $\phi$ belongs to $H^\infty$.

Proof. Lemmas 1 and 2 imply the theorem trivially.

\textbf{Corollary 1.} Suppose that $H^2 = H^2(T^N)$. If $\phi$ is a function in $L^\infty$ and Lat $\mathcal{A} \subseteq \text{Lat } T_\phi$ then $\phi$ belongs to $H^\infty$.

Proof. $\bar{K}$ is an invariant subspace under multiplications by the coordinates functions $z_1, \ldots, z_n$, $\cap \{z_1^{\ell_1} \cdots z_n^{\ell_n}K \ ; \ (\ell_1, \ldots, \ell_n) \geq (0, \ldots, 0)\}$ is a reducing subspace and so $\cap z_1^{\ell_1} \cdots z_n^{\ell_n}K = \chi_E L^2$ for some characteristic function $\chi_E$. Since $\chi_E L^2$ is orthogonal to $\bar{H}^2$, $\chi_E = 0$ and so $\{0\} = \cap z_1^{\ell_1} \cdots z_n^{\ell_n}K = \cap \{\bar{q}K \ ; \ q \in I(H^\infty)\}$.

\textbf{Corollary 2.} Suppose that $H^2 = H^2(\Omega)$. If $\phi$ is a function in $L^\infty$ and Lat $\mathcal{A} \subseteq \text{Lat } T_\phi$ then $\phi$ belongs to $H^\infty$.

Proof. Let $Z$ be the Ahlfors function for $\Omega$ then $|Z| = 1$ on $\partial \Omega = \overline{X}$ (see [3]). $\bigcap_{n=0}^\infty \bar{Z}^nK$ is invariant under the multiplications by $Z$ and $\bar{H}^\infty$. Since $H^\infty$ is a weak* maximal subalgebra of $L^\infty$, $\bigcap_{n=0}^\infty \bar{Z}^nK = \chi_\overline{E}L^2$. Since $\chi_\overline{E}L^2$ is orthogonal to $H^2$, $\chi_\overline{E} = 0$ and so $\cap \{\bar{q}K \ ; \ q \in I(H^\infty)\} = \{0\}$.

\textbf{Corollary 3.} Let $A$ be a Dirichlet algebra (see [4]). If $\phi$ is a function in $L^\infty$ and Lat $\mathcal{A} \subseteq \text{Lat } T_\phi$ then $\phi$ belongs to $H^\infty$. 

4
Proof. Since \( H^\infty \) is a uniform algebra which has the annulus property ([2],[6]) on a totally disconnected space, by [2, Theorem 1] the set of quotients of inner functions is norm dense in the set of unimodular functions in \( L^\infty \). In this situation, \( \bar{K} = \bar{H}^2 \) and \( Y = \cap \{ \bar{q}K \mid q \in I(H^\infty) \} \subset \bar{H}^2 \). \( \bar{q}Y = Y \) for any \( q \) in \( I(H^\infty) \) and so \( \bar{q}_1\bar{q}_2Y \subseteq Y \) for any \( q_1, q_2 \) in \( I(H^\infty) \). Hence \( \phi Y \subseteq Y \) for any unimodular function \( \phi \) in \( L^\infty \). Hence \( Y = \chi_E L^2 \) for the characteristic function \( \chi_E \) for some set \( E \). Since \( Y \subset \bar{H}^2 \), \( Y \) must be \( \{0\} \).

**Proposition 1.** Suppose that \( H^2 = L^2_\alpha(D) \), \( \phi \) is a function in \( L^\infty \) and \( \text{Lat } A \subseteq \text{Lat } T_\phi \). Then the following are valid.

1. \( \phi \) belongs to \( L^2_\alpha(D) + N^2 \).
2. If \( \phi = f + \ell \) where \( f \in H^\infty \) and \( \ell \in N^2 \) then \( \text{Lat } T_\ell \supseteq \text{Lat } A \).

Proof. (1) Since \( zL^2_\alpha \in \text{Lat } T_\phi \) by hypothesis, \( \mathcal{C} \in \text{Lat } T_\phi^* = \text{Lat } T_\phi \) and so \( \bar{\phi} = \bar{c} + \bar{k} \) where \( c \in \mathcal{C} \) and \( k \in zL^2_\alpha(D) + N^2 \). Hence \( \phi \in L^2_\alpha(D) + N^2 \). (2) If \( \phi = f + \ell \) and \( M \in \text{Lat } A \) then \( \phi M \subset M + \bar{K} \). Hence \( \bar{f} + \bar{\ell} g = \bar{f}g + \ell g \in M + \bar{K} \) for any \( g \in M \). Since \( \bar{f}g \in M \), \( \ell g \in M + \bar{K} \) for any \( g \in M \) and so \( \ell M \subset M + \bar{K} \). Thus \( M \in \text{Lat } T_\ell \).

