The e-multiplicity and addition-deletion theorems for multiarrangements

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December 23, 2006

Abstract

The addition-deletion theorems for hyperplane arrangements, which were originally shown in [T1], provide useful ways to construct examples of free arrangements. In this article, we prove addition-deletion theorems for multiarrangements. A key to the generalization is the definition of a new multiplicity, called the *e-multiplicity*, of a restricted multiarrangement. We compute the e-multiplicities in many cases. Then we apply the addition-deletion theorems to various arrangements including supersolvable arrangements and the Coxeter arrangement of type A_3 to construct free and non-free multiarrangements.

0 Introduction

Let $\mathcal{A}$ be a *hyperplane arrangement*, or simply an *arrangement*. In other words, $\mathcal{A}$ is a finite collection of hyperplanes in an ℓ-dimensional vector space V over a field $\mathbb{K}$. A *multiarrangement*, which was introduced by Ziegler in [Z], is a pair $(\mathcal{A}, m)$ consisting of a hyperplane arrangement $\mathcal{A}$ and a *multiplicity* $m : \mathcal{A} \to \mathbb{Z}_{>0}$. Define $|m| = \sum_{H\in\mathcal{A}} m(H)$. A multiarrangement $(\mathcal{A}, m)$ such that $m(H) = 1$ for all $H \in \mathcal{A}$ is just a hyperplane arrangement, and is sometimes called a *simple arrangement*.

Let $\{x_1, \ldots, x_\ell\}$ be a basis for V^*. Then $S := \mathrm{Sym}(V^*) \simeq \mathbb{K}[x_1, \ldots, x_\ell]$. When each $H \in \mathcal{A}$ contains the origin, we say that $\mathcal{A}$ is *central*. Throughout this article, assume that every arrangement is central. Let $\mathrm{Der}_{\mathbb{K}}(S)$ denote the set of $\mathbb{K}$-linear derivations from S to itself. For each $H \in \mathcal{A}$ we choose a defining form α_H. Following Ziegler [Z], we define an S-module $D(\mathcal{A}, m)$ of a multiarrangement $(\mathcal{A}, m)$ by

$$D(\mathcal{A}, m) = \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(\alpha_H) \in \alpha_H^{m(H)} S \text{ for all } H \in \mathcal{A}\}.$$

If $D(\mathcal{A}, m)$ is a free S-module we say that $(\mathcal{A}, m)$ is a *free multiarrangement*. When $(\mathcal{A}, m)$ is simple, the module coincides with the usual module $D(\mathcal{A})$ of logarithmic derivations (e.g., [OT, 4.1]). Thus free multiarrangements generalize free arrangements.

*Supported by 21st Century COE Program "Mathematics of Nonlinear Structures via Singularities" Hokkaido University.

†Supported in part by Japan Society for the Promotion of Science.

‡Supported by NSF grant # 0600893 and the NSF Japan program.

When $(\mathcal{A}, m)$ is a free multiarrangement we define the *exponents* of $(\mathcal{A}, m)$, denoted by $\exp(\mathcal{A}, m)$, to be the multiset of degrees of a homogeneous basis $\{\theta_1, \dots, \theta_\ell\}$ for $D(\mathcal{A}, m)$:

$$\exp(\mathcal{A}, m) := (\deg(\theta_1), \dots, \deg(\theta_\ell)),$$

where $\deg(\theta_i) := \deg(\theta_i(\alpha))$ for some linear form α with $\theta_i(\alpha) \neq 0$. Then the multiset $\exp(\mathcal{A}, m)$ does not depend upon choice of basis.

In his groundbreaking paper [Z], Ziegler writes "...the theory of multiarrangements and their freeness is not yet in a satisfactory state. In particular, we do not know any addition/deletion theorem ..." It is exactly the subject of this article. Namely, we generalize the *addition-deletion* theorems for simple arrangements [T1] to multiarrangements in this article. Let $(\mathcal{A}, m)$ be a nonempty multiarrangement and $\ell \geq 2$. Fix a hyperplane $H_0 \in \mathcal{A}$ and let α_0 be a defining form for H_0. To state the addition-deletion theorems for multiarrangements we need to define the deletion $(\mathcal{A}', m')$ and the restriction $(\mathcal{A}'', m'')$. First, we define the *deletion* as follows:

Definition 0.1

(i) If $m(H_0) = 1$, then $\mathcal{A}' := \mathcal{A} \setminus \{H_0\}$ and $m'(H) = m(H)$ for all $H \in \mathcal{A}'$.

(ii) If $m(H_0) \geq 2$, then $\mathcal{A}' := \mathcal{A}$ and for $H \in \mathcal{A}' = \mathcal{A}$, we define

$$m'(H) = \begin{cases} m(H) & \text{if } H \neq H_0, \\ m(H_0) - 1 & \text{if } H = H_0. \end{cases}$$

Next we define the restriction $(\mathcal{A}'', m'')$. Let

$$\mathcal{A}'' = \{H_0 \cap K \mid K \in \mathcal{A} \setminus \{H_0\}\},$$

which is an arrangement on H_0. We, however, have more than one choice to define a multiplicity m''. The definition of a suitable multiplicity m'' is crucial. The canonical definition is probably

$$m''(X) = \sum_{\substack{K \in \mathcal{A} \setminus \{H_0\} \\ K \cap H_0 = X}} m(K),$$

which is purely combinatorial and was used in [Y1, Y2, Z] effectively. In this article, however, in order to serve our purposes, we introduce a new multiplicity m^*, called the *e-multiplicity*, whose definition is algebraic rather than combinatorial.

For $X \in \mathcal{A}''$ define

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subset H\} \quad \text{and} \quad m_X = m \mid_{\mathcal{A}_X} .$$

Choose a coordinate system $(x_1, \dots, x_\ell)$ so that X is defined by $x_1 = x_2 = 0$. Let ∂_{x_i} denote $\dfrac{\partial}{\partial x_i}$ $(1 \leq i \leq \ell)$ throughout this article. By Proposition 2.1 we will see that $D(\mathcal{A}_X, m_X)$ has a basis

$$\theta_X, \psi_X, \partial_{x_3}, \partial_{x_4}, \dots, \partial_{x_\ell}, \tag{0.1}$$

such that $\theta_X \notin \alpha_0 \operatorname{Der}_{\mathbb{K}}(S)$ and $\psi_X \in \alpha_0 \operatorname{Der}_{\mathbb{K}}(S)$.

Definition 0.2
The e-multiplicity $m^* : \mathcal{A}'' \to \mathbb{Z}_{>0}$ *is defined by* $m^*(X) := \deg \theta_X$ $(X \in \mathcal{A}'')$. *Then define the restriction by* $(\mathcal{A}'', m^*)$.

For $(\mathcal{A}, m)$ and $H_0 \in \mathcal{A}$ we say the collection $(\mathcal{A}, m)$, the deletion $(\mathcal{A}', m')$ and the restriction $(\mathcal{A}'', m^*)$ is a *triple*.

Remark 0.3
When $(\mathcal{A}, m)$ *is simple the Euler derivation can be chosen as* θ_X. *In this case,* $m^* \equiv 1$, *so* $(\mathcal{A}'', m^*)$ *is simple.*

For $\theta \in D(\mathcal{A}, m)$ define $\overline{\theta} \in D(\mathcal{A}'')$ by $\overline{\theta}(\overline{f}) := \overline{\theta(f)}$ for $\overline{f} \in \overline{S} := S/\alpha_0 S$, where $\overline{f}$ is the image of an element $f \in S$ by the canonical projection $S \to \overline{S}$. In Proposition 2.2 we obtain an exact sequence

$$0 \longrightarrow D(\mathcal{A}', m') \xrightarrow{\alpha_0 \cdot} D(\mathcal{A}, m) \xrightarrow{\pi} D(\mathcal{A}'', m^*),$$

where $\alpha_0 \cdot$ denotes the multiplication by α_0 and $\pi(\theta) = \overline{\theta}$.

