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On the cusped fan in a planar portrait of a manifold

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Abstract As a way to draw a picture of a manifold in the plane, we consider to take the image of the manifold through a stable map. We call the image, paired with the critical values, the *planar portrait* of a manifold. The *cusped fan* is its basic local configuration. In this article, we focus on the fibreing structure over the cusped fan, and give its characterisation. As application, the source manifolds of certain planar portraits are characterised, and stable maps of closed manifolds such as the projective plane kP^2 ($k = \mathbf{R}, \mathbf{C}, \mathbf{H}$), regular complex toric surfaces, and some sphere bundles over spheres *etc.* are constructed. As a biproduct, we obtain an infinite to one correspondence of projections up to right-left equivalence, of a fixed manifold to a planar portrait. Further applications on characterising manifolds by planar portraits are left to our forthcoming papers.

1 Introduction

How can one draw a picture of a closed manifold on the plane, and what can one see from it ? In this article, we make an attempt to draw and read pictures by using stable maps. For a stable map $f : M \rightarrow \mathbf{R}^2$, M a smooth closed manifold with dimension two or more, we call the pair $(f(M), f(S_f))$ up to diffeomorphism of \mathbf{R}^2 a *planar portrait* of M through f , where S_f denotes the set of singular points. Since the second factor $f(S_f)$ is a curve possibly with cuspidal points and normal crossings, the planar portraits have enough and not too much complexity to be regarded a picture of M . See Figure 1 for samples of planar portraits. We note that stable maps from closed n -manifolds ($n \geq 2$) to 2-manifolds are known to be generic as smooth maps (Mather [Ma]).

It is not straight to relate the planar portraits to manifolds. In actual, infinitely many manifolds may admit a common planar portrait. As a fundamental relation, a classical result by R. Thom [Th] shows that the number of cusps in a planar portrait is mod 2 congruent to the Euler characteristic $\chi(M)$ of M . H. Levine [L2] studied the indices of critical points of the composed Morse function $\gamma \circ f : M \rightarrow \mathbf{R}^2 \rightarrow \mathbf{R}$, where γ is a generic linear projection, viewed from the location of $f(S_f)$. R. Pignoni [Pi] also obtained

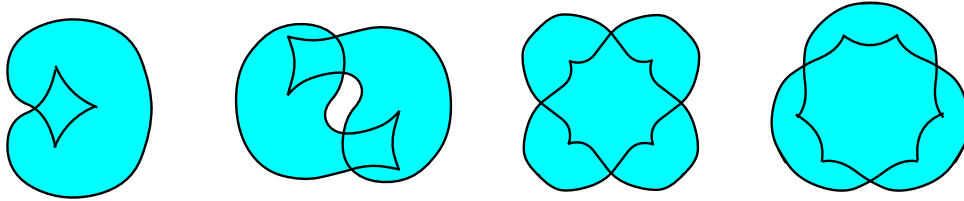


Figure 1: Planar portraits of certain closed manifolds

an Euler characteristic formula using $f(S_f)$ in case M is a surface. On the other hand, some examples show that finer topological properties are carried to the planar portraits (see *e.g.*, applications to Corollary 2.4 in Section 2). The purpose of this article is to give a step to read views of maps and manifolds from the planar portraits, and to offer samples of planar portraits as aids in seeking the suitable description for the relation between the planar portraits and the manifolds.

To read the planar portraits, one is required tasks of two kinds: to know the fibring structures over suitably piecified planar portraits, and to know the glueings of manifold pieces over the subdivided planar portraits. In this article, we deal with the first part and give a characterisation of the fibring structure over a basic piece named the *cusped fan* (See Figure 2, which is strictly defined in Section 2). In short, the fibring structure is uniquely determined up to right-left equivalence by the index of the cusp and further it is a perturbation of a twice folding projection $(|z|^2, |w|^2)$ of $D^p \times D^q$, where p and q depend on the index of the cusp (Theorem 2.2. The twice folding projection is not stable in itself, as seen easily). In the 2-dimensional case, what the result means is easily understood: see Figure 3.

The result is applied effectively to both reading and drawing planar portraits. For the first, a tool for reading is given, and admissible manifolds for certain planar portraits are detected (Corollary 2.4 and its application). For the second, lifting of Morse functions to stable maps into \mathbf{R}^2 , and perturbations of orbit (quotient) maps of certain group actions are considered as tools for drawing. In actual, planar portraits of projective planes kP^2 , $k = \mathbf{R}, \mathbf{C}$ or \mathbf{H} , regular complex toric surfaces, and sphere bundles over spheres with cross-sections, are constructed. As a biproduct we can show that a certain planar portrait is realised by infinitely many maps, even if the source manifold and the indices of singularities are fixed. Some further applications are left to our proceeding papers [K2],[K3], *etc.*, for compactness.

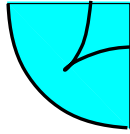


Figure 2: The cusped fan

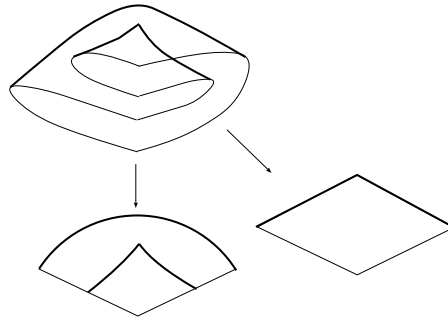


Figure 3: The cusped fan and the twice folding projection, in dimension 2

The organization of this article is as follows. In the next section, we introduce the cusped fan, states the main theorem, and give its simple applications. Through the proceeding four sections we prove the theorem, but technical calculations are left to the appendix. In Section 7, we give some applications of the theorem to the construction of stable maps, and also give a biproduct on the infiniteness of projections to a planar portrait. One can skip to Section 7 after he read Section 2 to get the outline. Henceforth, manifolds and maps are assumed to be smooth unless otherwise mentioned.

2 The cusped fan

A smooth map $f : M^n \rightarrow \mathbf{R}^2$ ($n \geq 2$) is called a *stable map* if for any small perturbation f' of f in $C^\infty(M, \mathbf{R}^2)$ equipped with Whitney C^∞ topology, $f' = l \circ f \circ r$ holds for some self-diffeomorphisms r and l of M and \mathbf{R}^2 , respectively. Two maps f and f' in this relation are called *right-left equivalent*. For a stable map $f : M^n \rightarrow \mathbf{R}^2$ ($n \geq 2$), the set of singular points S_f is made up of isolated *cusps* and arcs of *folds* between the cusps. It is known that in a

small neighbourhood centered at a fold point $P \in S_f$, the map f is right-left equivalent to the normal form (F) bellow, where a and b are non-negative integers (In case $a = 0$ (*resp.* $b = 0$), the terms x and $|x|^2$ (*resp.* y and $|y|^2$) should be omitted).

$$(F) : (u, x, y) \mapsto (u, |x|^2 - |y|^2), u \in \mathbf{R}, x \in \mathbf{R}^a, y \in \mathbf{R}^b$$

Similarly, a cusp point $P \in S_f$ has the normal form (C) bellow, where a and b are non-negative integers (In case $a = 0$ (*resp.* $b = 0$), the terms y and $|y|^2$ (*resp.* z and $|z|^2$) should be omitted).

$$(C) : (u, x, y, z) \mapsto (u, x^3 - 3ux + |y|^2 - |z|^2), u, x \in \mathbf{R}, y \in \mathbf{R}^a, z \in \mathbf{R}^b$$

The *absolute index* of a cusp is the maximum of $\{a, b\}$ in (C) (See [L1] for the original, coordinate free definition). As easily seen by (F) and (C), the set of singular points S_f is a 1 dimensional proper submanifold of M , and the restriction of f to S_f except for the cusps is an immersion. Moreover the image of the immersion has at most normal crossings and do not pass through the image of the cusps. Refer to *e.g.*, [GG] and [L3] for basic notions on stable maps.

