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# AR AND MA REPRESENTATION OF PARTIAL AUTOCORRELATION FUNCTIONS, WITH APPLICATIONS

AKIHIKO INOUE

ABSTRACT. We prove a representation of the partial autocorrelation function (PACF), or the Verblunsky coefficients, of a stationary process in terms of the AR and MA coefficients. We apply it to show the asymptotic behaviour of the PACF. We also propose a new definition of short and long memory in terms of the PACF.

## 1. INTRODUCTION

Let  $\{X_n : n \in \mathbf{Z}\}$  be a real, zero-mean, weakly stationary process, defined on a probability space  $(\Omega, \mathcal{F}, P)$  with spectral measure not of finite support, which we shall simply call a *stationary process*. We write  $\{\gamma_n : n \in \mathbf{Z}\}$  for the autocovariance function of  $\{X_n\}$ :  $\gamma_n := E[X_n X_0]$  for  $n \in \mathbf{Z}$ . For  $\{X_n\}$ , we have another sequence  $\{\alpha_n\}_{n=0}^\infty$  called the *partial autocorrelation function* (PACF), where  $\alpha_0 := \gamma_0$ ,  $\alpha_1 := \gamma_1/\gamma_0$ , and for  $n \geq 2$ ,  $\alpha_n$  is the correlation coefficient of the two residuals obtained from  $X_0$  and  $X_n$  by regressing on the intermediate values  $X_1, \dots, X_{n-1}$  (see §2 below).

The autocovariance function  $\{\gamma_n\}$  is positive definite, and the inequalities that this positive definiteness imposes may be inconvenient in some contexts. By contrast, the PACF  $\{\alpha_n\}_{n=0}^\infty$  gives an *unrestricted parametrization*, in that the only inequalities restricting the  $\alpha_n$  are the obvious ones implied by their being correlation coefficients, i.e.,  $\alpha_n \in [-1, 1]$ , or  $(-1, 1)$  in the non-degenerate case relevant here. This result is due to Barndorff-Nielsen and Schou [BS], Ramsey [Ra] in the time-series context. See also Dégerine [De], and for extensions to the non-stationary case, Dégerine and Lambert-Lacroix [DL]. However, in the context of mathematical analysis — specifically, the theory of orthogonal polynomials on the unit circle (OPUC) — the result dates back to 1935-6 to work of Verblunsky [V1, V2], where the PACF appears as the sequence of *Verblunsky coefficients*. For a survey of OPUC, see Simon [Si2], and for a textbook treatment, [Si3] (analytic theory), [Si4] (spectral theory). One of our main purposes here is to emphasize the importance of the PACF: “This knows everything”, a remark we owe to Yukio Kasahara.

The question thus arises of a ‘dictionary’, allowing one to pass between statements on the covariance  $\{\gamma_n : n \in \mathbf{Z}\}$ , or the spectral measure  $\mu$  defined by  $\gamma(n) = \int_{-\pi}^{\pi} e^{ik\theta} \mu(d\theta)$ , and the PACF  $\{\alpha_n\}_{n=0}^\infty$ ; see the last paragraph of [Si3], page 3, where this is stated as perhaps “the most central” question in OPUC. A prototype of such a result is *Baxter’s theorem* (see [Ba1, Ba2]; see also Theorem

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4.1 below and [Si3, Chapter 5]). Such a link exists, in the shape of the Levinson (or Szegő–Levinson–Durbin) algorithm, due to Szegő [Sz] in 1939 (Szegő recursion), Levinson [L] in 1947, Durbin [Du] in 1960; for a textbook account, see Pourahmadi [P], §7.2. For the history of Szegő recursion (Theorem 1.5.2 of [Si3]), see [Si3], page 69. But while very useful numerically, the Levinson algorithm is less suitable for theoretical studies, such as one of the questions that motivates us here — the behaviour of the PACF for large lags (that is, large  $n$ ).  $\square$

In this paper, to fill the gap above at least partially, we introduce a representation of the PACF which is given only in terms of another sequence  $\{\beta_n\}_{n=0}^\infty$  defined by

$$(*) \quad \beta_n, \text{ or } \beta(n), := \sum_{v=0}^{\infty} c_v a_{v+n} \quad (n = 0, 1, \dots)$$

(see Inoue and Kasahara [IK1, page 8], [IK2, (2.23)]), where the two sequences  $\{c_n\}_{n=0}^\infty$  and  $\{a_n\}_{n=0}^\infty$  have statistical interpretations as the coefficients of the MA( $\infty$ ) and AR( $\infty$ ) representations of the process, respectively (see (MA) and (AR) below). They also have analytic interpretations as the coefficients in the Maclaurin expansions of the *Szegő function*  $D(z)$  occurring in the theory of OPUC and its associate  $-1/D(z)$ , both of which are *outer functions* in Beurling’s sense (see §2 below). For background, see [Si3], or — from a statistical point of view — Grenander and Szegő [GS], Rozanov [Ro], Ibragimov and Rozanov [IR].

We are particularly interested in the asymptotics of  $\alpha_n$  as  $n \rightarrow \infty$ , and the representation of  $\alpha_n$  is useful in investigating them since the representation enables us to study  $\alpha_n$  directly via  $\beta_n$  or  $a_n$  and  $c_n$ . In a number of the specific examples of processes with *long memory* we treat, here and in [I2], [I3], [IK1], we observe behaviour of the form

$$(d/n) \quad \alpha_n \sim d/n \quad (n \rightarrow \infty)$$

(by the representation of  $\alpha_n$ , we are able to improve the estimate of this type in [I2, I3, IK1], where only  $|\alpha_n|$  was considered — we were unable to determine its sign). In  $(d/n)$ , and throughout the paper,  $a_n \sim b_n$  as  $n \rightarrow \infty$  means  $\lim_{n \rightarrow \infty} a_n/b_n = 1$ . On the one hand,  $(d/n)$  seems very special behaviour, if we begin by specifying our model via the PACF, since by Verblunsky’s theorem any value in  $(-1, 1)$  can be taken by any  $\alpha_n$ . On the other hand, it is more usual in practice to specify our model with long memory by other means, such as the AR and MA coefficients, and here such behaviour seems to be typical of the (quite broad) classes of example where one can carry out the computations and obtain an explicit asymptotic expression for  $\alpha_n$ , such as the fractional ARIMA (FARIMA) models studied in [I3, IK1] and Theorem 2.5 below. In this connection, we note that in recent work of Simon [Si1], the case  $(d/n)$  is described as ‘a prototypical example’. See also [Si3], [Si4] where [Si1] is developed further, and the papers by Golinskii and Ibragimov [GI], Damanik and Killip [DK] that inspired it. We note also that a generalization of  $(d/n)$ , in which ‘asymptotic to’ is replaced by ‘of the same order of magnitude as’, is obtained as the conclusion in work of Ibragimov and Solev [IS], under conditions on the spectral density.

We denote by  $H$  the real Hilbert space spanned by  $\{X_k : k \in \mathbf{Z}\}$  in  $L^2(\Omega, \mathcal{F}, P)$ . The norm of  $H$  is given by  $\|Y\| := E[Y^2]^{1/2}$ . For  $n, m \in \mathbf{N}$  with  $n \leq m$ , we denote by  $H_{[n,m]}$ ,  $H_{(-\infty,m]}$  and  $H_{[m,\infty)}$  the subspaces of  $H$  spanned by  $\{X_n, \dots, X_m\}$ ,  $\{X_k : k \leq m\}$ , and  $\{X_k : k \geq n\}$ , respectively. The proof of the representation of

the PACF is based on the approach introduced in [I2] which combines von Neumann's alternating projection theorem (see [P, Theorem 9.20]) and the following *intersection of past and future* property of  $\{X_n\}$ :

$$(IPF) \quad H_{(-\infty, n]} \cap H_{[1, \infty)} = H_{[1, n]} \quad (n = 1, 2, \dots).$$

This approach is also useful in continuous-time models; see Inoue and Nakano [IN] and the references therein. A useful sufficient condition for (IPF) is that  $\{X_n\}$  is purely non-deterministic (PND) (see §2 below) and has spectral density  $\Delta(\cdot)$  such that  $\int_{-\pi}^{\pi} d\theta/\Delta(\theta) < \infty$  (see [I2], Theorem 3.1), which itself is a discrete-time version of the Seghler–Dym theorem [S], [Dy]. This theorem itself originates in work of Levinson and McKean [LM]. Naturally, (IPF) is closely related to the property

$$(CND) \quad H_{(-\infty, 0]} \cap H_{[1, \infty)} = \{0\}$$

called *complete non-determinism*; see Bloomfield et al. [BJH]. In fact, a stationary process is completely non-deterministic if and only both (PND) and (IPF) are satisfied (see [IK2], Theorem 2.3).

In §2, we state the main results, including the representation of the PACF and its asymptotic behaviour of the type  $(d/n)$ . Sections 3–5 are devoted to their proofs. In §6, we give a further application of the representation, that is, the asymptotics for the PACF of processes with regularly varying covariance functions. In §7, we close the paper with results for ARMA processes.

## 2. MAIN RESULTS

As stated in §1, let  $H$  be the real Hilbert space spanned by  $\{X_k : k \in \mathbf{Z}\}$  in  $L^2(\Omega, \mathcal{F}, P)$ , which has inner product  $(Y_1, Y_2) := E[Y_1 Y_2]$  and norm  $\|Y\| := (Y, Y)^{1/2}$ . Also, for an interval  $I \subset \mathbf{Z}$ , we write  $H_I$  for the closed subspace of  $H$  spanned by  $\{X_k : k \in I\}$  and  $H_I^\perp$  for the orthogonal complement of  $H_I$  in  $H$ . Let  $P_I$  and  $P_I^\perp$  be the orthogonal projection operators of  $H$  onto  $H_I$  and  $H_I^\perp$ , respectively. Thus  $P_I^\perp Y = Y - P_I Y$  for  $Y \in H$ . The projection  $P_I Y$  stands for the best linear predictor of  $Y$  based on the observations  $\{X_k : k \in I\}$ , and  $P_I^\perp Y$  for its prediction error.

