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# Atomic decomposition for the weighted Besov / Triebel-Lizorkin spaces with $A_p^{\mathrm{loc}}$ weights

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**Abstract**

The aim of this paper is to give a natural definition of the weighted Besov spaces and the weighted Triebel-Lizorkin spaces. The highlight of this paper is that we form an atomic decomposition for the class even wider than the class $A_p$ due to Muckenhoupt.

**Keywords** Besov space, Triebel-Lizorkin space, atom, decomposition of functions

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## 1 Introduction

The class $A_1$ is a class of weights that has been studied for a long time. The pioneer of this field is Muckenhoupt as is well-known [**?**]. However, in Riemannian geometry we need to deal with weights of exponential growth. That is, we often encounter weights such as $E(x) = e^{|x|}$. Since the $A_1$-condition $ME \le c\,E$ fails, we have to say that the $A_1$-condition is superfluous. In [?, ?] Lemarié and Rychkov introduced a wider class of weights called $A_1$-local class and generally $A_p$-local class. Historically speaking, the research has been done for the weight satisfying

$$0 < w(x) \le C\,\exp(C\,|x-y|^\beta)w(y) \tag{1}$$

for all $x, y \in \mathbb{R}^n$. When $0 < \beta < 1$, Schmeisser and Triebel investigated the weighted function space as above in [?]. If $\beta = 1$, then it was investigated by Schott [?]. The class we are going to investigate covers the one satisfying (??).

By "weight" we mean a locally integrable and positive function. Given a weight $w$, we write $w(Q) := \int_Q w(x)\,dx$. Before we present the definition of the $A_p^{\rm loc}$-weights, let us make a view of the classical $A_1$-weights. Let $M$ be the Hardy-Littlewood maximal operator defined by

$$Mf(x) := \sup_{\substack{Q\,:\,\text{cube}\\ x\in Q}} \frac{1}{|Q|}\int_Q |f(y)|\,dy.$$

Let $1 \le p < \infty$. The $A_p$-constant is a constant given for a weight $w$ which is defined as

$$A_p(w) := \sup_{Q\,:\,\text{cube}} \left(\frac{w(Q)}{|Q|}\right)\cdot\left(\frac{w^{-\frac{1}{p-1}}(Q)}{|Q|}\right)^{\frac{1}{p-1}}$$

when $1 < p < \infty$ and

$$A_p(w) := \operatorname{esssup}_{x\in\mathbb{R}^n}\frac{Mw(x)}{w(x)}$$

when $p = 1$. The $A_p$-class is a class of weights $w$ characterized by the condition

$$\int_{\mathbb{R}^n} Mf(x)^p w(x)\,dx \le c_w \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx, \ \text{ for all } f$$

when $1 < p < \infty$ and

$$\int_{\{Mf>\lambda\}} w(x)\,dx \le \frac{c_w}{\lambda}\int_{\mathbb{R}^n} |f(x)| w(x)\,dx, \ \text{ for all } f \text{ and } \lambda > 0$$

when $p = 1$. Here the constant $c_w$ essentially depends on $n$, $p$ and the $A_p$-constant of $w$. In fact Buckley [?] gave the sharp estimation $c_w \le C_{n,p}\,A_p(w)^{\frac{p}{p-1}}$ when $1 < p < \infty$, where $C_{n,p}$ is a constant depending only on $n$ and $p$. Let $0 < p < \infty$. Then it is convenient that we introduce

$$\|f\,:\,L^w_p\| := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{\frac{1}{p}}.$$

Recall that we have defined $A_\infty = \bigcup_{1\le p<\infty} A_p$ in the classical case. For the property of these classes of weights we refer to [?].

The class $A_p^{\rm loc}$ deals with the local maximal operators whose definition is given by

$$M_{\le r}f(x) := \sup_{\substack{Q\,:\,\text{cube}\\ x\in Q,\,\ell(Q)\le r}} \frac{1}{|Q|}\int_Q |f(y)|\,dy,\ r\ge 1,$$

where $\ell(Q)$ means the side length of $Q$. For the sake of simplicity we write $M_{\rm loc} := M_{\le 1}$. We say that a weight $w$ satisfies the $A_1^{\rm loc}$-condition, if

$$A_1^{\rm loc}(w) := \sup_{x\in\mathbb{R}^n} \frac{M_{\rm loc}w(x)}{w(x)} < \infty.$$