A *cusped fan* is the pair of a quadrant and a curve of two connected components with one cuspidal singular point pictured in Figure 2. It can appear as a subpiece in a planar portrait $P = (f(M), f(S_f))$, where f is a stable map, as formed by a combination of a cusp with folds of definite types $(u, |x|^2)$. In constructing or in reading a planar portrait, it is often convenient and useful to consider piecifications by duplicated configurations (folds and folds, cusp and folds, cusp and cusp) of critical values. These configurations are reduced to combinations of the four in Figure 4, and the cusped fan is essentially the third case. We note that fibreing structures for the first two can be followed by using CPN (coordinatized product neighbourhood) in [L3] (where it is defined for dimension four).

Definition 2.1 Let X be a compact manifold with boundary, or possibly with corner. A map $f : X \rightarrow \mathbf{R}^2$ is a *fibrewise cut* of a stable map if there exist an open manifold \hat{X} which contains X as a proper submanifold, and a stable map $\hat{f} : \hat{X} \rightarrow \mathbf{R}^2$ which have the three properties bellow:

1. The restriction $\hat{f}|_X$ coincides with f ,
2. There exists a finite collection $\lambda_i, i = 1, \dots, m$, of smooth plane arcs such that the intersections of λ_i with other λ_j and with $\hat{f}(S_{\hat{f}})$ are transverse, and
3. X is obtained from \hat{X} by cutting it along $\hat{f}^{-1}(\lambda_1) \cup \dots \cup \hat{f}^{-1}(\lambda_m)$.

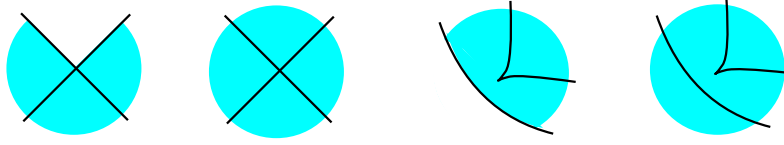


Figure 4: The duplicated local configurations in planar portraits

For a fibrewise cut of a stable map f , we define the set of singularities by $S_f = S_{\hat{f}} \cap X$, and call the pair $(f(X), f(S_f))$ the planar portrait of X , as before.

Our main result can be now stated as bellow:

Theorem 2.2 *Let X be a compact connected n -dimensional manifold with corner, and $f : X \rightarrow \mathbf{R}^2$ a fibrewise cut of a stable map which has the planar portrait $(f(X), f(S_f))$ as in Figure 2 (the cusped fan). Then,*

1. X is diffeomorphic to $D^{a+1} \times D^{b+1}$ as a manifold with corner, where a is the absolute index of the cusp and $b = n - 2 - a$.
2. f is determined uniquely up to right-left equivalence, by a given a .
3. We identify X with $\{|z| \leq 1, |w| \leq 1\}$, where $z = (z_0, \dots, z_a) \in \mathbf{R}^{a+1}$ and $w = (w_0, \dots, w_b) \in \mathbf{R}^{b+1}$. Then f is right-left equivalent to the map

$$(z, w) \mapsto (|z|^2 + 2\varepsilon k(z)w_0, |w|^2 + 2\varepsilon k(w)z_0)$$

for any positive small constant ε , where $k(t) = 1 - |t|^2$.

Remark 2.3 Identify X with $D^{a+1} \times D^{b+1}$ by 1. Then the boundaries $S^a \times D^{b+1}$, $D^{a+1} \times S^b$ and the corner $S^a \times S^b$ of X are the inverse images of the two edges and the vertex of the quadrant $f(M)$, respectively, by definition of a fibrewise cut, or by a direct calculation using 3.

Bellow is a simple application of the theorem. See section 7, for more applications.

Corollary 2.4 (*Bending and Tucking lemma*) *Let $P = (h(X), h(S_h))$ be a planar portrait listed in Figure 5, where $h : X \rightarrow \mathbf{R}^2$ is a fibrewise cut of a stable map. Then X is diffeomorphic to the product $\Sigma^{n-1} \times D^1$ (in case of*

a, b, c, d), $D^{n-1} \times S^1$ or $D^{n-1} \tilde{\times} S^1$ (the non-trivial bundle) (in case of e), or to $D^1 \times S^1$ or $D^1 \tilde{\times} S^1$ (in case of f, g), where Σ^{n-1} is an $(n-1)$ -homotopy sphere in the gamma group Γ_{n-1} . Conversely for each P , every diffeomorphism types of X above can be realised.

We can apply it to see, for example, that the source manifolds of the planar portraits in Figure 6 are all diffeomorphic to the total space of a Σ^{n-1} bundle over S^1 , where Σ^{n-1} is a homotopy sphere in Γ_{n-1} , and conversely, each planar portrait is realised at least by $\Sigma^{n-1} \times S^1$ and by the non-trivial bundle $S^{n-1} \tilde{\times} S^1$. The source manifold of the second planar portrait in Figure 1 is also diffeomorphic to a Σ^{n-1} bundle over S^1 , and the third one to either a torus or a Klein bottle. Further the first one in the figure is a planar portrait of $\mathbf{R}P^2$ which is realised by combining the planar portrait e of a Möbius strip with a cusped fan. In a similar way, various planar portraits can be constructed by combining the pieces listed in Figure 5 and the cusped fans.

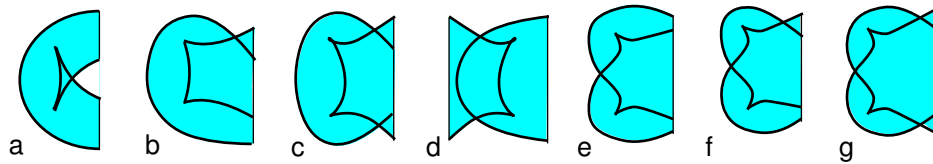


Figure 5: Bendings and tuckings in planar portraits

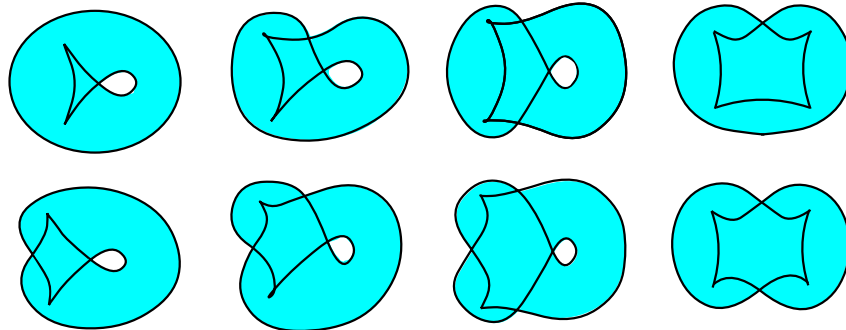


Figure 6: Planar portraits of Σ^{n-1} bundles over S^1

Proof of Corollary 2.4 For the first three cases, one can move h by homotopy to cancel the crossings of $h(S_h)$, since these crossings are caused by folds of definite types, and thus each fibre near the crossing point is disjoint. Hence we can move P to the union of two cusped fans attached

along an edge. For the case d, we can move P again to the same one as above, by a half twist of the rectangular region enclosed by the cuspidal arcs and a side edge of P by a homotopy of h .

