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# Garsia-Haiman modules for hook partitions and Green polynomials with two variables

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## Abstract

We consider Garsia-Haiman modules for the symmetric groups, a doubly graded generalization of Springer modules. Our main interest lies in singly graded submodules of a Garsia-Haiman module. We show that these submodules satisfy a certain combinatorial property, and verify that this property is implied by a behavior of Macdonald polynomials at roots of unity.

## 1 Introduction

The Garsia-Haiman modules are doubly graded modules for the symmetric groups, introduced by A. Garsia and M. Haiman [GH] to prove Macdonald's positivity conjecture [M1]. These modules are defined for partitions of positive integers  $n$ , denoted by  $D_\mu$ . The dimension of  $D_\mu$  is given by  $n!$  whenever  $\mu$  is a partition of  $n$  [H2]. As this fact implies, the Garsia-Haiman modules  $D_\mu$  are isomorphic to the coinvariant algebra  $R_n$  of  $S_n$ , the left regular representation of  $S_n$ .

Let  $D_\mu = \bigoplus_{r,s} D_\mu^{r,s}$  be the homogeneous decomposition of  $D_\mu$ . In this paper, we concern with module structures of homogeneous components  $D_\mu^{r,s}$ . For each  $s$ , let

$$D_\mu^{*,s} = \bigoplus_r D_\mu^{r,s}$$

be the direct sum of homogeneous components of  $D_\mu$  whose second coordinates of degrees are  $s$ . When  $s = 0$ , a fundamental fact of the Kostka-Macdonald coefficients [M, VI, (8.12)] implies that  $D_\mu^{*,0}$  is isomorphic to the Springer module  $R_\mu$  [Sp1, Sp2] corresponding to  $\mu$ .

In [Mt2], the author considers a certain combinatorial property of the Springer modules, and give an interpretation for the property in terms of representation theory of the symmetric groups. For related works, see [BLM, KW, Mt1, MN1, MN2, Sh, St]. The aim of this paper is to show a similar result for the Garsia-Haiman module  $D_\mu$  in the case where the corresponding partition  $\mu$  is a hook. Let  $l$  be a positive integer not larger than the leg length of the hook  $\mu$ , and let

$$D_\mu^{*,s}(k;l) := \bigoplus_{r \equiv k \pmod{l}} D_\mu^{r,s},$$

be the direct sum of homogeneous components of  $D_\mu^{*,s}$  whose degrees are congruent to  $k$  modulo  $l$  for each  $k = 0, 1, \dots, l-1$ . Let  $W_\mu(l)$  be the subgroup of  $S_n$  constructed in [Mt2]. For a fixed  $s$ , we show that there exist  $W_\mu(l)$ -modules  $Z_\mu^{*,s}(k; l)$  ( $k = 0, 1, \dots, l-1$ ) of equal dimension satisfying

$$D_\mu^{*,s}(k; l) \cong_{S_n} \text{Ind}_{W_\mu(l)}^{S_n} Z_\mu^{*,s}(k; l) \quad (1.1)$$

for each  $k$ . This fact implies that the submodules  $D_\mu^{*,s}(k; l)$  have the same dimension (if  $s$  is fixed).

Suppose that  $\mu$  is a hook of the form  $(h, 1^m)$  ( $h > 1$ ),  $l$  an integer such that  $1 \leq l \leq m$ , and  $m = lq + k$  for some integers  $q, k$  such that  $0 \leq k \leq l-1$ . Let  $r = h + k$ . In this case, if  $C_l$  is the cyclic subgroup of length  $l$  and  $S_r$  is the symmetric group of  $r$  letters, then those  $l$  isomorphisms (1.1) are equivalent to a single  $S_n \times C_l$ -isomorphism

$$D_\mu^{*,s} \cong \text{Ind}_{S_r}^{S_n} D_{\hat{\mu}}^{*,s} \quad (1.2)$$

where  $D_{\hat{\mu}}$  is the Garsia-Haiman module corresponding to  $\hat{\mu} = (h, 1^k)$ . To show (1.2), we consider a behavior of a Green polynomial  $\tilde{X}_\rho^\mu(q, t)$  with two variables at roots of unity. Let  $\zeta$  be an  $l$ -th root of unity with  $\zeta \neq 1$ . Then (1.2) is equivalent to a recursive formula which shows that the Green polynomial  $\tilde{X}_\rho^\mu(q, t)$  at  $t = \zeta$  is a scalar multiple of the Green polynomial  $\tilde{X}_{\hat{\rho}}^{\hat{\mu}}(q, t)$  at  $t = \zeta$ . Moreover, the scalar factor is a non-negative integer, which coincides with the cardinality of a set of permutations. The proof of the recursive formula is based on a behavior of modified Macdonald polynomials at roots of unity.

Thorough out this paper, we mainly follow [M] for notation on partitions and symmetric functions. The ground field is the complex number field  $\mathbf{C}$ .