The partial autocorrelation function (PACF)  $\{\alpha_n\}_{n=0}^\infty$  of  $\{X_n\}$  is defined by

$$(PACF) \quad \alpha_0 := \gamma_0, \quad \alpha_n := U_n/V_n \quad (n = 1, 2, \dots),$$

where  $U_1 := (X_1, X_0)$ ,  $V_1 := \|X_1\|^2$  and

$$\begin{aligned} U_n &:= (P_{[1, n-1]}^\perp X_n, P_{[1, n-1]}^\perp X_0) \quad (n = 2, 3, \dots), \\ V_n &:= \|P_{[1, n-1]}^\perp X_n\|^2 \quad (n = 2, 3, \dots) \end{aligned}$$

(cf. Brockwell and Davis [BD, §3.4 and §5.2]). We have  $V_n > 0$  for  $n \geq 1$  since we have assumed that the spectral measure  $\mu$  of  $\{X_n\}$  has infinite support;  $X_1, X_2, X_3, \dots$  are linearly independent since no nonzero trigonometric polynomial vanishes  $\mu$  a.e. Also, since  $\|X_0\| = \|X_1\|$  and  $\|P_{[1, n-1]}^\perp X_0\| = \|P_{[1, n-1]}^\perp X_n\|$  for  $n \geq 2$ ,  $\alpha_n$  is actually the correlation coefficient between the residuals  $P_{[1, n-1]}^\perp X_0$  and  $P_{[1, n-1]}^\perp X_n$  (resp.  $X_0$  and  $X_n$ ) for  $n \geq 2$  (resp.  $n = 1$ ).

A stationary process  $\{X_n\}$  is said to be *purely nondeterministic* (PND) if

$$(PND) \quad \bigcap_{n=-\infty}^{\infty} H_{(-\infty, n]} = \{0\}$$

or, equivalently, there exists a positive even and integrable function  $\Delta(\cdot)$  on  $(-\pi, \pi)$  such that

$$\gamma_n = \int_{-\pi}^{\pi} e^{in\theta} \Delta(\theta) d\theta \quad (n \in \mathbf{Z}), \quad \int_{-\pi}^{\pi} |\log \Delta(\theta)| d\theta < \infty$$

(see [BD, §5.7], [Ro, Chapter II] and [GS, Chapter 10]). We call  $\Delta(\cdot)$  the *spectral density* of  $\{X_n\}$ . Using  $\Delta(\cdot)$ , we define the *Szegö function*  $D(\cdot)$  by

$$D(z) := \sqrt{2\pi} \exp \left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \Delta(\theta) d\theta \right\} \quad (z \in \mathbf{C}, |z| < 1).$$

The function  $D(z)$  is an outer function in the Hardy space  $H^2$  of class 2 over the unit disk  $|z| < 1$ . Using  $D(\cdot)$ , we define the MA coefficients  $c_n$  by

$$D(z) = \sum_{n=0}^{\infty} c_n z^n \quad (|z| < 1),$$

and the AR coefficients  $a_n$  by

$$-\frac{1}{D(z)} = \sum_{n=0}^{\infty} a_n z^n \quad (|z| < 1)$$

(see [I2, §4] and [IK2, §2.2] for background). Both  $\{c_n\}$  and  $\{a_n\}$  are real sequences, and  $\{c_n\}$  is in  $l^2$ . The coefficients  $c_n$  and  $a_n$  are actually those that appear in the following MA( $\infty$ ) and AR( $\infty$ ) representations, respectively, of  $\{X_n\}$  (under suitable condition such as  $\{a_n\} \in l^1$  for the latter):

$$(MA) \quad X_n = \sum_{j=-\infty}^n c_{n-j} \xi_j \quad (n \in \mathbf{Z}),$$

$$(AR) \quad \sum_{j=-\infty}^n a_{n-j} X_j + \xi_n = 0 \quad (n \in \mathbf{Z}),$$

where  $\{\xi_k\}$  is the innovation process defined by  $\xi_k = \epsilon_k / \|\epsilon_k\|$  with  $\epsilon_k$  the prediction error when we predict  $X_k$  from the whole past  $\{X_m : m \leq k-1\}$ , i.e.,  $\epsilon_k = P_{(-\infty, k-1]}^\perp X_k$  (cf. [IK2, Section 2]). From (MA), we have the following equality:

$$(2.1) \quad \gamma_n = \sum_{v=0}^{\infty} c_v c_{|n|+v} \quad (n \in \mathbf{Z}).$$

We wish to derive a representation of  $\alpha_n$  which is given only in terms of  $\beta(\cdot)$  defined by (\*). For this purpose, we consider the following two conditions (BC) and  $(O(1/n))$ :

(BC) The process  $\{X_n\}$  has summable autocovariance function  $\{\gamma_n\}$ , i.e.,  $\sum_{n=-\infty}^{\infty} |\gamma_n| < \infty$ , and *positive* spectral density  $\Delta(\cdot)$ , i.e.,  $\min_{\theta \in [-\pi, \pi]} \Delta(\theta) > 0$ .

$(O(1/n))$   $\{X_n\}$  is PND, and satisfies both  $\sum_0^\infty |a_n| < \infty$  and

$$(2.2) \quad \sum_{v=0}^{\infty} |c_v a_{n+v}| = O(1/n) \quad (n \rightarrow \infty).$$

Notice that if  $\{\gamma_n\} \in l^1$ , then  $\{X_n\}$  has continuous spectral density  $\Delta(\theta) = (2\pi)^{-1} \sum_{-\infty}^{\infty} \gamma_k e^{-ik\theta}$ . We will see (Theorem 4.1) that (BC) holds if and only if

$\{X_n\}$  is PND,  $\{a_n\} \in l^1$  and  $\{c_n\} \in l^1$ . We also see that  $(O(1/n))$  holds for many interesting processes including the FARIMA( $p, d, q$ ) processes with  $0 < d < 1/2$  which we consider below. The condition  $(L(d, \ell))$  below implies  $(O(1/n))$  (see Proposition 5.1 below). ARMA processes satisfy both (BC) and  $(O(1/n))$ .

*Remarks.* 1. Condition (BC) requires that the process have *summable autocovariance* and *positive spectral density*. It could thus be denoted (SP). We call it (BC) instead to emphasize its role as *Baxter's condition*. *Baxter's theorem* ([Ba1, Ba2]; see also Theorem 4.1 below) gives the equivalence of (BC) with summability of the PACF (i.e.,  $\{\alpha_n\} \in l^1$ ) subject only to the very weak condition that the spectral measure  $\mu$  has infinite support. See Chapter 5 of [Si3], where Baxter's theorem is discussed in detail and proved.

2. Several different definitions of *short memory* (or its negation, *long memory*) are in current use. See for example Section 2 of the survey paper Baillie [Bai] for details and references. The most standard definition is that  $\{X_n\}$  has long memory (resp. short memory) if  $\sum_{k=-\infty}^{\infty} |\gamma_k| = \infty$  (resp.  $< \infty$ ); see Beran [Be, page 6], [BD, §13.2]. The fact that Baxter's theorem is so powerful and useful suggests the possibility of using Baxter's condition to define short memory in a new sense: call the process *short memory* if Baxter's condition holds, *long memory* otherwise.

3. The difference between these approaches to short and long memory is well illustrated by the fractional ARIMA (or FARIMA) processes (see below for definitions), studied in e.g. [I3, IK1] and Theorems 2.4 and 2.5 below. The two main cases  $d \in (-1/2, 0)$  and  $d \in (0, 1/2)$  behave in the *same* way from the point of view of asymptotics of PACF ( $d/n$  for each; see Theorem 2.5 below) and prediction error ( $d^2/n$  for each; see (2.25) below) — but in *different* ways from the point of view of summability of the autocovariance function. Our contention is that the PACF  $\{\alpha_n\}$ , and/or the prediction error  $\{\delta(n)\}$ , are more informative about the essence of long-range dependence — the rate at which the information in the remote past decays with time — than the autocovariance function  $\{\gamma_n\}$  which is usually used here.

4. Wu [W], §3 has a definition of 'long-range dependence' which allows  $\{\gamma_n\} \in l^1$ . He uses the Zygmund class of slowly varying functions; see Bingham et al. [BGT, §1.5.3], Zygmund [Z, V.2].

Under (BC) or  $(O(1/n))$ , we define, for  $n = 0, 1, \dots$ ,

$$(2.3) \quad d_1(n) = \beta(n),$$

$$(2.4) \quad d_2(n) = \sum_{m_1=0}^{\infty} \beta(m_1 + n)\beta(m_1 + n),$$

and, for  $k = 3, 4, \dots$ ,

$$(2.5) \quad \begin{aligned} d_k(n) = & \sum_{m_{k-1}=0}^{\infty} \beta(m_{k-1} + n) \sum_{m_{k-2}=0}^{\infty} \beta(m_{k-1} + m_{k-2} + n) \\ & \cdots \sum_{m_2=0}^{\infty} \beta(m_3 + m_2 + n) \sum_{m_1=0}^{\infty} \beta(m_2 + m_1 + n)\beta(m_1 + n), \end{aligned}$$

the sums converging absolutely (see Proposition 4.3 below).

Here is the representation of the PACF  $\{\alpha_n\}$ .

**Theorem 2.1.** *We assume either (BC) or  $(O(1/n))$ . Then, for  $n = 1, 2, \dots$ ,*

$$(2.6) \quad U_n = (c_0)^2 \sum_{k=1}^{\infty} d_{2k-1}(n),$$

$$(2.7) \quad V_n = (c_0)^2 \left\{ 1 + \sum_{k=1}^{\infty} d_{2k}(n) \right\},$$

$$(2.8) \quad \alpha_n = \frac{\sum_{k=1}^{\infty} d_{2k-1}(n)}{1 + \sum_{k=1}^{\infty} d_{2k}(n)},$$

*all the sums converging absolutely.*

If  $\{X_n\}$  is PND, then  $V_n \downarrow \|P_{(-\infty, -1]}^{\perp} X_0\|^2 = (c_0)^2$ , whence  $\alpha_n = U_n/V_n \sim (c_0)^{-2} U_n$ , as  $n \rightarrow \infty$  (see [I2, §2]). Thus an immediate consequence of Theorem 2.1 is the next corollary.