Let $1<p<\infty$. A weight $w$ satisfies the $A_p^{\rm loc}$-condition, if

$$A_p^{\rm loc}(w) := \sup_{\substack{Q\,:\,\text{cube}\\ x\in Q,\,\ell(Q)\le 1}} \left(\frac{w(Q)}{|Q|}\right)\cdot\left(\frac{w^{-\frac{1}{p-1}}(Q)}{|Q|}\right)^{p-1} < \infty.$$

Let $1\le p<\infty$. Below we denote by $A_p^{\rm loc}$ the set of all weights $w$ for which the $A_p^{\rm loc}$-constant $A_p^{\rm loc}(w)$ is finite. As is the case in the classical theory, define

$$A_\infty^{\rm loc} := \bigcup_{1\le p<\infty} A_p^{\rm loc}.$$

From the definition it is easy to see

$$A_1^{\rm loc}\subset A_p^{\rm loc}\subset A_q^{\rm loc}\subset A_\infty^{\rm loc},$$

whenever $1\le p\le q<\infty$.

The aim of this paper is to form the atomic decomposition of the functions in the Besov spaces and the Triebel-Lizorkin spaces weighted by an $A_\infty^{\rm loc}$-weight. Let us recall the definition of the non-weighted Besov spaces and the non-weighted Triebel-Lizorkin spaces. We refer to [**?**] for exhaustive details. Following the notation in [**?**], we define $\mathbb{N}_0 := \{0,1,2,\ldots\}$. Let $\tau_0,\tau\in\mathcal{S}$ be the functions satisfying

$$\chi_{B(2)}\le\tau_0\le\chi_{B(4)} \quad\text{and}\quad \chi_{B(4)\setminus B(2)}\le\tau\le\chi_{B(8)\setminus B(1)},$$

where $B(r) := \{y\in\mathbb{R}^n : |y_1|^2+|y_2|^2+\ldots+|y_n|^2<r^2\}$. Here and below we denote by $\chi_A$ the indicator function of a set $A\subset\mathbb{R}^n$. For $j\in\mathbb{N}$ we set $\tau_j(x) := \tau(2^{-j+1}x)$. Let $0<p<\infty, 0<q\le\infty$ and $s\in\mathbb{R}$. Then we define

$$\|f\,:\,B_{p,q}^s\| := \left(\sum_{j\in\mathbb{N}_0} 2^{jsq}\|\mathcal{F}^{-1}(\tau_j\cdot\mathcal{F}f)\,:\,L_p\|^q\right)^{\frac{1}{q}}$$

$$\|f\,:\,F_{p,q}^s\| := \left\|\left(\sum_{j\in\mathbb{N}_0} 2^{jsq}|\mathcal{F}^{-1}(\tau_j\cdot\mathcal{F}f)|^q\right)^{\frac{1}{q}}\,:\,L_p\right\|.$$

Therefore it seems natural that we define the norms by

$$\|f\,:\,B^{s,w}_{p,q}\| = \left(\sum_{j\in\mathbb{N}_0} 2^{jsq}\|\mathcal{F}^{-1}(\tau_j\cdot\mathcal{F}f)\,:\,L^w_p\|^q\right)^{\frac{1}{q}}$$
$$\|f\,:\,F^{s,w}_{p,q}\| = \left\|\left(\sum_{j\in\mathbb{N}_0} 2^{jsq}|\mathcal{F}^{-1}(\tau_j\cdot\mathcal{F}f)|^q\right)^{\frac{1}{q}}\,:\,L^w_p\right\|$$

for $f\in\mathcal{S}'$ under the same notation. However, this definition does not work. The primary concern about this definition is that the definition is independent of the admissible choice of $\tau$ and $\tau_0$. This is an obstacle in our definition above. Therefore, to overcome this difficulty we need to alter our point of view.

Furthermore, it is not enough to guarantee the completeness of the function spaces, if we stick to the Schwartz distribution $\mathcal{S}'$. We need to consider $\mathcal{S}_e$ and $\mathcal{S}'_e$ instead. $\mathcal{S}_e$ is a set defined as follows:

$$\mathcal{S}_e := \{\phi\in C^\infty\,:\,q_N(\phi)<\infty \text{ for all } N\in\mathbb{N}\},$$

where the seminorm $q_N$ is given by

$$q_N(\phi) := \sup_{\substack{\alpha\in\mathbb{N}_0\\ |\alpha|\le N}}\left(\sup_{x\in\mathbb{R}^n} e^{N|x|}|\partial^\alpha\phi(x)|\right).$$

The family $\{q_N\}_{N\in\mathbb{N}}$ topologizes $\mathcal{S}_e$. Then $\mathcal{S}_e$ carries a structure of a locally convex space whose topology is induced by the family $\{q_N\}_{N\in\mathbb{N}}$. $\mathcal{S}'_e$ is defined as the topological dual of $\mathcal{S}_e$.