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## 2 Preliminaries

Let  $\mu = (\mu_i)$  be a partition, and  $\mu'$  its conjugate. The length of  $\lambda$  is written by  $l(\lambda)$ . We set  $|\mu|$  to be the total sum of the components of  $\mu$ . In this case, we call  $\mu$  a partition of  $n$ , and write  $\mu \vdash n$ . The integer  $n$  is called the size of  $\mu$ . Let  $m_i(\mu)$  be the multiplicity of  $i$  in the partition  $\mu$ , and  $M_\mu$  the maximum multiplicity of  $\mu$ . For a partition  $\mu = (\mu_i) = (i^{m_i})$ , we set  $n(\mu) = \sum_{i \geq 1} (i-1)\mu_i$  and  $z_\mu = \prod_{i \geq 1} i^{m_i} m_i!$ . We identify a partition with its Young diagram. In this point of view, a partition of the form  $(h, 1^m)$  is called a hook. The leg length of a hook  $(h, 1^m)$  is defined to be  $m$  (if  $h > 1$ ). It is clear that  $M_\mu$  gives the leg length when  $\mu$  is a hook  $(h, 1^m)$  ( $h > 1$ ).

Let  $q, t$  be indeterminates and  $\mathbf{Q}$  the rational number field. Let  $\Lambda$  denote the ring of symmetric functions over  $\mathbf{Z}$ , and let  $\Lambda_{q,t} := \Lambda \otimes_{\mathbf{Z}} \mathbf{Q}(q, t)$ . The homogeneous decomposition of

$\Lambda$  is denoted by  $\Lambda = \bigoplus_{n \geq 0} \Lambda^n$ , and the space  $\Lambda_{q,t}^n = \Lambda^n \otimes_{\mathbf{Z}} \mathbf{Q}(q, t)$  gives the  $n$ -th homogeneous component of  $\Lambda_{q,t}^n$ . Let  $p_\lambda(x)$  denotes the power-sum function corresponding to a partition  $\lambda$ . Then the set  $\{p_\lambda(x) \mid |\lambda| = n\}$  gives a (homogeneous) basis of  $\Lambda^n$ . We define an inner product on  $\Lambda_{q,t}$  by

$$\langle p_\lambda(x), p_\mu(x) \rangle_{q,t} := \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1 - q^i}{1 - t^i},$$

where  $\delta_{\lambda\mu}$  denotes the Kronecker delta. When  $q = t = 0$ , this gives the usual inner product  $\langle \cdot, \cdot \rangle$  defined by  $\langle p_\lambda(x), p_\mu(x) \rangle = \delta_{\lambda\mu} z_\lambda$ .

Let  $x = (x_1, x_2, \dots)$  and  $y = (y_1, y_2, \dots)$  be two (infinite) sequences of indeterminates, and let

$$\Pi(x, y; q, t) := \prod_{i,j} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty},$$

where the symbol  $(a; q)_\infty$  stands for the infinite product  $\prod_{r=0}^{\infty} (1 - aq^r)$ . Then it is known [M, VI, (2.7)] that:

**Proposition 1** *For  $\mathbf{Q}(q, t)$ -basis  $\{u_\lambda\}_{|\lambda|=n}$  and  $\{v_\lambda\}_{|\lambda|=n}$  of  $\Lambda_{q,t}^n$ , the following conditions are equivalent:*

1.  $\{u_\lambda\}$  and  $\{v_\lambda\}$  are dual to each other with respect to the inner product  $\langle \cdot, \cdot \rangle_{q,t}$ ,
2.  $\sum_\lambda u_\lambda(x) v_\lambda(y) = \Pi(x, y; q, t)$ .

Note that, when  $q = t = 0$ , this shows that two homogeneous basis  $\{u_\lambda\}, \{v_\lambda\}$  of  $\Lambda^n$  are dual to each other with respect to  $\langle \cdot, \cdot \rangle$  if and only if  $\sum_\lambda u_\lambda(x) v_\lambda(y) = \prod_{i,j} 1/(1 - x_i y_j)$ .

Let  $h_r(x)$  denote the  $r$ -th complete symmetric function, and  $p_l(x)$  the  $l$ -th power-sum function as usual. We denote by  $(p_l \circ h_r)(x)$  the *plethysm* [M, I, 8] of  $h_r(x)$  by  $p_l(x)$ . It is known [M, I, (2,14)'] and (8.4)] that

$$(p_l \circ h_r)(x) = \sum_{\lambda \vdash r} z_\lambda^{-1} p_{l\lambda}(x), \tag{2.1}$$

where  $l\lambda := (l\lambda_i)$  for  $\lambda = (\lambda_i)$ .

Let  $n$  be a positive integer, and  $S_n$  the symmetric group of  $n$  letters. For each partition  $\mu \vdash n$ , we can define a doubly graded  $S_n$ -module, called the *Garsia-Haiman module* [GH] corresponding to  $\mu$ . We denote its (doubly) grading by

$$D_\mu = \bigoplus_{r,s} D_\mu^{r,s}$$

where each homogeneous component  $D_\mu^{r,s}$  is  $S_n$ -stable. The dimension of  $D_\mu$  is known to be  $n!$  [H2]. As an  $S_n$ -module,  $D_\mu$  is isomorphic to the left regular representation  $\mathbf{C}[S_n]$  of  $S_n$

for each  $\mu \vdash n$ . We can see [H2, 2.2, 3.7] that the highest degrees in  $r$  and  $s$  of  $D_\mu$  are  $n(\mu)$  and  $n(\mu')$  respectively. For each  $s = 0, 1, \dots, n(\mu')$ , we define

$$D_\mu^{*,s} := \bigoplus_{r=0}^{n(\mu)} D_\mu^{r,s},$$

which is a singly graded submodule of  $D_\mu$ .