**Corollary 2.2.** *We assume either (BC) or  $(O(1/n))$ . Then,*

$$(2.9) \quad \alpha_n \sim \sum_{k=1}^{\infty} d_{2k-1}(n) \quad (n \rightarrow \infty).$$

We turn to the results on the asymptotic behaviour of  $\alpha_n$  as  $n \rightarrow \infty$ . We write  $\mathcal{R}_0$  for the class of slowly varying functions at infinity: the class of positive, measurable  $\ell$ , defined on some neighborhood  $[A, \infty)$  of infinity, such that

$$(2.10) \quad \lim_{x \rightarrow \infty} \ell(\lambda x)/\ell(x) = 1 \quad \text{for all } \lambda > 0$$

(see [BGT, Chapter 1]). For  $\ell \in \mathcal{R}_0$  and  $d \in (0, 1/2)$ , we consider the following condition as a standard one for processes with *long memory* (see [IK2, (A2)]):

(L( $d, \ell$ ))  $\{X_n\}$  is PND and  $\{c_n\}$  and  $\{a_n\}$  satisfy, respectively,

$$(2.11) \quad c_n \sim n^{-(1-d)} \ell(n) \quad (n \rightarrow \infty),$$

$$(2.12) \quad a_n \sim n^{-(1+d)} \frac{1}{\ell(n)} \frac{d \sin(\pi d)}{\pi} \quad (n \rightarrow \infty).$$

The condition (L( $d, \ell$ )) implies

$$(2.13) \quad \gamma_n \sim n^{-(1-2d)} \ell(n)^2 B(d, 1-2d) \quad (n \rightarrow \infty)$$

(see [IK2, (2.22)]), whence  $\{\gamma_n\} \notin l^1$ , where  $B(\cdot, \cdot)$  denotes the beta integral, i.e.,

$$B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad (p, q > 0).$$

Here is a result of the type  $(d/n)$  for processes with long memory.

**Theorem 2.3.** *For  $\ell \in \mathcal{R}_0$  and  $d \in (0, 1/2)$ , (L( $d, \ell$ )) implies the asymptotic behaviour  $(d/n)$  of the PACF.*

For example, the FARIMA( $p, d, q$ ) model with  $0 < d < 1/2$ , which is regarded as a standard parametric model with long memory and which we consider below, satisfies (L( $d, \ell$ )) for some constant function  $\ell$  (see Kokoszka and Taquu [KT, Corollary 3.1]), whence, by this theorem, its PACF has the asymptotic behaviour  $(d/n)$ . We see another class satisfying (L( $d, \ell$ )) in Section 6. We can write the property  $(d/n)$  as  $d = \lim_{n \rightarrow \infty} n\alpha_n$ , suggesting how to estimate the parameter  $d$ , which is important in a process with long memory. See [IK1, §5] for numerical calculation.

We consider the FARIMA. For  $d \in (-1/2, 1/2)$  and  $p, q \in \mathbf{N} \cup \{0\}$ , a stationary process  $\{X_n\}$  is said to be a (causal and invertible) fractional ARIMA( $p, d, q$ ) (or FARIMA( $p, d, q$ )) process if it has a spectral density  $\Delta(\cdot)$  of the form

$$(2.14) \quad \Delta(\theta) = \frac{1}{2\pi} \frac{|\Theta(e^{i\theta})|^2}{|\Phi(e^{i\theta})|^2} |1 - e^{i\theta}|^{-2d} \quad (-\pi < \theta < \pi),$$

where  $\Phi(z)$  and  $\Theta(z)$  are polynomials with real coefficients of degrees  $p, q$ , respectively, satisfying the following condition:

$$(2.15) \quad \begin{array}{l} \Phi(z) \text{ and } \Theta(z) \text{ have no common zeros, and have no zeros} \\ \text{in the closed unit disk } \{z \in \mathbf{C} : |z| \leq 1\}. \end{array}$$

The fractional ARIMA model was introduced independently by Granger and Joyeux [GJ] and Hosking [Ho]. See [Be, §2.5] and [BD, §13.2] for textbook treatments and [KT] for a formulation in terms of the backward shift operator  $B$ . If  $d \in (-1/2, 1/2) \setminus \{0\}$ , then the MA coefficient  $c_n$ , AR coefficients  $a_n$  and autocovariance function  $\{\gamma_n\}$  of the FARIMA( $p, d, q$ ) process  $\{X_n\}$  satisfy

$$(2.16) \quad c_n \sim n^{-(1-d)} \frac{K_1}{\Gamma(d)} \quad (n \rightarrow \infty),$$

$$(2.17) \quad a_n \sim n^{-(1+d)} \frac{\Gamma(d)}{K_1} \cdot \frac{d \sin(\pi d)}{\pi} \quad (n \rightarrow \infty),$$

$$(2.18) \quad \gamma_n \sim n^{-(1-2d)} \frac{(K_1)^2 \Gamma(1-2d) \sin(\pi d)}{\pi} \quad (n \rightarrow \infty),$$

where

$$(2.19) \quad K_1 := \Theta(1)/\Phi(1)$$

(see [KT, Corollary 3.1] and [I3, §3]). In particular, if  $0 < d < 1/2$ , then  $\{X_n\}$  has long memory. On the other hand, if  $d = 0$ , then  $\{X_n\}$  reduces to the ordinary ARMA( $p, q$ ) process, and each of  $c_n, a_n$  and  $\gamma_n$  decays exponentially fast as  $n \rightarrow \infty$  (see [BD, Chapter 3] and §7 below).

Theorems 2.1 and 2.3 cover FARIMA with  $d \in (0, 1/2)$  but not the case  $d \in (-1/2, 0)$ . However, the FARIMA( $0, d, 0$ ) with  $d \in (-1/2, 1/2)$  has the PACF given by

$$(2.20) \quad \alpha_n = \frac{d}{n-d} \quad (n = 1, 2, \dots)$$

(see [Ho, Theorem 1(f)] as well as [BD, (13.2.10)]), which suggests that  $(d/n)$  may also hold even if  $d \in (-1/2, 0)$ , and this is actually the case as we show below.

For FARIMA with  $-1/2 < d < 0$ , we define

$$\phi_n := \begin{cases} -a_0 & (n = 0), \\ a_{n-1} - a_n & (n = 1, 2, \dots). \end{cases}$$

and

$$\psi_n := - \sum_{k=n+1}^{\infty} c_k \quad (n = 0, 1, \dots).$$

Notice that  $\phi_n$  here corresponds to  $-\phi_n$  in [IK1]. We define  $q \in (1/2, 1)$  by

$$q := 1 + d.$$



Then, by [IK1, Lemma 4.1] and [KT, Corollary 3.1] (see also [I3, Lemma 2.1]), we have

$$\begin{aligned}\psi_n &\sim n^{-(1-q)} \frac{K_1}{\Gamma(q)} \quad (n \rightarrow \infty), \\ \phi_n &\sim -n^{-(1+q)} \frac{\Gamma(q)}{K_1} \cdot \frac{q \sin(\pi q)}{\pi} \quad (n \rightarrow \infty),\end{aligned}$$

where  $K_1$  is as in (2.19). We define

$$\beta_-(n) := \sum_{v=0}^{\infty} \psi_v \phi_{v+n+1} \quad (n = 0, 1, \dots).$$

Notice that  $\beta_-(n)$  corresponds to  $-\beta(n)$  in [IK1]. We define, for  $n = 0, 1, \dots$ ,

$$(2.21) \quad d_1(n) = \beta_-(n),$$

$$(2.22) \quad d_2(n) = \sum_{m_1=0}^{\infty} \beta_-(m_1+n) \beta_-(m_1+n),$$

and, for  $k = 3, 4, \dots$ ,

$$(2.23) \quad \begin{aligned}d_k(n) &= \sum_{m_{k-1}=0}^{\infty} \beta_-(m_{k-1}+n) \sum_{m_{k-2}=0}^{\infty} \beta_-(m_{k-1}+m_{k-2}+n) \\ &\quad \cdots \sum_{m_2=0}^{\infty} \beta_-(m_3+m_2+n) \sum_{m_1=0}^{\infty} \beta_-(m_2+m_1+n) \beta_-(m_1+n),\end{aligned}$$

the sums converging absolutely (see [IK2, Theorem 3.3]).

The representation of the PACF of FARIMA with  $d \in (-1/2, 0)$  is given by the next theorem.

**Theorem 2.4.** *Let  $p, q \in \mathbf{N} \cup \{0\}$  and  $d \in (-1/2, 0)$ , and let  $\{X_n\}$  be a fractional ARIMA  $(p, d, q)$  process. Then, for  $n = 1, 2, \dots$ , the representations (2.6)–(2.8) of  $U_n, V_n$  and  $\alpha_n$  hold with all the sums converging absolutely.*

Here is the result of the type  $(d/n)$  for FARIMA.

**Theorem 2.5.** *Let  $p, q \in \mathbf{N} \cup \{0\}$  and  $d \in (-1/2, 1/2) \setminus \{0\}$ , and let  $\{X_n\}$  be a fractional ARIMA  $(p, d, q)$  process. Then the PACF has the asymptotics  $(d/n)$ .*

This last theorem as well as Theorem 2.3 is an improvement of earlier work [I2, I3, IK1] for long-memory or FARIMA processes, asserting that

$$(2.24) \quad |\alpha_n| \sim |d|/n \quad (n \rightarrow \infty).$$

Notice that while the earlier result cannot distinguish between the  $d$  and  $-d$  cases (that is, between positive and negative differencing), Theorems 2.3 and 2.5 can. In [I2, I3, IK1], the asymptotic behaviour of mean-squared prediction error of the type

$$(2.25) \quad \delta(n) \sim d^2/n \quad (n \rightarrow \infty)$$

was first derived and then used in Tauberian arguments to prove (2.24), where

$$\delta(n) := \frac{\|P_{[-n,0]}^\perp X_1\|^2 - \|P_{(-\infty,0]}^\perp X_1\|^2}{\|P_{(-\infty,0]}^\perp X_1\|^2} \quad (n = 1, 2, \dots).$$

The proofs of Theorems 2.3 and 2.5, which are based on the representation of the PACF, are more direct and much simpler than those of the earlier result.

