Free and non-free multiplicity on the arrangement of type $A_3 - 1$

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Abstract

We give the first complete classification of free and non-free multiplicities on an arrangement, called the arrangement of type $A_3 - 1$, which admits both of them.

0 Introduction

Let V be an ℓ-dimensional vector space over a field $\mathbb{K}$, $\{x_1, \ldots, x_\ell\}$ be a basis for the dual vector space V^* and $S := \mathrm{Sym}(V^*) \simeq \mathbb{K}[x_1, \ldots, x_\ell]$. $\mathrm{Der}_{\mathbb{K}}(S)$ denotes the S-module of $\mathbb{K}$-linear derivations of S, i.e., $\mathrm{Der}_{\mathbb{K}}(S) = \bigoplus_{i=1}^{\ell} S \cdot \partial_{x_i}$. We say a non-zero element $\theta = \sum_{i=1}^{\ell} f_i \partial_{x_i} \in \mathrm{Der}_{\mathbb{K}}(S)$ is *homogeneous of degree p* if f_i is zero or homogeneous of degree p for each i.

A *hyperplane arrangement* $\mathcal{A}$ (or simply an *arrangement*) is a finite collection of affine hyperplanes in V. If each hyperplane in $\mathcal{A}$ contains the origin, we say that $\mathcal{A}$ is *central*. In this article we assume that all arrangements are central. A *multiplicity* m on an arrangement $\mathcal{A}$ is a map $m : \mathcal{A} \to \mathbb{Z}_{>0}$ and a pair $(\mathcal{A}, m)$ is called a *multiarrangement*. $|m|$ denotes the sum of the multiplicity $\sum_{H \in \mathcal{A}} m(H)$. When $m \equiv 1$, $(\mathcal{A}, m)$ is the same as a hyperplane arrangement $\mathcal{A}$ and sometimes called a *simple arrangement*. For each hyperplane $H \in \mathcal{A}$ fix a linear form $\alpha_H \in \mathcal{A}$ such that $\ker(\alpha_H) = H$. Put

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$Q(\mathcal{A},m) := \prod_{H\in\mathcal{A}} \alpha_H^{m(H)}$. The main object in this article is a *logarithmic derivation module* $D(\mathcal{A},m)$ of $(\mathcal{A},m)$ defined by

$$D(\mathcal{A},m) := \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) | \theta(\alpha_H) \in S \cdot \alpha_H^{m(H)} \text{ (for all } H \in \mathcal{A})\}.$$

A multiarrangement $(\mathcal{A},m)$ is called *free* if $D(\mathcal{A},m)$ is a free S-module of rank ℓ. If $(\mathcal{A},m)$ is free, then there exists a homogeneous free basis $\{\theta_1,\ldots,\theta_\ell\}$ for $D(\mathcal{A},m)$. Then we define the *exponents* of a free multiarrangement $(\mathcal{A},m)$ by $\exp(\mathcal{A},m) := (\deg(\theta_1),\ldots,\deg(\theta_\ell))$. The exponents are independent of a choice of a basis.

Originally, a multiarrangement was defined by Ziegler in [Z] and used effectively in the studies of hyperplane arrangements, e.g., in [Y1] and [Y2]. However, very few have been known about the freeness and non-freeness of multiarrangements. Recently, some theorems to consider the freeness of multiarrangements are developed in [ATW1] and [ATW2]. In these papers, a concept of free multiplicity is introduced. For a simple arrangement $\mathcal{A}$, we say a multiplicity m on $\mathcal{A}$ is *free* if the multiarrangement $(\mathcal{A},m)$ is free. For example, every multiplicity is free on Boolean arrangements and no multiplicity is free on a generic arrangement which consists of four planes in a three-dimensional vector space (see [ATW2]). However, on an arrangement which admits both free and non-free multiplicity, only a partial classification of multiplicities are known. For example, Coxeter arrangements of type A_3 admits both free and non-free multiplicity, but such a classification is not known. Hence to consider the behavior, geometry and combinatorics of free and non-free multiplicities in $\mathbb{Z}_{>0}^{|\mathcal{A}|}$ is a new problem in the study of arrangements. In this article, we give the first complete classification of the freeness of all multiplicities on such an arrangement. Let us fix $\ell = 3$ and a basis $\{x,y,z\}$ for V^*.