Let  $\mu$  be a partition of  $n$ . Let  $\text{char}_{q,t} D_\mu$  be the graded character of  $D_\mu$ , i.e., for a partition  $\rho \vdash n$ , we define

$$\text{char}_{q,t} D_\mu(\rho) := \sum_{r=0}^{n(\mu)} \sum_{s=0}^{n(\mu')} q^s t^r \text{char} D_\mu^{r,s}(\rho),$$

where  $\text{char} D_\mu^{r,s}(\rho)$  is the character value of the  $S_n$ -module  $D_\mu^{r,s}$  on the conjugacy class of cycle type  $\rho$  in  $S_n$ .

### 3 Modified Macdonald polynomials at roots of unity

Let  $n$  be a positive integer and  $\lambda \vdash n$ . Let  $P_\lambda(x; q, t)$  be the Macdonald polynomial corresponding to  $\lambda$  and  $J_\lambda(x; q, t)$  its integral form. We define

$$Q_\lambda(x; q, t) := \langle P_\lambda(x; q, t), P_\lambda(x; q, t) \rangle_{q,t}^{-1} P_\lambda(x; q, t)$$

for each partition  $\lambda$ , which is also called a Macdonald polynomial. By the definition, it is easy to see that  $\{P_\lambda(x; q, t)\}$  and  $\{Q_\lambda(x; q, t)\}$  are dual to each other with respect to  $\langle \cdot, \cdot \rangle_{q,t}$ , i.e.,

$$\sum_{\lambda} P_\lambda(x; q, t) Q_\lambda(y; q, t) = \pi(x, y; q, t).$$

We now consider the following modification of symmetric functions. Let  $f(x)$  be a symmetric function with a parameter  $t$ . Since the ring of symmetric functions is generated by power-sum functions  $\{p_r(x) | r \geq 1\}$  as an algebra,  $f(x)$  is written in the form  $F(p_1(x), p_2(x), \dots)$  for some polynomial  $F = F(\xi_1, \xi_2, \dots)$ . We then introduce an automorphism of the ring of symmetric functions as follows:

$$\tilde{f}(x) := F\left(\frac{p_1(x)}{1-t}, \frac{p_2(x)}{1-t^2}, \dots, \frac{p_k(x)}{1-t^k}, \dots\right),$$

which coincides with a substitution of variables  $\tilde{f}(x) = f(x')$  where  $x' = \{x_i t^{j-1} | i, j \geq 1\}$  (c.f., [M, p. 234, ex. 7]). Clearly, the inverse of  $f \mapsto \tilde{f}$  is given by

$$\hat{f}(x) = F((1-t)p_1(x), (1-t^2)p_2(x), \dots, (1-t^k)p_k(x), \dots).$$

In  $\lambda$ -ring notation, these two automorphisms are written by  $\tilde{f}(x) = f(x/(1-t))$ ,  $\hat{f}(x) = f((1-t)x)$ .

Now it is possible to prove [DM] that

$$\Pi \left( x, \frac{1-q}{1-t}y; q, t \right) = \prod_{i,j} \frac{1}{1-x_i y_j}.$$

This shows that  $\{Q_\lambda((1-q)x/(1-t); q, t)\}$  is the dual of  $\{P_\lambda(x; q, t)\}$  with respect to the usual inner product  $\langle \cdot, \cdot \rangle$ . We denote  $Q_\lambda((1-q)x/(1-t); q, t)$  by  $Q'_\lambda(x; q, t)$ .

**Proposition 2**  $\langle P_\lambda(x; q, t), Q'_\mu(x; q, t) \rangle = \delta_{\lambda\mu}$  for any two partitions  $\lambda, \mu$ .

Let  $l$  be a positive integer which satisfies  $l \leq M_\mu$ , and suppose that  $m_i = m_i(\mu) \geq l$  for some  $i$ . Define  $\mu \setminus (i^l)$  to be the partition  $(1^{m_i} \cdots (i-1)^{m_i} i^{m_i-l} (i+1)^{m_{i+1}} \cdots n^{m_n})$ . In other words,  $\mu \setminus (i^l)$  is a partition whose Young diagram is obtained by subtracting a  $i \times l$  rectangle from the Young diagram of  $\mu$ . The following two results are fundamental to our argument. Let

$$\tilde{H}_\mu(x; q, t) = t^{n(\mu)} \tilde{J}_\mu(x; q, t^{-1}) \quad (3.1)$$

be the transformed Macdonald polynomials introduced by Haglund, Haiman and Loehr.

**Proposition 3 (Descouens-Morita-Thibon)** *If  $m_r(\mu) \geq l$  for some  $r$ , then it follows that :*

1.  $\tilde{H}_\mu(x; q, \zeta) = \tilde{H}_{(r^l)}(x; q, \zeta) \tilde{H}_{\mu \setminus (r^l)}(x; q, \zeta)$ ,
2.  $\tilde{H}_{(r^l)}(x; q, \zeta) = \left\{ \prod_{i=1}^r (1 - q^{il}) \right\} (p_l \circ h_r) \left( \frac{x}{1-q} \right)$ .

These identities are called the *factorization formula*, and the *plethystic formula*, respectively. The proof of these formulas will be found in [DM].

