ON THE WAVE OPERATORS FOR THE CRITICAL NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We prove that for the L^2-critical nonlinear Schrödinger equations, the wave operators and their inverse are related explicitly in terms of the Fourier transform. We discuss some consequences of this property. In the one-dimensional case, we show a precise similarity between the L^2-critical nonlinear Schrödinger equation and a nonlinear Schrödinger equation of derivative type.

1. Introduction

We consider the defocusing, L^2-critical, nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = |u|^{4/n}u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n. \tag{1.1}$$

We consider two types of initial data:

$$\text{Asymptotic state: } U_0(-t)u(t)\big|_{t=\pm\infty} = u_\pm, \quad \text{where } U_0(t) = e^{i\frac{t}{2}\Delta}. \tag{1.2}$$

$$\text{Cauchy data at } t=0: \ u_{|t=0} = u_0. \tag{1.3}$$

It is well known that for data $u_\pm, u_0 \in \Sigma = H^1 \cap \mathcal{F}(H^1)$, where

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx,$$

(1.1)–(1.2) has a unique, global, solution $u \in C(\mathbb{R};\Sigma)$ ([GV79], see also [Caz03]). Its initial value $u_{|t=0}$ is the image of the asymptotic state under the action of the wave operator:

$$u_{|t=0} = W_\pm u_\pm.$$

Similarly, (1.1)–(1.3) possesses asymptotic states:

$$\exists u_\pm \in \Sigma, \quad \|U_0(-t)u(t) - u_\pm\|_\Sigma \underset{t\to\pm\infty}{\longrightarrow} 0: \quad u_\pm = W_\pm^{-1}u_0.$$

Global well-posedness properties show that the wave operators are homeomorphisms on Σ. Besides this point, very few properties of these operators are known. The main result of this paper (proved in §2) shows that the wave operators and their inverses are easily related in terms of the Fourier transform:

Theorem 1.1. *Let $n \geqslant 1$. The following identity holds on Σ:*

$$\mathcal{F}\circ W_\pm^{-1} = W_\mp \circ \mathcal{F}. \tag{1.4}$$

In particular, if $\mathcal{C}$ denotes the conjugation $f \mapsto \overline{f}$, then we have:

$$W_\pm^{-1} = (\mathcal{C}\mathcal{F})^{-1}\, W_\pm\, (\mathcal{C}\mathcal{F})\,. \tag{1.5}$$

2000 *Mathematics Subject Classification.* 35B33; 35B40; 35Q55.

Using continuity properties of the flow map associated to (1.1), we infer the following result in §3:

Corollary 1.2. *The result of Theorem 1.1 still holds when Σ is replaced*

- *Either by $\mathcal{F}(H^1)$,*
- *Or by a neighborhood of the origin in $L^2(\mathbb{R}^n)$, for* (1.1) *as well as for its focusing counterpart, $i\partial_t u + \frac{1}{2}\Delta u = -|u|^{4/n}u$,*
- *Or by $L^2_r(\mathbb{R}^n)$ for $n \geqslant 3$, the set of* radial, *square integrable functions.*

Remark 1.3. The usual conjecture on (1.1) implies that the result of Theorem 1.1 is expected to remain valid when Σ is replaced by $L^2(\mathbb{R}^n)$ (but not for the focusing counterpart of (1.1), for which finite time blow-up may occur in H^1).

Remark 1.4. So far, the existence of wave operators on $\mathcal{F}(H^1)$ is not known. Similarly, asymptotic completeness in H^1 remains an open problem. Theorem 1.1 shows that the fact that these two problems are simultaneously open is not merely a technical point: they are exactly related by (1.4). This aspect is also reminiscent of the main result in [BC06].