The key idea by Rychkov is that we use the local means in [**?**] and considered the subspace as a subset of $\mathcal{S}'_e$. By way of the local means, Rychkov succeeded in proving the validness of the seminorms. If we place ourselves in his framework of $\mathcal{S}'_e$, then we are able to prove that the function spaces are complete.

Now let us present the correct definition due to Rychkov. Below for $s\in\mathbb{R}$ we denote by $[s]$ the largest integer that does not exceed $s$.

**Definition 1.1.** [**?**] Suppose that the parameters $p,q$ and $s$ satisfy

$$0<p<\infty,\ 0<q\le\infty \text{ and } s\in\mathbb{R}.$$

Let $\phi_0\in\mathcal{S}$ with

$$\chi_{B(1)}\le\phi_0\le\chi_{B(2)}.$$

For $j\in\mathbb{N}$, we now set

$$\psi_j(x) := 2^{jn}\psi(2^jx)-2^{(j-1)n}\psi(2^{j-1}x).$$

Assume that $\phi_0$ is taken so that $L_{\phi_1} \geq [s]$. Let $f \in \mathcal{S}'_e$. Then define

$$\|f \,:\, B^{s,w}_{p,q}\| := \left( \sum_{j \in \mathbb{N}_0} 2^{jsq} \|\phi_j * f \,:\, L^w_p\|^q \right)^{\frac{1}{q}}$$

$$\|f \,:\, F^{s,w}_{p,q}\| := \left\| \left( \sum_{j \in \mathbb{N}_0} 2^{jsq} |\phi_j * f|^q \right)^{\frac{1}{q}} \,:\, L^w_p \right\|.$$

$B^{s,w}_{p,q}$ and $F^{s,w}_{p,q}$ are the sets of all elements $f \in \mathcal{S}'_e$ such that $\|f \,:\, B^{s,w}_{p,q}\|$ and $\|f \,:\, F^{s,w}_{p,q}\|$ are finite respectively.

Having set down the definition of the weighted function spaces, we now describe the organization of this paper. In Section **??** we collect some results of weights and the maximal functions as well as the ones from [**?**]. In Section **??** we formulate and prove the atomic decomposition, which is a heart in this paper. As an application we reconsider local means discussed in [**?**]. As a corollary of the main theorem, we give a simple but helpful remark on the local means in Section **??**. Finally Section **??** is oriented to applications, the multiplier result and the diffeomorphic properties.

# 2 Preliminaries

In this section we make a review of the class of $A^{\mathrm{loc}}_p$-weights and the maximal operators.

## 2.1 The weight class $A^{\mathrm{loc}}_\infty$

As we have seen in Introduction, the definition of $A^{\mathrm{loc}}_p$ can be obtained by replacing "$\sup\limits_{\substack{Q\,:\,\mathrm{cube} \\ x \in Q}}$" by "$\sup\limits_{\substack{Q\,:\,\mathrm{cube} \\ x \in Q,\, \ell(Q) \leq 1}}$". We begin with the relation of $A^{\mathrm{loc}}_p(w)$ and the maximal operator $M_{\leq r}$.

**Proposition 2.1.** [**?**, **?**] *Let $1 < p < \infty$ and $w \in A^{\mathrm{loc}}_p$. Then there exist two constants $c_0 = c_0(p,n) > 0$ and $c_1 = c_1\left(p, n, A^{\mathrm{loc}}_p(w)\right) > 0$ such that*

$$\int_{\mathbb{R}^n} M_{\leq r} f(x)^p w(x)\,dx \leq c_0\, r^{-\frac{np^2}{p-1}} \exp(c_1\, r^n) A^{\mathrm{loc}}_p(w)^{\frac{p}{p-1}} \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx \qquad (2)$$

*for all $f$ with $\|f : L^w_p\| < \infty$ and $r \geq 1$.*

Proposition **??** was essentially proved by Rychkov [Ry]. As is mentioned, Buckley gave the sharp estimate of the operator norm of $M$. By virtue of Buckley's result, we can obtain the present form.

It might be helpful to see that $e^{c|x|} \in A_p^{\rm loc} \setminus A_p$ for $c > 0$ and $1 \le p \le \infty$. Next, keeping this example in mind, let us see the speed of the growth. Rychkov essentially showed the following proposition. Applying Proposition **??**, we can improve the estimate of the constant in the right-hand side of inequality (**??**) in Proposition **??**.