Roughly speaking, the addition-deletion theorems state that the freeness of any two of the triple, under a condition concerning their exponents, imply the freeness of the third. The following four addition-deletion theorems are the multiarrangement versions of Theorems 4.46 (1), 4.49, 4.46 (2), and 4.50 in [OT]. The ideas behind the proofs are very similar to those in [OT]. However, because of the indispensability of the e-multiplicity, we include the proofs.

Theorem 0.4
If $(\mathcal{A}, m)$ *and* $(\mathcal{A}', m')$ *are both free, then there exists a basis* $\{\theta_1, \ldots, \theta_\ell\}$ *for* $D(\mathcal{A}', m')$ *such that, for some* $k \in \{1, \ldots, \ell\}$, $\{\theta_1, \ldots, \theta_{k-1}, \alpha_0\theta_k, \theta_{k+1}, \ldots, \theta_\ell\}$ *is a basis for* $D(\mathcal{A}, m)$.

Theorem 0.5 (Deletion)
Assume that $(\mathcal{A}, m)$ *and* $(\mathcal{A}'', m^*)$ *are both free and* $\exp(\mathcal{A}'', m^*) \subset \exp(\mathcal{A}, m)$. *Then* $(\mathcal{A}', m')$ *is also free.*

Theorem 0.6 (Restriction)
Assume that $(\mathcal{A}, m)$ *and* $(\mathcal{A}', m')$ *are both free. Take a basis* $\{\theta_1, \ldots, \theta_k, \ldots, \theta_\ell\}$ *for* $D(\mathcal{A}', m')$ *as in Theorem 0.4. Then* $\{\overline{\theta_1}, \ldots, \overline{\theta_{k-1}}, \overline{\theta_{k+1}}, \ldots, \overline{\theta_\ell}\}$ *is a basis for* $D(\mathcal{A}'', m^*)$.

Theorem 0.7 (Addition)
Assume that $(\mathcal{A}', m')$ *and* $(\mathcal{A}'', m^*)$ *are both free and* $\exp(\mathcal{A}'', m^*) \subset \exp(\mathcal{A}', m')$. *Then* $(\mathcal{A}, m)$ *is also free.*

Summarizing these results we follow Cartier [C] to obtain the following addition-deletion theorem for multiarrangements.

Theorem 0.8 (Addition-Deletion)
Let $(\mathcal{A}, m)$ *be a nonempty multiarrangement in an* ℓ*-dimensional vector space* V, $H_0 \in \mathcal{A}$ *and let* $(\mathcal{A}, m), (\mathcal{A}', m'),$ $(\mathcal{A}'', m^*)$ *be the triple with respect to* H_0. *Then any two of the following statements imply the third:*

(i) $(\mathcal{A}, m)$ *is free with* $\exp(\mathcal{A}, m) = (d_1, \ldots, d_\ell)$.

(ii) $(\mathcal{A}', m')$ *is free with* $\exp(\mathcal{A}', m') = (d_1, \ldots, d_\ell - 1)$.

(iii) $(\mathcal{A}'', m^*)$ *is free with* $\exp(\mathcal{A}'', m^*) = (d_1, \ldots, d_{\ell-1})$.

Applying Addition Theorem 0.7 repeatedly, we can inductively construct the following class of free multiarrangements.

Definition 0.9
The class $\mathcal{IFM}$ of inductively free multiarrangements is the smallest class of multiarrangements which satisfies the following two conditions.

(1) The empty arrangement $\emptyset_\ell$ in an ℓ-dimensional vector space is contained in $\mathcal{IFM}$ for $\ell \geq 0$.

(2) For a multiarrangement $(\mathcal{A}, m)$, if there exists $H \in \mathcal{A}$ such that $(\mathcal{A}', m') \in \mathcal{IFM}$, $(\mathcal{A}'', m^*) \in \mathcal{IFM}$, and $\exp(\mathcal{A}', m') \supset \exp(\mathcal{A}'', m^*)$, then $(\mathcal{A}, m) \in \mathcal{IFM}$.

Remark 0.10
The intersection of the class of $\mathcal{IFM}$ with the class of simple arrangements is equal to the class of inductively free arrangements, $\mathcal{IF}$ [OT, Definition 4.53].

The outline of this article is as follows. In Section 1, we introduce some definitions and results in arrangement theory which will be used later. In Section 2, we prove Theorem 0.4 and Deletion Theorem 0.5. In Section 3, we prove Restriction Theorem 0.6 and Addition Theorem 0.7. In Section 4, we compute explicit values of the e-multiplicities in many cases. Applying the addition-deletion theorems together with the computations, in Section 5, we find multiplicities m such that the multiarrangement $(\mathcal{A}, m)$ is free for various arrangements $\mathcal{A}$ including supersolvable arrangements and the Coxeter arrangement of type A_3.

1 Preliminaries

In this section we fix some notation and introduce some results about multiarrangements which will be used later. For hyperplane arrangement theory, we refer the reader to [OT]. For a multiarrangement $(\mathcal{A}, m)$, define

$$Q(\mathcal{A}, m) := \prod_{H \in \mathcal{A}} \alpha_H^{m(H)}.$$

The S-module $\mathrm{Der}_{\mathbb{K}}(S)$ of $\mathbb{K}$-linear S-derivations has the natural basis:

$$\mathrm{Der}_{\mathbb{K}}(S) = \bigoplus_{i=1}^{\ell} S\partial_{x_i}.$$

We say a nonzero element $\theta = \sum_{i=1}^{\ell} f_i \partial_{x_i} \in \mathrm{Der}_{\mathbb{K}}(S)$ is *homogeneous of degree* p if f_i is zero or a homogeneous polynomial of degree p in S for $1 \leq i \leq \ell$. Recall the S-submodule

$$D(\mathcal{A}, m) = \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(\alpha_H) \in S \cdot \alpha_H^{m(H)} \ (\forall H \in \mathcal{A})\}$$

of $\mathrm{Der}_{\mathbb{K}}(S)$ and a multiarrangement $(\mathcal{A}, m)$ is *free* if $D(\mathcal{A}, m)$ is a free S-module. The fact that the module $D(\mathcal{A}, m)$ is reflexive (e.g., see Theorem 5 in [Z]) implies the following proposition.

Proposition 1.1
A multiarrangement $(\mathcal{A}, m)$ is free for any multiplicity m whenever $r(\mathcal{A}) := \mathrm{codim}_V(\bigcap_{H\in\mathcal{A}} H) \le 2$.

For $\theta_1, \ldots, \theta_\ell \in D(\mathcal{A}, m)$, we define the $(\ell \times \ell)$-matrix $M(\theta_1, \ldots, \theta_\ell)$ as the matrix whose (i,j)-entry is $\theta_j(x_i)$. In general, it is difficult to determine whether a given multiarrangement is free or not. However, using the following criterion (see Theorem 8 in [Z] and Theorem 4.19 in [OT]), we can verify that a candidate for a basis is actually a basis.

Theorem 1.2 (Saito-Ziegler's criterion)
Let $\theta_1, \ldots, \theta_\ell$ be derivations in $D(\mathcal{A}, m)$. Then $\{\theta_1, \ldots, \theta_\ell\}$ forms a basis for $D(\mathcal{A}, m)$ if and only if

$$\det M(\theta_1, \ldots, \theta_\ell) \in \mathbb{K}^* \cdot Q(\mathcal{A}, m).$$

In particular, if $\theta_1, \ldots, \theta_\ell$ are all homogeneous, then $\{\theta_1, \ldots, \theta_\ell\}$ forms a basis for $D(\mathcal{A}, m)$ if and only if the following two conditions are satisfied:

(i) $\theta_1, \ldots, \theta_\ell$ *are independent over* S.