### 3. INNER PRODUCTS OF PREDICTION ERRORS

In this section, we derive some expansions of  $V_n$  and  $U_n$  that we need to prove the representation of the PACF. The key is to extend [I2, Theorem 4.1] properly.

For  $n, k \in \mathbf{N}$ , we define the orthogonal projection operator  $P_n^k$  by

$$(3.1) \quad P_n^k := \begin{cases} P_{(-\infty, n-1]} & (k = 1, 3, 5, \dots), \\ P_{[1, \infty)} & (k = 2, 4, 6, \dots). \end{cases}$$

It should be noticed that  $\{P_n^k : k = 1, 2, \dots\}$  is merely an alternating sequence of projection operators, first to the subspace  $H_{(-\infty, n-1]}$ , then to  $H_{[1, \infty)}$ , and so on.

**Theorem 3.1.** *Let  $Y_1, Y_2 \in H$ . We assume that  $\{X_n\}$  is CND.*

(1) *We have*

$$(3.2) \quad \begin{aligned} (Y_1, Y_2) &= ((P_1^1)^\perp Y_1, (P_1^1)^\perp Y_2) \\ &+ \sum_{k=1}^{\infty} ((P_1^{k+1})^\perp P_1^k \cdots P_1^1 Y_1, (P_1^{k+1})^\perp P_1^k \cdots P_1^1 Y_2). \end{aligned}$$

(2) *We have, for  $n = 2, 3, \dots$ ,*

$$(3.3) \quad \begin{aligned} (P_{[1, n-1]}^\perp Y_1, P_{[1, n-1]}^\perp Y_2) &= ((P_n^1)^\perp Y_1, (P_n^1)^\perp Y_2) \\ &+ \sum_{k=1}^{\infty} ((P_n^{k+1})^\perp P_n^k \cdots P_n^1 Y_1, (P_n^{k+1})^\perp P_n^k \cdots P_n^1 Y_2). \end{aligned}$$

If we put  $Y_1 = Y_2$  in Theorem 3.1(2), then it reduces to [I2, Theorem 4.1].

*Proof.* (1) The orthogonal decompositions

$$\begin{aligned} H &= H_{(-\infty, 0]}^\perp \oplus H_{(-\infty, 0]}, \\ H &= H_{[1, \infty)}^\perp \oplus H_{[1, \infty)} \end{aligned}$$

of  $H$  imply the orthogonal decompositions

$$(3.4) \quad I_H = P_{(-\infty, 0]}^\perp \oplus P_{(-\infty, 0]},$$

$$(3.5) \quad I_H = P_{[1, \infty)}^\perp \oplus P_{[1, \infty)}$$

of the identity map  $I_H$ , respectively. Repeated use of (3.4) and (3.5) yields, for  $m = 2, 3, \dots$ ,

$$\begin{aligned} (Y_1, Y_2) &= ((P_1^1)^\perp Y_1, (P_1^1)^\perp Y_2) \\ &+ \sum_{k=1}^{m-1} ((P_1^{k+1})^\perp P_1^k \cdots P_1^1 Y_1, (P_1^{k+1})^\perp P_1^k \cdots P_1^1 Y_2) + R_1^m, \end{aligned}$$

where  $R_1^m := (P_1^m \cdots P_1^1 Y_1, P_1^m \cdots P_1^1 Y_2)$ . Since (CND) and von Neumann's alternating projection theorem (see [P, Theorem 9.20]) imply

$$\text{s-lim}_{m \rightarrow \infty} P_1^m \cdots P_1^1 = 0,$$

we have  $\lim_{m \rightarrow \infty} R_1^m = 0$ , whence (3.2).

(2) For  $n = 2, 3, \dots$ , we have the orthogonal decompositions

$$H_{[1, n-1]}^\perp = H_{(-\infty, n-1]}^\perp \oplus \left( H_{[1, n-1]}^\perp \cap H_{(-\infty, n-1]} \right),$$

$$H_{[1, n-1]}^\perp = H_{[1, \infty)}^\perp \oplus \left( H_{[1, n-1]}^\perp \cap H_{[1, \infty)} \right)$$

of  $H_{[1, n-1]}^\perp$ , which in turn imply the orthogonal decompositions

$$(3.6) \quad P_{[1, n-1]}^\perp = P_{(-\infty, n-1]}^\perp \oplus P_{[1, n-1]}^\perp P_{(-\infty, n-1]},$$

$$(3.7) \quad P_{[1, n-1]}^\perp = P_{[1, \infty)}^\perp \oplus P_{[1, n-1]}^\perp P_{[1, \infty)}$$

of  $P_{[1, n-1]}^\perp$ , respectively. Using (3.6) and (3.7) repeatedly, we find that, for  $m = 2, 3, \dots$ ,

$$\begin{aligned} \left( P_{[1, n-1]}^\perp Y_1, P_{[1, n-1]}^\perp Y_2 \right) &= \left( (P_n^1)^\perp Y_1, (P_n^1)^\perp Y_2 \right) \\ &+ \sum_{k=1}^{m-1} \left( (P_n^{k+1})^\perp P_n^k \cdots P_n^1 Y_1, (P_n^{k+1})^\perp P_n^k \cdots P_n^1 Y_2 \right) + R_n^m, \end{aligned}$$

where  $R_n^m := (P_{[1, n-1]}^\perp P_n^m \cdots P_n^1 Y_1, P_{[1, n-1]}^\perp P_n^m \cdots P_n^1 Y_2)$ . From (IPF) implied by (CND) (see [IK2, Theorem 2.3]) and the alternating projection theorem, we get

$$\text{s-lim}_{m \rightarrow \infty} P_n^m \cdots P_n^1 = P_{[1, n-1]},$$

whence

$$\lim_{m \rightarrow \infty} \| P_{[1, n-1]}^\perp P_n^m \cdots P_n^1 Y_i \| = \| P_{[1, n-1]}^\perp P_{[1, n-1]} Y_i \| = 0 \quad (i = 1, 2).$$

Thus  $\lim_{m \rightarrow \infty} R_n^m = 0$ , so that (3.3) follows.  $\square$

*Remark.* If  $\{X_n\}$  is CND, then by the same arguments as above we see that

$$P_{[1, n-1]}^\perp = (P_n^1)^\perp + (P_n^2)^\perp P_n^1 + (P_n^3)^\perp P_n^2 P_n^1 + \cdots.$$

Assuming (PND), we define

$$b_j^m := \sum_{k=0}^m c_k a_{j+m-k} \quad (m, j \in \mathbf{N} \cup \{0\}).$$

Notice that  $b_j^m$  here is equal to that in [IK1] but it corresponds to  $b_{j-1}^{m+1}$  in [I2, I3].

Recall  $U_n$  and  $V_n$  from §2. Here are their representations in terms of the AR and MA coefficients.

**Theorem 3.2.** *We assume (PND) and  $\sum_{n=0}^\infty |a_n| < \infty$ . Then, for  $n = 1, 2, \dots$ ,*

$$(3.8) \quad U_n = (c_0)^2 \sum_{k=1}^\infty \sum_{p=0}^\infty d_k(n, p) d_{k-1}(n, p),$$

$$(3.9) \quad V_n = (c_0)^2 \sum_{k=0}^\infty \sum_{p=0}^\infty d_k(n, p)^2,$$

where, for  $n \in \mathbf{N}$  and  $p \in \mathbf{N} \cup \{0\}$ ,  $d_0(n, p) := \delta_{p0}$ ,

$$(3.10) \quad d_1(n, p) := \sum_{m_1=0}^\infty a_{n+m_1+p} c_{m_1},$$

and, for  $k = 2, 3, \dots$ ,

$$(3.11) \quad d_k(n, p) := \sum_{m_{k-1}=0}^{\infty} a_{n+m_{k-1}} \sum_{m_{k-2}=0}^{\infty} b_{n+m_{k-2}}^{m_{k-1}} \cdots \sum_{m_1=0}^{\infty} b_{n+m_1}^{m_2} \sum_{v=0}^{\infty} b_{n+p+v}^{m_1} c_v,$$

all the sums converging absolutely.

*Proof.* (Compare the proof of [I2, Theorem 4.5].) Notice that  $\{X_n\}$  is CND (see [I2], Proposition 4.2 and Theorem 3.1). Hence, it follows from Theorem 3.1 that, for  $n = 1, 2, \dots$ ,

$$(3.12) \quad U_n = ((P_n^2)^\perp P_n^1 X_n, (P_n^2)^\perp X_0) + \sum_{k=2}^{\infty} ((P_n^{k+1})^\perp P_n^k \cdots P_n^1 X_n, (P_n^{k+1})^\perp P_n^k \cdots P_n^2 X_0),$$

$$(3.13) \quad V_n = \|(P_n^1)^\perp X_n\|^2 + \sum_{k=1}^{\infty} \|(P_n^{k+1})^\perp P_n^k \cdots P_n^1 X_n\|^2.$$

Let  $n \in \mathbf{N}$ . Suppose that  $k$  is even and  $\geq 2$ . By [I2, Theorem 4.4], we have, for  $n = 1, 2, \dots$  and  $m = 0, 1, \dots$ ,

$$P_{(-\infty, n-1]} X_{m+n} = \sum_{j=0}^{\infty} b_{n+j}^m X_{-j} \quad (\text{mod } H_{[1, n-1]} \text{ if } n \geq 2),$$

$$P_{[1, \infty)} X_{-m} = \sum_{j=0}^{\infty} b_{n+j}^m X_{j+n} \quad (\text{mod } H_{[1, n-1]} \text{ if } n \geq 2),$$

whence

$$\begin{aligned} P_n^k \cdots P_n^1 X_n &= c_0 \sum_{m_{k-1}=0}^{\infty} a_{n+m_{k-1}} \sum_{m_{k-2}=0}^{\infty} b_{n+m_{k-2}}^{m_{k-1}} \\ &\cdots \sum_{m_1=0}^{\infty} b_{n+m_1}^{m_2} \sum_{m_0=0}^{\infty} b_{n+m_0}^{m_1} X_{m_0+n} \quad (\text{mod } H_{[1, n-1]} \text{ if } n \geq 2). \end{aligned}$$