Using the asymptotic expansion of the wave operators near the origin, we prove in §4 (with an extension in Appendix A):

Corollary 1.5. *Let $n \geqslant 1$. For every $\phi \in L^2(\mathbb{R}^n)$, we have:*

$$\int_0^{\pm\infty} e^{it\frac{|x|^2}{2}} \mathcal{F}\left(|U_0(t)\phi|^{4/n} U_0(t)\phi\right) dt = \int_0^{\pm\infty} U_0(t)\left(|U_0(-t)\widehat{\phi}|^{4/n} U_0(-t)\widehat{\phi}\right) dt.$$

Finally, in space dimension $n = 1$, we relate the wave operators for (1.1) with the wave operators for the nonlinear Schrödinger equation of derivative type

$$i\partial_t \psi + \frac{1}{2}\partial_x^2 \psi = i\lambda \partial_x \left(|\psi|^2\right)\psi, \quad \lambda \in \mathbb{R}. \tag{1.6}$$

This equation appears as a model to study the nonlinear self-modulation for the Benjamin-Ono equation [Tan82]. For a more general nonlinear Schrödinger equation of derivative type (see e.g. [KT94, Tsu94] for the Cauchy problem related to similar equations),

$$i\partial_t \psi + \frac{1}{2}\partial_x^2 \psi = i\lambda |\psi|^2 \partial_x \psi + i\mu \psi^2 \partial_x \overline{\psi},$$

it is proved in [Oza96] that a short range scattering theory is available for $\lambda, \mu \in \mathbb{R}$ if and only if $\lambda = \mu$: we recover (1.6). This is apparently the only cubic, gauge invariant nonlinearity in space dimension one, for which a short range scattering theory is available. More precisely, for (1.6)–(1.2), the wave operators $\Omega_\pm : u_\pm \mapsto u(0)$ are well defined from X_ε to $H^2(\mathbb{R})$, where

$$X_\varepsilon = \{\phi \in H^4 \cap \mathcal{F}(H^4) \quad ; \quad \left\|(1+\xi^2)\widehat{\phi}\right\|_{L^\infty} < \varepsilon\},$$

and $\varepsilon > 0$ is sufficiently small. The following result shows that the nonlinearity in (1.6) should be thought of as the quintic case (1.1). This result goes in the same spirit as the approach followed in [OT98].

Theorem 1.6. *Let $\lambda \in \mathbb{R}$. Consider the quintic, focusing or defocusing, equation*

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = \frac{\lambda^2}{2}|u|^4 u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \quad \mu \in \mathbb{R}, \tag{1.7}$$

with associated wave operators $W_\pm(\mu)$ for small L^2 data. For $\phi\in L^2(\mathbb{R})$, define

$$(N^\lambda_\pm\phi)(x)=\phi(x)\exp\left(\pm i\lambda\int_{-\infty}^x|\phi(y)|^2dy\right).$$

- *If ψ solves* (1.6)*, then $N^\lambda_-(\psi)$ solves* (1.7)*.*
- *If u solves* (1.7)*, then $N^\lambda_+(u)$ solves* (1.6)*.*
- *The following identity holds when all terms are well-defined:*

$$\mathcal{F}\circ\Omega_\pm^{-1}=N^\lambda_-\circ\mathcal{F}\circ W_\pm^{-1}\circ N^\lambda_+=\left(N^\lambda_+\right)^{-1}\circ\mathcal{F}\circ W_\pm^{-1}\circ N^\lambda_+.$$
$$\Omega_\pm\circ\mathcal{F}^{-1}=\left(N^\lambda_+\right)^{-1}\circ W_\pm\circ\mathcal{F}^{-1}\circ N^\lambda_+.$$

This result is checked by elementary computations, so we leave out its proof.

