



Title	Remarks on analytic smoothing effect for the Schrödinger equation
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Citation	Hokkaido University Preprint Series in Mathematics, 822, 1-16
Issue Date	2006
DOI	10.14943/83972
Doc URL	<a href="http://hdl.handle.net/2115/69630">http://hdl.handle.net/2115/69630</a>
Type	bulletin (article)
File Information	pre822.pdf



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# Remarks on analytic smoothing effect for the Schrödinger equation

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Dedicated to Professor Masatake Miyake on the occasion of his sixtieth birthday

## Abstract

We study analytic smoothing effect of solutions to the Schrödinger equation with Cauchy data decaying exponentially at infinity. The domain of analyticity in the space variables of solutions is described under weight conditions on the data in the terms of the corresponding supporting functions. The domain of analyticity in the time variable is characterized by means of weight conditions of Gaussian type on the data. A generalization of various isometrical identities related to the analytic smoothing effect is introduced.

## 1 Introduction

We consider the regularity of wavefunctions given by solutions to the Cauchy problem for the free Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = 0,$$

where  $u : \mathbb{R} \times \mathbb{R}^n \ni (t, x) \mapsto u(t, x) \in \mathbb{C}$ ,  $\partial_t = \partial/\partial t$ , and  $\Delta$  is the Laplacian in  $\mathbb{R}^n$ . For any Cauchy data  $\phi \in L^2(\mathbb{R}^n)$  at  $t = 0$ , the equation is solved as  $u(t) = U(t)\phi$  by means of the free Schrödinger group  $U(t) = \exp(i\frac{t}{2}\Delta)$  in  $L^2(\mathbb{R}^n)$ .

In this paper we study smoothing properties of the wavefunctions  $u$  away from the initial time  $t = 0$ . There is a large literature on this issue (see for instance [AHS], [HK1], [HK2], [HS1], [HS2], [MRZ], [Nak], [OYY], [RZ1], [RZ2], [T1], [T2] and references therein). One of the basic tool is given by the generators of Galilei transformations  $J = J(t) = x + it\nabla = U(t)xU(-t)$ . On the basis of the relation  $J(t)U(t) = U(t)x$ , we see that for any  $\phi \in \langle x \rangle^{-1}L^2$  the corresponding solution  $u(t) = U(t)\phi$  satisfies  $\partial_j u(t) \in \langle x \rangle L^2$  for any  $t \neq 0$ ,  $j = 1, \dots, n$ , where  $\langle x \rangle = (1 + x^2)^{1/2}$ ,  $x^2 = x_1^2 + \dots + x_n^2$  for  $x = (x_1, \dots, x_n)$ .

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Similarly, for any  $m \in \mathbb{N}$  we see that the condition  $\phi \in \langle x \rangle^{-m} L^2$  implies that  $\partial^\alpha u(t) \in \langle x \rangle^{|\alpha|} L^2$  for any  $t \neq 0$  and any multi-index  $\alpha$  with  $|\alpha| \leq m$ . This implies in particular that for any  $\phi \in \bigcap_{m \geq 1} \langle x \rangle^{-m} L^2$  the corresponding solution  $u(t) = U(t)\phi$  satisfies  $u(t) \in \bigcap_{m \geq 1} H_{loc}^m \subset C^\infty$  for any  $t \neq 0$ . This means that

solutions become smooth instantly if the data decay faster than polynomials in the sense of  $L^2$ , even though there is no regularity in the data.

There arises a natural question what happens if the data decay exponentially in  $L^2$ . The first study in this direction was done by Hayashi and Saitoh [HS1] in one space dimension. Roughly speaking, they proved that for any data of exponential decay in  $L^2$  the corresponding solution is analytic on a strip including the real line with width proportional to  $|t|$  for  $t \neq 0$  and that for any data of Gaussian decay the corresponding solution is entire for  $t \neq 0$ . The method of proof depends on a power series expansion.

The purpose of this paper is to generalize the theory in any space dimension by a different approach. Our method of proof depends on the Fourier-Laplace transform. As a result, a simple relation is given between domains of analyticity in space and exponential decay conditions on the Cauchy data in terms of corresponding supporting functions. Moreover, domains of analyticity in time are characterized on the basis of Gaussian decay conditions on the Cauchy data.

## 2 Analyticity of solutions for Schrödinger equations

Our argument begins by describing analytic continuations of solutions for Schrödinger equations. For  $t \in \mathbb{R}^n$  let  $U(t)$  be the free propagator, *i.e.*

$$U(t) = \exp(i\frac{t}{2}\Delta) = \mathfrak{F}^{-1} \exp(-i\frac{t}{2}|\xi|^2)\mathfrak{F},$$

where  $\mathfrak{F}$  denotes the Fourier transform. For  $t \in \mathbb{R} \setminus \{0\}$ ,  $U(t)\phi$  with  $\phi \in L^1 + L^2$  has the representation

$$(U(t)\phi)(x) = (M(t)D(t)\mathfrak{F}M(t)\phi)(x),$$

where  $\mathfrak{F}$  is understood to be a bounded operator from  $L^1 + L^2$  to  $L^\infty + L^2$ ,

$$\begin{aligned} (M(t)\psi)(x) &= \exp\left(\frac{i}{2t}|x|^2\right)\psi(x), \\ (D(t)\psi)(x) &= (it)^{-n/2}\psi(t^{-1}x) \end{aligned}$$

with

$$(it)^{-n/2} = \begin{cases} \left( \frac{1+i}{\sqrt{2}} \sqrt{t} \right)^{-n} & \text{if } t > 0, \\ \left( \frac{1-i}{\sqrt{2}} \sqrt{-t} \right)^{-n} & \text{if } t < 0. \end{cases}$$

(See [C], [SS], for example.)