Definition 0.1
An arrangement $\mathcal{A}$ is called the arrangement of type $A_3 - 1$ if it is defined by

$$Q(\mathcal{A}) = xy(x-y)(x-z)(y-z).$$

This is a free arrangement with $\exp(\mathcal{A}) = (1,2,2)$, but Ziegler proved that the constant multiplicity $m \equiv 2$ is not a free multiplicity. So the arrangement of type $A_3 - 1$ admits both free and non-free multiplicity. Since this arrangement is close to the Coxeter arrangement of type A_3 and consists of only five planes, it is natural to consider the classification of multiplicities from the viewpoint of freeness. Our classification is as follows:

Theorem 0.2
Let $\mathcal{A}$ be the arrangement of type A_3-1 and $m=[a,b,c,d,e]$ be a multiplicity on $\mathcal{A}$ defined by

$$Q(\mathcal{A},m)=(y-z)^a y^b (x-y)^c x^d (x-z)^e.$$

Then m is a free multiplicity if and only if $c \geq a+e-1$ or $c \geq b+d-1$.

Remark 0.3
Theorem 0.2 implies, if we identify all the multiplicities on the arrangement of type A_3-1 with $\mathbb{Z}_{>0}^5=\{(a,b,c,d,e)\}$ (i.e., the moduli space of multiplicities on $\mathcal{A}$), then the set of free multiplicities consists of three chambers of the complement of the arrangement in $\mathbb{Z}_{>0}^5$ defined by

$$(c-a-e+\frac{1}{2})(c-b-d+\frac{1}{2})=0.$$

Also note that a choice of such an arrangement is not unique.

The organization of this article is as follows. In Section 1 we introduce some results and notation which will be used in this article. In Section 2 we prove Theorem 0.2.

1 Preliminaries

In this section we fix some notation and introduce some results. To prove Theorem 0.2, we use the following two results:

Theorem 1.1 ([ATW1], Corollary 4.6)
If a multiarrangement $(\mathcal{A},m)$ is free, then $GMP(k)=LMP(k)$ $(1 \leq k \leq \ell)$, where $GMP(k)$ is the k-th global mixed product of $(\mathcal{A},m)$ and $LMP(k)$ is the k-th local mixed product of $(\mathcal{A},m)$.

Theorem 1.2 ([ATW2], Theorem 5.10)
Let $(\mathcal{A},m)$ be a multiarrangement such that $\mathcal{A}$ is supersolvable with a filtration $\mathcal{A}=\mathcal{A}_r \supset \mathcal{A}_{r-1} \supset \cdots \supset \mathcal{A}_2 \supset \mathcal{A}_1$ and $r \geq 2$. Let m_i denote the multiplicity $m|_{\mathcal{A}_i}$ and $\exp(\mathcal{A}_2,m_2)=(d_1,d_2,0,\ldots,0)$. Assume that for each $H' \in \mathcal{A}_d \setminus \mathcal{A}_{d-1}$, $H'' \in \mathcal{A}_{d-1}$ $(d=3,\ldots,r)$ and $X:=H'\cap H''$, it holds that

$$\mathcal{A}_X=\{H',H''\}$$

or that

$$m(H'') \geq \sum_{X\subset H \in (\mathcal{A}_d\setminus\mathcal{A}_{d-1})} m(H)-1.$$

Then $(\mathcal{A}, m)$ *is free with*

$$\exp(\mathcal{A}, m) = (d_1, d_2, |m_3| - |m_2|, \ldots, |m_r| - |m_{r-1}|, 0, \ldots, 0).$$