## 4 Green polynomials with two variables

Let  $\mu, \rho$  be partitions of the same size. As in [M, VI, 8], we define the Green polynomials  $X_\rho^\mu(q, t)$  with two variables  $q, t$  by

$$J_\mu(x; q, t) = \sum_{\rho} z_\rho(t)^{-1} X_\rho^\mu(q, t) p_\rho(x),$$

where  $z_\rho(t) = z_\rho \prod_{i=1}^{l(\rho)} (1 - t^{\rho_i})^{-1}$  for  $\rho = (\rho_i)$ .

**Lemma 4** *For two partitions  $\mu, \rho$ , we have*

$$X_\rho^\mu(q, t) = \langle \tilde{J}_\mu(x; q, t), p_\rho(x) \rangle,$$

where  $\langle \cdot, \cdot \rangle$  is the usual inner product.

*Proof.* We have

$$\begin{aligned}
\tilde{J}_\mu(x; q, t) &= J_\mu\left(\frac{x}{1-t}; q, t\right) \\
&= \sum_{\rho} z_{\rho}^{-1} \prod_{i=1}^{l(\rho)} (1-t^{\rho_i}) X_{\rho}^{\mu}(q, t) p_{\rho}(x) \prod_{i=1}^{l(\rho)} (1-t^{\rho_i})^{-1} \\
&= \sum_{\rho} z_{\rho}^{-1} X_{\rho}^{\mu}(q, t) p_{\rho}(x).
\end{aligned}$$

The remaining part of the proof follows from the orthogonality of  $\{p_{\lambda}(x)\}$  [M, I, (4.7)].  $\square$

For any two partitions  $\mu, \rho$  of the same size, we define

$$\tilde{X}_{\rho}^{\mu}(q, t) := t^{n(\mu)} X_{\rho}^{\mu}(q, t^{-1}),$$

which is also called a Green polynomial. It follows immediately from (3.1) that

$$\tilde{X}_{\rho}^{\mu}(q, t) = \langle \tilde{H}_{\mu}(x; q, t), p_{\rho}(x) \rangle. \quad (4.1)$$

Let  $K_{\lambda\mu}(q, t)$  be the Kostka-Macdonald coefficient  $(\lambda, \mu \vdash n)$  [M, VI, (8.11)], and let

$$\tilde{K}_{\lambda}(q, t) := t^{n(\mu)} K_{\lambda\mu}(q, t^{-1}).$$

Then it follows [M, VI, (8.20)] that

$$\tilde{X}_{\rho}^{\mu}(q, t) = \sum_{\lambda} \chi_{\rho}^{\lambda} \tilde{K}_{\lambda\mu}(q, t), \quad (4.2)$$

where  $\chi_{\rho}^{\lambda}$  is the value of the irreducible character of  $S_n$  corresponding to  $\lambda$ , evaluated on the conjugacy class of cycle type  $\rho$ .

As a consequence of Haiman's  $n!$  theorem [H2], we have

$$\tilde{K}_{\lambda\mu}(q, t) = \sum_{r,s} q^s t^r [\text{char} D_{\mu}^{r,s} : \chi^{\lambda}], \quad (4.3)$$

where  $[\text{char} D_{\mu}^{r,s} : \chi^{\lambda}]$  stands for the multiplicity of  $\chi^{\lambda}$  in the submodule  $D_{\mu}^{r,s}$ . This shows that  $\tilde{K}_{\lambda\mu}(q, t)$  is a polynomial in  $q, t$  with non-negative integer coefficients. Thus  $\tilde{X}_{\rho}^{\mu}(q, t)$  is a polynomial in  $q, t$  with integer coefficients. By (4.2) and (4.3), we have

$$\tilde{X}_{\rho}^{\mu}(q, t) = \text{char}_{q,t} D_{\mu}(\rho).$$

At the end of this section, we consider a behavior of Green polynomials  $\tilde{X}_{\rho}^{\mu}(q, t)$  at roots of unity for  $\mu$  is a hook. Let  $\mu = (h, 1^m)$  be a hook ( $h > 1$ ), and let  $l$  be an integer such that  $2 \leq l \leq m$ . We may assume that  $l > 1$ , since the case  $l = 1$  is trivial for our aim.

Suppose that  $m = lq + k$  for non-negative integers  $q, k$  with  $k \leq l - 1$ , and let  $\hat{\mu} = (h, 1^k) \vdash r$  ( $r = h + k$ ). Let  $\zeta$  be a primitive  $l$ -th root of unity. Recall [M, p. 75, ex.3] that

$$\langle p_k f, g \rangle = \left\langle f, \frac{\partial}{\partial p_k} g \right\rangle, \quad f, g \in \Lambda. \quad (4.4)$$

For any two partition  $\lambda, \mu$ , we denote the partition  $(i^{m_i(\lambda)+m_i(\mu)})$  by  $\lambda \cup \mu$ .