It is therefore natural to define

$$u(t, \zeta) = \exp\left(\frac{i}{2t}\zeta^2\right) \cdot (it)^{-n/2} (\mathfrak{F}M(t)\phi)\left(\frac{\zeta}{t}\right) \quad (2.1)$$

for  $t \in \mathbb{R} \setminus \{0\}$  and  $\zeta \in \mathbb{C}^n$  as an analytic continuation of  $U(t)\phi$ , where  $\zeta^2 = \zeta \cdot \zeta = \sum_{j=1}^n \zeta_j^2$ . To describe the analyticity of (2.1) in  $\zeta$ , we introduce

**Definition 1** For any bounded convex set  $\Omega \subset \mathbb{R}^n$  with  $0 \in \text{Int } \Omega$ , its supporting function  $\gamma_\Omega$  is defined by

$$\gamma_\Omega(x) = \sup \{ x \cdot p; p \in \Omega \}$$

for  $x \in \mathbb{R}^n$ .

**Example 1**  $\gamma_\Omega(x) = a|x|$  for  $\Omega = \{x \in \mathbb{R}^n; |x| \leq a, \}$ .

$$\gamma_\Omega(x) = a \sum_{j=1}^n |x_j| \text{ for } \Omega = \{x \in \mathbb{R}^n; -a \leq x_j \leq a, j = 1, \dots, n\}.$$

**Proposition 1** Let  $\Omega \subset \mathbb{R}^n$  be a bounded convex set satisfying  $0 \in \text{Int } \Omega$ . Let  $\phi$  satisfy  $e^{\gamma_\Omega} \phi \in L^1 + L^\infty$ . Then for any  $t \in \mathbb{R} \setminus \{0\}$ , the function  $\zeta \mapsto u(t, \zeta)$  defined by (2.1) is an analytic continuation of  $U(t)\phi$  on  $\mathbb{R}^n + it(\text{Int } \Omega)$ .

**Remark 1** If  $e^{\gamma_\Omega} \phi \in L^1 + L^\infty$ , then  $\phi \in L^1$ . See the proof of Lemma 1 below.

Proposition 1 follows from the following lemma.

**Lemma 1** Let  $\Omega \subset \mathbb{R}^n$  be a bounded convex set with  $0 \in \text{Int } \Omega$  and let  $\gamma_\Omega$  be the supporting function of  $\Omega$ . Suppose that  $e^{\gamma_\Omega} \phi \in L^1 + L^\infty$  and let  $\mathfrak{F}\phi$  be the Fourier transform given by

$$(\mathfrak{F}\phi)(\zeta) = (2\pi)^{-n/2} \int \exp(-i\zeta \cdot x) \phi(x) dx,$$

where  $\zeta = \xi + i\eta \in \mathbb{C}^n$  and  $\zeta \cdot x = \xi \cdot x + i\eta \cdot x \in \mathbb{C}$ . Then  $\mathfrak{F}\phi$  is analytic on  $\mathbb{R}^n + i(\text{Int } \Omega)$ .

*Proof.* Assume that  $\zeta_0 = \xi_0 + i\eta_0 \in \mathbb{R}^n + i(\text{Int}\Omega)$  and that  $\delta > 0$  satisfies  $B(\eta_0; \delta) \subset \text{Int}\Omega$ . Then,  $\eta + \frac{\delta}{2|x|}x \in \Omega$  for each  $x \in \mathbb{R}^n \setminus \{0\}$  and  $\eta \in B(\eta_0, \delta/2)$ . As  $(\eta + (\delta/2|x|)x) \cdot x \leq \gamma_\Omega(x)$  by definition, we see that

$$\eta \cdot x \leq \gamma_\Omega(x) - (\delta/2)|x|$$

for any  $x \in \mathbb{R}^n$ . Therefore

$$\begin{aligned} |x_j \exp(-i\zeta \cdot x)\phi| &\leq |x| \exp(\eta \cdot x)|\phi| \\ &\leq |x| \exp(\gamma_\Omega(x) - (\delta/2)|x|)|\phi| \\ &\leq |x| \exp(-(\delta/2)|x|)|e^{\gamma_\Omega}\phi| \in L^1 \end{aligned}$$

since  $e^{\gamma_\Omega}\phi \in L^1 + L^\infty$  and  $|x| \exp(-(\delta/2)|x|) \in L^1 \cap L^\infty$ . Similarly,  $\exp(-i\zeta \cdot x)\phi \in L^1$ . By differentiating under the integral sign,

$$\frac{\partial}{\partial \zeta} \mathfrak{F}\phi = 0$$

around  $\zeta_0$ . By Looman-Menchoff's theorem (see [Nar] for example)  $\mathfrak{F}\phi$  is analytic on  $\mathbb{R}^n + i(\text{Int}\Omega)$ .

**Remark 2** If  $e^{a|x|^2}\phi \in L^1 + L^\infty$  for some  $a > 0$ ,  $e^{R|x|}\phi$  belongs to  $L^1$  for any  $R > 0$ . Lemma 1 ensures that  $\mathfrak{F}\phi$  is an entire function provided  $e^{a|x|^2}\phi \in L^1 + L^\infty$ .