Since we restrict to (PND),  $\{X_n\}$  has no deterministic component in the Wold decomposition and it permits the moving-average representation (MA), where the orthonormal system  $\{\xi_j : j \in \mathbf{Z}\}$  of  $H$  satisfies

$$H_{(-\infty, m]} = H_{(-\infty, m]}(\xi) \quad (m \in \mathbf{Z})$$

with  $H_{(-\infty, m]}(\xi)$  being the closed subspace of  $H$  spanned by  $\{\xi_j : -\infty < j \leq m\}$  (see [Ro, Chapter II], [BD, §5.7]). Since

$$P_{(-\infty, n-1]}^\perp X_{m+n} = \sum_{j=0}^m c_{m-j} \xi_{j+n} \quad (m = 0, 1, \dots),$$

we have

$$\begin{aligned} (P_n^{k+1})^\perp P_n^k \cdots P_n^1 X_n &= c_0 \sum_{m_{k-1}=0}^{\infty} a_{n+m_{k-1}} \sum_{m_{k-2}=0}^{\infty} b_{n+m_{k-2}}^{m_{k-1}} \\ &\cdots \sum_{m_1=0}^{\infty} b_{n+m_1}^{m_2} \sum_{m_0=0}^{\infty} b_{n+m_0}^{m_1} \sum_{j=0}^{m_0} c_{m_0-j} \xi_{j+n}, \end{aligned}$$

so that

$$((P_n^{k+1})^\perp P_n^k \cdots P_n^1 X_n, \xi_{p+n}) = \begin{cases} c_0 d_k(n, p) & (p = 0, 1, \dots), \\ 0 & (p = -1, -2, \dots). \end{cases}$$

Arguing similarly,

$$(\xi_{p+n}, (P_n^{k+1})^\perp P_n^k \cdots P_n^2 X_0) = \begin{cases} c_0 d_{k-1}(n, p) & (p = 0, 1, \dots), \\ 0 & (p = -1, -2, \dots). \end{cases}$$

Thus, from the Parseval equality, we get

$$(3.14) \quad \begin{aligned} & ((P_n^{k+1})^\perp P_n^k \cdots P_n^1 X_n, (P_n^{k+1})^\perp P_n^k \cdots P_n^2 X_0) \\ &= (c_0)^2 \sum_{p=0}^{\infty} d_k(n, p) d_{k-1}(n, p), \end{aligned}$$

$$(3.15) \quad \|(P_n^{k+1})^\perp P_n^k \cdots P_n^1 X_n\|^2 = (c_0)^2 \sum_{p=0}^{\infty} d_k(n, p)^2.$$

Similarly, we have (3.14) and (3.15) for  $k$  odd, and also

$$(3.16) \quad ((P_n^2)^\perp P_n^1 X_n, (P_n^2)^\perp X_0) = (c_0)^2 d_1(n, 0) = (c_0)^2 \sum_{p=0}^{\infty} d_1(n, p) d_0(n, p),$$

$$(3.17) \quad \|(P_n^1)^\perp X_n\|^2 = (c_0)^2 = (c_0)^2 \sum_{p=0}^{\infty} d_0(n, p)^2.$$

The assertions (3.8) and (3.9) now follow if we substitute (3.14) and (3.16) into (3.12), and (3.15) and (3.17) into (3.13).  $\square$

We write  $\sum^{\infty-}$  to indicate that the sum does not necessarily converge absolutely, i.e.,  $\sum_{k=m}^{\infty-} := \lim_{M \rightarrow \infty} \sum_{k=m}^M$ . We need the next variant of Theorem 3.2 when we consider the fractional ARIMA( $p, d, q$ ) processes with  $-1/2 < d < 0$ .

**Theorem 3.3.** *We assume (PND). Then the representations (3.8) and (3.9) still hold if all the summations  $\sum^\infty$  in (3.10) and (3.11) are replaced by  $\sum^{\infty-}$  and if  $\sum_0^\infty |a_n| < \infty$  is replaced by the two conditions  $\sum_0^\infty |c_n| < \infty$  and  $\sum_0^\infty |a_n|^2 < \infty$ .*

*Proof.* By [I2], Proposition 4.2 and Theorem 3.1, the conditions (PND) and  $\{a_n\} \in l^2$  imply (CND). Moreover, by [IK1, Proposition 2.1], we have, for  $n \in \mathbf{N}$  and  $m \in \mathbf{N} \cup \{0\}$ ,

$$P_{(-\infty, n-1]} X_{m+n} = \sum_{j=0}^{\infty-} b_{n+j}^m X_{-j} \quad (\text{mod } H_{[1, n-1]} \text{ if } n \geq 2),$$

$$P_{[1, \infty)} X_{-m} = \sum_{j=0}^{\infty-} b_{n+j}^m X_{j+n} \quad (\text{mod } H_{[1, n-1]} \text{ if } n \geq 2).$$

Using these equalities, we can prove the theorem as in the proof of Theorem 3.2. We omit the details.  $\square$

#### 4. PROOF OF THEOREM 2.1

First we give some necessary and sufficient conditions for (BC).

**Theorem 4.1.** *For a stationary process  $\{X_n\}$ , the following conditions are equivalent:*

- (1)  $\{X_n\}$  is PND and satisfies both  $\sum_0^\infty |a_n| < \infty$  and  $\sum_0^\infty |c_n| < \infty$ ;
- (2)  $\{X_n\}$  has a positive continuous spectral density and satisfies  $\sum_0^\infty |c_n| < \infty$ ;
- (3)  $\{X_n\}$  has a positive continuous spectral density and satisfies  $\sum_0^\infty |a_n| < \infty$ ;
- (4)  $\{X_n\}$  satisfies (BC);
- (5)  $\{X_n\}$  has summable PACF:  $\sum_0^\infty |\alpha_n| < \infty$ .

*Proof.* Notice that if  $\{X_n\}$  has a positive continuous spectral density  $\Delta(\cdot)$  on  $[-\pi, \pi]$ , then it is PND since  $\int_{-\pi}^\pi |\log \Delta(\theta)| d\theta < \infty$  holds.

Suppose (1). We write  $D(e^{i\theta})$  for the nontangential limit of  $D(z)$ , i.e.,

$$D(e^{i\theta}) = \lim_{r \rightarrow 1-0} D(re^{i\theta}) = \sum_{n=0}^\infty c_n e^{in\theta} \quad (-\pi \leq \theta \leq \pi).$$

Since  $\{c_n\} \in l^1$  implies the continuity of  $D(e^{i\theta})$ , the spectral density  $\Delta(\theta)$  is also continuous by the equality  $\Delta(\theta) = 2\pi |D(e^{i\theta})|^2$ . Letting  $r \rightarrow 1-0$  in

$$\left( \sum_{n=0}^\infty c_n r^n e^{in\theta} \right) \left( \sum_{n=0}^\infty a_n r^n e^{in\theta} \right) = -1 \quad (-\pi \leq \theta \leq \pi),$$

we obtain

$$\left( \sum_{n=0}^\infty c_n e^{in\theta} \right) \left( \sum_{n=0}^\infty a_n e^{in\theta} \right) = -1 \quad (-\pi \leq \theta \leq \pi).$$

This implies that  $D(e^{i\theta})$ , whence  $\Delta(\theta)$ , has no zeros on  $[-\pi, \pi]$ . Thus  $\Delta(\cdot)$  is positive, whence (2) and (3) follow.

Suppose (3). In the same way as above, we have

$$\frac{1}{\Delta(\theta)} = \frac{1}{2\pi} \left| \sum_{n=0}^\infty a_n e^{in\theta} \right|^2 \quad (-\pi \leq \theta \leq \pi).$$

This implies  $\sum_{n=0}^\infty a_n e^{in\theta} \neq 0$  for every  $\theta \in [-\pi, \pi]$ . By Wiener's theorem for absolutely convergent Fourier series (cf. Lemma 11.6 in [Ru]), we obtain  $\{c_n\} \in l^1$  (cf. Berg [Berg], page 493). Thus (1) follows. The proof of the implication (2)  $\Rightarrow$  (1) is similar.

By (2.1), (2) implies (4). Conversely, we assume (4). Then we have  $\{a_n\} \in l^1$ , whence (3), by the arguments in Baxter [Ba2], pp. 139–140, which involve the Wiener–Lévy theorem.

The equivalence between (4) and (5) is Baxter's theorem ([Ba1, Ba2]; see also [Si3, Chapter 5]). This completes the proof.  $\square$

We put

$$B(n) := \sum_{v=0}^\infty |c_v a_{n+v}| \quad (n \in \mathbf{N} \cup \{0\}).$$

For  $n, k, u, v \in \mathbf{N} \cup \{0\}$ , we define  $D_k(n, u, v)$  recursively by

$$\begin{cases} D_0(n, u, v) := \delta_{uv}, \\ D_{k+1}(n, u, v) := \sum_{w=0}^\infty B(n+v+w) D_k(n, u, w) \end{cases}$$

(see [IK2, §2.3]). We have, for example,

$$D_3(n, u, v) = \sum_{v_1=0}^{\infty} \sum_{v_2=0}^{\infty} B(n+v+v_1)B(n+v_1+v_2)B(n+v_2+u).$$

**Proposition 4.2.** *We assume either (BC) or  $(O(1/n))$ . Then, for  $k, n, v \in \mathbf{N} \cup \{0\}$ ,*

$$\sum_{u=0}^{\infty} D_k(n, u, v) < \infty \quad \text{and} \quad \sum_{u=0}^{\infty} D_k(n, u, v)^2 < \infty,$$

*respectively. In particular, we have  $D_k(n, u, v) < \infty$  for  $k, n, u, v \in \mathbf{N} \cup \{0\}$ .*

In view of Theorem 4.1, we can prove Proposition 4.2 in the same way as that of [IK2, Lemma 2.7], whence we omit it.