For a cube $Q$ and $r > 0$, $r\,Q$ denotes the cube concentric with $Q$ and such that $l(r\,Q) = r\,l(Q)$.

**Proposition 2.2.** [?] *Let $w \in A_p^{\rm loc}$ with $p > 1$. Then we have that for all $r \ge 1$ and all cubes $Q$ with $\ell(Q) \le 1$,*

$$w(r\,Q) \le c_0\, r^{-\frac{np}{p-1}} \exp(c_1\, r^n)\, A_p^{\rm loc}(w)^{\frac{p}{p-1}}\, w(Q),$$

*where $c_0, c_1 > 0$ are the constants appearing in Proposition* **??**.

*Proof.* By virtue of the definition of $M_{\le r} f$ and Proposition **??**, we have

$$\begin{aligned} w(r\,Q) &= \int_{\mathbb{R}^n} \chi_{r\,Q}(x)^p\, w(x)\, dx \\ &\le \int_{\mathbb{R}^n} \{r^n\, M_{\le r}(\chi_Q)(x)\}^p\ w(x)\, dx \\ &\le c_0\, r^{-\frac{np}{p-1}} \exp(c_1\, r^n)\, A_p^{\rm loc}(w)^{\frac{p}{p-1}} \int_{\mathbb{R}^n} \chi_Q(x)^p\, w(x)\, dx \\ &= c_0\, r^{-\frac{np}{p-1}} \exp(c_1\, r^n)\, A_p^{\rm loc}(w)^{\frac{p}{p-1}}\, w(Q). \end{aligned}$$

Thus, the proof is now complete. □

The following proposition is a tool that allows us to reduce the matter to the theory of $A_\infty$-weights.

**Proposition 2.3.** [?] *Let $Q$ be a cube with $\ell(Q) = 1$ and $w \in A_p^{\rm loc}$ with $1 < p < \infty$. Then there exists $\overline{w} \in A_p$ such that*

$$A_p(\overline{w}) \le c\, A_p^{\rm loc}(w) \text{ and } \overline{w} = w \text{ on } Q.$$

## 2.2 Local reproducing formula

In this section we consider the local reproducing formula.

Given a function $g$ on $\mathbb{R}^n$, $L_g$ denotes the maximal number such that $g$ has vanishing moments up to order $L_g$, i.e., $\displaystyle\int_{\mathbb{R}^n} x^\alpha g(x)\, dx = 0$ for all $\alpha \in \mathbb{N}_0{}^n$ with $|\alpha| \le L_g$. If no moments of $g$ vanish, then we put $L_g := -1$. Rychkov proved the following reproducing formula.

**Proposition 2.4.** [?] *Suppose that* $\phi_0 \in C_c^\infty$ *with* $\int \phi_0 \neq 0$. *Set*

$$\phi_j(x) := 2^{jn}\phi_0(2^j x) - 2^{(j-1)n}\phi_0(2^{j-1}x)$$

*for* $j \in \mathbb{N}$. *Let* $L \in \mathbb{N}$. *Then there exists* $\psi_0 \in C_c^\infty$ *such that* $L_{\psi_1} \geq L$ *and*

$$f = \sum_{j\in\mathbb{N}_0} \psi_j * \phi_j * f \ \text{ in } \mathcal{S}_e' \tag{3}$$

*for all* $f \in \mathcal{S}_e'$. *Here we have set* $\psi_j(x) := 2^{jn}\psi_0(2^j x) - 2^{(j-1)n}\psi_0(2^{j-1}x)$ *for* $j \in \mathbb{N}$.

In [?] Rychkov actually proved (**??**) for $f \in D'$, where the convergence takes place in $D'$. In this paper we have converted the assertion to the form of our convenience.

### 2.3 Maximal estimates

Next, we consider the maximal estimate. It was helpful to use Proposition **??** when we prove that the definition of the non-weighted Besov spaces $B^s_{p,q}$ and the non-weighted Triebel-Lizorkin spaces $F^s_{p,q}$ do not depend on the choice of $\tau_0$ and $\tau_1$ in Introduction. It is helpful to introduce a notation. For a measurable function $f$, $1 \leq r < \infty$ and $0 < \eta < \infty$ we define

$$M^{(\eta)}f(x) := M\left(|f|^\eta\right)(x)^{\frac{1}{\eta}},\ M^{(\eta)}_{\leq r}f(x) := M_{\leq r}\left(|f|^\eta\right)(x)^{\frac{1}{\eta}} \text{ and } M^{(\eta)}_{\mathrm{loc}}f := M^{(\eta)}_{\leq 1}f.$$