2. Proof of Theorem 1.1

The proof of Theorem 1.1 relies on a series of lemmas, which are stated, and proved, in a slightly different fashion in [Tsu85]. Introduce the transform Ψ acting on function of (t,x) as:

$$(\Psi u)(t,x)=\frac{1}{(it)^{n/2}}e^{i\frac{|x|^2}{2t}}u\left(\frac{-1}{t},\frac{x}{t}\right),\quad\text{for } t\neq 0. \tag{2.1}$$

Lemma 2.1. *For $n\geqslant 1$ and $\phi\in L^2(\mathbb{R}^n)$, we have:*

$$\lim_{t\to\pm\infty}\left\|U_0(t)\mathcal{F}^{-1}\phi(\cdot)-(\Psi\phi)(t,\cdot)\right\|_{L^2}=0.$$

Proof. We recall the standard decomposition of the free group, for $t\neq 0$:

$$U_0(t)=\mathcal{M}_tD_t\mathcal{F}\mathcal{M}_t,$$

where $\mathcal{M}_t$ is the multiplication by $e^{i|x|^2/(2t)}$, and D_t is the dilation operator

$$D_t\phi(x)=\frac{1}{(it)^{n/2}}\phi\left(\frac{x}{t}\right).$$

Noting that $\Psi\phi=\mathcal{M}D\phi$, Plancherel formula yields:

$$\left\|U_0(t)\mathcal{F}^{-1}\phi(\cdot)-(\Psi\phi)(t,\cdot)\right\|_{L^2}=\left\|\left(M_t-1\right)\mathcal{F}^{-1}\phi(\cdot)\right\|_{L^2}.$$

Since $|M_t(x)-1|\lesssim|x|/\sqrt{t}$, the lemma follows for $\phi\in H^1(\mathbb{R}^n)$. By density, we infer the result for $\phi\in L^2(\mathbb{R}^n)$. $\square$

Lemma 2.2. *Let $v=\Psi u$. Suppose that there exist $\psi_\pm\in L^2(\mathbb{R}^n)$ such that*

$$\|v(t)-\psi_\pm\|_{L^2}\underset{t\to\pm 0}{\longrightarrow}0.$$

Then u has asymptotic states in L^2:

$$\|U_0(-t)u(t)-\mathcal{F}^{-1}R\psi_\mp\|_{L^2}\underset{t\to\pm\infty}{\longrightarrow}0,$$

where R stands for the symmetry with respect to the origin, $(R\phi)(x)=\phi(-x)$.

Proof. We note that Ψ is almost an involution: $\Psi^2=R$. Therefore, $u=\Psi Rv$:

$$\begin{aligned}U_0(-t)u(t)-\mathcal{F}^{-1}R\psi_\mp&=U_0(-t)\Psi Rv\left(\frac{-1}{t}\right)-\mathcal{F}^{-1}R\psi_\mp\\&=U_0(-t)\Psi R\left(v\left(\frac{-1}{t}\right)-\psi_\mp\right)+\left(U_0(-t)\Psi-\mathcal{F}^{-1}\right)R\psi_\mp.\end{aligned}$$

Taking the L^2 norm, we infer:

$$\left\|U_0(-t)u(t)-\mathcal{F}^{-1}R\psi_\mp\right\|_{L^2}\leqslant\left\|v\left(\frac{-1}{t}\right)-\psi_\mp\right\|_{L^2}+\|\Psi R\psi_\mp-U_0(t)\mathcal{F}^{-1}R\psi_\mp\|_{L^2}.$$

The first term of the right-hand side goes to zero as $t\to\pm\infty$ by assumption. The second term goes to zero by Lemma 2.1. □

Lemma 2.3. *Let $v=\Psi u$. Suppose that $u\in C([-T,T];L^2)$ for some $T>0$, and $u_{|t=0}=u_0\in L^2(\mathbb{R}^n)$. Then*

$$\left\|U_0(-t)v(t)-\mathcal{F}^{-1}u_0\right\|_{L^2}\underset{t\to\pm\infty}{\longrightarrow}0.$$

Proof. Since $U_0(-t)=U_0(t)^{-1}$, we have

$$U_0(-t)v(t)=\mathcal{M}_t^{-1}\mathcal{F}^{-1}D_t^{-1}\mathcal{M}_t^{-1}v(t)=\mathcal{M}_t^{-1}\mathcal{F}^{-1}u\left(\frac{-1}{t}\right).$$