Recall  $\beta(n)$  from (\*) and  $d_k(n, p)$  from Theorem 3.2.

**Proposition 4.3.** *We assume either (BC) or  $(O(1/n))$ . Then we have, for  $n \in \mathbf{N}$  and  $p \in \mathbf{N} \cup \{0\}$ ,*

$$(4.1) \quad d_1(n, p) = \beta(n+p),$$

$$(4.2) \quad d_2(n, p) = \sum_{m_1=0}^{\infty} \beta(m_1+n)\beta(m_1+n+p),$$

*and, for  $k = 3, 4, \dots$ ,*

$$(4.3) \quad \begin{aligned} d_k(n, p) &= \sum_{m_{k-1}=0}^{\infty} \beta(m_{k-1}+n) \sum_{m_{k-2}=0}^{\infty} \beta(m_{k-1}+m_{k-2}+n) \\ &\cdots \sum_{m_2=0}^{\infty} \beta(m_3+m_2+n) \sum_{m_1=0}^{\infty} \beta(m_2+m_1+n)\beta(m_1+n+p), \end{aligned}$$

*the sums converging absolutely.*

*Proof.* By Proposition 4.2, we can use the Fubini–Tonelli theorem to exchange the order of sums in (3.11), and we get (4.2) and (4.3) as in the proof of [I2, Theorem 4.6].  $\square$

**Proposition 4.4.** *We assume either (BC) or  $(O(1/n))$ . Then, for  $i, j \in \mathbf{N}$ ,*

$$(4.4) \quad \sum_{p=0}^{\infty} d_i(n, p)d_j(n, p) = d_{i+j}(n, 0) \quad (n = 1, 2, \dots).$$

*Proof.* For simplicity, we give details for the case  $i = j = 4$  only. The general case can be treated in the same way. From Proposition 4.3, we have, for  $n = 1, 2, \dots$  and  $p = 0, 1, \dots$ ,

$$(4.5) \quad \begin{aligned} d_4(n, p) &= \sum_{m_1=0}^{\infty} \beta(m_1+n) \sum_{m_2=0}^{\infty} \beta(m_1+m_2+n) \\ &\quad \sum_{m_3=0}^{\infty} \beta(m_2+m_3+n)\beta(m_3+p+n). \end{aligned}$$

By Proposition 4.2 and the Fubini theorem,

$$(4.6) \quad \begin{aligned} d_4(n, p) &= \sum_{m_3=0}^{\infty} \beta(m_3+p+n) \sum_{m_2=0}^{\infty} \beta(m_2+m_3+n) \\ &\quad \sum_{m_1=0}^{\infty} \beta(m_1+m_2+n)\beta(m_1+n). \end{aligned}$$

Writing  $(m_7, m_6, m_5)$  for  $(m_1, m_2, m_3)$  in (4.5), we get

$$(4.7) \quad d_4(n, p) = \sum_{m_7=0}^{\infty} \beta(m_7 + n) \sum_{m_6=0}^{\infty} \beta(m_6 + m_7 + n) \sum_{m_5=0}^{\infty} \beta(m_5 + m_6 + n) \beta(p + m_5 + n).$$

From (4.6), (4.7) and the Fubini theorem,

$$\begin{aligned} \sum_{p=0}^{\infty} d_4(n, p) d_4(n, p) &= \sum_{m_4=0}^{\infty} d_4(n, m_4) d_4(n, m_4) \\ &= \sum_{m_4=0}^{\infty} \left\{ \sum_{m_7=0}^{\infty} \beta(m_7 + n) \sum_{m_6=0}^{\infty} \beta(m_6 + m_7 + n) \sum_{m_5=0}^{\infty} \beta(m_5 + m_6 + n) \beta(m_4 + m_5 + n) \right\} \\ &\quad \times \left\{ \sum_{m_3=0}^{\infty} \beta(m_3 + m_4 + n) \sum_{m_2=0}^{\infty} \beta(m_2 + m_3 + n) \sum_{m_1=0}^{\infty} \beta(m_1 + m_2 + n) \beta(m_1 + n) \right\}, \end{aligned}$$

which is equal to

$$\begin{aligned} &\sum_{m_7=0}^{\infty} \beta(m_7 + n) \sum_{m_6=0}^{\infty} \beta(m_6 + m_7 + n) \sum_{m_5=0}^{\infty} \beta(m_5 + m_6 + n) \\ &\quad \sum_{m_4=0}^{\infty} \beta(m_4 + m_5 + n) \sum_{m_3=0}^{\infty} \beta(m_3 + m_4 + n) \sum_{m_2=0}^{\infty} \beta(m_2 + m_3 + n) \\ &\quad \sum_{m_1=0}^{\infty} \beta(m_1 + m_2 + n) \beta(m_1 + n) \\ &= d_8(n, 0). \end{aligned}$$

Thus the desired result for  $i = j = 4$  follows.  $\square$

*Proof of Theorem 2.1.* By Proposition 4.3, we see that

$$d_k(n) = d_k(n, 0) \quad (k, n \in \mathbf{N}).$$

From this, Proposition 4.4 and Theorem 3.2, the theorem follows.  $\square$

## 5. PROOFS OF THEOREMS 2.3–2.5

**Proposition 5.1.** *For  $d \in (0, 1/2)$  and  $\ell \in \mathcal{R}_0$ , we assume  $(L(d, \ell))$ .*

(1) *It holds that*

$$\beta_n \sim \frac{\sin(\pi d)}{\pi} n^{-1} \quad (n \rightarrow \infty).$$

(2) *The condition  $(O(1/n))$  holds. More precisely, we have*

$$\sum_{v=0}^{\infty} |c_v a_{n+v}| \sim \frac{\sin(\pi d)}{\pi} n^{-1} \quad (n \rightarrow \infty).$$

(3) *For  $s \geq 0$  and  $u \geq 0$ , it holds that*

$$\beta([ns] + [nu] + n) \sim \frac{\sin(\pi d)}{\pi(s + u + 1)} n^{-1} \quad (n \rightarrow \infty).$$

(4) *For  $r \in (1, \infty)$ , there exists  $N_1 \in \mathbf{N}$  such that*

$$(5.1) \quad |\beta([ns] + [nu] + n)| \leq \frac{r \sin(\pi d)}{\pi(s + u + 1)} n^{-1} \quad (s \geq 0, u \geq 0, n \geq N_1).$$



*Proof.* The assertions (1) and (2) follow from [II, Proposition 4.3]. Since we have  $[ns] + [nu] + n \sim n(s + u + 1)$  as  $n \rightarrow \infty$ , (3) follows from (1). Let  $r \in (1, \infty)$ . Then  $n/([ns] + [nu] + n) \rightarrow 1/(s + u + 1)$  as  $n \rightarrow \infty$ , uniformly in  $s \geq 0$  and  $u \geq 0$  (cf. [BGT, Theorem 1.5.2]), so that there exists  $N_2 \in \mathbf{N}$  such that

$$\frac{1}{([ns] + [nu] + n)} \leq \frac{r^{1/2}}{n(s + u + 1)} \quad (s \geq 0, u \geq 0, n \geq N_2),$$

while, from (1), there exists  $N_3 \in \mathbf{N}$  such that

$$|\beta_n| \leq \frac{r^{1/2} \sin(\pi d)}{\pi n} \quad (n \geq N_3).$$

If we put  $N_1 := \max(N_2, N_3)$ , then, for  $s \geq 0, u \geq 0, n \geq N_1$ , we have

$$|\beta([ns] + [nu] + n)| \leq \frac{r^{1/2} \sin(\pi d)}{\pi([ns] + [nu] + n)} \leq \frac{r \sin(\pi d)}{\pi(s + u + 1)} n^{-1}.$$

Thus (4) follows.  $\square$

Recall  $d_k(n)$  from (2.3)–(2.5). For  $k = 1, 2, \dots$ , we define the constant  $\tau_k$ , which is equal to  $f_k(0)$  in [IK2], by

$$\tau_1 = \frac{1}{\pi}, \quad \tau_2 = \frac{1}{\pi^2} \int_0^\infty \frac{ds_1}{(s_1 + 1)(s_1 + 1)} = \frac{1}{\pi^2},$$

and, for  $k = 3, 4, \dots$ ,

$$\tau_k = \frac{1}{\pi^k} \int_0^\infty ds_{k-1} \cdots \int_0^\infty ds_1 \frac{1}{(s_{k-1} + 1)} \left\{ \prod_{m=1}^{k-2} \frac{1}{(s_{m+1} + s_m + 1)} \right\} \frac{1}{(s_1 + 1)}.$$

**Proposition 5.2.** For  $d \in (0, 1/2)$  and  $\ell \in \mathcal{R}_0$ , we assume  $(L(d, \ell))$ .

(1) For  $r \in (1, \infty)$  and  $N_1 \in \mathbf{N}$  satisfying (5.1),

$$(5.2) \quad |d_k(n)| \leq n^{-1} \{r \sin(\pi d)\}^k \tau_k \quad (u \geq 0, k \in \mathbf{N}, n \geq N_1).$$

(2) For  $k \in \mathbf{N}$  and  $u \geq 0$ ,

$$(5.3) \quad d_k(n) \sim n^{-1} \{\sin(\pi d)\}^k \tau_k \quad (n \rightarrow \infty).$$

*Proof.* Let  $k \geq 3$  and write

$$\begin{aligned} d_k(n) &= \int_0^\infty ds_{k-1} \cdots \int_0^\infty ds_1 \beta([s_{k-1}] + n) \\ &\quad \times \left\{ \prod_{m=1}^{k-2} \beta([s_{m+1}] + [s_m] + n) \right\} \times \beta([s_1] + n + [nu]) \\ &= n^{k-1} \int_0^\infty ds_{k-1} \cdots \int_0^\infty ds_1 \beta([ns_{k-1}] + n) \\ &\quad \times \left\{ \prod_{m=1}^{k-2} \beta([ns_{m+1}] + [ns_m] + n) \right\} \times \beta([ns_1] + n). \end{aligned}$$