For details and notation of these theorems, see [ATW1] and [ATW2]. Note that the arrangement of type $A_3 - 1$ is supersolvable. Theorem 1.1 is used to show the non-freeness of a multiarrangement. To apply it, we need some elementary results on number theory. From now on, let us assume $\ell = 3$ and fix a coordinate system $\{x, y, z\}$ for V^*. For the rest of this article we only consider the 2nd mixed products. Hence $LMP(\mathcal{A}, m)$ denotes the 2nd local mixed product of $(\mathcal{A}, m)$, and $GMP(\mathcal{A}, m)$ the 2nd global mixed product of $(\mathcal{A}, m)$ if it is free. In other words,

$$LMP(\mathcal{A}, m) = \sum_{X \in L(\mathcal{A})_2} d_1^X d_2^X,$$

where $L(\mathcal{A})_2$ consists of elements in the intersection lattice $L(\mathcal{A})$ of $\mathcal{A}$ (e.g., see [OT]) such that $\operatorname{codim}_V(X) = 2$ and $\exp(\mathcal{A}_X, m|_{\mathcal{A}_X}) = (d_1^X, d_2^X, 0, \ldots, 0)$ for $X \in L(\mathcal{A})_2$. Moreover, if $(\mathcal{A}, m)$ is free with $\exp(\mathcal{A}, m) = (d_1, d_2, d_3)$, then

$$GMP(\mathcal{A}, m) = d_1 d_2 + d_2 d_3 + d_3 d_1.$$

Sometimes for the triple of integers (d_1, d_2, d_3), $GMP(d_1, d_2, d_3)$ denotes $d_1 d_2 + d_2 d_3 + d_3 d_1$. Let us agree that $(d_1, d_2, d_3)_\leq$ denotes the integers d_1, d_2, d_3 with $d_1 \leq d_2 \leq d_3$.

Lemma 1.3
Let us put $m_0 := \max\{m(H) | H \in \mathcal{A}\}$ *for a free multiarrangement* $(\mathcal{A}, m)$ *with* $\exp(\mathcal{A}, m) = (d_1, d_2, d_3)_\leq$. *Then* $d_3 \geq m_0$.

Proof. We may assume that $m_0 = m(\{x = 0\})$. If $d_3 < m_0$, then all elements in $D(\mathcal{A}, m)$ can be expressed as $f_y \partial_y + f_z \partial_z$ for $f_y, f_z \in S$, which contradicts to the freeness of $(\mathcal{A}, m)$. $\square$

For a rational number $\alpha \in \mathbb{Q}$, let $\lceil \alpha \rceil$ denote the smallest integer which is larger than or equal to α, and $\lfloor \alpha \rfloor$ the largest integer which is smaller than or equal to α.

Lemma 1.4
Let $(\mathcal{A}, m)$ *be a free multiarrangement with* $\exp(\mathcal{A}, m) = (d_1, d_2, d_3)_\leq$ *and* $(\mathcal{B}, m')$ *be a free submultiarrangement of* $(\mathcal{A}, m)$ *with* $\exp(\mathcal{B}, m') = (e_1, e_2, e_3)_\leq$. *Put* $n := |m| - |m'|$ *and assume that*

$$e_3 \geq \left\lceil \frac{|m|}{3} \right\rceil, \ e_2 \geq \left\lfloor \frac{|m| - e_3}{2} \right\rfloor.$$

Then $GMP(d_1,d_2,d_3) \le GMP(e_1+n, e_2, e_3)$.

Proof. Since $D(\mathcal{A},m) \subset D(\mathcal{B},m')$, there exist non-negative integers α,β such that

$$d_3 = e_3+\alpha,\ d_2 = e_2+\beta,\ d_1 = e_1+n-\alpha-\beta.$$

Hence

$$\begin{aligned} GMP(e_1+n,e_2,e_3) - GMP(d_1,d_2,d_3) &= (e_1+n)e_2 + (e_1+n)e_3 + e_2e_3 \\ &\quad -\{(e_1+n-\alpha-\beta)(e_2+\beta) \\ &\quad +(e_1+n-\alpha-\beta)(e_3+\alpha) + (e_2+\beta)(e_3+\alpha)\} \\ &= \alpha^2 + \alpha(e_3-e_1-n) + \beta(e_2+\alpha+\beta-e_1-n). \end{aligned}$$