In the view of (2.1), it is natural to define

$$u(\tau, \zeta) = \exp\left(\frac{i}{2\tau}\zeta^2\right) \cdot (i\tau)^{-n/2} (\mathfrak{F}M(\tau)\phi) \left(\frac{\zeta}{\tau}\right) \quad (2.2)$$

for  $\tau \in \mathbb{C} \setminus \{0\}$  and  $\zeta \in \mathbb{C}^n$  as an analytic continuation of  $u(t, \zeta)$  defined by (2.1). In fact:

**Proposition 2** *Let  $a > 0$  and let  $\phi$  satisfy  $e^{a|x|^2}\phi \in L^1 + L^\infty$ . Then the function  $(\tau, \zeta) \mapsto u(\tau, \zeta)$  is analytic on  $(\mathbb{C} \setminus B(\frac{i}{4a}, \frac{1}{4a})) \times \mathbb{C}^n$ , where*

$$\overline{B\left(\frac{i}{4a}, \frac{1}{4a}\right)} = \left\{ \tau \in \mathbb{C}; \left| \tau - \frac{i}{4a} \right| \leq \frac{1}{4a} \right\}.$$

**Remark 3**  $u$  defined by (2.2) can be a double-valued function out of the factor  $(i\tau)^{-n/2}$  provided  $n$  is odd. To be more specific,  $U(t)\phi$  is connected with  $-U(t)\phi$  through a mutual continuation on  $(\mathbb{C} \setminus \overline{B(\frac{i}{4a}, \frac{1}{4a})}) \times \mathbb{C}^n$  provided  $n$  is odd.

For the proof of Proposition 2, we introduce

**Lemma 2** Let  $\phi$  satisfy  $e^{a|x|^2} \phi \in L^1 + L^\infty$  for some  $a > 0$  and let

$$(\Phi(\tau))(x) = \exp(i\tau|x|^2) \phi(x).$$

Then the function

$$\{\tau \in \mathbb{C}; \Im\tau > -a\} \times \mathbb{C}^n \ni (\tau, \zeta) \mapsto (\mathfrak{F}\Phi(\tau))(\zeta) \in \mathbb{C}$$

is analytic.

*Proof.* As for the analyticity with respect to  $\zeta$  we have only to check  $e^{R|\cdot|} \Phi(\tau) \in L_x^1$  for any  $R > 0$  because of Lemma 1.

Let  $\tau = t + is$ . Then,

$$\begin{aligned} e^{R|x|} (\Phi(\tau))(x) &= \exp(R|x| + i\tau|x|^2) \phi(x) \\ &= \exp(it|x|^2) \cdot \exp(R|x| - (s+a)|x|^2) \cdot e^{a|x|^2} \phi(x). \end{aligned}$$

Since  $e^{a|x|^2} \phi \in L^1 + L^\infty$ ,  $e^{R|\cdot|} \Phi(\tau)$  belongs to  $L_x^1$  provided  $\Im\tau > -a$ . Thus  $\mathfrak{F}(\Phi(\tau))(\zeta)$  is analytic in  $\zeta$ .

The analyticity with respect to  $\tau$  is given by the fact that  $\frac{\partial}{\partial \bar{\tau}} \mathfrak{F}(\Phi(\tau)) = 0$ , which is shown by differentiating under the integral sign. Here we use the estimate

$$\begin{aligned} \left| \frac{\partial}{\partial t} \Phi(\tau)(x) \right| &= \left| \frac{\partial}{\partial s} \Phi(\tau)(x) \right| \\ &= |x|^2 |\Phi(\tau)(x)| \leq |x|^2 \exp((b-a)|x|^2) \cdot e^{a|x|^2} \phi(x) \end{aligned}$$

for  $0 < b < a$  and  $\tau \in \mathbb{C}$  with  $\Im\tau > -b$ . Hence we have shown the analyticity of  $\mathfrak{F}(\Phi(\tau))(\zeta)$ .

Lemma 2 yields the following Lemma 3 since  $\Im(1/2\tau) > -a$  if and only if  $|\tau - i/(4a)| > 1/(4a)$ . Proposition 2 follows directly from Lemma 3.

**Lemma 3** Let  $e^{a|x|^2} \phi \in L^1 + L^\infty$  for  $a > 0$ . Then the function

$$\left( \mathbb{C} \setminus \overline{B\left(\frac{i}{4a}, \frac{1}{4a}\right)} \right) \times \mathbb{C}^n \ni (\tau, \zeta) \mapsto (\mathfrak{F}M(\tau)\phi)\left(\frac{\zeta}{\tau}\right) \in \mathbb{C}$$

is analytic.

**Remark 4** It is possible to make  $(\mathfrak{F}M(\tau)\phi)\left(\frac{\zeta}{\tau}\right)$  with  $e^{a|x|^2} \phi \in L^1 + L^\infty$  not be analytic on the outside of the domain mentioned in Lemma 3. The integral

$$\int_0^\infty \exp(-py^2 - qy) dy$$

converges if and only if  $(p, q) \in \{(p, q); \Re p > 0\} \cup \{(p, q); \Re p = 0, \Im p \neq 0, \Re q \geq 0\} \cup \{(0, q); \Re q > 0\}$ . Hence, for  $\phi_a(x) = e^{-a|x|^2}$ ,

$$(\mathfrak{F}M(\tau)\phi_a)\left(\frac{\zeta}{\tau}\right) = \int_{\mathbb{R}^n} \exp\left(-i\frac{\zeta}{\tau} \cdot x + i\frac{|x|^2}{2\tau} - \lambda|x|^2\right) dx$$

converges if and only if

$$\begin{aligned} (\tau, \zeta) \in & \left( \mathbb{C} \setminus \overline{B\left(\frac{i}{4a}, \frac{1}{4a}\right)} \right) \times \mathbb{C}^n \\ & \cup \left\{ (t + is, \xi + i\eta); \frac{s}{2(t^2 + s^2)} = a, s\xi_j = t\eta_j, j = 1, \dots, n \right\}. \end{aligned}$$

Namely,  $(\mathfrak{F}M(\tau)\phi_0)\left(\frac{\zeta}{\tau}\right)$  with  $\tau \in \overline{B\left(\frac{i}{4a}, \frac{1}{4a}\right)}$  is not analytic even if it converges.