Let $0 < \eta < \infty$, $E$ a measurable set with $|E| > 0$, and $f$ a positive function. Then we write

$$m^{(\eta)}_E(f) := \left(\frac{1}{|E|}\int_E f^\eta\right)^{\frac{1}{\eta}}.$$

**Proposition 2.5.** [?] *Let* $f \in \mathcal{S}'$ *be a distribution such that the support of its Fourier transform is engulfed by a ball of radius* $r > 0$. *Then for* $0 < \eta < \infty$ *there exists* $c_\eta$ *independent of* $f$ *and* $r$ *such that*

$$\sup_{y\in\mathbb{R}^n} \frac{|f(x-y)|}{1+|ry|^{\frac{n}{\eta}}} \leq c_\eta\, M^{(\eta)}f(x).$$

However, Proposition **??** does not work in this present situation. Because we are convoluting functions with compact support, Proposition **??** does not do. Instead of Proposition **??** we use the following maximal operator and the key estimate.

**Definition 2.6.** [?] Let $f \in \mathcal{S}_e'$, $\nu \in \mathbb{N}_0$ and $A, B \geq 0$. Then define

1. $m_{\nu,A,B}(y) := (1+2^\nu|y|)^A\, 2^{B|y|}$
2. $\phi^*_{\nu,A,B}f(x) := \sup_{y\in\mathbb{R}^n} \dfrac{|\phi_\nu * f(x-y)|}{m_{\nu,A,B}(y)}$.

Instead of Propsoition **??**, we use the following pointwise estimate. To formulate our desired pointwise estimate, we need to introduce another definition. Let $f$ be a locally integrable function and $B > 0$. Then define

$$K_B f(x) := \int_{\mathbb{R}^n} e^{-B|x-y|} f(y)\, dy,$$

provided the integral converges. Here and below we pick $\phi_0$ such that $\displaystyle\int \phi_0 \neq 0$. Define

$$\phi_j(x) := 2^{jn}\phi_0(2^j x) - 2^{(j-1)n}\phi_0(2^{j-1}x)$$

for $j \in \mathbb{N}$. Keeping to the notations above, Rychkov proved the following.

**Proposition 2.7.** [?, Lemma 2.10] *Let $0 < \eta < \infty$. Then there exists a constant $c > 0$ such that*

$$\phi^*_{\nu,A,B} f(x)^\eta \le c \sum_{k=\nu}^{\infty} 2^{(\nu-k)(A\eta-n)} \left\{ M^{(\eta)}_{\mathrm{loc}}[\phi_k * f](x)^\eta + K_{B\eta}[|\phi_k * f|^\eta](x) \right\},$$

*provided $A > n/\eta$ and $B \ge 0$.*

Finally let us see the estimates of the maximal operator and the operator $K_B$.

**Definition 2.8** (Weighted vector-valued norm)**.** Let $w \in A^{\mathrm{loc}}_\infty$, $0 < p < \infty$, $0 < q \le \infty$ and $\{f_j\}_{j\in\mathbb{N}_0}$ a sequence of measurable functions. Then define

$$\|\{f_j\}_{j\in\mathbb{N}_0} \,:\, l_q(L^w_p)\| := \left( \sum_{j\in\mathbb{N}_0} \|f_j \,:\, L^w_p\|^q \right)^{\frac{1}{q}}$$

$$\|\{f_j\}_{j\in\mathbb{N}_0} \,:\, L^w_p(l_q)\| := \left\| \left( \sum_{j\in\mathbb{N}_0} |f_j|^q \right)^{\frac{1}{q}} \,:\, L^w_p \right\|.$$

Under this notation our key estimates can be formulated as follows:

**Proposition 2.9.** [?, Lemma 2.10] *Let $0 < p < \infty, 0 < q \le \infty, 1 \le u < \infty$, $w \in A^{\mathrm{loc}}_u$, $\{f_j\}_{j\in\mathbb{N}_0}$ a sequence of measurable functions and $f \in \mathcal{S}'_e$.*