Note that an arc of S_h abutting the cusp consists of folds of definite types. This implies that one edge of each cusped fan has inverse image diffeomorphic to $D^{n-1} \times S^0$, and hence X is the union of two $D^{n-1} \times D^1$ attached along $S^{n-2} \times D^1$, by the theorem. Namely, X is diffeomorphic to a D^1 bundle over Σ , where Σ is the union of two D^{n-1} attached along boundary. Its diffeomorphism class is either that of a trivial bundle or the non-trivial one. On the other hand, the restriction $h|_{\partial X}$ is a Morse function of $(n-1)$ dimensional manifold with four critical points of indices either 0 or $n-1$, which implies that ∂X is the disjoint union $\Sigma \amalg \Sigma$. Therefore X is in actual the trivial bundle $\Sigma \times D^1$.

In the last three cases, we can move P of f and g to e, similarly as before. On the other hand, e is separated into the union of two degenerated cusped fans along the cut line which passes through the crossing point. Since the crossing is caused again by definite folds, we can see that the inverse image of each piece above is also $D^{n-1} \times D^1$. Hence X is the union of them attached along $D^{n-1} \times S^0$, or it is a total space of a D^{n-1} bundle over S^1 . On the other hand, the normal form of the fold of definite types shows that the attaching diffeomorphism in $\text{Diff}(D^{n-1})$ preserves the norms when D^{n-1} is regarded a unit disc in \mathbf{R}^{n-1} , and hence it can be linearised, or moved to an $O(n-1)$ element by isotopy. Hence X has the required diffeomorphism types. Note that $n=2$ in the case f and g, since both arcs abutting a cusp are consisting of definite folds, and hence both boundary of the invese image of each cusped fan are diffeomorphic to the disjoint union $D^{n-1} \amalg D^{n-1}$. The construction part is easy (refer to Section 7). \square

3 A reduction to the smoothed case

We start to prove Theorem 2.2. To show assertion 1 and 2, we will prove a corner removed version instead. Suppose that \tilde{X} is a compact connected n -manifold, $n \geq 2$ with boundary, and $h : \tilde{X} \rightarrow \mathbf{R}^2$ a fibrewise cut of a stable map with one cusp of absolute index a , such that the planar portrait $(h(\tilde{X}), h(S_h))$ is as illustrated in Figure 7: $h(\tilde{X})$ is the half-disc and $h(S_h)$ is the bolded curves consisting of two components. For simplicity, we assume that $h(\tilde{X}) = \{x^2 + y^2 \leq 1, x \leq 0\}$ and set $L = \{0\} \times [-1, 1]$. Note that $\partial \tilde{X} = h^{-1}(L)$, by the definition of a fibrewise cut.

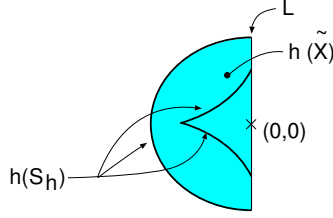


Figure 7: Planar portrait by the straightening h of f

It is clear that \tilde{X} and h can be obtained from X and f in Theorem 2.2 by straightening of corner. We can back to f conversely, by restricting h to the inverse image of a quadrant with vertex $(0,0)$. The assertions are hence reduced to the proposition bellow.

Proposition 3.1 *Let $h : \tilde{X} \rightarrow \mathbf{R}^2$ be a fibrewise cut of a stable map with its planar portrait as in Figure 7. Then ;*

1. \tilde{X} is diffeomorphic to D^n , and $\partial\tilde{X}$ is divided into two pieces $S^a \times D^{b+1}$ and $D^{a+1} \times S^b$ along the fibre $h^{-1}(0,0)$, where a is the absolute index of the cusp and $b = n - 2 - a$.
2. h is determined uniquely up to right-left equivalence, by a .

In this section, we prove 1, and in the preceding section we do 2. The first half of 1 is an easy application of [L2]. Hence we omit the proof. To show the second half, we regard $h|\partial\tilde{X}$ as a Morse function onto L by the lemma bellow.

Lemma 3.2 (Levine) *Let $f : M \rightarrow \mathbf{R}^2$ be a stable map and λ an arc which is transverse to $f(S_f)$. Then the restriction $f|f^{-1}(\lambda)$ can be considered a Morse function onto λ , by composing an inclusion $\lambda \rightarrow \mathbf{R}$.*

Proof This is just an analogy of [L3], Proposition 2 in 1.3, in general dimensions. \square

Note that $h|\partial\tilde{X} : \partial\tilde{X} \rightarrow L \subset \mathbf{R}$ is a Morse function of the $(n-1)$ -sphere with four critical points. By choosing the orientation of L well, we may assume that their indices are $0, a, a+1$ and $n-1$, lined in this order ([L1], Lemma(2) in 3.2), and hence $\partial\tilde{X}$ has a handlebody decomposition

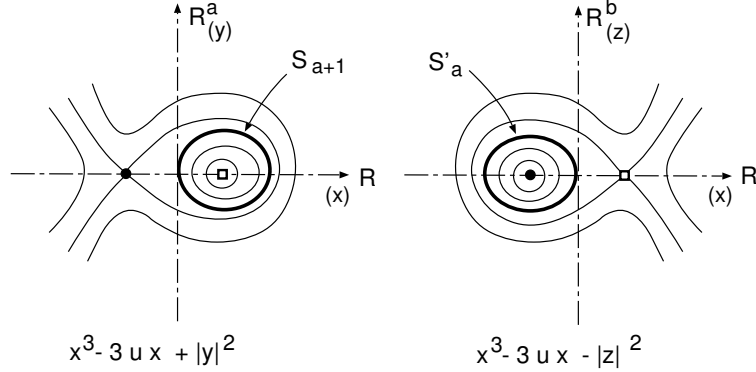


Figure 8: Attached and detached spheres in a neighbourhood of a cusp

$H_0 \cup H_a \cup H_{a+1} \cup H_{n-1}$. We show here that on the middle regular level $\partial(H_0 \cup H_a)$, the sphere $S'_a = \{0\} \times \partial D^{n-1-a}$ in $H_a = D^a \times D^{n-1-a}$ (the attached sphere of the surgery of index a), and the sphere $S_{a+1} = \partial D^{a+1} \times \{0\}$ in $H_{a+1} = D^{a+1} \times D^{n-2-a}$ (the detached sphere of the surgery of index $a+1$) meet transversely at one point. Note that $H_0 \cup H_a$ is a total space of an S^a bundle over D^{b+1} in general (recall that $n-1-a = b+1$). But under the above transversality, it is known to be a trivial bundle, by standard argument in cancellation of handles for $H_0 \cup H_a \cup H_{a+1}$. Hence $H_0 \cup H_a$ is diffeomorphic to $S^a \times D^{b+1}$, and $H_{a+1} \cup H_{n-1}$ to $D^{a+1} \times S^b$, by duality.