A bounded operator \( B \) is called reflexive if whenever \( C \) is a bounded operator and \( \text{Lat } B \subseteq \text{Lat } C \) then \( C \) belongs to the closed algebra (in weak operator topology) generated by \( B \). When \( B \) is subnormal, it is known that \( B \) is reflexive [7]. Hence if \( f \) is a nonzero function in \( H^\infty \) and \( \text{Lat } T_\phi \supseteq \text{Lat } T_f \) then \( T_\phi \) belongs to the closed algebra generated by \( T_f \). Hence \( T_\phi \) belongs to \( A \). Usually \( \text{Lat } A \nsubseteq \text{Lat } T_f \) and so this does not answer Question 1. However if there exists a function \( f \) in \( H^\infty \) such that \( \text{Lat } T_f = \text{Lat } A \) then the above result about subnormal operators answers Question 1. Hence when \( H^2 = H^2(T) \), if \( \text{Lat } T_\phi \supseteq \text{Lat } A \) then \( T_\phi \) belongs to \( A \) because \( \text{Lat } T_z = \text{Lat } A \). Therefore Corollary 1 is not new for \( N = 1 \). Similarly Question 1 can be answered for \( H^2 = L^2_\alpha(D) \). Hence Proposition 1 is a very weak result.

§3. \( \text{Lat } T_\phi \subset \not\subset \text{Lat } A \)

In this section we study Question 2. Theorem 2 shows that Question 2 can be answered positively for the disc algebra. In fact, it gives a few results for more general uniform algebras about Question 2.

**Lemma 3.** Let \( Q \) be a function in \( I(H^\infty) \). Then \( \bar{K} = \sum_{n=0}^\infty (\bar{K} \ominus \bar{Q} \bar{K}) \bar{Q}^n \oplus \bigcap_{n=0}^\infty \bar{Q}^n \bar{K} \).

Proof. Since \( |Q| = 1 \) a.e. and \( \bar{Q} \bar{K} \subset \bar{K}, \bar{Q} \) is an isometry on \( \bar{K} \). Hence this is well known and called a Wold decomposition.
Theorem 2. Suppose that \( \text{Lat} \, T_\phi \subset \text{Lat} \, \mathcal{A} \). If \( M \in \text{Lat} \, T_\phi \) and \( \cap \{ \bar{Q}^n \mathcal{K} \mid Q \in \mathcal{I} \} = \{0\} \) for some subset \( \mathcal{I} \) in \( I(H^\infty) \) then there exists a nonconstant \( Q \) in \( \mathcal{I} \) such that \( M \cap (H^2 \ominus QH^2) \neq \langle 0 \rangle \) or \( \phi M \subseteq M \).

Proof. If \( M \in \text{Lat} \, T_\phi \) then by Lemma 1 there exist \( f \in M, g \in M \) and \( k \in K \) such that \( \phi f = g + \bar{k} \). If \( \phi M \not\subseteq M \) then we may assume that \( k \neq 0 \). For any fixed \( Q \in \mathcal{I} \), by Lemma 3 \( \bar{k} = \sum_{n=0}^\infty k_n \bar{Q}^n + k_\infty \)

where \( k_n \in \bar{K} \ominus Q\bar{K} \) (\( n = 0, 1, 2, \cdots \)) and \( k_\infty \in \bigcap_{n=0}^\infty \bar{Q}^n \bar{K} \). Then \( Q\bar{k} = Qk_0 + \sum_{n=1}^\infty k_n \bar{Q}^{n-1} + Qk_\infty \) and by Lemma 1 \( Q\bar{k} \) belongs to \( M + \bar{K} \) because \( \phi f = g + \bar{k} \) and \( QM \subseteq M \).

Suppose that there does not exist a nonconstant function \( Q \) in \( \mathcal{I} \) such that \( M \cap (H^2 \ominus QH^2) \neq \langle 0 \rangle \). Then we will get a contradiction. By what was proved above, \( Qk_0 \) belongs to \( M \cap (H^2 \ominus QH^2) = \{0\} \). Hence \( k_0 = 0 \). Next we consider \( Q^2 \bar{k} \) and then \( k_1 = 0 \) follows. Proceeding similarly we can show that \( \bar{k} = k_\infty \). By hypothesis, this implies that \( \bar{k} \equiv 0 \) because \( Q \) is arbitrary in \( \mathcal{I} \). This contradiction implies that there exists \( Q \in \mathcal{I} \) such that \( M \cap (H^2 \ominus QH^2) \neq \langle 0 \rangle \).