1. *Assume that $0 < \eta < \min\left(1, \dfrac{p}{u}\right)$. Then there exists a constant $c > 0$ independent of $\{f_j\}_{j\in\mathbb{N}_0}$ such that*

$$\left\| \left\{ M^{(\eta)}_{\mathrm{loc}} f_j \right\}_{j\in\mathbb{N}_0} \,:\, l_q(L^w_p) \right\| \le c \left\| \{f_j\}_{j\in\mathbb{N}_0} \,:\, l_q(L^w_p) \right\|, \tag{4}$$

$$\left\| \left\{ M^{(\eta)}_{\mathrm{loc}} f_j \right\}_{j\in\mathbb{N}_0} \,:\, L^w_p(l_q) \right\| \le c \left\| \{f_j\}_{j\in\mathbb{N}_0} \,:\, L^w_p(l_q) \right\|. \tag{5}$$

2. *Assume that $B$ is sufficiently large. Then there exists a constant $c > 0$ independent of $\{f_j\}_{j\in\mathbb{N}_0}$ such that*

$$\left\|\{K_B f_j\}_{j\in\mathbb{N}_0} : l_q(L_p^w)\right\| \le c \left\|\{f_j\}_{j\in\mathbb{N}_0} : l_q(L_p^w)\right\|,$$
$$\left\|\{K_B f_j\}_{j\in\mathbb{N}_0} : L_p^w(l_q)\right\| \le c \left\|\{f_j\}_{j\in\mathbb{N}_0} : L_p^w(l_q)\right\|.$$

3. *Assume that $\phi_0$ satisfies $\int \phi_0 \neq 0$. Set $\phi_j(x) := 2^{jn}\phi_0(2^j x) - 2^{(j-1)n}\phi_0(2^{j-1}x)$ for $j \in \mathbb{N}$. Then there exists a constant $c > 0$ independent of $f$ such that*

$$\left\|2^{\nu s}\phi^*_{\nu,A,B} f : l_q(L_p^w)\right\| \le c \left\|2^{\nu s}\phi_\nu * f : l_q(L_p^w)\right\|, \tag{6}$$
$$\left\|2^{\nu s}\phi^*_{\nu,A,B} f : L_p^w(l_q)\right\| \le c \left\|2^{\nu s}\phi_\nu * f : L_p^w(l_q)\right\|. \tag{7}$$

By using Propositions **??** and **??**, Rychkov proved that the definitions of the function spaces $B^{s,w}_{p,q}$ and $F^{s,w}_{p,q}$ do not depend on the choice of $\phi_0$.

# 3 Atomic decomposition

Having set down the preliminaries, we now formulate the atomic decomposition theorem. Let us begin by fixing the notation of the dyadic cubes. Let $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. Then we adopt the definition below to define the dyadic cube $Q_{\nu m}$.

$$Q_{\nu m} := \prod_{j=1}^{n} \left[\frac{m_j}{2^\nu}, \frac{m_j+1}{2^\nu}\right].$$

Let us denote by $\|f : C\|$ the sup-norm of a continuous funciton $f$.

**Definition 3.1.** [?] Let

$$d, p, s \in \mathbb{R},\ K, L \in \mathbb{Z},\ d > 1,\ 0 < p < \infty,\ K \ge 0 \text{ and } L \ge -1.$$

1. Let $m \in \mathbb{Z}^n$. $a \in C^K$ is said to be an atom centered at $Q_{0m}$, if $a$ is supported on $d\,Q_{0m}$ and the following differential inequality is fulfilled:

$$\|\partial^\alpha a : C\| \le 1 \text{ for all } \alpha \in {\mathbb{N}_0}^n \text{ with } |\alpha| \le K.$$

2. Let $\nu \in \mathbb{N}$ and $m \in \mathbb{Z}^n$. $a \in C^K$ is said to be an atom centered at $Q_{\nu m}$, if $a$ is supported on $d\,Q_{\nu m}$ and $a$ satisfies the following moment condition as well as the differential inequality as above:

$$\|\partial^\alpha a : C\| \le 2^{-\nu\left(s-\frac{n}{p}\right)+\nu|\alpha|} \text{ for all } \alpha \in {\mathbb{N}_0}^n \text{ with } |\alpha| \le K$$

and

$$\int_{\mathbb{R}^n} x^\beta a(x)\,dx = 0 \text{ for all } \beta \in {\mathbb{N}_0}^n \text{ with } |\beta| \le L. \tag{8}$$

If $L = -1$, then it will be understood that the moment condition (**??**) above is empty.

3. Define

$$\mathbf{Atom}_0 := \{\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^n} \;:\; \text{each } a_{\nu m} \text{ is an atom centered at } Q_{\nu m}\}$$

and

$$\mathbf{Atom} := \{\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^n} \;:\; \{c_0\, a_{\nu m}\}_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^n} \in \mathbf{Atom}_0 \text{ for some } c_0 > 0\}\,.$$

Next, we give two sequence spaces to describe the condition of the coefficients.