(ii) $\sum_{i=1}^{\ell} \deg(\theta_i) = \sum_{H\in\mathcal{A}} m(H)$.

Let V_i be vector spaces over $\mathbb{K}$ and $(\mathcal{A}_i, m_i)$ be multiarrangements in V_i $(i = 1, 2)$. Let us define their product $(\mathcal{A}_1 \times \mathcal{A}_2, m_1 \times m_2)$ in the vector space $V_1 \oplus V_2$ by the following manner:

$$\begin{aligned} \mathcal{A}_1 \times \mathcal{A}_2 &:= \{H_1 \oplus V_2 \mid H_1 \in \mathcal{A}_1\} \cup \{V_1 \oplus H_2 \mid H_2 \in \mathcal{A}_2\}, \\ (m_1 \times m_2)(H_1 \oplus V_2) &:= m_1(H_1), \\ (m_1 \times m_2)(V_1 \oplus H_2) &:= m_2(H_2). \end{aligned}$$

The following Lemma is a special case of Lemma 1.4 in [ATW].

Lemma 1.3

$$D(\mathcal{A}_1 \times \mathcal{A}_2, m_1 \times m_2) \simeq S \cdot D(\mathcal{A}_1, m_1) \oplus S \cdot D(\mathcal{A}_2, m_2),$$

where $S = \mathrm{Sym}((V_1 \oplus V_2)^*)$.

We will use the following lemma in this article repeatedly. For the proof see [OT, Theorem 4.42] for example.

Lemma 1.4
Let $M = \oplus_{i=0}^{\infty} M_i$ be a free graded S-module with a homogeneous basis $\eta_1, \ldots, \eta_\ell$. Suppose $\deg \eta_i = d_i$ $(1 \le i \le \ell)$ with $d_1 \le \cdots \le d_\ell$. Assume that there exist elements $\theta_1, \ldots, \theta_k$ $(1 \le k \le \ell)$ in M which satisfy the following two conditions:

(i) $\deg(\theta_i) = d_i$ $(i = 1, \ldots, k)$.

(ii) $\theta_i \notin S\theta_1 + S\theta_2 + \ldots + S\theta_{i-1}$ $(i = 1, \ldots, k)$.

Then $\theta_1, \ldots, \theta_k$ can be extended to a basis for M.

2 Deletion

In this section we prove Theorem 0.4 and Deletion Theorem 0.5. We use the notation $(d_1, \ldots, d_\ell)_\leq$ to indicate $d_1 \leq \cdots \leq d_\ell$.

Proof of Theorem 0.4. Let $(d_1, \ldots, d_\ell)_\leq$ be the exponents of $(\mathcal{A}', m')$ and $(d_1, \ldots, d_{k-1}, e_k, \ldots, e_\ell)_\leq$ be the exponents of $(\mathcal{A}, m)$ such that $e_k \neq d_k$. Choose a basis $\{\theta_1, \ldots, \theta_{k-1}, \psi_k, \ldots, \psi_\ell\}$ for $D(\mathcal{A}, m)$ with $\deg(\theta_i) = d_i$ and $\deg(\psi_i) = e_i$. Because $\theta_1, \ldots, \theta_{k-1}$ are contained in $D(\mathcal{A}', m')$ and satisfy the two conditions in Lemma 1.4, we can find a basis $\{\theta_1, \ldots, \theta_{k-1}, \theta_k, \ldots, \theta_\ell\}$ for $D(\mathcal{A}', m')$ with $\deg(\theta_i) = d_i$ $(k \leq i \leq \ell)$. Since $\alpha_0\theta_k \in D(\mathcal{A}, m)$,

$$\alpha_0\theta_k = \sum_{i=1}^{k-1} a_i\theta_i + \sum_{i=k}^{\ell} b_i\psi_i \ (a_i, b_i \in S).$$

Given that $\theta_1, \ldots, \theta_k$ are independent over S, there exists some j, $j \geq k$ such that $b_j \neq 0$. Hence,

$$\deg(\alpha_0\theta_k) = d_k + 1 \geq \deg(\psi_j) \geq \deg(\psi_k) = e_k.$$

Moreover, since $\psi_k \in D(\mathcal{A}', m')$,

$$\psi_k = \sum_{i=1}^{k-1} a_i\theta_i + \sum_{i=k}^{\ell} b_i\theta_i \ (a_i, b_i \in S).$$

A similar argument as the above implies

$$\deg(\psi_k) = e_k \geq \deg(\theta_j) \geq \deg(\theta_k) = d_k.$$

The assumption that $e_k \neq d_k$ implies that $e_k = d_k + 1$. Noting that $\deg(\alpha_0\theta_k) = d_k + 1 = e_k$, Lemma 1.4 shows that the elements $\{\theta_1, \ldots, \theta_{k-1}, \alpha_0\theta_k\}$, which are contained in $D(\mathcal{A}, m)$, can be extended to a basis $\{\theta_1, \ldots, \theta_{k-1}, \alpha_0\theta_k, \theta'_{k+1}, \ldots, \theta'_\ell\}$ for $D(\mathcal{A}, m)$. Then Theorem 1.2 implies $\{\theta_1, \ldots, \theta_{k-1}, \theta_k, \theta'_{k+1}, \ldots, \theta'_\ell\}$ is a basis for $D(\mathcal{A}', m')$. $\square$

Let $(\mathcal{A}, m)$ be a multiarrangement and $H_0 \in \mathcal{A}$. Recall the restriction

$$\mathcal{A}'' = \{H_0 \cap K \mid K \in \mathcal{A} \setminus \{H_0\}\},$$

which is an arrangement on H_0. Let $X \in \mathcal{A}''$. Note that $(\mathcal{A}_X, m_X)$ can be decomposed into a direct product of a multiarrangement in $\mathbb{K}^2$ and the empty arrangement in $X \simeq \mathbb{K}^{\ell-2}$. Choose a coordinate system $(x_1, \ldots, x_\ell)$ so that $\alpha_0 = x_1$ and $X = \{x_1 = x_2 = 0\}$.

Proposition 2.1
We may choose a basis

$$\theta_X, \psi_X, \partial_{x_3}, \ldots, \partial_{x_\ell}$$

for $D(\mathcal{A}_X, m_X)$ such that $\theta_X \notin \alpha_0 \operatorname{Der}_{\mathbb{K}}(S)$ and $\psi_X \in \alpha_0 \operatorname{Der}_{\mathbb{K}}(S)$.

Proof. Let $(\mathcal{A}'_X, m'_X)$ be the deletion of $(\mathcal{A}_X, m_X)$ with respect to H_0. Then $(\mathcal{A}_X, m_X)$ and $(\mathcal{A}'_X, m'_X)$ are both free by Proposition 1.1. It follows from Lemma 1.3 and Theorem 0.4 that there exists a homogeneous basis $\{\theta_1, \theta_2, \partial_{x_3}, , \ldots, \partial_{x_\ell}\}$

for $D(\mathcal{A}'_X, m'_X)$ such that $\{\theta_1, x_1\theta_2, \partial_{x_3}, \ldots, \partial_{x_\ell}\}$ is a basis for $D(\mathcal{A}_X, m_X)$. Define $\theta_X := \theta_1$ and $\psi_X := x_1\theta_2$. It suffices to show that $\theta_1 \notin x_1 \operatorname{Der}_{\mathbb{K}}(S)$. If $\theta_X = \theta_1 \in x_1 \operatorname{Der}_{\mathbb{K}}(S)$ then $\theta'_1 := \theta_1/x_1 \in D(\mathcal{A}'_X, m'_X)$. This contradicts the assumption that $\{\theta_1 = x_1\theta'_1, \theta_2, \partial_{x_3}, \ldots, \partial_{x_\ell}\}$ is a basis for $D(\mathcal{A}'_X, m'_X)$. $\square$

Using the derivation θ_X in Proposition 2.1 we may define the e-multiplicity

$$m^*(X) = \deg \theta_X$$

as in Definition 0.2.