Corollary 4. Suppose that \( H^2 = H^2(T^N) \), \( \phi \) is a function in \( L^\infty \) and \( \text{Lat} \, T_\phi \subset \text{Lat} \, \mathcal{A} \). If \( M \in \text{Lat} \, T_\phi \) and \( M \neq \langle 0 \rangle \) then \( M \) contains a nonzero function which is \((N-1)\)-variable. Hence if \( N = 1 \) then \( M = H^2 \).

Proof. It is known that if \( \phi M \subseteq M \) then \( \phi \in H^\infty \). Hence we may assume that \( \phi M \not\subseteq M \). Put \( \mathcal{I} = \{ z_1, \cdots, z_N \} \) then \( \mathcal{I} \) satisfies the condition of Theorem 2. By Theorem 2, there exists \( z_j \) such that \( 1 \leq j \leq N \) and \( (H^2 \ominus z_j H^2) \cap M \neq \{0\} \). Since \( H^2 \ominus z_j H^2 = H^2(z'_j, T^{N-1}) \) where \( z = (z_j, z'_j) \), \( M \) contains a nonzero \((N-1)\)-variable function.

Corollary 5. Suppose that \( H^2 = H^2(\Omega) \), \( \text{Lat} \, T_\phi \subset \text{Lat} \, \mathcal{A} \) and \( Z \) is the Alfors function for \( \Omega \) (see [3]). If \( M \in \text{Lat} \, T_\phi \) and \( M \neq \langle 0 \rangle \) then \( M \cap (H^2 \ominus ZH^2) \neq \langle 0 \rangle \).

Proof. Put \( \mathcal{I} = \{ Z \} \) then \( \mathcal{I} \) satisfies the condition of Theorem 2. It is known that if \( \phi M \subseteq M \) then \( \phi \in H^\infty \). Hence we may assume that \( \phi M \not\subseteq M \).

Proposition 2. If \( T_\phi \) is subnormal and \( \text{Lat} \, T_\phi \subseteq \text{Lat} \, \mathcal{A} \) then \( T_\phi \) commutes with \( \mathcal{A} \) and so \( T_\phi f = P(\phi f) \) (\( f \in H^\infty \)) for some \( \phi_0 \) in \( H^2 \). If \( \mathcal{A} \) is a uniform algebra which approximates in modulus on \( X \) then \( \phi \) belongs to \( H^2 \cap L^\infty \).

Proof. If \( T_\phi \) is subnormal and \( \text{Lat} \, T_\phi \subseteq \text{Lat} \, \mathcal{A} \) then it is known [7] that \( \mathcal{A} \) is contained in the closed algebra generated by \( T_\phi \). Hence \( T_\phi \) commutes with \( \mathcal{A} \). Let \( \phi_0 = \)
$T_\phi f = T_\phi T_f 1 = T_f T_\phi 1 = P(\phi_0 f)$ for $f \in H^\infty$. Since $\|\phi_0 f\|_2 \leq \|T_\phi\| \|f\|_2$ ($f \in H^\infty$),

$$\left| \int_X \phi_0 f g dm \right| \leq \|\phi\|_\infty \|f\|_2 \|g\|_2 \quad (f, g \in H^\infty).$$

Hence

$$\left| \int_X \phi_0 |f|^2 \, dm \right| \leq \|\phi\|_\infty \|f^2\|_1.$$ 

Since $A$ approximates in modulus on $X$, $\phi_0$ belongs to $H^2 \cap L^\infty$. It is easy to see that $\phi = \phi_0$.

**Corollary 6.** Suppose that $H^2 = H^2(T^N)$ or $H^2 = H^2(\Omega)$. If $T_\phi$ is subnormal then $\text{Lat } T_\phi \subset \text{Lat } A$ or $\phi$ belongs to $H^\infty$.