Therefore,

$$\left\|U_0(-t)v(t)-\mathcal{F}^{-1}u_0\right\|_{L^2}\leqslant\left\|u\left(\frac{-1}{t}\right)-u_0\right\|_{L^2}+\left\|\left(\mathcal{M}_{-t}-1\right)\mathcal{F}^{-1}u_0\right\|_{L^2}.$$

The first term of the right-hand side goes to zero as $t\to\pm\infty$ by assumption. So does the second, by the standard argument recalled in the proof of Lemma 2.1. □

Proof of Theorem 1.1. Let $u_0\in\Sigma$: there exists a unique solution $u\in C(\mathbb{R};\Sigma)$ to (1.1)–(1.3). Set $v=\Psi u$. Because of the conformal invariance for (1.1), v solves the same equation as u, for $t\neq0$:

$$i\partial_t v+\frac{1}{2}\Delta v=|v|^{4/n}v,\quad (t,x)\in\mathbb{R}\setminus\{0\}\times\mathbb{R}^n.$$

Lemma 2.3 shows that

$$\left\|U_0(-t)v(t)-\mathcal{F}^{-1}u_0\right\|_{L^2}\underset{t\to\pm\infty}{\longrightarrow}0.$$

Let $w_\pm$ denote the solutions to the scattering problems:

$$i\partial_t w_\pm+\frac{1}{2}\Delta w_\pm=|w_\pm|^{4/n}w_\pm\quad;\quad U_0(-t)w_\pm(t)\big|_{t=0}=\mathcal{F}^{-1}u_0.$$

By uniqueness for (1.1)–(1.2), we see that

$$v(t,x)=\begin{cases}w_-(t,x) & \text{for } t<0,\\ w_+(t,x) & \text{for } t>0.\end{cases}$$

In particular,

$$\|v(t)-w_\pm(0)\|_{L^2}\underset{t\to\pm0}{\longrightarrow}0.$$

From Lemma 2.2, u has asymptotics states, given by:

$$\|U_0(-t)u(t)-\mathcal{F}^{-1}Rw_\mp\|_{L^2}\underset{t\to\pm\infty}{\longrightarrow}0,$$

that is, $u_\pm=\mathcal{F}^{-1}Rw_\mp$. We infer:

$$\mathcal{F}\circ W_\pm^{-1}u_0=\mathcal{F}u_\pm=Rw_\mp=RW_\mp\mathcal{F}^{-1}u_0.$$

Since (1.1) is invariant by R, $RW_\mp\mathcal{F}^{-1}u_0=W_\mp R\mathcal{F}^{-1}u_0=W_\mp\mathcal{F}u_0$. This yields (1.4). The identity (1.5) follows from (1.4) and from the identity

$$W_\pm=\mathcal{C}\circ W_\mp\circ\mathcal{C},$$

which was noticed in [CW92] (see also [Caz03]). □

3. Proof of Corollary 1.2

The first case follows by density, since $W_\pm$ are defined and continuous on $H^1(\mathbb{R}^n)$ [GV85] (see also [Gin97] for a simplified presentation), and since $W_\pm^{-1}$ are defined and continuous on $\mathcal{F}(H^1)$ [GOV94, GV79, Tsu85].

For the second case, existence of wave operators, their asymptotic completeness, and continuity properties, were proved by T. Cazenave and F. Weissler [CW89]. We note that Corollary 1.2 can be proved in this case like Theorem 1.1, provided that we work in a sufficiently small neighborhood of the origin in $L^2(\mathbb{R}^n)$.