In Section 0 we defined the map $\pi : D(\mathcal{A}, m) \to D(\mathcal{A}'')$ by $\pi(\theta) = \overline{\theta}$ for $\theta \in D(\mathcal{A}, m)$. Note that π is well-defined because $\overline{\theta(f)} = \overline{\theta(g)}$ if $f - g \in \alpha_0 S$. Let $\alpha_0 \cdot : D(\mathcal{A}', m') \to D(\mathcal{A}, m)$ be the multiplication map by α_0.

Proposition 2.2
We have an exact sequence

$$0 \longrightarrow D(\mathcal{A}', m') \xrightarrow{\alpha_0 \cdot} D(\mathcal{A}, m) \xrightarrow{\pi} D(\mathcal{A}'', m^*).$$

Proof. The injectivity of $\alpha_0 \cdot$ and the exactness at $D(\mathcal{A}, m)$ are both obvious. So it suffices to show that $\pi(\theta)$ lies in $D(\mathcal{A}'', m^*)$ for $\theta \in D(\mathcal{A}, m)$. Let $X \in \mathcal{A}''$. Note that $D(\mathcal{A}, m) \subseteq D(\mathcal{A}_X, m_X)$. We use the notation in the proof of Proposition 2.1. Moreover, by Lemma 1.4, we may assume $\theta_X(x_i) \in \mathbb{K}[x_1, x_2]$ $(i = 1, 2)$. Thus we obtain $\theta_X(x_2) \in (x_1, x_2^{m^*(X)})S$, or equivalently $\pi(\theta_X)(\overline{x_2}) \in \overline{x_2}^{m^*(X)}\overline{S}$. Because $\pi(\psi_X) = 0$ and $\pi(\partial_{x_i})(\overline{x_2}) = 0$ for $3 \le i \le \ell$, we have $\pi(\theta)(\overline{x_2}) \in \overline{x_2}^{m^*(X)}\overline{S}$ for all $\theta \in D(\mathcal{A}, m)$. $\square$

To show Deletion Theorem 0.5, we need the following lemma.

Lemma 2.3
Let $(\mathcal{A}'', m^*)$ be a free multiarrangement with exponents $(d_1, \ldots, d_{\ell-1})_{\le}$. Assume the elements $\theta_1, \ldots, \theta_k$ $(1 \le k \le \ell - 1)$ in $D(\mathcal{A}, m)$ satisfy the following two conditions:

(i) $\deg(\theta_i) = d_i$ $(i = 1, \ldots, k-1)$.

(ii) $\deg(\theta_k) < d_k$.

Then there exists p, $1 \le p \le k$ such that

$$\theta_p \in S\theta_1 + \ldots + S\theta_{p-1} + \alpha_0 D(\mathcal{A}', m'). \tag{2.1}$$

Proof. Assume that for all i, $1 \le i \le k$, condition (2.1) is not true. Then $\overline{\theta}_1, \ldots, \overline{\theta}_{k-1}$ satisfy the two conditions in Lemma 1.4. So $\overline{\theta}_1, \ldots, \overline{\theta}_{k-1}$ can be extended to a basis for $D(\mathcal{A}'', m^*)$. Since $\deg(\overline{\theta_k}) < d_k$,

$$\overline{\theta_k} = \sum_{i=1}^{k-1} \overline{a_i}\,\overline{\theta_i}$$

for $a_i \in S$. This implies $\theta_k \in S\theta_1 + \ldots + S\theta_{k-1} + \alpha_0 D(\mathcal{A}', m')$, which is a contradiction. $\square$

Proof of Deletion Theorem 0.5. Put

$$\begin{aligned}\exp(\mathcal{A}, m) &= (d_1, \ldots, d_\ell)_\leq, \\ \exp(\mathcal{A}'', m^*) &= (d_1, \ldots, d_{k-1}, d_{k+1}, \ldots, d_\ell)_\leq.\end{aligned}$$

We may assume that $d_k < d_{k+1}$ or $k = \ell$. First assume $d_k < d_{k+1}$. Take a basis $\{\theta_1, \ldots, \theta_\ell\}$ for $D(\mathcal{A}, m)$ with $\deg(\theta_i) = d_i$. Since $\deg(\theta_k) = d_k < d_{k+1}$, Lemma 2.3 shows that there exists some p, $1 \leq p \leq k$ such that

$$\theta_p \in S\theta_1 + \cdots + S\theta_{p-1} + \alpha_0 D(\mathcal{A}', m').$$

Hence we may assume that $\theta_p \in \alpha_0 D(\mathcal{A}', m')$. Then Theorem 1.2 implies $\{\theta_1, \ldots, \theta_{p-1}, \theta_p/\alpha_0, \theta_{p+1}, \ldots, \theta_\ell\}$ is a basis for $D(\mathcal{A}', m')$. Next assume $k = \ell$, then $\exp(\mathcal{A}'', m^*) = (d_1, \ldots, d_{\ell-1})$. If $\overline{\theta_i} \in \bar{S}\bar{\theta}_1 + \cdots + \bar{S}\bar{\theta}_{i-1}$ for some i, $1 \leq i \leq \ell - 1$, then we can use the same argument as above. If $\overline{\theta_i} \notin \bar{S}\bar{\theta}_1 + \cdots + \bar{S}\bar{\theta}_{i-1}$ for all i, $1 \leq i \leq \ell - 1$, then Lemma 1.4 shows $\{\overline{\theta_1}, \ldots, \overline{\theta_{\ell-1}}\}$ is a basis for $D(\mathcal{A}'', m^*)$. Hence

$$\theta_\ell \in S\theta_1 + \cdots + S\theta_{\ell-1} + \alpha_0 D(\mathcal{A}', m'),$$

and the same argument as above completes the proof. $\square$

3 Addition and Restriction

In this section we prove Restriction Theorem 0.6 and Addition Theorem 0.7. First, for each $X \in \mathcal{A}''$, let us fix a hyperplane $H_X \in \mathcal{A} \setminus \{H_0\}$ such that $\overline{H_X} := H_0 \cap H_X = X$. Let m_0 denote $m(H_0)$. Recall the definition of $\theta_X, \psi_X \in D(\mathcal{A}_X, m_X)$ in Proposition 2.1. Denote

$$e_X := \deg(\theta_X) \text{ and } d_X := \deg(\psi_X).$$

Lemma 3.1
Let $(\mathcal{A}, m)$ be a multiarrangement in $\mathbb{K}^2$ with exponents (d, e). Fix a line $H_0 = \{\alpha_0 = 0\} \in \mathcal{A}$. By Theorem 0.4, there exists a basis $\{\theta, \psi\}$ for $D(\mathcal{A}, m)$ such that $\deg(\theta) = e$, $\deg(\psi) = d$ and that $\overline{\theta} \neq 0$, $\overline{\psi} = 0$. Then $d - m_0 \geq 0$.

Proof. We may assume that $S \simeq \mathbb{K}[x_1, x_2]$ and $\alpha_0 = x_1$. If $\psi(x_1) = 0$, then $\theta(x_1) \neq 0$ and Theorem 1.2 implies $Q(\mathcal{A}, m) \in \mathbb{K}^* \cdot \theta(x_1)\psi(x_2)$. Also we have $x_1|\psi(x_2)$ and $x_1^{m_0}|\theta(x_1)$. This implies $x_1^{m_0+1}|Q(\mathcal{A}, m)$, which is a contradiction. So we may assume that $\psi(x_1) \neq 0$. Therefore, $x_1^{m_0}|\psi(x_1)$ and thus $\deg(\psi) = d \geq m_0$. $\square$

Proposition 3.2
For all $X \in \mathcal{A}''$, we have $d_X - m_0 \geq 0$.