To see the transversality of S'_a and S_{a+1} , we may assume that L is very close to the cusp image $h(P)$. In actual, the Morse function $h|_{h^{-1}(L)}$ changes only up to isotopy, when the line L is moved close to $h(P)$ in keeping the transversality with $h(S_h)$. Now by the normal form (C), we can take a common neighbourhood of the middle two critical points of $h|\partial L$ and a system of coordinates, in which the Morse function is written by $x^3 - 3ux + |y|^2 - |z|^2$ for a positive small u . It is easy to check that, by isotopy moves, the spheres S'_a and S_{a+1} can be placed in the hyper plane $|y| = 0$ and $|z| = 0$, respectively, as shown in Figure 8. In the figure, the dot and the box indicate the two critical points on the x axis, one with index a and the other with $a+1$, respectively. Clearly these spheres meet transversely at the origin $(x, y, z) = (0, 0, 0)$. \square

4 Uniqueness

In this section, we prove 2 of Proposition 3.1. Denote by P the unique cusp of h , and take a small neighbourhood U of P . We can take a system of

coordinates (u, x, y, z) centered at P such that on U , h takes the normal form (C) in Section 2.

We modify CPN in [L3] Proposition 1 in 2.2 by putting boundary conditions to it, as follows.

Proposition 4.1 *Let $g(u, x, y, z), u, x \in \mathbf{R}, y \in \mathbf{R}^a, z \in \mathbf{R}^b$ be the normal form (C) for the cusp. For positive small numbers ε and λ , there exist a product neighbourhood $I \times J$, $I = [-\varepsilon, \varepsilon]$, $J = [-\lambda, \lambda]$ of the origin $0 \in \mathbf{R}^2$ and a closed neighbourhood W of the origin $0 \in \mathbf{R}^n$ diffeomorphic to $D^{n-2} \times I \times J$ such that the following properties hold (refer to Figure 9) : Let $W_u, u \in I$ be the u -slice $W \cap (\{u\} \times \mathbf{R} \times \mathbf{R}^a \times \mathbf{R}^b)$, and let $g_u(x, y, z)$ be the function of W_u given by $x^3 - 3ux + |y|^2 - |z|^2$. Then ;*

1. $\{\varepsilon\} \times J$ has two transverse intersections with the critical value set $g(S_g)$,
2. $g(W_u) = \{u\} \times J$,
3. On the boundary of W corresponding to $D^{n-2} \times I \times \{\lambda\}$, each $g_u, u \in I$ is regular and takes constantly λ ,
4. On the boundary of W corresponding to $D^{n-2} \times I \times \{-\lambda\}$, each $g_u, u \in I$ is regular and takes constantly $-\lambda$,
5. Each fibre $g_u^{-1}(t), u \in I, t \in J$ is transverse to the boundary of W corresponding to $\partial D^{n-2} \times I \times J$.

We call W a *cubic neighbourhood* of the cusp. Its proof is similar to that in [K1], pp.347–348, where the special case $a = b = 1$ is treated, and hence we omit it. In the following, we regard W a subneighbourhood of the cusp P in the neighbourhood U .

Lemma 4.2 *The connected component of S_h which contains the cusp P has a compact neighbourhood N such as bellow :*

1. N is diffeomorphic to the cubic neighbourhood W so that the boundary $N \cap \partial X$ is identified with the boundary W_ε of W , and
2. The restriction $h|N$ is right-left equivalent to the restriction $g|W$, where g is the normal form of a cusp.

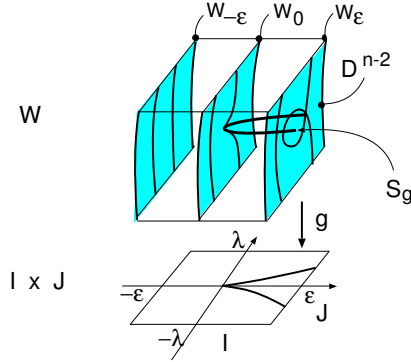


Figure 9: The cubic neighbourhood W

Proof For a small neighbourhood of P in S_h , the cubic neighbourhood W itself deserves as the neighbourhood N . It is not difficult to enlarge such N to the whole neighbourhood of S_h , as seen bellow. Take a linear projection $\gamma : \mathbf{R}^2 \rightarrow \mathbf{R}$ so that $\gamma \circ h$ is a Morse function with $\gamma \circ h(P) = 0$ and with $\gamma(L) = 1$ (refer to [L2]). Let W_u be the u -slice of W as in Proposition 4.1. We may assume that $\gamma \circ h(W_u) = u$ for $u \in [-\epsilon, \epsilon]$, for a positive small number ϵ . Since the projection h restricted to the right side of $\gamma^{-1}(\epsilon)$ is simply right-left equivalent to the product map $h|(\gamma \circ h)^{-1}(\epsilon) \times \text{id}$, where id is the identity map of $[\epsilon, 1]$, it is easy to enlarge W to the required neighbourhood N of the whole component of S_h . \square

Take the neighbourhood N of S_h as in Lemma 4.2, and we temporarily replace h inside N with the projection corresponding to $D^{n-2} \times I \times J \rightarrow I \times J$ of the cubic neighbourhood W . Denote by $H : \tilde{X} \rightarrow \mathbf{R}^2$ the resultant. It is again a fibrewise cut of a stable map but only with definite folds $(u, x_1, x_2, \dots, x_{n-1}) \mapsto (u, x_1^2 + x_2^2 + \dots + x_{n-1}^2)$ as singular points. It is clearly unique up to right-left equivalence, since it can be shrunk preserving the right-left equivalence class, to a small neighbourhood of a definite fold point. On the other hand, N has intersections with some regular fibres of H so that it cuts D^{n-2} from each fibre. Hence any self-diffeomorphism of \tilde{X} which preserves the fibres of H is made into the identity map on a neighbourhood of N , by moving it if necessary by a H -fibre preserving isotopy.

Denote by N_e the closure of $\tilde{X} - N$. The restriction $h|N_e$ is unique up to right-left equivalence. In actual, H is right-left equivalent to the map $H_0 = (s^2 + |t|^2, s) : B \rightarrow \mathbf{R}^2$, where B is the disc $s^2 + |t|^2 \leq 1$ of \mathbf{R}^n , and at the same time, the restriction $H|N$ is identified with the restriction $H_0|B'$, where B' is a small cubic collar neighbourhood of the boundary point

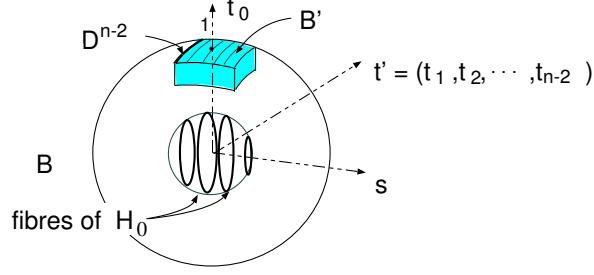


Figure 10: The neighbourhood B' in the ball B

$s = 0, t = (1, 0, \dots, 0)$ given by $(1 - \lambda)^2 \leq s^2 + |t|^2 \leq 1, |s| \leq \varepsilon$ and $|t'| \leq \mu$ for positive small numbers ε, λ and μ , where $t' = (t_1, t_2, \dots, t_{n-2})$ (See Figure 10). Hence $h|N_e$ is right-left equivalent to the map $H_0|_{\overline{B - B'}}$. Recall also that $h|N$ is uniquely determined up to right-left equivalence, by a positive integer a (2 of Lemma 4.2). We now fix the integer a .