**Theorem 5** *Let  $j$  be an integer with  $1 \leq j \leq l - 1$ , and suppose that  $\zeta^j$  is a primitive  $k$ -th root of unity ( $k|l$ ). If  $\tilde{X}_\rho^\mu(q, \zeta^j) \neq 0$ , then  $\rho$  should be of the form  $(k^e) \cup \nu$  for some  $\nu \vdash r$  and  $e = ql/k$ . Moreover, for  $\rho = (k^e) \cup \nu$  ( $\nu \vdash r$ ), we have*

$$\tilde{X}_\rho^\mu(q, \zeta^j) = k^e e! \binom{e + m_k(\nu)}{e} \tilde{X}_\nu^{\hat{\mu}}(q, \zeta^j).$$

*Proof.* It follows from Proposition 3 that

$$\tilde{H}_\mu(x; q, \zeta^j) = \tilde{H}_{\hat{\mu}}(x; q, \zeta^j) \left\{ \tilde{H}_{(1^k)}(x; q, \zeta^j) \right\}^{lq/k},$$

and

$$\tilde{H}_{(1^k)}(x; q, \zeta^j) = (1 - q^k)(p_k \circ h_1) \left( \frac{x}{1 - q} \right).$$

From (2.1), we have  $(p_k \circ h_1)(x) = z_{(1)}^{-1} p_k(x)$ , hence  $\tilde{H}_{(1^k)}(x; q, \zeta^j) = p_k(x)$ . Therefore it follows from (4.1) that

$$\tilde{X}_\rho^\mu(q, \zeta^j) = \langle p_k(x)^{lq/k} \tilde{H}_{\hat{\mu}}(x; q, \zeta^j), p_\rho(x) \rangle. \quad (4.5)$$

By (4.4), (4.5) equals

$$\left\langle \tilde{H}_{\hat{\mu}}(x; q, \zeta^j), \left( p \frac{\partial}{\partial p_k(x)} \right)^{lq/k} p_\rho(x) \right\rangle.$$

Hence the condition  $\tilde{X}_\rho^\mu(q, \zeta^j) \neq 0$  forces  $(k^{lq/k}) \subset \rho$ .

Suppose that  $\rho = (k^{lq/k}) \cup \nu$  for some  $\nu \vdash r$ . Then we have

$$\begin{aligned} \tilde{X}_\rho^\mu(q, \zeta^j) &= \left\langle \tilde{H}_{\hat{\mu}}(x; q, \zeta^j), \left( k \frac{\partial}{\partial p_k(x)} \right)^{lq/k} p_{(k^{lq/k}) \cup \nu}(x) \right\rangle \\ &= k^{lq/k} (m_k(\nu) + lq/k) (m_k(\nu) + lq/k - 1) \cdots (m_k(\nu) + 1) \left\langle \tilde{H}_{\hat{\mu}}(x; q, \zeta^j), p_\nu \right\rangle \\ &= k^{lq/k} (lq/k)! \binom{m_k(\nu) + lq/k}{lq/k} \left\langle \tilde{H}_{\hat{\mu}}(x; q, \zeta^j), p_\nu \right\rangle. \end{aligned}$$

Since  $\tilde{X}_\nu^{\hat{\mu}}(q, \zeta^j) = \left\langle \tilde{H}_{\hat{\mu}}(x; q, \zeta^j), p_\nu \right\rangle$ , it follows that

$$\tilde{X}_\rho^\mu(q, \zeta^j) = k^{lq/k} (lq/k)! \binom{m_k(\nu) + lq/k}{lq/k} \tilde{X}_\nu^{\hat{\mu}}(q, \zeta^j),$$

where  $e = lq/k$ . □



## 5 Garsia-Haiman modules for hook partitions

Let  $\mu$  be a hook  $(h, 1^m)$ , and let  $n = h + m$ . Let  $l$  be a positive integer with  $l \leq m$ , and suppose that  $m = lq + k$  for non-negative integers  $q, k$  with  $k \leq l - 1$ . For a fixed integer  $s$  with  $0 \leq s \leq n(\mu')$ , let  $D_\mu^{*,s}$  denote the singly graded submodule  $\bigoplus_{r=0}^{n(\mu)} D_\mu^{r,s}$  of  $D_\mu$ . For each  $k = 0, 1, \dots, l - 1$ , we define

$$D_\mu^{*,s}(k; l) := \bigoplus_{r \equiv k \pmod{l}} D_\mu^{r,s}.$$

In this section, we show that these submodules  $D_\mu^{*,s}(k; l)$  are induced from modules of equal dimension for some subgroup  $W_\mu(l)$  of  $S_n$ , and consequently they have the same dimension.

Let  $\zeta$  be a primitive  $l$ -th root of unity. Set  $r = n - lq$  and  $\hat{\mu} = (h, 1^k)$  which is a hook partition of  $r$ . Let  $S_{\{lq+1, lq+2, \dots, n\}}$  be the symmetric group of the last  $r$  letters. This is a subgroup of  $S_n$  isomorphic to  $S_r$ . Let  $W_\mu(l)$  be a subgroup  $C_l \times S_r$  of  $S_n$ , where  $C_l$  is a cyclic group of order  $l$  generated by the cyclic permutation  $a = (1, 2, \dots, l)$ . For each integer  $k = 0, 1, \dots, l - 1$ , let  $\varphi_l^{(k)}$  be the irreducible representation of  $C_l$ , which maps the generator  $a$  to  $\zeta^k$ . For each  $k = 0, 1, \dots, l - 1$ , we define a  $W_\mu(l)$ -module  $Z_\mu^{*,s}(k; l)$  by

$$Z_\mu^{*,s}(k; l) := \bigoplus_{r=0}^{n(\hat{\mu})} \varphi_l^{(k-r)} \otimes D_{\hat{\mu}}^{r,s}$$

Note that the dimension of  $Z_\mu^{*,s}(k; l)$  coincides with  $\dim D_{\hat{\mu}}^{*,s}$  for all  $k$ , i.e.,  $\dim Z_\mu^{*,s}(k; l)$  does not depend on  $k$  if  $s$  is fixed, since the degrees of  $\varphi_l^{(k)}$ 's are all one.