Proof. Since $(\mathcal{A}_X, m_X)$ can be decomposed into a direct product of a multiarrangement in $\mathbb{K}^2$ and the empty arrangement in $X \simeq \mathbb{K}^{\ell-2}$, Lemma 1.3 and Lemma 3.1 complete the proof. $\square$

By Proposition 3.2, we make the following key definition.

Definition 3.3
Define a polynomial $B = B(\mathcal{A}'', m^*)$ by

$$B(\mathcal{A}'', m^*) := \alpha_0^{m_0-1} \prod_{X \in \mathcal{A}''} \alpha_{H_X}^{d_X - m_0}.$$

Lemma 3.4
For any $\theta \in D(\mathcal{A}', m')$, we have $\theta(\alpha_0) \in (\alpha_0^{m_0}, B(\mathcal{A}'', m^*))$.

Proof. Take any $X \in \mathcal{A}''$ and consider the S-module $D(\mathcal{A}'_X, m'_X)$, which contains $D(\mathcal{A}', m')$ as a submodule. Since X is of codimension two and $\ell \geq 2$, $(\mathcal{A}'_X, m'_X)$ is free with exponents $(e_X, d_X - 1, 0, \ldots, 0)$. By Proposition 2.1, we have basis elements θ_X and ψ'_X for $D(\mathcal{A}'_X, m'_X)$ such that $\deg(\theta_X) = e_X$, $\deg(\psi'_X) = d_X - 1$ and that θ_X and $\alpha_0 \psi'_X$ are basis elements for $D(\mathcal{A}_X, m_X)$. First we show $D(\mathcal{A}'_X, m'_X)\alpha_0 := \{\theta(\alpha_0) |\, \theta \in D(\mathcal{A}'_X, m'_X)\} \subseteq (\alpha_0^{m_0}, \alpha_0^{m_0-1}\alpha_{H_X}^{d_X - m_0})$. We may assume that $\alpha_0 = x_1$ and $\alpha_{H_X} = x_2$. Then $\{\theta_X, \psi'_X, \partial_{x_3}, \ldots, \partial_{x_\ell}\}$ is a basis for $D(\mathcal{A}'_X, m'_X)$. Since $\theta_X(x_1) \in x_1^{m_0} S$ and $\partial_{x_i}(x_1) = 0$ $(2 \leq i \leq \ell)$, it suffices to show $\psi'_X(x_1) \in (x_1^{m_0}, x_1^{m_0-1} x_2^{d_X - m_0})$. We may assume that ψ'_X is a derivation of $\mathbb{K}[x_1, x_2]$ by Lemma 1.3. Thus there exist $f \in \mathbb{K}[x_1, x_2]$ and $g \in \mathbb{K}[x_2]$ such that

$$\psi'_X(x_1) = x_1^{m_0-1}(x_1 f(x_1, x_2) + g(x_2)).$$

Note that $\deg(\psi'_X(x_1)) = d_X - 1$, so $\deg(g(x_2)) = d_X - m_0$. Hence

$$\psi'_X(x_1) \in (x_1^{m_0}, x_1^{m_0-1} x_2^{d_X - m_0}).$$

So we have

$$\begin{aligned} D(\mathcal{A}', m')\alpha_0 \subseteq \bigcap_{X \in \mathcal{A}''} D(\mathcal{A}'_X, m'_X)\alpha_0 \quad &\subseteq \quad \bigcap_{X \in \mathcal{A}''} (\alpha_0^{m_0}, \alpha_0^{m_0-1}\alpha_{H_X}^{d_X - m_0}) \\ &= \quad (\alpha_0^{m_0}, \alpha_0^{m_0-1} \prod_{X \in \mathcal{A}''} \alpha_{H_X}^{d_X - m_0}). \end{aligned}$$

□

Proof of Restriction Theorem 0.6. Recall that we have a basis $\{\theta_1, \ldots, \theta_k, \ldots, \theta_\ell\}$ for $D(\mathcal{A}', m')$ such that $\{\theta_1, \ldots, \alpha_0\theta_k, \ldots, \theta_\ell\}$ is a basis for $D(\mathcal{A}, m)$. Noting that

$$|m| = \sum_{H \in \mathcal{A}} m(H) = m_0 + \sum_{X \in \mathcal{A}''} (e_X + d_X - m_0),$$

we have

$$\deg(B(\mathcal{A}'', m^*)) = m_0 - 1 + \sum_{X \in \mathcal{A}''} (d_X - m_0) = |m| - 1 - \sum_{X \in \mathcal{A}''} e_X = |m'| - |m^*|.$$

Assume that $\deg(\theta_k) < \deg(B(\mathcal{A}'', m^*))$. Then Lemma 3.4 implies that $\theta_k(\alpha_0) \in \alpha_0^{m_0} S$. This is equivalent to $\theta_k \in D(\mathcal{A}, m)$, which contradicts Theorem 0.4. Hence, $\deg(\theta_k) \geq \deg(B(\mathcal{A}'', m^*))$. This inequality implies

$$\begin{aligned} \sum_{i \neq k} \deg(\theta_i) \quad &= \quad |m'| - \deg(\theta_k) \\ &\leq \quad |m'| - \deg(B(\mathcal{A}'', m^*)) = |m'| - (|m'| - |m^*|) = |m^*|. \end{aligned}$$

To complete the proof by using Theorem 1.2, it suffices to show $\{\overline{\theta_1}, \ldots, \overline{\theta_{k-1}}, \overline{\theta_{k+1}}, \ldots, \overline{\theta_\ell}\}$ is independent over $\overline{S}$. Assume that there exist $a_i \in S$ $(i = 1, \ldots, k-1, k+1, \ldots, \ell)$ such that $\sum_{i \neq k} \overline{a_i}\overline{\theta_i} = 0$. This implies that there exists some $\theta \in \mathrm{Der}_{\mathbb{K}}(S)$ such that

$$\sum_{i \neq k} a_i \theta_i = \alpha_0 \theta.$$

Since $\theta_1, \ldots, \theta_{k-1}, \theta_{k+1}, \ldots, \theta_\ell$ lie in $D(\mathcal{A}, m)$, we can see that $\theta \in D(\mathcal{A}', m')$, and this implies $\overline{a_i} = 0$ for all i. □

Proof of Addition Theorem 0.7. Denote

$$\begin{aligned} \exp(\mathcal{A}', m') &= (d_1, \ldots, d_\ell)_\leq, \\ \exp(\mathcal{A}'', m^*) &= (d_1, \ldots, d_{k-1}, d_{k+1}, \ldots, d_\ell)_\leq. \end{aligned}$$

Choose a basis $\{\theta_1, \ldots, \theta_\ell\}$ for $D(\mathcal{A}', m')$ such that $\deg(\theta_i) = d_i$ $(i = 1, \ldots, \ell)$. We may assume that $d_k < d_{k+1}$ or $k = \ell$. Note that $\deg(B(\mathcal{A}'', m^*)) = |m'| - |m^*| = d_k$ in this case. Hence, Lemma 3.4 implies any θ_j satisfying $\deg(\theta_j) < \deg(B(\mathcal{A}'', m^*)) = d_k$ is contained in $D(\mathcal{A}, m)$. First assume that $d_k < d_{k+1}$. Consider the following condition:

(3.1) For all θ_j with $\deg(\theta_j) = d_k$, it holds that $\theta_j \in D(\mathcal{A}, m)$.