The last case follows from the recent paper by T. Tao, M. Visan and X. Zhang [TVZ]. The proof of Corollary 1.2 then relies on asymptotic completeness (in the same space), along with continuous dependence upon the initial data. For $n \geqslant 3$, let $X = L^2_r(\mathbb{R}^n)$; X is invariant under the action of the Fourier transform. For $\phi \in X$, let ϕ_j be a sequence in Σ, converging to ϕ in X. Define $u_j^\pm$ as the solutions to:

$$i\partial_t u_j^\pm + \frac{1}{2}\Delta u_j^\pm = |u_j^\pm|^{4/n} u_j^\pm \quad ; \quad U_0(-t)u_j^\pm(t)\big|_{t=\pm\infty} = \widehat{\phi}_j.$$

There exists $u_{0j}^\pm = u_j^\pm(0) = W_\pm \widehat{\phi}_j \in \Sigma$. Since $u_{0j}^\pm = \mathcal{F}W_\mp^{-1}\phi_j$ from Theorem 1.1, the results in [CW89, TVZ] imply that there exists $u_0^\pm \in X$ such that $\|u_{0j}^\pm - u_0^\pm\|_{L^2} \to 0$ as $j \to \infty$. Let $u^\pm$ solve

$$i\partial_t u^\pm + \frac{1}{2}\Delta u^\pm = |u^\pm|^{4/n} u^\pm \quad ; \quad u^\pm_{|t=0} = u_0^\pm.$$

We have

$$\begin{aligned}\|U_0(-t)u^\pm(t) - \widehat{\phi}\|_{L^2} \leqslant{} & \left\|U_0(-t)\left(u^\pm(t) - u_j^\pm(t)\right)\right\|_{L^2} + \|U_0(-t)u_j^\pm(t) - \widehat{\phi}_j\|_{L^2} \\ & + \|\phi_j - \phi\|_{L^2}.\end{aligned}$$

The global well-posedness for (1.1) in X implies

$$\limsup_{t\to\pm\infty} \|U_0(-t)u^\pm(t) - \widehat{\phi}\|_{L^2} \leqslant F\left(\|u_0^\pm - u_{0j}^\pm\|_{L^2}\right) + \|\phi_j - \phi\|_{L^2},$$

where F is a continuous function such that $F(0) = 0$. Finally, by letting $j \to \infty$, we see that $u^\pm$ solves

$$i\partial_t u^\pm + \frac{1}{2}\Delta u^\pm = |u^\pm|^{4/n} u^\pm \quad ; \quad U_0(-t)u^\pm(t)\big|_{|t=\pm\infty}\widehat{\phi}.$$

Let $\mathcal{V}$ be a neighborhood of ϕ in L^2. From [CW89], we see by Strichartz estimates and a bootstrap argument that the problem (1.1)–(1.2) is well-posed in $L^\infty(]-\infty, -T]; \mathcal{V})$ (we consider only the minus sign for simplicity) for some $T > 0$ possibly depending on $\mathcal{V}$. By uniqueness, we infer

$$\exists W_\pm \widehat{\phi} = u_0^\pm.$$

Since under our assumptions, $W_\mp^{-1}$ are homeomorphisms on X, we also have:

$$u_0^\pm = \lim_{j\to\infty} u_{0j}^\pm = \lim_{j\to\infty} \mathcal{F}W_\mp^{-1}\phi_j = \mathcal{F}W_\mp^{-1} \lim_{j\to\infty}\phi_j = \mathcal{F}W_\mp^{-1}\phi,$$

hence Corollary 1.2.

4. Proof of Corollary 1.5

Corollary 1.5 is a consequence of Theorem 1.1 and of the asymptotic expansion of the wave operators near the origin in L^2:

Proposition 4.1. *Let $n \geqslant 1$ and $\phi \in L^2(\mathbb{R}^n)$. Then for $\varepsilon > 0$ sufficiently small $W_\pm(\varepsilon^{n/4}\phi)$ and $W_\pm^{-1}(\varepsilon^{n/4}\phi)$ are well defined in $L^2(\mathbb{R}^n)$, and, as $\varepsilon \to 0$:*

$$W_\pm\left(\varepsilon^{\frac{n}{4}}\phi\right) = \varepsilon^{\frac{n}{4}}\phi \mp i\varepsilon^{1+\frac{n}{4}}\int_0^{\pm\infty} U_0(-t)\left(|U_0(t)\phi|^{4/n}U_0(t)\phi\right)dt + \mathcal{O}\left(\varepsilon^{2+\frac{4}{n}}\right),$$

$$W_\pm^{-1}\left(\varepsilon^{\frac{n}{4}}\phi\right) = \varepsilon^{\frac{n}{4}}\phi \pm i\varepsilon^{1+\frac{n}{4}}\int_0^{\pm\infty} U_0(-t)\left(|U_0(t)\phi|^{4/n}U_0(t)\phi\right)dt + \mathcal{O}\left(\varepsilon^{2+\frac{4}{n}}\right).$$