**Definition 3.2.** Given a doubly indexed sequence $\lambda = \{\lambda_{\nu m}\}_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^n}$, define

$$\|\lambda \,:\, b^w_{p,q}\| := \left\| \left\{ \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\chi^{(p)}_{\nu m} \right\}_{\nu\in\mathbb{N}_0} \,:\, l_q(L^w_p) \right\|$$

$$\|\lambda \,:\, f^w_{p,q}\| := \left\| \left\{ \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\chi^{(p)}_{\nu m} \right\}_{\nu\in\mathbb{N}_0} \,:\, L^w_p(l_q) \right\|.$$

Here $\chi^{(p)}_{\nu m} := 2^{\frac{\nu n}{p}}\chi_{Q_{\nu m}}$. It is convenient to write $a^w_{p,q}$ to denote either $b^w_{p,q}$ or $f^w_{p,q}$.

We are to show that any element in $A^{s,w}_{p,q}$ can be decomposed into the sum given by

$$\sum_{\nu\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}a_{\nu m}.$$

Let us begin by showing that the sum makes sense at least in $\mathcal{S}'_e$, provided $K$ and $L$ in Definition **??** are large enough. Below following [**?**] for example, let us set

$$\sigma_{pq} := \frac{n}{\min(1,p,q)} - n \quad \text{and} \quad \sigma_p := \sigma_{pp}.$$

**Lemma 3.3.** *Suppose that the parameters $p, q$ and $s$ satisfy*

$$0 < p < \infty,\ 0 < q \le \infty \text{ and } s \in \mathbb{R}.$$

*Assume that $w \in A^{\mathrm{loc}}_u$ with $1 \le u < \infty$. Let $K, L \in \mathbb{Z}$ satisfy*

$$K \ge (1+[s])_+ \quad \text{and} \quad L \ge \max\left(-1, [\sigma_p - s + (u-1)n]\right).$$

*Let $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^n} \in \mathbf{Atom}_0$ and $\lambda = \{\lambda_{\nu m}\}_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^n} \in a^w_{p,q}$. Then the series*

$$\lim_{P\to\infty}\sum_{\nu=0}^{P}\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}a_{\nu m}$$

*converges in $\mathcal{S}'_e$.*

*Proof.* By virtue of the Minkowski inequality it is trivial that

$$l_{\min(p,q)}(L_p^w) \subset L_p^w(l_q) \subset l_{\max(p,q)}(L_p^w)$$

in the sense of the continuous embedding. Therefore, let us assume that $a = b$, that is, we concentrate on the case of Besov type.

First let us check that the sum

$$\sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}$$

converges in the topology of $\mathcal{S}_e'$ for every fixed $\nu \in \mathbb{N}_0$. To check this, we note

$$|\lambda_{\nu m}| \le c\, w(Q_{\nu m})^{-1/p}.$$

Now we invoke Proposition **??**. Then we obtain

$$w(Q_{\nu m}) \ge c \exp(-N_0\, 2^{-\nu}|m|) w(2^{\nu+2}|m| Q_{\nu m}) \ge c \exp(-N_0\, 2^{-\nu}|m|) w(Q_{\nu 0}),$$

where $N_0$ and $c$ are numbers depending on the weight $w$. Therefore, the coupling

$$\left\langle \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}, \phi \right\rangle = \sum_{m \in \mathbb{Z}^n} \int_{\mathbb{R}^n} \lambda_{\nu m} a_{\nu m}(x) \phi(x)\, dx$$

converges absolutely for all $\phi \in \mathcal{S}_e$.

In view of this we have only to prove

$$\lim_{P \to \infty} \sum_{\nu=0}^{P} \left( \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \right)$$

converges in $\mathcal{S}_e'$.

Pick a test function $\phi \in \mathcal{S}_e$ again. Then by virtue of the moment condition we have

$$\left\langle \sum_{\nu=0}^{P} \left( \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \right), \phi \right\rangle = \sum_{\nu=0}^{P} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \int_{\mathbb{R}^n} a_{\nu m}(x) \phi_{\nu m}(x)\, dx,$$

where $\phi_{\nu m}$ is given by

$$\phi_{\nu m}(x) := \phi(x) - \left( \sum_{|\beta| \le L} \frac{\partial^\beta \phi(2^{-\nu} m)}{\beta!} (x - 2^{-\nu} m)^\beta \right).$$