Applying Proposition 5.1 and the dominated convergence theorem to this, we obtain (5.2) and (5.3). The cases  $k = 1, 2$  can be treated in a similar fashion.  $\square$

**Proposition 5.3.** For  $k = 1, 2, \dots$ , we have  $\tau_k \leq \pi^{-2}$ .

*Proof.* Let  $T$  be the linear bounded operator on  $L^2((0, \infty), du)$  defined by

$$Tg(u) := \int_0^\infty \frac{1}{u+v} g(v) dv.$$

Then, by Hilbert's theorem (cf. [HLP, Theorems 316 and 317]), the operator norm  $\|T\|$  is equal to  $\pi$ . Hence, for the inner product  $(\cdot, \cdot)$  of  $L^2((0, \infty), du)$  and  $f(x) := 1/(1+x)$ , we have  $\tau_k \leq \pi^{-k} (f, T^{k-2} f) \leq \pi^{-2}$ , yielding the proposition.  $\square$

*Proof of Theorem 2.3.* To apply the dominated convergence theorem, we choose  $r$  so that  $0 < r \sin(\pi d) < 1$ . Then, by Proposition 5.3,  $\sum_{k=1}^\infty \tau_{2k-1} \{r \sin(\pi d)\}^{2k-1} < \infty$ . Hence Proposition 5.2 and the dominated convergence theorem yield

$$(5.4) \quad \lim_{n \rightarrow \infty} n \sum_{k=1}^\infty d_{2k-1}(n) = \sum_{k=1}^\infty \tau_{2k-1} \sin^{2k-1}(\pi d).$$

By Lemma 5.4 below, the right-hand side is equal to  $d$ . Thus, by Corollary 2.2,  $(d/n)$  follows.  $\square$

**Lemma 5.4.** For  $|x| < 1$ , we have  $\sum_{k=1}^\infty \tau_k x^{2k-1} = \pi^{-1} \arcsin x$ .

*Proof.* (Compare the proof of [I2, Lemma 6.5].) For  $0 < d < 1/2$ , let  $\{Y_n : n \in \mathbf{Z}\}$  be a fractional ARIMA(0,  $d$ , 0) process such that  $E[(Y_0)^2] = \Gamma(1-2d)/\Gamma^2(1-d)$ . We denote by  $c'_n$ ,  $a'_n$ , and  $\alpha'_n$  the MA and AR coefficients, and PACF of  $\{Y_n\}$ , respectively. Then we have, for  $n = 0, 1, \dots$ ,

$$c'_n = \frac{\Gamma(n+d)}{\Gamma(n+1)\Gamma(d)}, \quad a'_n = \frac{\Gamma(n-d)d}{\Gamma(n+1)\Gamma(1-d)}$$

(see, e.g., [BD, §13.2]). We define  $d'_k(n)$  similarly. Then since  $\{Y_n\}$  satisfies  $(L(d, \ell'))$  with  $\ell' \equiv 1/\Gamma(d)$ , it follows from (5.4) that

$$\lim_{n \rightarrow \infty} n \sum_{k=1}^\infty d'_{2k-1}(n) = \sum_{k=1}^\infty \tau_{2k-1} \{\sin(\pi d)\}^{2k-1}.$$

However, since  $\alpha'_n = d/(n-d)$ , Corollary 2.2 gives

$$\lim_{n \rightarrow \infty} n \sum_{k=1}^\infty d'_{2k-1}(n) = \lim_{n \rightarrow \infty} n \alpha'_n = d.$$

Combining, we obtain  $\sum_1^\infty \tau_k \sin^{2k-1}(\pi d) = d$ . The lemma follows if we substitute  $\pi^{-1} \arcsin x$  with  $0 < x < 1$  for  $d$  and use analytic continuation.  $\square$

*Remark.* From Lemma 5.4, it follows that

$$\tau_k = \frac{1}{\pi} \cdot \frac{(2k-2)!}{2^{2k-2}((k-1)!)^2(2k-1)} \quad (k = 1, 2, \dots).$$

*Proof of Theorem 2.4.* Let  $\{X_n\}$  be a fractional ARIMA( $p, d, q$ ) process with spectral density (2.14) with (2.15). We assume that  $-1/2 < d < 0$ . Then (2.16) and (2.17) imply  $\{c_n\} \in l^1$  and  $\{a_n\} \in l^2$ , respectively, so that we can use Theorem 3.3. Let  $d_k(n, p)$  be as in Theorem 3.3. Recall  $\phi_n$ ,  $\psi_n$  and  $\beta_-(n)$  from §2. By [IK1, Theorem 3.3], we have the same conclusions as those in Proposition 4.3 with  $\beta(n)$  replaced by  $\beta_-(n)$ . Notice that, in [IK1, Theorem 3.3], the results are stated for  $n \geq 2$  but we can prove the case  $n = 1$  in the same way. The equality (4.4) holds

in the same way as the proof of Proposition 4.4. Hence the theorem follows from Theorem 3.3.  $\square$

*Proof of Theorem 2.5.* If  $0 < d < 1/2$ , then  $(d/n)$  follows immediately from Theorem 2.3. We assume that  $-1/2 < d < 0$ . Let  $q := d + 1 \in (1/2, 1)$  as in §2. Then, from (2.11), (2.12) and [I1, Proposition 4.3], it follows that

$$\beta_-(n) \sim -\frac{\sin(\pi q)}{\pi} n^{-1} \quad (n \rightarrow \infty).$$

Running through the same arguments as those in the proof of Theorem 2.3, we see that

$$\lim_{n \rightarrow \infty} n\alpha_n = \lim_{n \rightarrow \infty} n \sum_{k=1}^{\infty} d_{2k-1}(n) = -\sum_{k=1}^{\infty} \tau_{2k-1} \sin^{2k-1}(\pi q) = d.$$

Thus, again,  $(d/n)$  holds.  $\square$

## 6. MODEL WITH REGULARLY VARYING AUTOCOVARANCE FUNCTION

In this section, we apply the representation of PACF to a stationary process  $\{X_n\}$  which has regularly varying autocovariance function. We will also assume that  $\{X_n\}$  is PND and satisfies the following conditions (cf. [I2, §2]):

- (C1)  $c_n \geq 0$  for all  $n \geq 0$ ;
- (C2)  $\{c_n\}$  is eventually decreasing to zero;
- (A1)  $\{a_n\}$  is eventually decreasing to zero.

Notice that (C1) and (A1) imply  $\{a_n\} \in l^1$  (see [I2, Proposition 4.3]). In [I2], the extra condition

- (A2)  $\{a_n - a_{n+1}\}$  is eventually decreasing to zero

is also required but we do not need it here. By [I2, Theorem 7.3],  $\{X_n\}$  satisfies (C1)–(A1) (and also (A2)) if

- (RP) there exists a finite Borel measure  $\sigma$  on  $[0, 1)$   
such that  $\gamma_n = \int_0^1 t^{|n|} \sigma(dt)$  ( $n \in \mathbf{Z}$ ).

This property is called *reflection positivity* or *T-positivity*, which originates in quantum field theory; see, e.g., Osterwalder and Schrader [OS], Hegerfeldt [He] and Okabe [O]. A prototype of such a process is  $\{X_n\}$  with  $\gamma_n = (1 + |n|)^{-(1-2d)}$ ,  $-\infty < d < 1/2$ , which we consider in the Example below.

Let  $\ell \in \mathcal{R}_0$ , and choose a positive constant  $B$  so large that  $\ell(\cdot)$  is locally bounded on  $[B, \infty)$  (see [BGT, Corollary 1.4.2]). When we say  $\int_B^\infty \ell(s) ds/s = \infty$ , it means that  $\int_B^\infty \ell(s) ds/s = \infty$ . If so, then we define another slowly varying function  $\tilde{\ell}$  by

$$(6.1) \quad \tilde{\ell}(x) := \int_B^x \frac{\ell(s)}{s} ds \quad (x \geq B)$$

(see [BGT, §1.5.6]). The asymptotic behaviour of  $\tilde{\ell}(x)$  as  $x \rightarrow \infty$  does not depend on the choice of  $B$  since we have assumed that  $\int_B^\infty \ell(s) ds/s = \infty$ .

Here is the result on the asymptotic behaviour of  $\alpha_n$ .

**Theorem 6.1.** Let  $-\infty < d < 1/2$  and  $\ell \in \mathcal{R}_0$ . We assume (PND), (C1), (C2), (A1), and

$$(6.2) \quad \gamma_n \sim n^{2d-1}\ell(n) \quad (n \rightarrow \infty).$$

- (1) If  $0 < d < 1/2$ , then (d/n) holds;  
(2) if  $d = 0$  and  $\int^\infty \ell(s)ds/s = \infty$ , then

$$(6.3) \quad \alpha_n \sim n^{-1} \frac{\ell(n)}{2\tilde{\ell}(n)} \quad (n \rightarrow \infty);$$

- (3) if  $d = 0$  with  $\int^\infty \ell(s)ds/s < \infty$  or  $-\infty < d < 0$ , then

$$(6.4) \quad \alpha_n \sim \frac{n^{2d-1}\ell(n)}{\sum_{-\infty}^\infty \gamma_k} \quad (n \rightarrow \infty).$$

This theorem is an improvement of [I2, Theorem 2.1] where only  $|\alpha(n)|$  is considered with additional assumption (A2).

We need some propositions to prove the theorem above.

**Proposition 6.2.** Let  $\ell \in \mathcal{R}_0$ . If  $\int^\infty \ell(s)ds/s = \infty$ , then  $\ell(n)/\tilde{\ell}(n)$  tends to 0 as  $n \rightarrow \infty$ . If  $\int^\infty \ell(s)ds/s < \infty$ , then  $\ell(n)$  tends to 0 as  $n \rightarrow \infty$ .

Proposition 6.2 follows immediately from [BGT, Proposition 1.5.9a].

**Proposition 6.3.** Let  $\ell \in \mathcal{R}_0$  and  $-\infty < d \leq 0$ . We assume (PND), (C1), (C2), (A1), and (6.2).