### 3 An identity related to analytic solutions for Schrödinger equations

We shall next consider identities by using the description of analytic continuations of  $U(t)\phi$ .

**Theorem 1** *Let  $a > 0$  and let  $\phi$  satisfy  $e^{a|x|^2}\phi \in L^2$ . Then  $U(t)\phi$  with  $t \neq 0$  has an analytic continuation  $u(t, \cdot)$  defined on  $\mathbb{C}^n$ , which satisfies that*

$$\begin{aligned} & (2\pi at^2)^{-n/2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \exp\left(-\frac{\eta^2}{2at^2}\right) \left| \exp\left(-i\frac{(\xi + i\eta)^2}{2t}\right) u(t, \xi + i\eta) \right|^2 d\xi d\eta \\ &= \sum_{\alpha \geq 0} \frac{(2a)^{|\alpha|}}{\alpha!} \|J^\alpha U(t)\phi\|_{L^2}^2 = \|e^{a|x|^2}\phi\|_{L^2}^2, \end{aligned} \quad (3.1)$$

where  $J^\alpha = J^\alpha(t) = \prod_{k=1}^n J_k(t)^{\alpha_k}$ ,  $J_k(t) = x_k + it\partial_k$  and  $\alpha = (\alpha_1, \dots, \alpha_n)$  is a multi-index.

*Proof.* For the sake of Lemma 3 the function  $U(t)\phi$  has an analytic continuation described by (2.2). Let  $I$  be the LHS of (3.1), i.e.,

$$I = (2\pi at^2)^{-n/2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \exp\left(-\frac{\eta^2}{2at^2}\right) \left| \exp\left(-i\frac{(\xi + i\eta)^2}{2t}\right) u(t, \xi + i\eta) \right|^2 d\xi d\eta.$$

This equals

$$\begin{aligned} & (2\pi a)^{-n/2} t^{-2n} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \exp\left(-\frac{\eta^2}{2at^2}\right) \left| (\mathfrak{F}M(t)\phi)\left(\frac{\xi + i\eta}{t}\right) \right|^2 d\xi d\eta \\ &= (2\pi a)^{-n/2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \exp\left(-\frac{\eta^2}{2a}\right) |(\mathfrak{F}M(t)\phi)(\xi + i\eta)|^2 d\xi d\eta. \end{aligned}$$

Let  $\psi(t) = M(t)\phi$  and let  $\Phi_\varepsilon(x, x'; \xi, \eta)$  be

$$(2\pi)^{-3n/2} a^{-n/2} \exp\left(-\varepsilon\xi^2 - \frac{\eta^2}{2a} - i\xi \cdot x + \eta \cdot x + i\xi \cdot x' + \eta \cdot x'\right) \psi(t)(x) \overline{\psi(t)(x')},$$

where  $\zeta \cdot \zeta' = \sum_{k=1}^n \zeta_k \zeta'_k$  for  $\zeta = (\zeta_1, \dots, \zeta_n)$ ,  $\zeta' = (\zeta'_1, \dots, \zeta'_n) \in \mathbb{C}^n$ . These satisfy that

$$\begin{aligned} & \iint_{\mathbb{R}^n \times \mathbb{R}^n} \left( \iint_{\mathbb{R}^n \times \mathbb{R}^n} \Phi_\varepsilon(x, x'; \xi, \eta) dx dx' \right) d\xi d\eta \\ &= (2\pi a)^{-n/2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \exp\left(-\varepsilon\xi^2 - \frac{\eta^2}{2a}\right) |\mathfrak{F}\psi(\xi + i\eta)|^2 d\xi d\eta. \end{aligned} \quad (3.2)$$

This converges to  $I$  as  $\varepsilon \searrow 0$ . Let the integral of (3.2) be denoted by  $I_\varepsilon$ .

As  $\Phi_\varepsilon(x, x'; \xi, \eta)$  belongs to  $L^1(\mathbb{R}^{4n})$ ,  $I_\varepsilon$  equals

$$\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \Phi_\varepsilon(x, x'; \xi, \eta) d\eta \right) d\xi \right) dx'$$

by Fubini's theorem.