Proof. A uniform algebra $A$ approximates in modulus on $X$, that is, for every positive continuous function $g$ on $X$ and $\varepsilon > 0$, there is an $f$ in $A$ with $|g - |f|| < \varepsilon$ if the set of unimodular elements of $A$ separates points of $X$ (see [6, Lemma 4.12]). Since the coordinate functions $z_1, \ldots, z_n$ separate $T^N$, the polydisc algebra approximates in modulus on $T^N$. If $T_\phi$ is subnormal on $H^2(T^N)$ and $\text{Lat } T_\phi \subset \text{Lat } A$ then by Proposition 2 $\phi$ belongs to $H^2(T^N) \cap L^\infty = H^\infty(T^N)$. If $A = H^\infty(\Omega)$ then by [3, Lemma 4.8] $I(H^\infty(\Omega))$ separates $X = \text{the maximal ideal space of } L^\infty(\partial D)$. Hence Corollary 6 for $H^2 = H^2(\Omega)$ follows from Proposition 2.

**Proposition 3.** If $\text{Lat } T_\phi \subset \text{Lat } A$, then $\text{Lat } T_\phi \cap \text{Lat } T_\phi \subset \text{Lat } A^* \cap \text{Lat } A$.

Proof. If $M \in \text{Lat } T_\phi^*$ then $M^\perp \in \text{Lat } T_\phi$ and so $M^\perp \in \text{Lat } A$ because $\text{Lat } T_\phi \subset \text{Lat } A$. Hence $M \in \text{Lat } A^*$ and so $\text{Lat } T_\phi^* \subset \text{Lat } A^*$.

By Proposition 3, when $\text{Lat } A^* \cap \text{Lat } A = \{(0), H^2\}$, if $\text{Lat } T_\phi \subset \text{Lat } A$ then $T_\phi$ does not have a nontrivial reducing subspace. Hence if $T_\phi$ is normal then $\text{Lat } T_\phi \not\subset \text{Lat } A$. Therefore it is important to know that $\text{Lat } A^* \cap \text{Lat } A = \{(0), H^2\}$, that is, $A$ is irreducible.

§4. $A^* \cap \text{Lat } A$

In this section we study Question 3. Theorem 3 shows that Question 3 can be answered positively for usual uniform algebras. Recall $A^* = \{T_\phi^* ; \phi \in H^\infty\}$.

**Theorem 3.** If $M \in \text{Lat } A^* \cap \text{Lat } A$ then $M \subset \chi_E L^2 \subset M + \tilde{K}$ where $E = \cup \{\text{supp } f ; f \in M\}$. Hence if $E = X$ then $M = H^2$.

Proof. If $\phi \in L^\infty$ then by the Stone-Weierstrass theorem for any $\varepsilon > 0$ there exist $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ in $H^\infty$ such that $\| \phi - \sum_{j=1}^n f_j g_j \|_\infty < \varepsilon$. Since $T_{f_j g_j} M \subset M$ for
\(j = 1, \cdots, n, T_\phi M \subset M\). By Lemma 1 \(\phi M \subset M \oplus \tilde{K}\). Thus \(\chi_EL^2 \subset M \oplus \tilde{K}\). If \(E = X\) then \(L^2 = M \oplus \tilde{K}\) and so \(M = H^2\).

\[\text{Corollary 7. Suppose that there does not exist a nonzero function in } H^2 \text{ such that } m(\{x \in X \mid f(x) = 0\}) \neq 0. \text{ If } M \in \text{Lat} \mathcal{A} \cap \text{Lat} \mathcal{A} \text{ then } M = \langle 0 \rangle \text{ or } H^2.\]

§5. \text{Lat} \(T_\phi \cap \text{Lat} \mathcal{A}\)

In this section we study Question 4. We don’t know whether \(\text{Lat} T_\phi \neq \{\langle 0 \rangle, H^2\}\). However we show that \(\text{Lat} T_\phi \cap \text{Lat} \mathcal{A} = \{\langle 0 \rangle, H^2\}\) if \(\phi \notin H^\infty\) and \(H^2 = H^2(T)\). For any \(M\) in \(\text{Lat} T_\phi\), put

\[K_M = \{k \in K \mid \bar{k} = \phi f - g \text{ for some } f \text{ and } g \in M\} \]

then \(K_M \subseteq K\) and \(\phi M \subset M + \bar{K}_M\) (see Lemma 1).

\[\text{Theorem 4. If } M \in \text{Lat} T_\phi \cap \text{Lat} \mathcal{A} \text{ then } K_M \times (H^2 \ominus M) \subseteq (H^\infty)^\perp \text{ and } T^*_k(H^\infty) \subseteq M \text{ for any } k \text{ in } K_M.\]