To show the uniqueness of h , consider h' which is another pasting of $h|N$ and $h|N_e$. Then we replace h' in N as before and denote by H' the resultant. Note that $H' = \psi \circ H \circ \varphi$ for self-diffeomorphisms ψ of \mathbf{R}^2 and φ of \tilde{X} . But as mentioned before, φ is the identity map in a neighbourhood of N . Hence we have $h' = \psi \circ h \circ \varphi$. \square

5 Jet transversalities

Though this and the proceeding sections, we prove 3 of Theorem 2.2. Since the uniqueness property 2 of the theorem is proved, we are enough to check that the map presented in 3 of the theorem is a fibrewise cut of a stable map over a cusped fan with its absolute index of cusp a . To save notation, we denote the polynomial described in 3 of the theorem also by f .

Set $\Delta = \{|z| \leq 1, |w| \leq 1\}$. To show the stability of f , we extend the domain slightly to an open neighbourhood $\hat{\Delta}$ of Δ . The stability of f is equivalent to the transversalities of jets $j^1 f \bar{\cap} \Sigma_i$ and $j^2 f \bar{\cap} \Sigma_{ij}$ (see [L1]), besides with a certain genericity condition on the mutual position of the critical values. In this section, we check the transversalities. Only the outlines are stated here and the proofs are given in Appendix. Throughout the section, we set

$$p = (1 - 2\varepsilon w_0)z_0,$$

$$\begin{aligned}
p_i &= (1 - 2\varepsilon w_0)z_i, \text{ for } i = 1, \dots, a, \\
q &= \varepsilon k(z), \\
r &= \varepsilon k(w), \\
s &= (1 - 2\varepsilon z_0)w_0, \text{ and} \\
s_j &= (1 - 2\varepsilon z_0)w_j, \text{ for } j = 1, \dots, b.
\end{aligned}$$

By straight calculation we can show:

Lemma 5.1 *Set $S = ps - qr$. The set of singular points S_f is defined by $S = 0, (z_1, z_2, \dots, z_a) = 0$, and $(w_1, w_2, \dots, w_b) = 0$.*

We can show further that (See Appendix for the proof):

Lemma 5.2 *S_f is a regular curve in $\hat{\Delta}$ with two connected components.*

Let $d^2f : E \rightarrow L^* \otimes G$ be the second differential of f defined in [L1], where $E = T\hat{\Delta}|_{S_1(f)}$, $G = \text{cok } df$, and $L = \ker df$. The lemma bellow implies the transversality of 1-jet (3.1, [L1]). See Appendix for the proof.

Lemma 5.3 *At every point in S_f , $\text{rk } df = 1$ and d^2f is surjective.*

Next we show the 2-jet transversality. Denote by $S_{11}(f)$ the points of S_f where the corank of $d^2f|_L$ is 1, by convention. It is well known that a point $P \in S_{11}(f)$ is a cusp.

Lemma 5.4 *Set $K = q \partial_{z_0} S - p \partial_{w_0} S$, where $\partial_t S$ stands for $\frac{\partial S}{\partial t}$. A singular point $P \in S_f$ is in $S_{11}(f)$ if and only if $K(P) = 0$.*

Lemma 5.5 *K restricted to S_f vanishes only at a single point P_D where $z_0 = w_0$.*

Lemma 5.6 *At $P_D \in S_{11}(f)$, the locus $K = 0$ is transverse to S_f .*

One can see the proofs of these lemmaes in Appendix. The 2-jet transversality follows the last one. For the absolute index of the cusp P_D , see Remark 8.1 in Appendix.

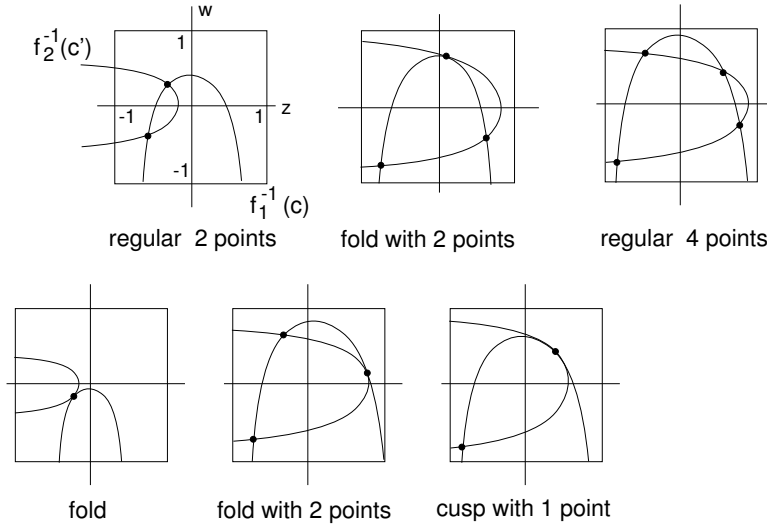


Figure 11: Fibres of f , in the 2-dimensional case

6 The planar portrait produced by f

The rest of proof is the check of the mutual position of the critical values and the condition of a stable cut. For a while until the global condition is checked, we continue to regard f defined over an open neighbourhood $\hat{\Delta}$ of Δ . Let C_0 and C_1 be the image of the component of S_f without, and with the cusp, respectively.

Lemma 6.1 *The two plane curves C_0 and C_1 are both simple and mutually disjoint.*

Proof By Lemma 5.1, we can reduce the setup to the two variable case $z = z_0, w = w_0$. Suppose that the curve $C_0 \cup C_1$ has a multiple point, say $f(Q) = f(Q') = (c, c')$ for some $Q, Q' \in S_f$. Note that $c, c' < 1$, since $S_f \cap \partial\Delta$ consists of the four points $(\pm 1, 0)$ and $(0, \pm 1)$, which have mutually different images. Write $f = (f_1, f_2)$. Since S_f is the locus of tangents of the two level curves of f_1 and f_2 , the two curves $f_1^{-1}(c)$ and $f_2^{-1}(c')$ have two or more tangent points. But this is impossible by the convexity of the curves (See Figure 11). \square

By the lemma, we see that $f|_{S_f - \{\text{cusp}\}}$ is an embedding and the cusp is not mapped onto the image of the other singular points. Hence the global condition of stability (see *eg.*, [GG]) holds and thus $f : \hat{\Delta} \rightarrow \mathbf{R}^2$ is stable.