We remark that

$$\begin{aligned} \int_{\mathbb{R}^n} \exp\left(-\frac{\eta^2}{2a}\right) e^{\eta \cdot (x+x')} d\eta &= (2a\pi)^{n/2} \exp\left(\frac{a}{2}(x+x')^2\right) \\ &= (2a\pi)^{n/2} \exp\left(-\frac{a}{2}(x-x')^2\right) e^{ax^2} e^{ax'^2} \end{aligned}$$

and that

$$\int_{\mathbb{R}^n} \exp(-\varepsilon\xi^2) e^{-i\xi \cdot (x-x')} d\xi = (\pi/\varepsilon)^{n/2} \exp\left(-\frac{1}{4\varepsilon}(x-x')^2\right).$$

These yield

$$\begin{aligned} & \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \Phi(x, x'; \xi, \eta) d\eta \right) d\xi \\ &= (4\pi\varepsilon)^{-n/2} \exp\left(-\left(\frac{1}{4\varepsilon} + \frac{a}{2}\right)(x-x')^2\right) e^{ax^2} \psi(t)(x) \overline{e^{ax'^2} \psi(t)(x')} \\ &= \left(\frac{1}{4\varepsilon} + \frac{a}{2}\right)^{-n/2} (4\varepsilon)^{-n/2} \overline{W_{(1/\varepsilon+2a)^{-1}}(x-x') f(t)(x')} f(t)(x), \end{aligned}$$

where  $f(t)(x) = e^{ax^2} \cdot \psi(t)(x)$  and  $W_\delta$  denotes the Gauss-Weierstrass kernel, *i.e.*,

$$W_\delta(x) = (4\pi\delta)^{-n/2} \exp\left(-\frac{x^2}{4\delta}\right).$$



(See [M] for example.) By setting  $\delta(\varepsilon) = \left(\frac{1}{\varepsilon} + 2a\right)^{-1}$

$$\begin{aligned} I_\varepsilon &= (\delta(\varepsilon)/\varepsilon)^{n/2} \int_{\mathbb{R}^n} \overline{W_{\delta(\varepsilon)} * f(t)(x)} \cdot f(t)(x) dx \\ &= (\delta(\varepsilon)/\varepsilon)^{n/2} \langle f(t), W_{\delta(\varepsilon)} * f(t) \rangle_{L^2}. \end{aligned}$$

Since  $f(t) \in L^2$ ,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} I_\varepsilon &= \lim_{\varepsilon \rightarrow 0} (\delta(\varepsilon)/\varepsilon)^{n/2} \langle f(t), f(t) \rangle_{L^2} \\ &= \int_{\mathbb{R}^n} |f(t)(x)|^2 dx \\ &= \int_{\mathbb{R}^n} e^{2ax^2} |\phi(x)|^2 dx. \end{aligned}$$

We thus have

$$I = \int_{\mathbb{R}^n} e^{2ax^2} |\phi(x)|^2 dx,$$

where the right hand side is rewritten as

$$\begin{aligned} \sum_{\alpha \geq 0} \frac{(2a)^{|\alpha|}}{\alpha!} \int_{\mathbb{R}^n} x^{2\alpha} |\phi(x)|^2 dx &= \sum_{\alpha \geq 0} \frac{(2a)^{|\alpha|}}{\alpha!} \|x^\alpha \phi\|_{L^2}^2 \\ &= \sum_{\alpha \geq 0} \frac{(2a)^{|\alpha|}}{\alpha!} \|U(t)x^\alpha U(-t)U(t)\phi\|_{L^2}^2 \\ &= \sum_{\alpha \geq 0} \frac{(2a)^{|\alpha|}}{\alpha!} \|J^\alpha(t)U(t)\phi\|_{L^2}^2 \end{aligned}$$

This completes the proof. □

## 4 Generalization

We shall generalize Theorem 1 by the following statement. It is convenient to use the notation

$$(\mathfrak{L}_\Omega)p(x) = \int_\Omega p(y)e^{2x \cdot y} dy.$$

**Theorem 2** *Let  $\Omega \subset \mathbb{R}^n$  be a convex open set with  $0 \in \text{Int}\Omega$ . Suppose that  $p \in L^1(\Omega)$  is non-negative and that*

$$\text{supp } p \cap \Omega = \Omega.$$

Suppose that  $\mathfrak{L}_\Omega p \in L^1_{loc}$  and assume that  $(\mathfrak{L}_\Omega p)|\phi|^2 \in L^1$ . Then,  $U(t)\phi$  with  $t \neq 0$  has an analytic continuation  $u(t, \cdot)$  defined on  $\mathbb{R}^n + it\Omega$ , which satisfies that

$$\begin{aligned} & |t|^{-n} \int_{t\Omega} q(\eta/t) \left( \int_{\mathbb{R}^n} \left| \exp\left(-i\frac{(\xi + i\eta)^2}{2t}\right) u(t, \xi + i\eta) \right|^2 d\xi \right) d\eta \\ &= \int_{\mathbb{R}^n} (\mathfrak{L}_\Omega q)(x) |\phi(x)|^2 dx \end{aligned} \quad (4.1)$$

for any  $q \in L^1(\Omega)$  with  $|q| \leq p$ .

For the proof of this theorem we prepare two properties. In the following lemma we assume that  $\Omega$  is a bounded convex open set with  $0 \in \Omega$  and that  $p$  is the same as above.

**Lemma 4** (1) For  $\rho > 0$  let  $\Omega_\rho$  be defined by the set

$$\left\{ y; \frac{1}{\rho}y \in \Omega \right\}.$$

Then,  $\gamma_{\Omega_\rho} = \rho\gamma_\Omega$ .

(2) Let  $\Omega_\rho$  be the same as above. Then, for each  $0 < \delta < 1$  there exists a positive constant  $C = C(\delta)$  such that

$$\mathfrak{L}_\Omega p \geq C \exp(2\gamma_{\Omega_{1-\delta}}).$$

*Proof.* The first part follows by the inequalities

$$y \cdot x = \rho \left( \frac{1}{\rho}y \cdot x \right) \leq \rho\gamma_\Omega(x) \quad \text{for } y \in \Omega_\rho$$

and

$$y \cdot x = \frac{1}{\rho}(\rho y \cdot x) \leq \frac{1}{\rho}\gamma_{\Omega_\rho}(x) \quad \text{for } y \in \Omega.$$

(See [DS] for example.)