By the assumption, $e_3-e_1-n \ge 0$. If $e_2-e_1-n+\alpha+\beta<0$, then the assumption implies that $e_2+1 = e_1+n$ and $\alpha=\beta=0$. In this case $(d_1,d_2,d_3) = (e_1+n,e_2,e_3)$. □

Lemma 1.5
Let $(\mathcal{A},m)$ *be a free multiarrangement with* $\exp(\mathcal{A},m) = (d_1,d_2,d_3)_\le$. *If* $\max\{m(H)|H\in\mathcal{A}\} = a \ge \left\lceil \frac{|m|}{3} \right\rceil$, *then*

$$GMP(d_1,d_2,d_3) \le GMP\left(a, \left\lceil \frac{|m|-a}{2}\right\rceil, \left\lfloor \frac{|m|-a}{2}\right\rfloor\right).$$

Proof. Lemma 1.3 implies that

$$d_3 = a+\alpha,\ d_2 = \frac{|m|-a-\alpha}{2}+\beta,\ d_1 = \frac{|m|-a-\alpha}{2}-\beta\ (\alpha\in\mathbb{Z},\ \beta\in\frac{1}{2}\mathbb{Z}).$$

Hence

$$GMP\left(a, \left\lceil \frac{|m|-a}{2}\right\rceil, \left\lfloor \frac{|m|-a}{2}\right\rfloor\right) - GMP(d_1,d_2,d_3) = o(-\frac{1}{4}) + \frac{3}{4}\alpha^2 + \beta^2 + \alpha(\frac{3}{2}a - \frac{1}{2}|m|),$$

where

$$o(-\frac{1}{4}) = \begin{cases} 0 & \text{if } |m|-a \text{ is even,} \\ -\frac{1}{4} & \text{if } |m|-a \text{ is odd.} \end{cases}$$

By the assumption, $\frac{3}{2}a-\frac{1}{2}|m| \ge 0$. If $\alpha>0$ or $\beta>0$, then $GMP\left(a, \left\lceil \frac{|m|-a}{2}\right\rceil, \left\lfloor \frac{|m|-a}{2}\right\rfloor\right) - GMP(d_1,d_2,d_3) \ge 0$. If $\alpha=\beta=0$, then there is nothing to prove. □

2 Proof of Theorem 0.2

In this section we prove Theorem 0.2. From now on, let $\mathcal{A}$ be the arrangement of type $A_3 - 1$ and $m = [a, b, c, d, e]$ a multiplicity on $\mathcal{A}$ as in the statement of Theorem 0.2. Moreover, let us introduce some notation. Let $\exp[a, c, e]$ (resp : $\exp[b, c, d]$) denote the exponents of a 2-multiarrangement defined by $(y-z)^a(x-y)^c(x-z)^e$ (resp : $y^b(x-y)^c x^d$). If we put $\exp[a, c, e] = (d_1, d_2)$ and $\exp[b, c, d] = (e_1, e_2)$, then $[a, c, e]$ denotes $d_1 \times d_2$ and $[b, c, d]$ denotes $e_1 \times e_2$. We say $[a, c, e]$ or $[b, c, d]$ is *balanced* if each integer of the three is less than the sum of the other two plus one, i.e., $a \leq c + e + 1$ and so on. It is known that $\exp[a, c, e] = (d_1, d_2)$ or $\exp[b, c, d] = (e_1, e_2)$ are determined by the multiplicities. For example, if $[a, c, e]$ is balanced, then $|d_1 - d_2| \leq 1$ and if $\max\{a, c, e\} = a$ and $[a, c, e]$ is not balanced, then $(d_1, d_2) = (a, c + e)$ (see [Waka]).

Now let us prove Theorem 0.2. First we show the condition in Theorem 0.2 is a sufficient condition.

Proposition 2.1
If $c \geq a + e - 1$ or $c \geq b + d - 1$, then $(\mathcal{A}, m)$ is free.