The following theorem is the main result of this paper.

**Theorem 6** *For each  $k = 0, 1, \dots, l - 1$ , we have*

$$D_\mu^{*,s}(k; l) \cong_{S_n} \text{Ind}_{W_\mu(l)}^{S_n} Z_\mu^{*,s}(k; l).$$

**Corollary 7** *For each fixed  $s \in \{0, 1, \dots, n(\mu')\}$ , we have  $\dim D_\mu^{*,s}(k; l) = (\dim D_{\hat{\mu}}^{*,s})/l$  for each  $k = 0, 1, \dots, l - 1$ .*

Since we have  $\tilde{K}_{\lambda\mu}(q, t) = \tilde{K}_{\lambda\mu'}(q, t)$  [H1, Proposition 2.5], it follows that, for each  $r = 0, 1, \dots, n(\mu')$ ,

$$D_\mu^{r,*} \cong D_{\mu'}^{*,r}$$

as graded  $S_n$ -modules. Let  $l'$  be a positive integer such that  $l' \leq M_{\mu'}$ , and

$$D_\mu^{r,*}(k'; l') := \bigoplus_{s \equiv k' \pmod{l'}} D_\mu^{r,s}$$

for each  $k' = 0, 1, \dots, l' - 1$  and  $r = 0, 1, \dots, n(\mu)$ .

**Proposition 8** *Let  $\mu$  be a hook,  $m$  a positive integer with  $l' \leq M_{\mu'}$ , and  $r = 0, 1, \dots, n(\mu)$  fixed. Then there exist  $W_{\mu'}(l')$ -modules  $Z_{\mu'}^{r,*}(k'; l')$  ( $k' = 0, 1, \dots, l' - 1$ ) of equal dimension such that*

$$D_{\mu}^{r,*}(k'; l') \cong \text{Ind}_{W_{\mu'}(l')}^{S_n} Z_{\mu'}^{r,*}(k'; l')$$

for each  $k'$ . In particular, we have  $\dim D_{\mu}^{r,*}(k'; l') = (\dim D_{\mu'}^{r,*})/l'$  for each  $k'$ .

In what follows, we shall see that Theorem 6 is equivalent to the existence of an  $S_n \times C_l$ -module isomorphism between  $D_{\mu}^{*,s}$  and  $\text{Ind}_{S_r}^{S_n} D_{\hat{\mu}}^{*,s}$ . This fact is originally suggested by T. Shoji in the case of the Springer modules. The  $S_n \times C_l$ -module structures are defined as follows. The  $S_n$ -module structures are natural ones. Let  $a$  be the generator of  $C_l$ . The action of  $a$  on  $D_{\mu}^{*,s}$  is defined by

$$ax = \zeta^r x, \quad x \in D_{\mu}^{r,s}.$$

Note that  $\text{Ind}_{S_r}^{S_n} D_{\hat{\mu}}^{*,s} = \bigoplus_{\sigma \in S_n/S_r} \sigma \otimes D_{\hat{\mu}}^{*,s}$ . Then the  $C_l$ -action on  $\text{Ind}_{S_r}^{S_n} D_{\hat{\mu}}^{*,s}$  is defined by

$$a(\sigma \otimes x) = \sigma a^{-1} \otimes ax$$

for  $\sigma \in S_n/S_r$  and  $x \in D_{\hat{\mu}}^{*,s}$ . The following theorem is proved in the next section.

**Theorem 9** *As  $S_n \times C_l$ -modules,  $D_{\mu}^{*,s} \cong \text{Ind}_{S_r}^{S_n} D_{\hat{\mu}}^{*,s}$  for each  $s = 0, 1, \dots, n(\mu')$ .*

We prove the equivalence of Theorem 6 and Theorem 9. Suppose that  $D_{\mu}^{*,s}$  is isomorphic to  $\text{Ind}_{S_r}^{S_n} D_{\hat{\mu}}^{*,s}$  as  $S_n \times C_l$ -module. we compare eigenspaces of these modules with respect to the action of  $a \in C_l$ . It is clear from the definition that the  $\zeta^k$ -eigenspace of the action of  $a$  in  $D_{\mu}^{*,s}$  is  $D_{\mu}^{*,s}(k; l)$ . On the other hand, we have

$$\begin{aligned} \text{Ind}_{S_r}^{S_n} D_{\hat{\mu}}^{*,s} &= \bigoplus_{\sigma \in S_n/S_r} \sigma \otimes D_{\hat{\mu}}^{*,s} \\ &= \bigoplus_{\sigma \in S_n/S_r} \bigoplus_{i=0}^{l-1} \sigma a^i \otimes D_{\hat{\mu}}^{*,s}. \end{aligned}$$

If we define  $b_k = \sum_{i=0}^{l-1} \zeta^{ki} a^i$ , then it follows that  $b_k a^{-1} = \zeta^k b_k$  for each  $k = 0, 1, \dots, l - 1$ . Thus we identify  $\mathbf{C}b_k$  with the representation space of  $\varphi_l^{(k)}$  for each  $k$ . Now we have

$$\text{Ind}_{S_r}^{S_n} D_{\hat{\mu}}^{*,s} = \bigoplus_{\sigma \in S_n/S_r} \bigoplus_{k=0}^{l-1} \sigma b_k \otimes D_{\hat{\mu}}^{*,s},$$

and it is clear from the definition of  $S_n \times C_l$ -action that the  $\zeta^k$ -eigenspace of  $a$  coincides with  $D_{\mu}^{*,s}(k; l)$ . These arguments can be traced back, which proves that Theorem 6 and Theorem 9 are equivalent.