The second part is shown by the convexity of  $\Omega$ . In fact, by definition for any  $x \in \mathbb{R}^n$  there is  $y_x \in \Omega_{1-\delta/2}$  such that

$$(1 - 2\delta/3)\gamma_\Omega(x) \leq y_x \cdot x \leq (1 - \delta/2)\gamma_\Omega(x)$$

for each  $x \in \mathbb{R}^n$ . Let  $C_1 = \sup\{\gamma_\Omega(-x)/\gamma_\Omega(x); x \neq 0\}$ . Since

$$\{\gamma_\Omega(-x)/\gamma_\Omega(x); x \neq 0\} = \{\gamma_\Omega(x)/\gamma_\Omega(-x); x \neq 0\},$$

$C_1$  is greater than or equal to 1. Let  $\delta_1 = \delta/(3C_1)$  and assume that  $z \in \Omega_{\delta_1}$ . Then

$$\begin{aligned}
(y_x + z) \cdot x &= y_x \cdot x - z \cdot (-x) \\
&\geq (1 - 2\delta/3)\gamma_\Omega(x) - \delta_1\gamma_\Omega(-x) \\
&\geq (1 - 2\delta/3 - C_1\delta_1)\gamma_\Omega(x) \\
&= (1 - \delta)\gamma_\Omega(x) = \gamma_{\Omega_{1-\delta}}(x).
\end{aligned}$$

The convexity of  $\Omega$  yields

$$y_x + \Omega_{\delta_1} \subset \Omega_{1-\delta/2+\delta_1} \Subset \Omega.$$

Therefore

$$\begin{aligned}
\int_{\Omega} p(y)e^{2x \cdot y} dy &\geq \int_{\Omega_{\delta_1}} p(y_x + z) \exp(2(y_x + z) \cdot x) dz \\
&\geq \int_{\Omega_{\delta_1}} p(y_x + z) \exp(2\gamma_{\Omega_{1-\delta}}(x)) dz \\
&\geq \exp(2\gamma_{\Omega_{1-\delta}}(x)) \int_{\Omega_{\delta_1}} p(y_x + z) dz \quad (4.2)
\end{aligned}$$

A lower bound for the integral in the rightest hand side of (4.2) is essentially given by Fatou's lemma. Since the closure of  $\Omega_{1-\delta/2}$  is compactly included by  $\Omega$ , there are a sequence  $\{y_{x_j}\} \subset \Omega_{1-\delta/2}$  and  $y_0 \in \Omega$  such that

$$\begin{aligned}
y_{x_j} &\rightarrow y_0 \\
\int_{\Omega_{\delta_1}} p(y_{x_j} + z) dz &\rightarrow \inf \left\{ \int_{\Omega_{\delta_1}} p(y_x + z) dz ; x \in \mathbb{R}^n \right\}.
\end{aligned}$$

Therefore for  $0 < \delta' < \delta_1$  sufficiently large  $j$  satisfies that  $y_{x_j} - y_0 \in \Omega_{\delta_1 - \delta'}$ . By using the convexity of  $\Omega$  this yields  $y_0 + \Omega_{\delta'} \subset y_{x_j} + \Omega_{\delta_1}$  for sufficiently large  $j$ . Hence

$$\int_{\Omega_{\delta_1}} p(y_{x_j} + z) dz \geq \int_{\Omega_{\delta'}} p(y_0 + z) dz$$

for sufficiently large  $j$ . Namely,

$$\inf \left\{ \int_{\Omega_{\delta_1}} p(y_x + z) dz ; x \in \mathbb{R}^n \right\} \geq \int_{\Omega_{\delta'}} p(y_0 + z) dz.$$

This with (4.2) yields

$$\int_{\Omega} p(y)e^{2x \cdot y} dy \geq \int_{\Omega_{\delta'}} p(y_0 + z) dz \cdot \exp(2\gamma_{\Omega_{1-\delta}}(x)).$$

We have thus shown the second part of Lemma 4.  $\square$

*Proof of Theorem 2.*

In the case that  $\Omega$  is bounded, by the virtue of Lemma 4

$$\exp(\gamma_{\Omega_1-\delta})\phi \in L^2$$

for  $0 < \delta < 1$ . Hence  $U(t)\phi$  has an analytic continuation defined on  $\mathbb{R}^n + it\Omega$ .

In the case that  $\Omega$  is unbounded, Lemma 4 replaced  $\Omega$  by  $\Omega \cap B_R$  yields the analyticity of  $U(t)\phi$  on  $\mathbb{R}^n + it\Omega \cap B_R$ , where  $B_R$  is the open ball with radius  $R$  centered at the origin. Here we use

$$\int_{\Omega} p(y)e^{2x \cdot y} dy \geq \int_{\Omega \cap B_R} p(y)e^{2x \cdot y} dy.$$

Namely,

$$(\mathfrak{L}_{\Omega \cap B_R} p) |\phi|^2 \in L^1.$$

It is thus shown by the arbitrariness of  $R$  that  $U(t)\phi$  has an analytic continuation  $u(t, \cdot)$  defined on  $\mathbb{R}^n + it\Omega$  if  $\Omega$  is unbounded. We remark that  $u(t, \zeta)$  can be described by (2.1) for  $\zeta \in \mathbb{R}^n + it\Omega$  even if  $\Omega$  is unbounded.