If (3.1) is true, then $\theta_1, \ldots, \theta_k \in D(\mathcal{A}, m)$. Applying Lemma 2.3, we can see that there exists p, $1 \leq p \leq k$ such that

$$\theta_p \in S\theta_1 + \ldots + S\theta_{p-1} + \alpha_0 D(\mathcal{A}', m').$$

Thus we may assume that $\theta_p \in \alpha_0 D(\mathcal{A}', m')$. This implies that $\alpha_0^{m_0} | \alpha_0 \cdot \det M(\theta_1, \ldots, \theta_p/\alpha_0, \ldots, \theta_\ell)$, which is a contradiction. So, there exists some p, $1 \leq p \leq k$ such that $\deg(\theta_p) = d_k$ and $\theta_p \notin D(\mathcal{A}, m)$. Let us put

$$\theta_p(\alpha_0) = f_p \alpha_0^{m_0} + c_p B(\mathcal{A}'', m^*)$$

for some $f_p, c_p \in S$. Since $\deg(\theta_p) = d_k = \deg(B(\mathcal{A}'', m^*))$ and $\theta_p \notin D(\mathcal{A}, m)$, we may assume that $c_p = 1$. Similarly, for $j \neq p$, put

$$\theta_j(\alpha_0) = f_j \alpha_0^{m_0} + c_j B(\mathcal{A}'', m^*)$$

for some $f_j, c_j \in S$. Define $\eta_j := \theta_j - c_j \theta_p$ $(j \neq p)$ and $\eta_p := \alpha_0 \theta_p$. Then $\eta_1, \ldots, \eta_\ell \in D(\mathcal{A}, m)$. Theorem 1.2 implies $\{\eta_1, \ldots, \eta_\ell\}$ is a basis for $D(\mathcal{A}, m)$.

Next assume that $k = \ell$. If (3.1) is true, then $\theta_1, \ldots, \theta_\ell \in D(\mathcal{A}, m)$, which is a contradiction. Hence there exists some p, $1 \leq p \leq \ell$ such that $\deg(\theta_p) = d_\ell = \deg(B(\mathcal{A}'', m^*))$ and $\theta_p \notin D(\mathcal{A}, m)$. Then the same argument as above completes the proof. □

4 e-multiplicities

To apply the addition-deletion theorems the computation of the e-multiplicities m^* of the restriction is crucial. In general, computating the e-multiplicities is difficult. On the other hand, using results from [Waka] and [WY] we can compute the e-multiplicities in the following cases.

Proposition 4.1
Let $X \in \mathcal{A}''$ where $\mathcal{A}''$ is the restriction to $H_0 \in \mathcal{A}$ and $m_0 = m(H_0)$. Suppose $k = |\mathcal{A}_X|$ and $m_1 = \max\{m(H) | H \in \mathcal{A}_X \backslash \{H_0\}\}$.

(1) If $k = 2$ then $m^*(X) = m_1$.

(2) If $2m_0 \geq |m_X|$ then $m^*(X) = |m_X| - m_0$.

(3) If $2m_1 \geq |m_X| - 1$ then $m^*(X) = m_1$.

(4) If $|m_X| \leq 2k - 1$ and $m_0 > 1$ then $m^*(X) = k - 1$.

(5) If $|m_X| \leq 2k - 2$ and $m_0 = 1$ then $m^*(X) = |m_X| - k + 1$.

(6) If $m_X \equiv 2$ then $m^*(X) = k$.

(7) If $k = 3$, $2m_0 \leq |m_X|$, and $2m_1 \leq |m_X|$ then $m^*(X) = \left\lfloor \frac{|m_X|}{2} \right\rfloor$.

Proof. Without loss of generality we can assume $\ell = 2$, $H_0 = \{x_1 = 0\}$, $Q(\mathcal{A}_X) = x_1 x_2 Q$, and $Q(\mathcal{A}_X, m_X) = x_1^{m_0} x_2^{m_1} \tilde{Q}$ for some $Q, \tilde{Q} \in S$. Then for case (1) we have $\theta_X = x_2^{m_1} \partial_{x_2}$ and $\psi_X = x_1^{m_0} \partial_{x_1}$ in the notation of Proposition 2.1. Thus, for case (1) we have $m^*(X) = m_1$. In case (2), if $2m_0 \geq |m_X|$ then the fact that $m^*(X) = |m_X| - m_0$ follows from [WY] because the smallest degree derivation is of the form $\theta = \frac{Q(\mathcal{A}_X, m_X)}{x_1^{m_0}} \partial_{x_2}$. Case (3) is similar to case (2). The only difference is that one of a basis element is of the form $\theta = \frac{Q(\mathcal{A}_X, m_X)}{x_2^{m_1}} \partial_{x_1}$ and $\overline{\theta} = 0$. Now, suppose that $|m_X| \leq 2k - 1$. Then the exponents are $(|m_X| - k + 1, k - 1)$ and $\varphi = \frac{Q(\mathcal{A}_X, m_X)}{Q(\mathcal{A}_X)} (x_1 \partial_{x_1} + x_2 \partial_{x_2})$ can be chosen as a basis element by [WY]. In case (4), φ is divisible by x_1 and hence $m^*(X) = k - 1$. In case (5), φ is not divisible by x_1. Since φ is a basis element of the smallest degree, we have $m^*(X) = \deg \varphi = |m_X| - k + 1$. In case (6), if $m_X \equiv 2$ then the exponents are (k, k) (see Proposition 5.4 in [SoT]). In case (7) the formula given by Wakamiko in [Waka] for the smallest degree generator is not divisible by x_1. Thus, in case (7) $m^*(X) = \left\lfloor \frac{|m_X|}{2} \right\rfloor$. $\square$

The next example shows that even when the exponents are combinatorially determined the e-multiplicities may depend on the position of the hyperplanes.

Example 4.2
Consider the class of two-dimensional multiarrangements $(\mathcal{A}_\xi, m)$ given by the defining polynomial $\tilde{Q}_\xi = x_1^4 x_2^3 (x_1 - x_2)(x_1 - \xi x_2)$ where $\xi \in \mathbb{K} - \{0, 1\}$. Then a basis for $D(\mathcal{A}_\xi, m)$ for all $\xi \in \mathbb{K} - \{0, 1\}$ is the following derivations

$$\theta_1 = x_1^4 \partial_{x_1} + \left[(1 + \xi(1 + \xi)) x_1 x_2^3 - \xi(1 + \xi) x_2^4 \right] \partial_{x_2}$$

and

$$\theta_2 = x_2^3 (x_1 - x_2)(x_1 - \xi x_2) \partial_{x_2}.$$

Suppose that $(\mathcal{A}_\xi, m)$ is of the form $(\mathcal{A}_X, m_X)$ for some $(\mathcal{A}, m)$ and $X \in \mathcal{A}''$ where $H_0 = \{x_1 = 0\}$. Then the basis $\{\theta_1, \theta_2\}$ shows that

$$m^*(X) = \begin{cases} 5 & \text{if } \xi = -1 \\ 4 & \text{otherwise} \end{cases}.$$

5 Applications

In this section, we apply the addition-deletion theorems together with the computations of the e-multiplicities in Proposition 4.1 to construct free and non-free multiarrangements.

Definition 5.1
Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a simple arrangement. Then $[m_1, \dots, m_n] \in \mathbb{Z}^n_{>0}$ is a free multiplicity for $\mathcal{A}$ if $(\mathcal{A}, m)$ is a free multiarrangement where $m(H_i) = m_i$ for all $1 \le i \le n$.

It is difficult to determine which multiplicities are free for a fixed simple arrangement. At least the following proposition provides an infinite number of free multiplicities for an arbitrary free arrangement.

Proposition 5.2
Let $\mathcal{A}$ be a free simple arrangement with $\exp(\mathcal{A}) = (1, d_2, \dots, d_\ell)$. Fix one hyperplane $H_0 \in \mathcal{A}$ and consider a multiarrangement $(\mathcal{A}, m)$ where m is defined by

$$m(H) = \begin{cases} 1 & \text{if } H \neq H_0, \\ m_0 & \text{if } H = H_0. \end{cases}$$

Then $(\mathcal{A}, m)$ is free with $\exp(\mathcal{A}, m) = (m_0, d_2, \dots, d_\ell)$.