We check that $f : \Delta \rightarrow \mathbf{R}^2$ is a fibrewise cut of a stable map. Recall that S_f is transverse to the boundary $\partial\Delta$. In actual, S_f lies in the z_0w_0 plane (Lemma 5.1), and the normal direction $(\partial_{z_0}S, \partial_{w_0}S)$ to S_f in the plane is not $(0, 1)$ at $(0, \pm 1)$, and is not $(1, 0)$ at $(\pm 1, 0)$, as seen in the proof of Lemma 5.2 in Appendix. Let λ and λ' be slightly extended arcs of $\{1\} \times [-2\varepsilon, 1]$ and $[-2\varepsilon, 1] \times \{1\}$, respectively. The transversality of them to $f(S_f)$ follows from the above transversality and the normal form (F) . Then it is easy to see that f is a stable cut.

To see that f produces a cusped fan, we are again enough to consider the two variable case (Lemma 5.1). By the normal form (C) , the cusp image lies in the interior of $f(\Delta)$, and hence $f(\Delta)$ is enclosed by C_0, λ and λ' . We have two candidates for such planar portraits, corresponding to the direction of the cusp: the cusp and nearby folds make an wedge, pointed to C_0 or to the opposite. Suppose that the second is the case. Then we see a contradiction occurs by counting the number of inverse points $f^{-1}(Q)$ for a regular value Q : it is two, if Q is near C_0 , inside the wedge, it should be greater by two, than that for the outside by the normal form (C) , hence it should be zero, outside the cuspidal edge. It is a contradiction, since the corner point $(1, 1)$ of the fan has four inverse points.

This ends the proof of Theorem 2.2.

7 Applications

In this section, we give some applications of Theorem 2.2. The composition $(z, w) \mapsto (|z|^2, |w|^2) \mapsto |z|^2 \pm |w|^2$ shows that the twice folding projection is a lift into \mathbf{R}^2 , of a Morse type critical point of a function. By applying the method in [L2], we see that the stable cut over a cusped fan is still so, as illustrated in Figure 12: The lines in the figure show the surjection $\gamma : \mathbf{R}^2 \rightarrow \mathbf{R}$ through which the Morse function is lifted. Let f produce the cusped fan. Then the tangent point of the line to $f(S_f)$ in the second figure corresponds to the critical point of $\gamma \circ f$ of index 0, and that in the last figure corresponds to the critical point of index $a + 1$, where a is the absolute index of the cusp, by Theorem 2.2. By using this property, we can give lifts of some Morse functions.

Example 7.1 Let f be a stable cut over a cusped fan, of any absolute index of cusp. We take two copies of f and make a double to obtain a stable map $\tilde{f} : S^n \rightarrow \mathbf{R}^2$, which produces the planar portrait as in Figure 13.

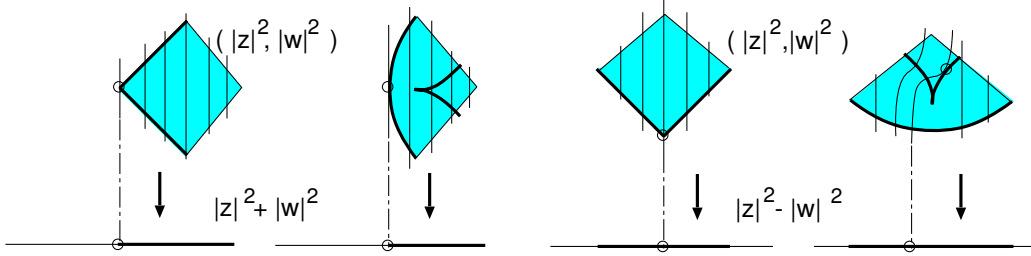


Figure 12: lifts of Morse critical points

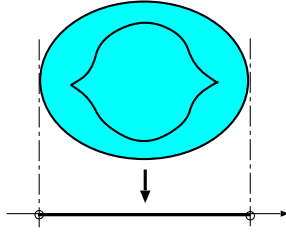


Figure 13: A lift of height function of the sphere

Clearly it is a lift of the height function of S^n given by the projection of the unit sphere in \mathbf{R}^{n+1} to the first coordinate. \square

Example 7.2 Let $g : kP^2 \rightarrow \mathbf{R}, k = \mathbf{R}, \mathbf{C}, \mathbf{H}$ be a Morse function defined by $g([x : y : z]) = \frac{a|x|^2 + b|y|^2 + c|z|^2}{|x|^2 + |y|^2 + |z|^2}$, where $0 < a < b < c$. It has three critical points at $[1 : 0 : 0], [0 : 1 : 0]$ and $[0 : 0 : 1]$, of indices $0, l$ and $2l$ with critical values a, b and c , respectively, where $l = 1, 2$ or 4 , corresponding to $k = \mathbf{R}, \mathbf{C}$ or \mathbf{H} , respectively.

Take a decomposition $kP^2 = H_0 \cup H_1 \cup H_2$, where H_0, H_1, H_2 are neighbourhoods of $[1 : 0 : 0], [0 : 1 : 0]$ and $[0 : 0 : 1]$ defined by $\{|y|, |z| \leq |x|\}, \{|z|, |x| \leq |y|\}$ and $\{|x|, |y| \leq |z|\}$, respectively. Identify each of them with $\{|s_i|, |t_i| \leq 1\}$ and hence to $D^l \times D^l$, where $s_0 = \frac{y}{x}, t_0 = \frac{z}{x}, s_1 = \frac{z}{y}, t_1 = \frac{x}{y}$, and $s_2 = \frac{x}{z}, t_2 = \frac{y}{z}$. On these pieces, g is right equivalent to $s_0^2 + t_0^2, s_1^2 - t_1^2$, and $-s_2^2 - t_2^2$, respectively. Take three copies of $f : D^l \times D^l \rightarrow \mathbf{R}^2$ in 3 of Theorem 2.2, and place the cusped fans on the plane as in Figure 14, left, in referring to Figure 12. It is easily seen from normal form 3 in Theorem 2.2 that these map pieces can be well pasted to form a map $\tilde{f} : kP^2 \rightarrow \mathbf{R}^2$ which

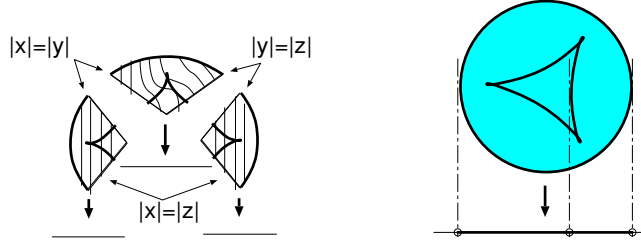


Figure 14: A planar portrait (right) of kP^2 , $k = \mathbf{R}, \mathbf{C}$ or \mathbf{H}

produce the planar portrait as in Figure 14, right, by moving the images up to isotopies, if necessary. It is clearly a lift of g , as shown in the figure. \square

The stable cut over the cusped fan can be also regarded a perturbation of the quotient map of the linear product actions of $O(p) \oplus O(q)$ to $D^p \times D^q$. By using this, we can obtain a perturbation of the moment map of a regular toric complex surface and the planar portrait produced by them.