## 6 Proof

This section is devoted to the proof of Theorem 9. Since we work on a field of characteristic zero, it is enough to show that

$$\text{char} D_{\mu}^{*,s}(w, a^j) = \text{char} \text{Ind}_{S_r}^{S_n} D_{\hat{\mu}}^{*,s}(w, a^j)$$

for every  $(w, a^j) \in S_n \times C_l$  and  $s = 0, 1, \dots, n(\mu')$ , which is equivalent to show

$$\sum_{s=0}^{n(\mu')} q^s \text{char} D_{\mu}^{*,s}(w, a^j) = \sum_{s=0}^{n(\mu')} q^s \text{char} \text{Ind}_{S_r}^{S_n} D_{\hat{\mu}}^{*,s}(w, a^j). \quad (6.1)$$

Note that

$$\text{Ind}_{S_r}^{S_n} D_{\hat{\mu}}^{*,s} = \bigoplus_{\sigma \in S_n/S_r} \sigma \otimes D_{\hat{\mu}}^{*,s}. \quad (6.2)$$

Here we identify  $S_n/S_r$  with the set of permutations  $\sigma = [\sigma_1, \sigma_2, \dots, \sigma_n]$  satisfying  $\sigma_{n-r+1} < \sigma_{n-r+2} < \dots < \sigma_n$ . If we suppose that

$$\sum_{s=0}^{n(\mu')} q^s \text{char} \text{Ind}_{S_r}^{S_n} D_{\hat{\mu}}^{*,s}(w, a^j) \neq 0, \quad (6.3)$$

then there exists  $s$  satisfying

$$\text{char}(\sigma \otimes D_{\hat{\mu}}^{*,s})(w, a^j) \neq 0 \quad (6.4)$$

for some  $\sigma \in S_n/S_r$ . If  $\mathcal{B}_r^s$  denotes a homogeneous basis of  $D_{\hat{\mu}}^{*,s}$ , then we have

$$\begin{aligned} \text{char}(\sigma \otimes D_{\hat{\mu}}^{*,s})(w, a^j) &= \sum_{x \in \mathcal{B}_r^s} (w, a^j)(\sigma \otimes x)|_{\sigma \otimes x} \\ &= \sum_{x \in \mathcal{B}_r^s} \zeta^{js} w \sigma a^{-j} \otimes x|_{\sigma \otimes x}. \end{aligned}$$

Hence (6.4) implies that  $w \sigma a^{-j} \equiv \sigma$  modulo  $S_r$ , which shows that  $w$  is conjugate (in  $S_n$ ) to an element of the form  $a^j \tau$  for some  $\tau \in S_r$ . To summarize:

**Lemma 10** *If the condition (6.3) holds, then  $w$  is conjugate to an element of  $W_{\mu}(l)$  of the form  $a^j \tau$  ( $\tau \in S_r$ ).*

Suppose that  $w = a^j \tau$  ( $\tau \in S_r$ ). For each  $\sigma \in S_n/S_r$ , let  $\pi_{\sigma}$  be an element of  $S_r$  satisfying  $w \sigma a^{-j} = \sigma \pi_{\sigma}$ . Then, by (6.3), it follows that

$$\text{char} \text{Ind}_{S_r}^{S_n} D_{\hat{\mu}}^{*,s}(w, a^j) = \sum_{\substack{\sigma \in S_n/S_r \\ w \sigma a^{-j} \equiv \sigma \pmod{S_r}}} \sum_{x \in \mathcal{B}_r^s} \sigma \otimes \pi_{\sigma} x|_{\sigma \otimes x}.$$

Now it is clear that  $\sum_{x \in \mathcal{B}_r^s} \sigma \otimes \pi_\sigma x|_{\sigma \otimes x} = \text{char} D_{\hat{\mu}}^{*,s}(\pi_\sigma)$ . Hence it follows that

$$\text{char} \text{Ind}_{S_r}^{S_n} D_{\hat{\mu}}^{*,s}(w, a^j) = \sum_{\substack{\sigma \in S_n/S_r \\ w\sigma a^{-j} \equiv \sigma \pmod{S_r}}} \text{char} D_{\hat{\mu}}^{*,s}(\pi_\sigma).$$

Since  $\tau$  and  $\pi_\sigma$  are conjugate in  $S_n$ , we have  $\text{char} D_{\hat{\mu}}^{*,s}(\pi_\sigma) = \text{char} D_{\hat{\mu}}^{*,s}(\tau)$  for each  $\sigma$ , which implies that

$$\text{char} \text{Ind}_{S_r}^{S_n} D_{\hat{\mu}}^{*,s}(w, a^j) = \#\{\sigma \in S_n/S_r | w\sigma a^{-j} \equiv \sigma \pmod{S_r}\} \text{char} D_{\hat{\mu}}^{*,s}(\tau).$$

Therefore, we have

$$\sum_{s=0}^{n(\mu')} q^s \text{char} \text{Ind}_{S_r}^{S_n} D_{\hat{\mu}}^{*,s}(w, a^j) = \#\{\sigma \in S_n/S_r | w\sigma a^{-j} \equiv \sigma \pmod{S_r}\} \tilde{X}_{\rho(\tau)}^{\hat{\mu}}(q).$$