Proof. By the remark above, if \(M \in \text{Lat} T_\phi \cap \text{Lat} \mathcal{A}\) then \(\phi M \subset M + \bar{K}_M\). If \(k \in K_M\) then by its definition there exist \(f\) and \(g\) such that \(\phi f = g + \bar{k}\). For any \(\ell \in H^\infty\), \(\phi f \ell = g \ell + \bar{k} \ell \in M + \bar{K}_M\) and so \(P(\bar{k} \ell) \in M\). Since

\[\bar{k} \ell = P(\bar{k} \ell) + (I - P)(\bar{k} \ell) \in M + \bar{K}_M,\]

if \(s \in H^2 \ominus M\) then \(\langle \bar{k} \ell, s \rangle = \langle P(\bar{k} \ell), s \rangle = 0\). Hence \(ks\) belongs to \((H^\infty)^\perp\) and so \(K_M \times (H^2 \ominus M) \subseteq (H^\infty)^\perp\). The above proof implies that \(T^*_k(H^\infty) \subseteq M\).

\[\text{Corollary 8. Suppose that } H^2 = H^2(\Omega), \mathcal{C} \setminus \Omega \text{ has } n \text{ components and } \phi \notin H^\infty. \text{ If } M \in \text{Lat} T_\phi \cap \text{Lat} \mathcal{A}\text{ then } \dim(H^2 \ominus M) \leq n.\]

Proof. By Theorem 4

\[K_M \times (H^2 \ominus M) \subseteq (H^\infty)^\perp \cap (\bar{H}^\infty)^\perp = (H^\infty + \bar{H}^\infty)^\perp \cap L^1\]

and \(\dim(H^\infty + \bar{H}^\infty)^\perp \cap L^1 = n\) because \(\mathcal{C} \setminus \Omega\) has \(n\) components. If \(K_M = \langle 0 \rangle\) then \(\phi M \subset M\). It is known [4] that \(L^\infty\) is generated by \(\phi\) and \(H^\infty\) in the weak* topology. Hence \(M \in \text{Lat} \mathcal{A} \cap \text{Lat} \mathcal{A}^* = \{\langle 0 \rangle, H^2\}\) by Corollary 7 and so \(M = H^2\). It is clear that if \(K_M \neq \langle 0 \rangle\) then \(\dim(H^2 \ominus M) \leq n\).

\[\text{Corollary 9. If } H^2 = H^2(T) \text{ and } \phi \notin H^\infty \text{ then } \text{Lat} T_\phi \cap \text{Lat} \mathcal{A} = \{\langle 0 \rangle, H^2\}.\]

Proof. When \(\Omega\) is the open unit disc, \(H^2(\Omega) = H^2(T)\) and so by Corollary 8 \(\text{Lat} T_\phi \cap \text{Lat} \mathcal{A} = \{\langle 0 \rangle, H^2\}.\)

8
Corollary 10. Let $A$ be a Dirichlet algebra. If $\phi \notin H^\infty$ then $\text{Lat} \ T_\phi \cap \text{Lat} \ A = \{\langle0\rangle, H^2\}$.

Proof. It is known that $(\bar{H}^\infty)^\perp \cap (H^\infty)^\perp = \langle0\rangle$. The corollary is a result of Theorem 4.

In general, it seems to be difficult to describe $\text{Lat} \ T_\phi \cap \text{Lat} \ A$. When $H^2 = H^2(\Omega)$ and $\bar{\phi} \in H^\infty$, $\text{Lat} \ T_\phi \cap \text{Lat} \ A = \{\langle0\rangle, H^2\}$ by Corollary 8. In fact, if $M \in \text{Lat} \ T_\phi \cap \text{Lat} \ A$ then $\bar{\phi}(H^2 \ominus M) \subseteq H^2 \ominus M$. Since dim$(H^2 \ominus M) < \infty$ by Corollary 8, $M$ must be equal to $H^2$. When $H^2 = H^2(T^2)$ and $\phi = \bar{z}$, $\text{Lat} \ T_\phi \cap \text{Lat} \ A = \{\langle0\rangle, qH^2(w, T); q = q(w) \text{ is a one variable inner function}\}$ where $z$ and $w$ are the independent variables on $T^2$. In fact, if $M \in \text{Lat} \ T_\phi \cap \text{Lat} \ A$ then $T^*_z M_1$ is orthogonal to $M$ where $M_1 = M \ominus zM$. Since $T^*_z M_1 \subseteq M$, $T^*_z M_1 = \langle0\rangle$ and so $M_1 \subset H^2(w, T)$. Corollary 10 shows that $\text{Lat} \ A^* \cap \text{Lat} \ A = \{\langle0\rangle, H^2\}$ if $A$ is a Dirichlet algebra.

References