Proof. The proof follows from the same perturbative analysis as in [Gér96] (see also [Car01] for the nonlinear Schrödinger equation). First, it follows from [CW89] that $W_\pm(\varepsilon^{n/4}\phi)$ and $W_\pm^{-1}(\varepsilon^{n/4}\phi)$ are well defined in $L^2(\mathbb{R}^n)$ for $\varepsilon > 0$ sufficiently small.

We prove the asymptotic formula for the minus sign, since the proof of the formula for the plus sign is similar. Consider u^ε solving:

$$i\partial_t u^\varepsilon + \frac{1}{2}\Delta u^\varepsilon = |u^\varepsilon|^{4/n}u^\varepsilon \quad ; \quad U_0(-t)u^\varepsilon(t)\big|_{t=-\infty} = \varepsilon^{n/4}\phi.$$

Plugging an expansion of the form $u^\varepsilon = \varepsilon^{n/4}(\varphi_0+\varepsilon\varphi_1+\varepsilon r^\varepsilon)$ into the above equation, and ordering in powers of ε, it is natural to impose the following conditions:

- Leading order: $\mathcal{O}(\varepsilon^{n/4})$.

$$i\partial_t\varphi_0 + \frac{1}{2}\Delta\varphi_0 = 0 \quad ; \quad U_0(-t)\varphi_0(t)\big|_{t=-\infty} = \phi.$$

- First corrector: $\mathcal{O}(\varepsilon^{1+n/4})$.

$$i\partial_t\varphi_1 + \frac{1}{2}\Delta\varphi_1 = |\varphi_0|^{4/n}\varphi_0 \quad ; \quad U_0(-t)\varphi_1(t)\big|_{t=-\infty} = 0.$$

The first equation yields

$$\varphi_0(t) = U_0(t)\phi.$$

From the second equation, we have:

$$\varphi_1(t) = -i\int_{-\infty}^t U_0(t-s)\left(|\varphi_0(s)|^{4/n}\varphi_0(s)\right)ds.$$

We also have:

$$i\partial_t r^\varepsilon + \frac{1}{2}\Delta r^\varepsilon = G\left(\varphi_0+\varepsilon\varphi_1+\varepsilon r^\varepsilon\right) - G(\varphi_0) \quad ; \quad U_0(-t)r^\varepsilon(t)\big|_{t=-\infty} = 0,$$

where $G(z) = |z|^{4/n}z$. Let $\gamma = 2+4/n$, and denote $L^r_{t,x} = L^r(]-\infty,-t]\times\mathbb{R}^n)$. Strichartz and Hölder estimates yield

$$\begin{aligned}
\|r^\varepsilon\|_{L^\gamma_{t,x}} &\lesssim \left\|\left(|\varphi_0|^{4/n}+|\varepsilon\varphi_1|^{4/n}+|\varepsilon r^\varepsilon|^{4/n}\right)\varepsilon\left(|\varphi_1|+|r^\varepsilon|\right)\right\|_{L^{\gamma'}_{t,x}} \\
&\lesssim \left(\|\varphi_0\|^{4/n}_{L^\gamma_{t,x}}+\|\varepsilon\varphi_1\|^{4/n}_{L^\gamma_{t,x}}+\|\varepsilon r^\varepsilon\|^{4/n}_{L^\gamma_{t,x}}\right)\left(\varepsilon\|\varphi_1\|_{L^\gamma_{t,x}}+\|\varepsilon r^\varepsilon\|_{L^\gamma_{t,x}}\right) \\
&\lesssim \left(\|\phi\|^{4/n}_{L^2}+\|\varepsilon r^\varepsilon\|^{4/n}_{L^\gamma_{t,x}}\right)\left(\varepsilon\|\phi\|_{L^2}+\|\varepsilon r^\varepsilon\|_{L^\gamma_{t,x}}\right). \\
&\lesssim \varepsilon\|\phi\|^{1+4/n}_{L^2}+\|\varepsilon r^\varepsilon\|^{1+4/n}_{L^\gamma_{t,x}}.
\end{aligned}$$