Suppose that the order of  $a^j$  is  $k$ , and let  $m = lq + c$  where  $0 \leq c \leq l - 1$ . Let  $e = lq/k$ . Then it is known [Mt1, Proposition 16] that

$$\#\{\sigma \in S_n/S_r | w\sigma a^{-j} \equiv \sigma \pmod{S_r}\} = k^e e! \binom{e + m_k(\rho(\tau))}{e}.$$

To summarize:

**Lemma 11** *Let  $w$  be an element of  $W_\mu(l)$  of the form  $a^j \tau$  ( $\tau \in S_r$ ). Then it follows that*

$$\sum_{s=0}^{n(\mu')} q^s \text{char} \text{Ind}_{S_r}^{S_n} D_{\hat{\mu}}^{*,s}(w, a^j) = k^e e! \binom{e + m_k(\rho(\tau))}{e} \tilde{X}_{\rho(\tau)}^{\hat{\mu}}(q).$$

On the other hand, since the graded character of  $D_\mu$  coincide with the Green polynomial  $\tilde{X}_\rho^\mu(q, t)$ , considering the action of  $C_l$ , it is not difficult to show that

$$\tilde{X}_{\rho(w)}^\mu(q, \zeta_l^j) = \sum_{s=0}^{n(\mu')} q^s \text{char} D_{\hat{\mu}}^{*,s}(w, a^j).$$

By Theorem 5, Lemma 10 and Lemma 11, we can see that the right hand side of (6.1) satisfies the same condition for the left hand side of (6.1), which completes the proof.

## 7 Factorization of Hilbert polynomials

Let  $\mu \vdash n$  be a partition. The Hilbert polynomial  $h_\mu(q, t)$  of the Garsia-Haiman module  $D_\mu$  is defined by

$$h_\mu(q, t) = \sum_{r,s} q^s t^r \dim D_\mu^{r,s}.$$

Since the dimension of  $D_\mu^{r,s}$  equals the character value of  $D_\mu^{r,s}$  evaluated on the identity element, the Hilbert polynomial  $h_\mu(q, t)$  coincides with the Green polynomial  $\tilde{X}_\rho^\mu(q, t)$  at  $\rho = (1^n)$ . Let  $c_s(t)$  be the coefficient of  $q^s$  in  $h_\mu(q, t)$ . This is a polynomial in  $t$  with non-negative integer coefficients, which gives the Hilbert polynomial of the singly graded submodule  $D_\mu^{*,s}$ . Similarly, the coefficient  $c'_r(q)$  of  $t^r$  gives the Hilbert polynomial of  $D_\mu^{r,*}$ .

We recall the following result [MN1, Lemma 3]. For a positive integer  $l$  and a indeterminate  $t$ , we define

$$[l]_t := \frac{1-t^l}{1-t} = 1 + t + \dots + t^{l-1}.$$

**Lemma 12** *Let  $f(t) = \sum_{i=0}^d c_i t^i$  be a polynomial in  $t$  with coefficients in a field of characteristic zero. Let  $l$  be an integer with  $l > 1$ , and  $a_k$  ( $k = 0, 1, \dots, l-1$ ) the sum of the coefficients of the polynomial  $f$  whose degrees are congruent to  $k$  modulo  $l$ . Let  $\zeta$  be a primitive  $l$ -th root of unity. Then the following conditions are equivalent:*

1.  $a_k = f(1)/l$  for all  $k = 0, 1, \dots, l-1$ ,
2.  $f(\zeta^k) = 0$  for all  $k = 1, 2, \dots, l-1$ ,
3.  $[l]_t$  divides  $f(t)$ .

**Proposition 13** *Let  $\mu = (h, 1^m)$  be a hook. Then there exists a polynomial  $G^\mu(q, t)$  with the variables  $q, t$  satisfying*

$$h_\mu(q, t) = \left\{ \prod_{i=1}^{M_\mu} [i]_t \prod_{j=1}^{M_{\mu'}} [j]_q \right\} G^\mu(q, t).$$

*Proof.* Let  $l$  be a positive integer with  $l \leq M_\mu$ , and  $s = 0, 1, \dots, n(\mu)$  arbitrarily fixed. By Corollary 7, the submodules  $D_\mu^{*,s}(k; l)$  ( $k = 0, 1, \dots, l-1$ ) have the same dimension. This shows that the sum of coefficients of the polynomial  $c_s(t)$  whose degrees are congruent to  $k$  ( $k = 0, 1, \dots, l-1$ ) modulo  $l$  do not depend on  $k$ , which is equivalent to the condition  $[l]_t | c_s(t)$ . Similarly, by Proposition 8, it follows that  $[m]_q | c'_r(q)$  for all positive integer  $m$  with  $m \leq M_{\mu'}$  and all  $r = 0, 1, \dots, n(\mu')$ .  $\square$

**Example 14**  $h_{(3,1,1)}(q, t) = [2]_t [2]_q (1 + 3q + 6q^2 + 3t + 4tq + 3tq^2 + 6t^2 + 3t^2q + t^2q^2)$ ,  
 $h_{(4,1,1)}(q, t) = [2]_t [2]_q [3]_q (1 + 3q + 6q^2 + 10q^3 + 4t + 6tq + 6tq^2 + 4tq^3 + 10t^2 + 6t^2q + 3t^2q^2 + t^2q^3)$ .

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