Example 7.3 For $\gamma \in \pi_1 SO(2) = \mathbf{Z}$, denote by $\tilde{\gamma}$ the diffeomorphism of $D^2 \times \partial D^2 \rightarrow \partial D^2 \times D^2$ defined by $\tilde{\gamma}(z, w) = (\bar{w}, w^\gamma z)$, where D^2 is regarded as a unit disc in the \mathbf{C} plane, and \bar{w} denotes the complex conjugate ($w_0, -w_1$) of $w = (w_0, w_1)$. It is well-known that a regular toric surface M is obtained by putting k -copies of $D^2 \times D^2$ together, where k is the Euler characteristic $\chi(M)$ of M , by using the diffeomorphisms $\tilde{\gamma}_i(z, w) = (\bar{w}, w^{\gamma_i} z)$, $i = 1, \dots, k$ for some $\gamma_1, \dots, \gamma_k \in \mathbf{Z}$.

Take k copies of f given in 3 of Theorem 2.2. By composing certain isotopy of \mathbf{R}^2 to f , we may assume that $\text{Im } f$ is a sector with vertex of angle $2\pi/k$. Then as before in Example 7.2, one can obtain a stable map $\tilde{f} : M \rightarrow \mathbf{R}^2$ with k cusps such that the planar portrait is obtained by pasting k cusped fans to form a disc (Figure 15, (b)).

We note that \tilde{f} is a perturbation of a moment map: Let $q : M \rightarrow P_k \subset \mathbf{R}^2$ be the map obtained by taking the unfolding parameter ε in f to be zero in the above construction, where P_k is a polygon with k edges (Figure 15,(a)). It is clearly a moment map, or the quotient map of the toric action. \square

Remark 7.4 For the projective plane kP^2 , the maps in Example 7.2 and in Example 7.3 are the same. Note that the Morse function g in Example

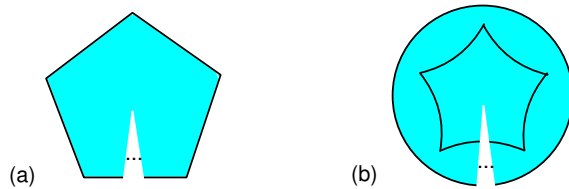


Figure 15: (a) image of the moment map, and (b) the planar portrait of a regular toric surface

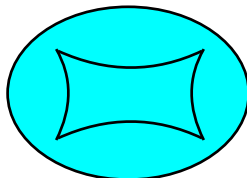


Figure 16: A common planar portrait of sphere bundles over spheres with cross sections

7.2 can be factorized as

$$[x : y : z] \mapsto \frac{1}{|x|^2 + |y|^2 + |z|^2} (a|x|^2, b|y|^2, c|z|^2) \mapsto g([x : y : z]),$$

where the first factor is a moment map of kP^2 with the image a triangle in \mathbf{R}^3 , which we regard a map into \mathbf{R}^2 . The map in Example 7.3 is the perturbation of it.

Bellow is a biproduct of this construction.

Corollary 7.5 *Let M be either $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$, where the latter denotes the total space of the non-trivial S^2 bundle over S^2 . Then M admits infinitely many stable maps into \mathbf{R}^2 with four cusps of the absolute index one, of the common planar portrait illustrated in Figure 16, such that they are inequivalent to each other up to right-left equivalence.*

Proof Let H_m be a Hirzebruch surface with Euler number m , which is a toric surface and obtained by pasting four copies of $D^2 \times D^2$ along their boundaries by using $\tilde{\gamma}_i, i = 1, \dots, 4$ with $\gamma_1 = m, \gamma_2 = 0, \gamma_3 = -m$, and $\gamma_4 = 0$. Since H_m is the total space of the S^2 bundle over S^2 with

Euler number m , and hence is diffeomorphic to $S^2 \times S^2$ if m is even, and to $S^2 \tilde{\times} S^2$ otherwise, we can obtain a stable map \tilde{f}_m of $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$, by applying the construction in Example 7.3 for each m . Bellow we show the inequivalence of these maps, for different positive m .

Take a pair of crossing straight lines in the plane so that it divides the planar portrait into four cusped fans. Along a line, M is divided into two B_0 and along the other, it is divided into B_m and B_{-m} , where B_k denotes the total space of the D^2 bundle over S^2 with Euler number k . Now take positive integers m and m' of the same parity but $m \neq m'$. If the two maps \tilde{f}_m and $\tilde{f}_{m'}$ are right-left equivalent, then B_m should be mapped by a diffeomorphism of M onto one of $B_0, B_{m'}$ or $B_{-m'}$, which is a contradiction. \square

We can generalise the construction of maps for Hirzebruch surfaces above as follows. Let M be the total space of an S^p bundle over S^q ($p, q \geq 1$) with the structure group $O(p+1)$. Assume that the bundle has a cross section. Then by dividing each fibre into a disc neighbourhood of the section and the rest, and further dividing the base space S^q into two discs, one can cut M into four pieces of $D^p \times D^q$. Then in a similar way as before, one can obtain stable maps of M to see that:

Corollary 7.6 *The total space of an S^p bundle over S^q ($p, q \geq 1$) with the structure group $O(p+1)$ that admits a cross section has the planar portrait made of four cusped fans as drawn in Figure 16. The absolute indices of the cusps are the same and is $\max\{p-1, q-1\}$.*

Remark 7.7 The converse, or a characterisation of such sphere bundles by the planar portrait P in Figure 16 do not hold, unfortunately. In actual, one can take a stable map of $\mathbf{C}P^2 \sharp \mathbf{C}P^2$ which produces P , by taking natural connected sum of two projections in Example 7.2 and then by performing a cusp elimination. On the other hand, the manifold admits no such bundle structures, since it is diffeomorphic to neither one of $S^1 \times S^3, S^1 \tilde{\times} S^3$ (the non-trivial S^3 bundle over S^1), $S^2 \times S^2, S^2 \tilde{\times} S^2$.

8 Appendix

Proof of Lemma 5.2 By Lemma 5.1, S_f is a curve in the $z_0 w_0$ plane. We may hence reduce the setup to the case of two variables $z = z_0, w = w_0$.

Note that S_f passes through the four points $(\pm 1, 0)$ and $(0, \pm 1)$ on $\partial\Delta$, at which S_f is regular as seen bellow. Set $A = \partial_{z_0} S$ and $B = \partial_{w_0} S$, where

$\partial_t = \frac{\partial}{\partial t}$. By calculation,

$$A = (1 - 2\varepsilon w)s - (1 - 2\varepsilon w)z \cdot 2\varepsilon w + 2\varepsilon zr.$$

Hence $A \neq 0$ at $(0, \pm 1)$, and by symmetry, $B \neq 0$ at $(\pm 1, 0)$.

On the other hand, $S = 0$ is transformed to $ZW = 1$ by a coordinate change $(z, w) \mapsto (Z, W)$ of $\text{Int}\Delta$, where $Z = F(z), W = F(w)$, and $F(t) = \frac{(1 - 2\varepsilon t)t}{\varepsilon(1 - t^2)}$. It is easy to see that one component of $\{S = 0\} \cap \text{Int}\Delta$ reaches to $(1, 0)$ and $(0, 1)$ when t moves toward ± 1 , while the other reaches to $(-1, 0)$ and $(0, -1)$. We hence obtain the lemma. \square

Proof of Lemma 5.3 One can check the first half easily, by calculation on Jacobi matrix J_f . Hence we omit it.