A bootstrap argument shows that for $0 < \varepsilon \ll 1$, $r^\varepsilon \in L^\gamma(\mathbb{R}\times\mathbb{R}^n)$, and

$$\|r^\varepsilon\|_{L^\gamma(\mathbb{R}\times\mathbb{R}^n)} \lesssim \varepsilon.$$

Using Strichartz estimates again, we infer:

$$\|r^\varepsilon\|_{L^\infty(\mathbb{R};L^2(\mathbb{R}^n))} \lesssim \varepsilon.$$

Considering u^ε at time $t = 0$ yields the first part of the proposition. The second part can be proven in the same way, but can also be inferred from the first part *via* Neumann series, since $W_\pm$ are small perturbations of the identity near the origin. □

Now Corollary 1.5 follows from Corollary 1.2 and Proposition 4.1, where we identify the terms of order $\varepsilon^{1+4/n}$.

Remark 4.2. Considering the asymptotic expansion of the wave operators and their inverse to higher order would yield other formulae, similar to Corollary 1.5. We have not written them, for they are more intricate (they involve several integrations in time), and we do not know if they can be of some interest.

Appendix A. Sub-critical case

In this appendix, we consider more generally the nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = |u|^{2\sigma}u, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^n, \tag{A.1}$$

in the sub-critical case $\sigma < 2/n$. Following the approach to prove Corollary 1.5, we have:

Proposition A.1. *Let $\sigma < 2/n$, with*

- $\sigma > 1/n$ *if* $n \leqslant 2$.
- $\sigma > 2/(n+2)$ *if* $n \geqslant 2$.

Then the following identities hold for every $\phi \in \Sigma$:

$$\int_0^{\pm\infty} e^{it\frac{|x|^2}{2}}\mathcal{F}\left(|U_0(t)\phi|^{2\sigma}U_0(t)\phi\right)dt = \int_0^{\pm\infty}|t|^{n\sigma-2}U_0(t)\left(|U_0(-t)\widehat{\phi}|^{2\sigma}U_0(-t)\widehat{\phi}\right)dt,$$

$$\int_0^{\pm\infty}|t|^{n\sigma-2} e^{it\frac{|x|^2}{2}}\mathcal{F}\left(|U_0(t)\phi|^{2\sigma}U_0(t)\phi\right)dt = \int_0^{\pm\infty}U_0(t)\left(|U_0(-t)\widehat{\phi}|^{2\sigma}U_0(-t)\widehat{\phi}\right)dt.$$

Sketch of the proof. Let u solving (A.1). Then $v = \Psi u$ solves

$$i\partial_t v + \frac{1}{2}\Delta v = |t|^{n\sigma-2}|v|^{2\sigma}v, \quad (t,x)\in\mathbb{R}\setminus\{0\}\times\mathbb{R}^n. \tag{A.2}$$

It follows from [CW92] (see also [Caz03]) that wave operators exist, are continuous and invertible, near the origin in Σ, both for (A.1) and (A.2). We can then mimic the proof of Theorem 1.1, with the remark that in Theorem 1.1, the operators $W_\pm^{-1}$ on the left-hand side are associated to u, while the operators $W_\mp$ on the right-hand side are associated to v.

Adapting Proposition 4.1 to the cases of (A.1) and (A.2) proceeds along the same lines as the estimates in [CW92]. This yields the first identity in the above proposition.

For the second, we simply notice that $\Psi^2 = R$, so that we can exchange the roles of u and v. □

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