- (1) If  $d = 0$  and  $\int^\infty \ell(s)ds/s = \infty$ , then

$$(6.5) \quad c_n \sim n^{-1}\ell(n)\{2\tilde{\ell}(n)\}^{-1/2} \quad (n \rightarrow \infty),$$

$$(6.6) \quad a_n \sim n^{-1}\ell(n)\{2\tilde{\ell}(n)\}^{-3/2} \quad (n \rightarrow \infty),$$

$$(6.7) \quad \beta_n \sim n^{-1}\ell(n)\{2\tilde{\ell}(n)\}^{-1} \quad (n \rightarrow \infty).$$

- (2) If  $d = 0$  with  $\int^\infty \ell(s)ds/s < \infty$  or  $-\infty < d < 0$ , then

$$(6.8) \quad c_n \sim n^{2d-1}\ell(n) \left\{ \sum_{-\infty}^\infty \gamma_k \right\}^{-1/2} \quad (n \rightarrow \infty),$$

$$(6.9) \quad a_n \sim n^{2d-1}\ell(n) \left\{ \sum_{-\infty}^\infty \gamma_k \right\}^{-3/2} \quad (n \rightarrow \infty),$$

$$(6.10) \quad \beta_n \sim n^{2d-1}\ell(n) \left\{ \sum_{-\infty}^\infty \gamma_k \right\}^{-1} \quad (n \rightarrow \infty).$$

*Proof.* The assertions (6.5) and (6.6) follow from [I2, Theorem 5.2]. Using them, we obtain (6.7) (see [I2, (6.19)]). The assertions (6.8) and (6.9) follow from [I2, Theorem 5.3]. From them, we get (6.10) (see the proof of [I2, Theorem 6.7]).  $\square$

**Proposition 6.4.** Let  $\ell \in \mathcal{R}_0$  and  $-\infty < d \leq 0$ . We assume (PND), (C1), (C2), (A1), and (6.2).

- (1) For every  $R \in (1, \infty)$ , there exists  $N \in \mathbf{N}$  such that

$$(6.11) \quad \left| \frac{\beta([ns] + [nu] + n)}{\beta(n)} \right| \leq \frac{R}{(s+u+1)} \quad (s \geq 0, u \geq 0, n \geq N).$$

- (2) For every  $r \in (0, 1)$ , there exists  $N \in \mathbf{N}$  such that

$$(6.12) \quad |\beta([ns] + [nu] + n)| \leq \frac{r}{\pi(s+u+1)} n^{-1} \quad (s \geq 0, u \geq 0, n \geq N).$$

*Proof.* By Proposition 6.3 and [BGT, Theorem 1.5.2], we have

$$\beta([ns] + [nu] + n)/\beta(n) \rightarrow (s + u + 1)^{2d-1} \quad (n \rightarrow \infty)$$

uniformly in  $s \geq 0$  and  $u \geq 0$ . Since

$$(s + u + 1)^{2d-1} \leq (s + u + 1)^{-1},$$

(1) follows. By Propositions 6.2 and 6.3, we have

$$(6.13) \quad \lim_{n \rightarrow \infty} n\beta_n = 0.$$

This and (1) show (2). □

Recall  $d_k(n)$  from (2.3)–(2.5) and  $\tau_k$  from §5.

**Proposition 6.5.** *Let  $\ell \in \mathcal{R}_0$  and  $-\infty < d \leq 0$ . We assume (PND), (C1), (C2), (A1), and (6.2).*

- (1) *Let  $r \in (0, 1)$  and  $R \in (1, \infty)$ . Choose  $N \in \mathbf{N}$  so that both (6.11) and (6.12) hold. Then  $|d_k(n)/\beta_n| \leq \pi R r^{k-1} \tau_k$  for  $k \in \mathbf{N}$ ,  $n \geq N$ .*
- (2) *For  $k \geq 2$ ,  $\lim_{n \rightarrow \infty} d_k(n)/\beta_n = 0$ .*

*Proof.* In the same way as the proof of (6.13), we see that  $(O(1/n))$  holds. We assume  $k \geq 3$ ; the cases  $k = 1, 2$  can be treated in a similar way. We write

$$\begin{aligned} \frac{d_k(n)}{\beta(n)} &= \int_0^\infty ds_{k-1} \cdots \int_0^\infty ds_1 \frac{\beta([ns_{k-1}] + n)}{\beta(n)} \\ &\quad \times \left\{ \prod_{m=1}^{k-2} n\beta([ns_{m+1}] + [ns_m] + n) \right\} \times n\beta([ns_1] + n). \end{aligned}$$

From this as well as (6.11) and (6.12), we get (1). The assertion (2) follows from (6.11)–(6.13) and the dominated convergence theorem. □

*Proof of Theorem 6.1.* Assume that  $0 < d < 1/2$ . Then, from [I2, Theorem 5.1],  $(L(d, \ell'))$  holds with  $\ell'(n) := [\ell(n)/B(d, 1 - 2d)]^{1/2}$ . Hence  $(d/n)$  follows immediately from Theorem 2.3.

Next, we assume  $-\infty < d \leq 0$ . By (A1) and (C1),  $\sum_{v=0}^\infty |c_v a_{v+n}| = \beta_n$  if  $n$  is large enough. Hence from (6.7), (6.10) and Proposition 6.2,  $(O(1/n))$  holds. By Proposition 6.5 and the dominated convergence theorem, we have

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^\infty d_{2k-1}(n)}{\beta_n} = 1 + \lim_{n \rightarrow \infty} \sum_{k=2}^\infty \frac{d_{2k-1}(n)}{\beta_n} = 1.$$

Therefore, by Corollary 2.2, we see (2.10). Thus (2) and (3) follow from (6.7) and (6.10), respectively. □

*Example.* Let  $-\infty < d < 1/2$ , and let  $\{X_n\}$  be a stationary process with autocovariance function of the form  $\gamma_n = (1 + |n|)^{-(1-2d)}$ . Then  $\{X_n\}$  satisfies (RP) (cf. [I2, Example in §7]). Let  $\{\alpha_n\}$  be the PACF of  $\{X_n\}$ . Applying Theorem 1.7 to  $\{X_n\}$ , we get the following result:

- (1) if  $0 < d < 1/2$ , then we have  $(d/n)$ .
- (2) if  $d = 0$ , then  $\alpha_n \sim 1/(2n \log n)$  as  $n \rightarrow \infty$ .
- (3) if  $-\infty < d < 0$ , then  $\alpha_n \sim n^{2d-1} \cdot [2\zeta(1 - 2d) - 1]^{-1}$  as  $n \rightarrow \infty$ .

Here  $\zeta(s)$  is the Riemann zeta function.

*Remarks.* 1. Recall by (2.3) that  $\beta_n$  is the first term on the right of (2.9). By the arguments above, we see that, for the processes treated in Theorem 6.1,

$$(6.14) \quad \lim_{n \rightarrow \infty} \frac{\alpha_n}{\beta_n} = \begin{cases} \frac{\sin(\pi d)}{\pi d} & (0 < d < 1/2), \\ 1 & (-\infty < d \leq 0). \end{cases}$$

We raise, and leave open, the question of how generally this happens.

2. We note that in Theorems 2.3, 2.5 and 6.1 we have also

$$\alpha_n \sim \frac{\gamma_n}{\sum_{k=-n}^n \gamma_k} \quad (n \rightarrow \infty)$$

(see [I2, §2 and §6] and [I3, §5] for the proofs). It would be interesting to know how generally this relation holds.

## 7. ARMA PROCESSES

In this section, we consider the fractional ARIMA( $p, 0, q$ ) processes, that is, the ARMA( $p, q$ ) processes. Let  $p, q \in \mathbf{N} \cup \{0\}$ , and let  $\Phi(z)$  and  $\Theta(z)$  be polynomials with real coefficients of degrees  $p, q$ , respectively, satisfying (2.15). Let  $\{X_n\}$  be an ARMA( $p, q$ ) process with spectral density

$$\Delta(\theta) = \frac{1}{2\pi} \frac{|\Theta(e^{i\theta})|^2}{|\Phi(e^{i\theta})|^2} \quad (-\pi < \theta < \pi).$$

We put  $R := 0$  if  $q = 0$  and

$$R := \max(1/|u_1|, \dots, 1/|u_q|) \quad \text{if } q \geq 1,$$

where  $u_1, \dots, u_q$  are the (complex) zeros of  $\Theta(z)$ :

$$\Theta(z) = \text{const.} \times (z - u_1) \cdots (z - u_q).$$

From the assumption (2.15), we see that  $|u_k| > 1$  for  $k = 1, \dots, q$ , whence  $R \in [0, 1)$ .

Let  $\{\alpha_n\}$  be the PACF of the ARMA( $p, q$ ) process  $\{X_n\}$ . The next theorem implies that  $\alpha_n$  decays exponentially as  $n \rightarrow \infty$ .

**Theorem 7.1.** *For every  $r > R$ , we have*

$$(7.1) \quad \alpha_n = O(r^n) \quad (n \rightarrow \infty).$$

*In particular,  $\alpha_n$  decays exponentially fast as  $n \rightarrow \infty$ .*

*Proof.* The Szegő function  $D(z)$  of  $\{X_n\}$  is given by  $D(z) = \Theta(z)/\Phi(z)$  for  $|z| < 1$ . Hence  $-\Phi(z)/\Theta(z) = \sum_{n=0}^{\infty} a_n z^n$ , so that, for every  $r > R$ ,

$$a_n = O(r^n) \quad (n \rightarrow \infty).$$

Treating  $D(z) = \Theta(z)/\Phi(z)$  similarly, we see that  $c_n$  also decays exponentially as  $n \rightarrow \infty$ , in particular,  $\{c_n\} \in l^1$ . Therefore, we see that, for every  $r > R$ ,

$$\beta_n = O(r^n) \quad (n \rightarrow \infty).$$

The assertion (7.1) follows easily from this and Theorem 2.1. Since  $r$  can be chosen so that  $R < r < 1$ , (7.1) implies exponential decay of  $\alpha_n$ .  $\square$

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, HOKKAIDO UNIVERSITY, SAPPORO 060-0810, JAPAN

*E-mail address:* inoue@math.sci.hokudai.ac.jp