Proof. Let $(\mathcal{A}, m), (\mathcal{A}', m')$ and $(\mathcal{A}'', m^*)$ be the triple with respect to H_0. Recall the restricted multiarrangement $(\mathcal{A}'', m'')$, where $m''(X) = |\mathcal{A}_X| - 1$ for all $X \in \mathcal{A}''$ which is defined by Ziegler in [Z]. It is proved in [Z] that if $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, d_2, \dots, d_\ell)$, then $(\mathcal{A}'', m'')$ is also free with $\exp(\mathcal{A}'', m'') = (d_2, \dots, d_\ell)$. Let $X \in \mathcal{A}''$. By Proposition 4.1 (2) and (4), $m^*(X) = |\mathcal{A}_X| - 1 = m''(X)$. To finish the proof, apply Addition Theorem 0.7. □

In the next example, we exhibit a free multiarrangement that is not inductively free by using Proposition 5.2.

Example 5.3
Recall the arrangement $\mathcal{A}$ in Example 4.59, based on a pentagon, in [OT], which is due to K. Brandt and J. Keaty. This arrangement is free with exponents $(1, 5, 5)$, but it is not inductively free. Fix $H_0 \in \mathcal{A}$ which is not the infinite hyperplane. Then by Proposition 5.2, the multiplicity m defined by

$$m(H) = \begin{cases} 1 & \text{if } H \neq H_0, \\ 2 & \text{if } H = H_0. \end{cases}$$

is a free multiplicity of $\mathcal{A}$ and $\exp(\mathcal{A}, m) = (2, 5, 5)$. Since $\mathcal{A}$ is not inductively free, to show $(\mathcal{A}, m)$ is not inductively free, it suffices to show that any deletion $(\mathcal{A}', m')$ with respect to $H \in \mathcal{A} \setminus \{H_0\}$ is not free. By Proposition 4.1 (1), (3) and (5), the restricted multiarrangement $(\mathcal{A}'', m^*)$ with respect to H has e-multiplicity $m^* = [2, 1, 1, 1, 1]$. Hence $\exp(\mathcal{A}'', m^*) = (2, 4)$. Now Deletion Theorem 0.5 implies that $(\mathcal{A}', m')$ is not free, so $(\mathcal{A}, m)$ is not inductively free.

Definition 5.4
An arrangement $\mathcal{A}$ is totally free (or totally non-free) if $(\mathcal{A}, m)$ is free (respectively non-free) for any multiplicity m on $\mathcal{A}$.

Remark 5.5
If $\ell \le 2$, then any arrangement is totally free by Proposition 1.1. Also, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are both totally free, then so is $\mathcal{A}_1 \times \mathcal{A}_2$ by Lemma 1.3. Consequently, any Boolean arrangement is totally free.

Example 5.6
Let $\mathcal{A}$ be an arrangement consisting of four generic hyperplanes in $\mathbb{K}^3$. Let

$$Q(\mathcal{A}, m) = x_1^a x_2^b x_3^c (x_1 + x_2 + x_3)^d$$

with $1 \le a \le b \le c \le d$. We will show that $\mathcal{A}$ is a totally non-free arrangement. Suppose that $(\mathcal{A}, m)$ is free with minimum $|m|$. Let $\exp(\mathcal{A}, m) = (d_1, d_2, d_3)_\le$. Let $H_0 = \{x_1 = 0\}$ and $\exp(\mathcal{A}'', m^*) = (e_1, e_2)_\le$. If $a = 1$, then $(\mathcal{A}', m')$ is Boolean with exponents (b, c, d). Thus $(e_1, e_2) \subset (b, c, d)$. This is a contradiction because $e_1 + e_2 = b + c + d$. So we may assume $2 \le a$.

Case 1. If $d_1 < e_1$, then $(\mathcal{A}', m')$ is free, which is a contradiction.

Case 2. If $d_1 = e_1$ and $d_2 \le e_2$, then $(\mathcal{A}', m')$ is free, which is a contradiction.

Case 3. If $d_1 = e_1$ and $d_2 > e_2$, then this is a contradiction because $a + b + c + d = d_1 + d_2 + d_3 > e_1 + e_2 + d_3 = b + c + d + d_3 \ge a + b + c + d$.

Case 4. If $d_1 > e_1$ and $b + c \le d$, then $d_1 > e_1 = b + c \ge a + b$. This is a contradiction because $x_1^a x_2^b (\partial_{x_1} - \partial_{x_2}) \in D(\mathcal{A}, m)$.

Case 5. If $d_1 > e_1$ and $b + c > d$, then $d_1 > e_1$ and $d_1 \ge e_1 + 1 \ge e_2$. This is a contradiction because $a + b + c + d = d_1 + d_2 + d_3 > e_1 + e_2 + d_3 = b + c + d + d_3 \ge a + b + c + d$.

Remark 5.7
In general, an arrangement can be neither totally free nor totally non-free (see Example 14 in [Z]). Also note that the example by Edelman and Reiner in [ER] is a non-free simple arrangement which admits a free multiplicity.

Let us consider supersolvable arrangements defined by Stanley in [St1]. (The following definition is equivalent to the original definition.)

Definition 5.8
An arrangement $\mathcal{A}$ is supersolvable *if there exists a filtration*

$$\mathcal{A} = \mathcal{A}_r \supset \mathcal{A}_{r-1} \supset \cdots \supset \mathcal{A}_2 \supset \mathcal{A}_1$$

such that

(1) $\operatorname{rank}(\mathcal{A}_i) = i$ $(i = 1, \dots, r)$.

(2) For any $H, H' \in \mathcal{A}_i$, there exists some $H'' \in \mathcal{A}_{i-1}$ such that $H \cap H' \subset H''$.

Remark 5.9
It is shown in [T3] that an arrangement is supersolvable if and only if it is fiber type.

Let us consider a multiarrangement $(\mathcal{A}, m)$ for a supersolvable arrangement $\mathcal{A}$. It is shown in [JT] and [St2] that $m \equiv 1$ is a free multiplicity. The following theorem gives another sufficient condition for m to be a free multiplicity.

Theorem 5.10
Let $(\mathcal{A}, m)$ be a multiarrangement such that $\mathcal{A}$ is supersolvable with a filtration $\mathcal{A} = \mathcal{A}_r \supset \mathcal{A}_{r-1} \supset \cdots \supset \mathcal{A}_2 \supset \mathcal{A}_1$ and $r \geq 2$. Let m_i denote the multiplicity $m|_{\mathcal{A}_i}$ and $\exp(\mathcal{A}_2, m_2) = (d_1, d_2, 0, \ldots, 0)$. Assume that for each $H' \in \mathcal{A}_d \backslash \mathcal{A}_{d-1}$, $H'' \in \mathcal{A}_{d-1}$ $(d = 3, \ldots, r)$ and $X := H' \cap H''$, it holds that

$$\mathcal{A}_X = \{H', H''\} \tag{5.1}$$

or that

$$m(H'') \geq \sum_{X \subset H \in (\mathcal{A}_d \backslash \mathcal{A}_{d-1})} m(H) - 1. \tag{5.2}$$

Then $(\mathcal{A}, m)$ is inductively free with

$$\exp(\mathcal{A}, m) = (d_1, d_2, |m_3| - |m_2|, \ldots, |m_r| - |m_{r-1}|, 0, \ldots, 0).$$