We next show the identity in question. Let  $I$  be the LHS of (4.1). By the representation of  $u$

$$\begin{aligned} I &= |t|^{-n} \int_{it\Omega} q(\eta/t) \left( \int_{\mathbb{R}^n} \left| (it)^{-n/2} (\mathfrak{F}\psi(t)) \left( \frac{\xi + i\eta}{t} \right) \right|^2 d\xi \right) d\eta \\ &= \int_{\Omega} q(\eta) \left( \int_{\mathbb{R}^n} |(\mathfrak{F}\psi(t))(\xi + i\eta)|^2 d\xi \right) d\eta, \end{aligned}$$

where  $\psi(t) = M(t)\phi$ . Let  $\Psi_{\varepsilon}(x, x'; \xi, \eta)$  be

$$(2\pi)^{-n} q(\eta) \exp(-\varepsilon\xi^2 - i\xi x + \eta x + i\xi x' + \eta x') \psi(t)(x) \overline{\psi(t)(x')}$$

and let  $I_{\varepsilon}$  be

$$\int_{\Omega} \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} \Psi_{\varepsilon}(x, x'; \xi, \eta) dx dx' \right) d\xi \right) d\eta.$$

Namely,

$$I_{\varepsilon} = \int_{\Omega} q(\eta) \left( \int_{\mathbb{R}^n} e^{-\varepsilon\xi^2} |(\mathfrak{F}\psi(t))(\xi + i\eta)|^2 d\xi \right) d\eta.$$

Since

$$\int_{\mathbb{R}^n} e^{-\varepsilon\xi^2} |(\mathfrak{F}\psi(t))(\xi + i\eta)|^2 d\xi \nearrow \int_{\mathbb{R}^n} |(\mathfrak{F}\psi(t))(\xi + i\eta)|^2 d\xi$$

as  $\varepsilon \searrow 0$ ,  $\lim_{\varepsilon \downarrow 0} I_\varepsilon$  coincides with  $I$ .

As  $\Psi_\varepsilon \in L^1(\mathbb{R}^{4n})$ ,

$$I_\varepsilon = \int_{\Omega} \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \Psi_\varepsilon(x, x'; \xi, \eta) d\xi \right) dx' \right) dx \right) d\eta$$

by Fubini's theorem. Since

$$\begin{aligned} \int_{\mathbb{R}^n} \exp(-\varepsilon\xi^2) e^{-i\xi \cdot (x-x')} d\xi &= (\pi/\varepsilon)^{n/2} \exp\left(-\frac{1}{4\varepsilon}(x-x')^2\right) \\ &= (2\pi)^n \overline{W_\varepsilon(x-x')}, \end{aligned}$$

we have

$$\begin{aligned} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \Psi_\varepsilon(x, x'; \xi, \eta) d\xi \right) dx' &= \int_{\mathbb{R}^n} q(\eta) e^{\eta \cdot (x+x')} \overline{W_\varepsilon(x-x') \cdot \psi(t)(x')} \cdot \psi(t)(x) dx' \\ &= q(\eta) \overline{W_\varepsilon * (g(\eta, t))}(x) (g(\eta, t))(x), \end{aligned}$$

where  $(g(\eta, t))(x) = e^{\eta \cdot x} \cdot (\psi(t))(x)$ . By Young's inequality the absolute value of

$$\langle g(\eta, t), W_\varepsilon * g(\eta, t) \rangle_{L^2}$$

is dominated by

$$\begin{aligned} \|g(\eta, t)\|_{L^2}^2 &= \int_{\mathbb{R}^n} |(g(\eta, t))(x)|^2 dx \\ &= \int_{\mathbb{R}^n} e^{2\eta \cdot x} |\phi(x)|^2 dx. \end{aligned}$$

Since  $q(\cdot) |g|^2(\cdot, t) \in L^1(\Omega \times \mathbb{R}^n)$  and

$$\int_{\mathbb{R}^n} \left( \int_{\Omega} q(\eta) |(g(\eta, t))(x)|^2 d\eta \right) dx = \int_{\mathbb{R}^n} (\mathfrak{L}_\Omega q)(x) |\phi(x)|^2 dx,$$

we have

$$\begin{aligned} \int_{\Omega} q(\eta) \|g(\eta, t)\|_{L^2}^2 d\eta &= \int_{\mathbb{R}^n} \left( \int_{\Omega} q(\eta) |(g(\eta, t))(x)|^2 d\eta \right) dx \\ &= \int_{\mathbb{R}^n} (\mathfrak{L}_\Omega q)(x) |\phi(x)|^2 dx. \end{aligned}$$

We also obtain

$$\begin{aligned} \int_{\Omega} p(\eta) \|g(\eta, t)\|_{L^2}^2 d\eta &= \int_{\mathbb{R}^n} \left( \int_{\Omega} p(\eta) |(g(\eta, t))(x)|^2 d\eta \right) dx \\ &= \int_{\mathbb{R}^n} (\mathfrak{L}_\Omega p)(x) |\phi(x)|^2 dx. \end{aligned}$$

in the same way. Therefore  $\|g(\eta, t)\|_{L^2} < \infty$  for almost every  $\eta$  where  $p(\eta)$  is not zero. Hence

$$W_\varepsilon * g(\eta, t) \longrightarrow g(\eta, t) \quad \text{in } L^2 \quad \text{as } \varepsilon \rightarrow 0$$