Proof. Assume $c \geq a + e - 1$. Consider a supersolvable filtration $\mathcal{A}_3 \supset \mathcal{A}_2 \supset \mathcal{A}_1$ of $\mathcal{A}$ defined by

$$\begin{aligned} \mathcal{A}_1 : &= \{x = 0\}, \\ \mathcal{A}_2 : &= \{xy(x-y) = 0\}, \\ \mathcal{A}_3 : &= \{xy(x-y)(x-z)(y-z) = 0\}. \end{aligned}$$

To complete the proof, apply Theorem 1.2. The same argument is valid when $c \geq b + d - 1$. $\square$

Hence, for the rest of this section, we assume the following condition:

$$c \leq a + e - 2 \text{ and } c \leq b + d - 2. \tag{2.1}$$

Note that, in particular, $c < \left\lfloor \dfrac{|m|}{3} \right\rfloor$. We show that $(\mathcal{A}, m)$ is not free under the condition (2.1) by using Theorem 1.1 and related non-freeness criterion in [ATW1]. After an appropriate change of coordinates, we may assume that

$$a \geq e, \ b \geq d, \ \text{and } a \geq b.$$

Let us define a submultiarrangement $(\mathcal{B}, m')$ of $(\mathcal{A}, m)$ by

$$Q(\mathcal{B}, m') := (y-z)^a y^b (x-y)^c (x-z)^e.$$

Lemma 2.2
Assume that $a \geq \left\lceil \frac{|m|}{3} \right\rceil$ *and* $b \geq \left\lfloor \frac{|m|}{3} \right\rfloor$. *Then* $(\mathcal{A}, m)$ *is not free.*

Proof. Let us assume that $(\mathcal{A}, m)$ is free with $\exp(\mathcal{A}, m) = (d_1, d_2, d_3)$. Since $c+d+e \leq \left\lceil \frac{|m|}{3} \right\rceil$, $LMP(\mathcal{A}, m) = (a+b)(c+d+e)+ab+de$. Note that Theorem 1.2 implies $(\mathcal{B}, m')$ is free with $\exp(\mathcal{B}, m') = (a, c+e, b)$. By the assumption, $(\mathcal{B}, m')$ satisfies the condition of Lemma 1.4. Therefore $GMP(d_1, d_2, d_3) \leq GMP(a, b, e+d+e)$. Hence $LMP(\mathcal{A}, m) - GMP(d_1, d_2, d_3) \geq de > 0$, which contradicts to Theorem 1.1. $\square$

Lemma 2.3
Assume that $a \geq \left\lceil \frac{|m|}{3} \right\rceil$ *and* $b < \left\lfloor \frac{|m|}{3} \right\rfloor$. *Then* $(\mathcal{A}, m)$ *is not free.*

Proof. Assume that $(\mathcal{A}, m)$ is free with $\exp(\mathcal{A}, m) = (d_1, d_2, d_3)$. Put $LMP := LMP(\mathcal{A}, m)$ and $GMP := GMP(\mathcal{A}, m)$. Note that $\max\{a, b, c, d, e\} = a$ by the assumption and (2.1).

Case 1. When $b + d \leq \left\lfloor \frac{|m|}{3} \right\rfloor$ and $[a, c, e]$ is balanced. In this case, $\exp(\mathcal{B}, m') = (\exp[a, c, e], b)$. Hence the assumption and Lemma 1.4 imply $GMP(d_1, d_2, d_3) \leq GMP(\exp[a, c, e], b + d)$. So

$$\begin{aligned} LMP - GMP &\geq (a+e)(b+d) + [a,c,e] + [b,c,d] - \{(a+c+e)(b+d) + [a,c,e]\} \\ &= [b,c,d] - c(b+d) > 0 \text{ (by (2.1))}, \end{aligned}$$

which is a contradiction.

Case 2. When $b+d \leq \left\lfloor \frac{|m|}{3} \right\rfloor$, $[a, c, e] = a(c+e)$ and $c+e \geq b+d$. In this case $\exp(\mathcal{B}, m') = (a, c+e, b)$. So Lemma 1.4 implies that $GMP(d_1, d_2, d_3) \leq GMP(a, c+e, b+d)$. Hence

$$\begin{aligned} LMP - GMP &\geq a(b+c+d+e) + [b,c,d] + e(b+d) - \{a(b+c+d+e) + (c+e)(b+d)\} \\ &= [b,c,d] - c(b+d) > 0 \text{ (by (2.1))}, \end{aligned}$$

which is a contradiction.