Proof. Let us put $d_i := |m_i| - |m_{i-1}|$ $(i = 3, \ldots, r)$. We may assume that

$$\left\{ \prod_{i=1}^{d} x_i = 0 \right\} \subseteq \mathcal{A}_d$$

for all d, $1 \leq d \leq \ell$. We prove by an induction on r. When $r = 2$, there is nothing to prove. Assume $r \geq 3$ and $(\mathcal{A}_{r-1}, m_{r-1})$ is free with $\exp(\mathcal{A}_{r-1}, m_{r-1}) = (d_1, d_2, d_3, \ldots, d_{r-1}, 0, \ldots, 0)$. We show that $(\mathcal{A}_r, m_r)$ is free with $\exp(\mathcal{A}_r, m_r) = (d_1, d_2, d_3, \ldots, d_{r-1}, d_r, 0, \ldots, 0)$. Let $H_r \in \mathcal{A}_r \setminus \mathcal{A}_{r-1}$ and $(\mathcal{A}''_r, m^*_r)$ be the restricted multiarrangement with respect to H_r. Since $\mathcal{A}$ is supersolvable, $\mathcal{A}''_r = \mathcal{A}_{r-1}|_{H_r}$. Also, the conditions (5.1), (5.2), Proposition 4.1 (1) and (3) imply $m^*_r(X) = m_{r-1}(H)$ where $H \in \mathcal{A}_{r-1}$ and $X = H \cap H_r$. Hence $(\mathcal{A}''_r, m^*_r)$ is free with $\exp(\mathcal{A}''_r, m^*_r) = (d_1, d_2, \ldots, d_{r-1}, 0, \ldots, 0)$. To complete the proof, apply this argument and Addition Theorem 0.7 repeatedly. □

Theorem 5.10 gives many free multiplicities on supersolvable arrangements. For the remainder of this article, assume that $\ell = 3$ and we consider the following supersolvable multiarrangement $(\mathcal{A}, m)$.

Definition 5.11
The Coxeter multiarrangement of type A_3 can be defined by the following polynomial:

$$Q(\mathcal{A}, m) = x_1^{m_1}(x_1 - x_3)^{m_2}(x_1 - x_2)^{m_3} x_2^{m_4}(x_2 - x_3)^{m_5} x_3^{m_6}.$$

The filtration is given by

$$\begin{aligned} \mathcal{A}_1 : &= \{x_1 = 0\}, \\ \mathcal{A}_2 : &= \{x_1 x_2 (x_1 - x_2) = 0\}, \\ \mathcal{A}_3 : &= \{x_1 x_2 x_3 (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = 0\}. \end{aligned}$$

It is shown in [Sa1], [Sa2] and [T4] that $m = [m_1 \ldots, m_6] = [m, m, m, m, m, m]$ for $m \in \mathbb{Z}_{>0}$ is a free multiplicity of $\mathcal{A}$. Now we obtain the following corollary by using Theorem 5.10.

Corollary 5.12
Let $(\mathcal{A}, m)$ be the Coxeter multiarrangement of type A_3. Assume that $m_1 \geq \max\{m_3, m_4\}$, $m_1 \geq m_2 + m_6 - 1$, $m_4 \geq m_5 + m_6 - 1$ and $m_3 \geq m_2 + m_5 - 1$. Then $(\mathcal{A}, m)$ is inductively free with

$$\begin{aligned} &\exp(\mathcal{A}, m) \\ &= \begin{cases} (\lfloor \frac{m_1+m_3+m_4}{2} \rfloor, \lceil \frac{m_1+m_3+m_4}{2} \rceil, m_2+m_5+m_6) & \text{if} \quad m_1 \leq m_3+m_4-1, \\ (m_1, m_3+m_4, m_2+m_5+m_6) & \text{if} \quad m_1 > m_3+m_4-1. \end{cases} \end{aligned}$$

Remark that for any $H \in \mathcal{A}$ the e-multiplicity on H can be calculated by Proposition 4.1 (1), (2), (3) and (7).

Example 5.13
Let $\mathcal{A}$ be a Coxeter arrangement of type A_3. Then the multiplicity $m := [1,1,2,2,1,1]$ is free by Theorem 5.10. However, the multiplicity $k := [2,1,1,1,2,1]$ is not free. Assume k is free. It is shown in [Sa1] and [Sa2] that $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1,2,3)$. Also, Proposition 5.2 implies $m_0 := [2,1,1,1,1,1]$ is free with $\exp(\mathcal{A}, m_0) = (2,2,3)$. Then Theorem 0.4 implies $\exp(\mathcal{A}, k) = (2,2,4)$ or $(2,3,3)$. However, for the restricted multiarrangement $(\mathcal{A}'', k^*)$ with respect to $x_2 - x_3 = 0$, we can see that $k^* = [2,2,2]$. Hence $\exp(\mathcal{A}'', k^*) = (3,3)$, which contradicts Restriction Theorem 0.6. Hence $k = [2,1,1,1,2,1]$ is not a free multiplicity of $\mathcal{A}$. We note that the non-freeness criterion in [ATW] also shows that k is not a free multiplicity.

References

[ATW] T. Abe, H. Terao and M. Wakefield, The characteristic polynomial of a multiarrangement. arxiv math.AG/0611742.

[C] P. Cartier, Les arrangements d'hyperplans: un chapitre de géometrie combinatroire. In *Séminaire Bourbaki* 1980/81. Lecture Notes in Math. **901**, Springer Verlag, 1981, pp.1–22.

[ER] P. H. Edelman and V. Reiner, A counterexample to Orlik's conjecture. *Proc. Amer. Math. Soc.* **118** (1993), 927–929.

[JT] M. Jambu and H. Terao, Free arrangements of hyperplanes and supersolvable lattices. *Advances in Math.* **52** (1984), 248–258.

[OT] P. Orlik and H. Terao, *Arrangements of hyperplanes.* Grundlehren der Mathematischen Wissenschaften, **300**. Springer-Verlag, Berlin, 1992.

[Sa1] K. Saito, On the uniformization of complements of discriminant loci. In: *Conference Notes. Amer. Math. Soc. Summer Institute, Williamstown* (1975).

[Sa2] K. Saito, Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27** (1980), 265–291.

[SoT] L. Solomon and H. Terao, The double Coxeter arrangement. *Comment. Math. Helv.* 64 (1998), 237–258.

[St1] R. P. Stanley, Supersolvable lattices. *Algebra Universalis* 2 (1972), 197–217.

[St2] R. P. Stanley, T-free arrangements of hyperplanes. In: *Progress in Graph Theory.* Academic Press (1984), p. 539.

[T1] H. Terao, Arrangements of hyperplanes and their freeness I, II. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27** (1980), 293–320.

[T2] H. Terao, Generalized exponents of a free arrangement of hyperplanes and Shephard-Todd-Brieskorn formula. *Invent. math.* **63** (1981), 159–179.

[T3] H. Terao, Modular elements of lattices and topological fibration. *Advances in Math.* **62** (1986), 135–154.

[T4] H. Terao, Multiderivations of Coxeter arrangements. *Invent. math.* **148** (2002), 659–674.

[Waka] A. Wakamiko, On the Exponents of 2-Multiarrangements. to appear in *Tokyo. J. Math.*

[WY] M. Wakefield and S. Yuzvinsky, Derivations of an effective divisor on the complex projective line. arXiv:math.AG/0507323, to appear in *Trans. Amer. Math. Soc.*

[Y1] M. Yoshinaga, Characterization of a free arrangement and conjecture of Edelman and Reiner. *Invent. math.* **157** (2004), no.2, 449–454.

[Y2] M. Yoshinaga, On the freeness of 3-arrangements. *Bull. London Math. Soc.* **37** (2005), no. 1, 126–134.

[Z] G. M. Ziegler, Multiarrangements of hyperplanes and their freeness. in *Singularities* (Iowa City, IA, 1986), 345–359, Contemp. Math., **90**, Amer. Math. Soc., Providence, RI, 1989.

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