for almost every  $\eta$  with  $p(\eta) \neq 0$ . This yields

$$\langle g(\eta, t), W_\varepsilon * g(\eta, t) \rangle_{L^2} \longrightarrow \|g(\eta, t)\|_{L^2}^2 \quad \text{as } \varepsilon \rightarrow 0$$

for the same  $\eta$  above. Thus the estimate

$$|q(\eta) \langle g(\eta, t), W_\varepsilon * g(\eta, t) \rangle_{L^2}| \leq p(\eta) \|g(\eta, t)\|_{L^2}^2$$

with Lebesgue's convergence theorem yields

$$\begin{aligned} I_\varepsilon &= \int_{\Omega} q(\eta) \langle g(\eta, t), W_\varepsilon * g(\eta, t) \rangle_{L^2} d\eta \\ &\longrightarrow \int_{\Omega} q(\eta) \|g(\eta, t)\|_{L^2}^2 d\eta \\ &= \int_{\mathbb{R}^n} (\mathfrak{L}_\Omega q)(x) |\phi(x)|^2 dx \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

Therefore

$$I = \int_{\mathbb{R}^n} (\mathfrak{L}_\Omega q)(x) |\phi(x)|^2 dx.$$

This completes the proof.  $\square$

**Remark 5** By setting  $p(x) = q(x) = (2\pi a)^{-n/2} \exp(-x^2/(2a))$  and  $\Omega = \mathbb{R}^n$ , Theorem 1 reduces to Theorem 2.

**Remark 6** The shape of  $\Omega$  in Theorem 2 can be relaxed into a star-shaped domain with respect to the origin. For each  $x \in \Omega$  there is a bounded open convex set containing  $x$  and the origin. This together with Lemma 4 yields the existence of an analytic continuation  $u$  of  $U(t)\phi$  defined on  $\mathbb{R}^n + it\Omega$  satisfying (2.1) in view of the uniqueness theorem of analytic continuations. By the same argument as above we have the conclusion.

**Corollary 1** Let  $a > 0$  and suppose that

$$\left( \prod_{j=1}^n \frac{\sinh 2ax_j}{2ax_j} \right)^{1/2} \phi \in L^2$$

Then,

(i)  $U(t)\phi$  with  $t > 0$  has an analytic continuation  $u(t, \cdot)$  defined on  $\mathbb{R}^n + iQ_{at}$ , where

$$Q_{at} = (-at, at)^n = \{ \eta = (\eta_1, \dots, \eta_n); -at < \eta_j < at, j = 1, \dots, n \}.$$

(ii)  $u(t, \cdot)$  with  $t > 0$  satisfies the following identity.

$$\begin{aligned} & (2at)^{-n} \int_{Q_{at}} \left( \int_{\mathbb{R}^n} \left| \exp \left( -i \frac{(\xi + i\eta)^2}{2t} \right) u(t, \xi + i\eta) \right|^2 d\xi \right) d\eta \\ &= \sum_{\alpha \geq 0} \frac{(2a)^{2|\alpha|}}{\prod_{j=1}^n (2\alpha_j + 1)!} \|J^\alpha U(t)\phi\|_{L^2}^2 \\ &= \int_{\mathbb{R}^n} \left( \prod_{j=1}^n \frac{\sinh(2ax_j)}{2ax_j} \right) |\phi(x)|^2 dx. \end{aligned}$$

*Proof.* Let  $p(x) = |Q_a|^{-1}$ . Then

$$\int_{Q_a} p(y) e^{2x \cdot y} dy = \prod_{j=1}^n \frac{\sinh 2ax_j}{2ax_j} = \sum_{\alpha \geq 0} \frac{(2a)^{2|\alpha|}}{\prod_{j=1}^n (2\alpha_j + 1)!} x^{2\alpha}.$$

We have thus shown the corollary.  $\square$

**Remark 7** We have some other examples for  $\Omega$ ,  $p$ ,  $q$ ,  $\mathfrak{L}_\Omega p$  and  $\mathfrak{L}_\Omega q$  in Theorem 2 as follows:

(i)

$$\begin{aligned} \Omega &= Q_\pi = (-\pi, \pi)^n \\ p(x) &= \prod_{j=1}^n |\sin N_j x_j|, \\ q(x) &= \prod_{j=1}^n \sin N_j x_j, \\ (\mathfrak{L}_\Omega p)(x) &= \prod_{j=1}^n \frac{\sinh 2\pi x_j}{\tanh(\pi x_j / N_j)} \cdot \frac{2N_j}{4x_j^2 + N_j^2}, \\ (\mathfrak{L}_\Omega q)(x) &= \prod_{j=1}^n \frac{2(-1)^{N_j+1} N_j \sinh 2\pi x_j}{4x_j^2 + N_j^2}. \end{aligned}$$

(ii)

$$\begin{aligned}\Omega &= (-\pi, \pi)^n \\ p(x) &= \prod_{j=1}^n |\cos N_j x_j|, \\ q(x) &= \prod_{j=1}^n \cos N_j x_j, \\ (\mathfrak{L}_\Omega p)(x) &= \prod_{j=1}^n \frac{2 \sinh 2\pi x_j}{4x_j^2 + N_j^2} \left( 2x_j + \frac{N_j}{\sinh \pi x_j / N_j} \right), \\ (\mathfrak{L}_\Omega q)(x) &= \prod_{j=1}^n \frac{4(-1)^{N_j} x_j \sinh 2\pi x_j}{4x_j^2 + N_j^2}.\end{aligned}$$

Here  $N_j \in \mathbb{N}$  for  $j = 1, \dots, n$ . The proof depends on a direct calculation. Details are omitted.

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