Hence it suffices to show the non-freeness of the following two cases:

(1) $b + d \leq \left\lfloor \frac{|m|}{3} \right\rfloor$, $[a, c, e] = a(c+e)$ and $b + d > c + e$.

(2) $b + d > \left\lfloor \frac{|m|}{3} \right\rfloor$.

Note that the condition $b+d > \left\lfloor \frac{|m|}{3} \right\rfloor$ implies $[a,c,e] = a(c+e)$ and $b+d > c+e$.

Case 3. When $[a,c,e] = a(c+e)$, $b+d > c+e$ and $[b,c,d]$ is balanced. Lemma 1.5 and $\max\{a,b,c,d,e\} = a$ imply $GMP(d_1,d_2,d_3) \leq GMP(a, \left\lceil \frac{|m|-a}{2} \right\rceil, \left\lfloor \frac{|m|-a}{2} \right\rfloor)$. Hence

$$\begin{aligned}
LMP - GMP &\geq a(b+c+d+e) + [b,c,d] + e(b+d) \\
&\quad -\{a(b+c+d+e) + \left\lceil \frac{|m|-a}{2} \right\rceil \left\lfloor \frac{|m|-a}{2} \right\rfloor\} \\
&\geq e(b+d) + \frac{(b+c+d)^2}{4} - \frac{(b+c+d+e)^2}{4} - \frac{1}{4} \\
&= e(b+d) - \frac{e(b+c+d)}{2} - \frac{e^2}{4} - \frac{1}{4} \\
&= \frac{2e(b+d-c-e) + e^2 - 1}{4} > 0,
\end{aligned}$$

which is a contradiction.

Case 4. When $[a,c,e] = a(c+e)$, $b+d > c+e$, $b \geq c+d+2$ and $b < \frac{|m|-a}{2}$. Note that $b+d \geq \left\lceil \frac{|m|-a}{2} \right\rceil$ because of $b+d > c+e$. Hence $b \geq c+d+2$ implies

$$\begin{aligned}
LMP &= a(b+c+d+e) + e(b+d) + b(c+d) \\
&\geq a(b+c+d+e) + e(b+d) + \left\lceil \frac{|m|-a}{2} \right\rceil (b+c+d - \left\lceil \frac{|m|-a}{2} \right\rceil).
\end{aligned}$$

Since $GMP \leq GMP(a, \left\lceil \frac{|m|-a}{2} \right\rceil, \left\lfloor \frac{|m|-a}{2} \right\rfloor)$, we have

$$\begin{aligned}
LMP - GMP &\geq e(b+d) + \left\lceil \frac{|m|-a}{2} \right\rceil (b+c+d - \left\lceil \frac{|m|-a}{2} \right\rceil - \left\lfloor \frac{|m|-a}{2} \right\rfloor) \\
&= e(b+d - \left\lceil \frac{|m|-a}{2} \right\rceil) + \left\lceil \frac{|m|-a}{2} \right\rceil (b+c+d+e-(|m|-a)) \\
&= e(b+d - \left\lceil \frac{|m|-a}{2} \right\rceil).
\end{aligned}$$

Since $b+d \geq \left\lceil \frac{|m|-a}{2} \right\rceil$, the last equation is positive unless $b+d = \left\lceil \frac{|m|-a}{2} \right\rceil$ and $c+e = \left\lfloor \frac{|m|-a}{2} \right\rfloor$. In this case, the same argument as **Case 2** implies $LMP > GMP$.