To show that d^2f is surjective, note first that $k(z)$ and $k(w)$ are not simultaneously zero on S_f . In actual, suppose $k(z) = k(w) = 0$ at a singular point. Then $q = r = 0$, and the Jacobi matrix shows either $p, p_1, \dots, p_a = 0$ or $s, s_1, \dots, s_b = 0$. They imply $z = 0$ or $w = 0$. But it is a contradiction, since $k(0) = 1 \neq 0$. Bellow we show the surjectivity in $k(z) \neq 0$. The proof works in parallel in the other case.

Let J_f be the Jacobi matrix of f ;

$$J_f = 2 \cdot \begin{pmatrix} p & p_1 & \cdots & p_a & q & 0 & \cdots & 0 \\ r & 0 & \cdots & 0 & s & s_1 & \cdots & s_b \end{pmatrix}.$$

Since $q \neq 0$, we can change J_f to J' bellow, by a coordinate change.

$$J' = 2 \cdot \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ S & p_1 s & \cdots & p_a s & * & s_1 & \cdots & s_b \end{pmatrix}.$$

Hence at a singular point $P \in S_f$, d^2f has the matrix representation

$$H_P = \left(\begin{array}{c|c|c|c} A|_P & & B|_P & \\ \hline & C_1 & & \\ & & \ddots & \\ & & & C_a \\ \hline & & & D_1 \\ & & & & \ddots \\ & & & & & D_b \end{array} \right),$$

where \check{M} is the matrix obtained by removing the $(a+2)$ -th column from M . The $(1,1)$ component is K . Hence the lemma is obvious if $q \neq 0$. On the other hand, assume that $q = 0$, or $k(z) = 0$. Then P is not in $S_{11}(f)$ nor $K(P) = 0$, as calculation shows. \square

Remark 8.1 The absolute index of the cusp P_D is $\max\{a, b\}$, since in the representation H_L of $d^2f|L$ above, the a diagonals from the $(2,2)$ component and the rest b diagonals have opposite sign. We may assume that $a > b$, by exchanging z and w if necessary.

Proof of Lemma 5.5 We reduce the setup to the two variable case $z = z_0, w = w_0$ as before, since S_f is in the z_0w_0 plane. Note that $P \in S_{11}(f)$ is not on $\partial\Delta$, as seen by checking K at the four points $(\pm 1, 0), (0, \pm 1)$ on $S_f \cap \partial\Delta$. Hence by Lemma 5.4, we are enough to check the zeros of K in $\text{Int}\Delta$.

Denote by S_0 the connected component of S_f which passes through the region $z, w \leq 0$ of Δ , and S_1 the other. Each component has a unique diagonal point, since S_f is changed to $ZW = 1$ by a coordinate change in the proof of Lemma 5.2 and since the change preserves the diagonal set. Denote by P_D the diagonal point on S_1 . Note that $p^2 = q^2$ at P_D . But $p = -q$ is attained only at negative z , as shown by calculation. Hence $p = q$ at P_D , and thus K vanishes at P_D . We show below that it is the unique vanishing point of K in $\text{Int}\Delta$.

We divide $S_1 \cap \text{Int}\Delta$ into the upper and the lower part of P_D .

Claim 1: $0 < p < q$ on the upper part.

Proof. Note that p is positive, since so is z , and that $p < p_d = (1 - 2\varepsilon z)z$, since $z < w$. It is also easy to see that $p_d < q$ for $0 < z < \alpha$, where we set $P_D = (\alpha, \alpha)$. They imply the claim. \square

Claim 2: $0 < B < A$ on the upper part, where $A = \partial_z S$ and $B = \partial_w S$.

Proof. Note that

$$(A, B) = (\partial_z S, \partial_w S) \begin{pmatrix} \partial_z F(z) & 0 \\ 0 & \partial_w F(w) \end{pmatrix}.$$

The first factor of the right hand side is a gradient vector of S , which is perpendicular to the curve $ZW = 1$, and hence both elements have constant sign, on this piece. The diagonal elements in the last factor have also constant

sign. Hence so have both A and B . The sign is positive, as checked at a regular point $(1, 0)$.

On the other hand, we see that

$$B = (1 - 2\varepsilon z)(1 - 2\varepsilon w)z + 2\varepsilon w(\varepsilon z^2 - z + \varepsilon)$$

(refer to the equation on A in the proof of Lemma 5.2), and hence

$$\begin{aligned} A - B &= (w - z)((1 - 2\varepsilon z)(1 - 2\varepsilon w) + 2\varepsilon^2 zw - 2\varepsilon^2) \\ &= (w - z)(6\varepsilon^2 zw - 2\varepsilon(z + w) + 1 - 2\varepsilon^2). \end{aligned}$$

The last factor of above is positive on $\hat{\Delta}$: it defines a hyperbola

$$h(z, w) := 6(\varepsilon z - \frac{1}{3})(\varepsilon w - \frac{1}{3}) + \frac{1}{3} - 2\varepsilon^2 = 0$$

on the z - w plane, which have no intersection with $\hat{\Delta}$ for positive small ε , while $h(0, 0) > 0$. Hence we obtain the claim. \square

By the claims, $pB < qA$ on the upper part, and hence K do not vanish. By symmetry, K has no zeros on $S_1 \cap \text{Int}\Delta$ but for the diagonal point P_D . It is easily checked that K has no zeros on $\text{Int}\Delta \cap S_0$, by using the facts $p < 0, q > 0$ (since $z, w < 0$) while both A and B are negative. \square

Proof of Lemma 5.6 We can reduce the setup to the two variable case $z = z_0, w = w_0$, as before. Since the tangent direction of S_f at P_D is $(1, -1)$, we are enough to show $\partial_z K - \partial_w K \neq 0$ at P_D . Set $J_1 = (\partial_z K)_{P_D}$ and $J_2 = (\partial_w K)_{P_D}$.

Recall that $p = q$ at P_D , $\partial_z A = \partial_w B$ at P_D and that $\partial_w A = \partial_z B = \partial_{zw} S$. By using them,

$$\begin{aligned} J_1 &= \partial_z q \cdot A + p \cdot \partial_z A - \partial_z p \cdot A - p \cdot \partial_z B, \quad \text{and} \\ J_2 &= \partial_w q \cdot A + p \cdot \partial_z B - \partial_w p \cdot A - p \cdot \partial_z A. \end{aligned}$$

Hence

$$\begin{aligned} J_1 - J_2 &= (\partial_z q - \partial_z p - \partial_w q + \partial_w p)A + 2p(\partial_z A - \partial_z B) \\ &= -(1 + 2\varepsilon z)A + 2p\partial_z(A - B). \end{aligned}$$

On the other hand, by the equation on $A - B$ in the proof of Lemma 5.5, we see that

$$\partial_z(A - B)_{P_D} = -(6\varepsilon^2 z^2 - 4\varepsilon z + 1 - 2\varepsilon^2).$$

Substituting this to the equation on $J_1 - J_2$ above, and using $A = p(1 - 2\varepsilon z)$ at P_D , we obtain

$$J_1 - J_2 = -z(1 - 2\varepsilon z)(8\varepsilon^2 z^2 - 8\varepsilon z - 4\varepsilon^2 + 3).$$

It is now easy to see that $J_1 - J_2 \neq 0$ at P_D , since the last factor is positive on $|z| \leq 1$ for small ε . \square

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