Case 5. When $[a,c,e] = a(c+e)$, $b+d > c+e$, $b \geq c+d+2$ and $b \geq \dfrac{|m|-a}{2}$. Lemma 1.4 implies $GMP \leq GMP(a,b,c+d+e)$. Hence

$$LMP - GMP \geq de > 0,$$

and the proof is completed. $\square$

By Lemma 2.2 and 2.3, we may assume the following condition:

$$a,b,c,d,e < \left\lceil \frac{|m|}{3} \right\rceil . \tag{2.2}$$

Lemma 2.4
Assume that $b+d \leq \left\lceil \dfrac{|m|}{3} \right\rceil$ *(resp:* $a+e \leq \left\lceil \dfrac{|m|}{3} \right\rceil$*). Then* $(\mathcal{A}, m)$ *is not free.*

Proof. Assume that $(\mathcal{A}, m)$ is free with $\exp(\mathcal{A}, m) = (d_1, d_2, d_3)$. If $b+d \leq \left\lceil \dfrac{|m|}{3} \right\rceil$, then $a+c+e \geq \left\lfloor \dfrac{2}{3}|m| \right\rfloor$. Also, the condition (2.2) implies $[a,c,e]$ is balanced. So $GMP = GMP(d_1,d_2,d_3) \leq GMP(\exp[a,c,e], b+d)$ by Lemma 1.4. Thus

$$\begin{aligned} LMP(\mathcal{A},m) - GMP(\mathcal{A},m) &\geq (a+e)(b+d) + [a,c,e] + [b,c,d] \\ &\quad -\{[a,c,e] + (a+c+e)(b+d)\} \\ &= [b,c,d] - c(b+d) > 0 \text{ (by (2.1))}, \end{aligned}$$

which contradicts to Theorem 1.1 $\square$

Therefore we may assume that

$$\left\lceil \frac{|m|}{3} \right\rceil < a+e \text{ (resp} : b+d) < \left\lfloor \frac{2}{3}|m| \right\rfloor - 1. \tag{2.3}$$

Hence the next lemma completes the proof of Theorem 0.2.

Lemma 2.5
Under the conditions (2.1), (2.2) and (2.3), $(\mathcal{A}, m)$ *is not free.*

Proof. Assume that $(\mathcal{A}, m)$ is free with $\exp(\mathcal{A}, m) = (d_1, d_2, d_3)$. Note that $LMP := LMP(\mathcal{A}, m) = (a+e)(b+d) + [a,c,e] + [b,c,d]$. Put $GMP := GMP(\mathcal{A}, m)$.

Case 1. When $|m| = 3k$ $(k \in \mathbb{Z})$. By the assumptions, $a < k$, $b < k$, $a+e > k$, $b+d > k$. Hence, if we define a new multiplicity $\overline{m}$ by

$$Q(\mathcal{A}, \overline{m}) = (x-y)^c (y-z)^k y^k x^{b+d-k} (x-z)^{a+e-k},$$

then $LMP \geq LMP(\mathcal{A}, \overline{m}) = 3k^2 + (a+e-k)(b+d-k)$. So $GMP \leq GMP(k,k,k) = 3k^2 < LMP$, which is a contradiction.

Case 2. When $|m| = 3k+1$ ($k \in \mathbb{Z}$). By the assumptions, $a < k+1$, $b < k+1$, $a+e > k+1$, $b+d > k+1$. Hence, if we define a new multiplicity $\overline{m}$ by

$$Q(\mathcal{A}, \overline{m}) = (x-y)^c (y-z)^{k+1} y^k x^{b+d-k} (x-z)^{a+e-k-1},$$

then $LMP \geq LMP(\mathcal{A}, \overline{m}) = 3k^2 + 2k + (a+e-k-1)(b+d-k)$. So $GMP \leq GMP(k+1,k,k) = 3k^2 + 2k < LMP$, which is a contradiction.

Case 3. When $|m| = 3k+2$ ($k \in \mathbb{Z}$). By the assumptions, $a < k+1$, $b < k+1$, $a+e > k+1$, $b+d > k+1$. Hence, if we define a new multiplicity $\overline{m}$ by

$$Q(\mathcal{A}, \overline{m}) = (x-y)^c (y-z)^{k+1} y^{k+1} x^{b+d-k-1} (x-z)^{a+e-k-1},$$

then $LMP \geq LMP(\mathcal{A}, \overline{m}) = 3k^2 + 4k + 1 + (a+e-k-1)(b+d-k-1)$. So $GMP \leq GMP(k+1,k,k) = 3k^2 + 4k + 1 < LMP$, which is a contradiction. $\square$

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