By the mean value theorem we have

$$|\phi_{\nu m}(x)| \le c\, 2^{-\nu(L+1)} \left( \sup_{\substack{|\gamma| = L+1 \\ y \in d\, Q_{\nu m}}} |\partial^\gamma \phi(y)| \right)$$

for $x \in d\,Q_{\nu m}$. Thus, the pointwise estimate $|a_{\nu m}(x)| \le 2^{-\nu(s-n/p)},\ x \in \mathbb{R}^n$ yields

$$
\begin{aligned}
e^{N|x|}|a_{\nu m}(x)\phi_{\nu m}(x)| &\le c\,2^{-\nu\left(s-\frac{n}{p}+L+1\right)} \left( \sup_{\substack{|\gamma|=L+1 \\ y \in d\,Q_{\nu m}}} e^{N|y|}|\partial^\gamma \phi(y)| \right) \chi_{d\,Q_{\nu m}}(x) \\
&\le c\,2^{-\nu\left(s-\frac{n}{p}+L+1\right)} q_N(\phi) \chi_{d\,Q_{\nu m}}(x).
\end{aligned}
$$

Here $N$ is a constant chosen as large as we wish. Below let us assume that $N$ is sufficiently large, say, $N \gg 1$. Adding the estimate above over $m \in \mathbb{Z}^n$, we obtain

$$
\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m} a_{\nu m}(x)\phi_{\nu m}(x)| \le c\,2^{-\nu\left(s-\frac{n}{p}+L+1\right)} q_N(\phi) \cdot e^{-N|x|} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| \chi_{d\,Q_{\nu m}}(x).
$$

Let us write $Q(r) := \{x \in \mathbb{R}^n \,:\, \max(|x_1|, |x_2|, \dots, |x_n|) \le r\}$. Inserting this pointwise estimate, we have

$$
\begin{aligned}
&\sum_{m \in \mathbb{Z}^n} \int_{\mathbb{R}^n} |\lambda_{\nu m} a_{\nu m}(x)\phi_{\nu m}(x)|\,dx \\
&\le c\,2^{-\nu\left(s-\frac{n}{p}+L+1\right)} q_N(\phi) \int_{\mathbb{R}^n} e^{-N|x|} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| \chi_{d\,Q_{\nu m}}(x)\,dx \\
&\le c\,2^{-\nu\left(s-\frac{n}{p}+L+1\right)} q_N(\phi) \sum_{k=0}^{\infty} \exp(-2^{k-1}N) \int_{Q(2^k)} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| \chi_{d\,Q_{\nu m}}(x)\,dx \\
&\le c\,2^{-\nu\left(s-\frac{n}{p}+n+L+1\right)} q_N(\phi) \sum_{k=0}^{\infty} \exp(-2^{k-1}N) \int_{Q(2^k)} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi^{(1)}_{\nu m}(x) \right| dx.
\end{aligned}
$$

Here the constant $c > 0$ depends on $N$. We define two auxiliary constants $0 < \eta < 1$ and $1 < \mu < \infty$ by

$$
\eta := \frac{p}{1+(u-1)p} \quad \text{and} \quad \mu := \frac{p}{\eta} = 1 + (u-1)p.
$$

Denote by $\mu'$ the harmonic conjugate of $\mu$: $\mu' = \dfrac{\mu}{\mu-1}$. Then we have

$$
\eta\mu' = \frac{\eta\mu}{\mu-1} = \frac{1}{u-1}.
$$

Keeping this in mind, we estimate the integral in question:

$$
\begin{aligned}
\int_{Q(2^k)} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi^{(1)}_{\nu m}(x) \right| dx &\le \left( \int_{Q(2^k)} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi^{(\eta)}_{\nu m}(x) \right|^\eta dx \right)^{\frac{1}{\eta}} \\
&= 2^{-\frac{n\nu}{p}+\frac{n\nu}{\eta}} \left( \int_{Q(2^k)} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi^{(p)}_{\nu m}(x) \right|^\eta dx \right)^{\frac{1}{\eta}}.
\end{aligned}
$$

Applying the Hölder inequality, we obtain

$$\begin{aligned}
&\int_{Q(2^k)} \left| \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} \chi^{(1)}_{Q_{\nu m}}(x) \right| dx \\
&\le 2^{-\frac{n\nu}{p}+\frac{n\nu}{\eta}} \left( \int_{Q(2^k)} w(x)^{-\eta} \left| \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} \chi^{(p)}_{\nu m}(x) \right|^{\eta} w(x)^{\eta}\, dx \right)^{\frac{1}{\eta}} \\
&\le 2^{-\frac{n\nu}{p}+\frac{n\nu}{\eta}} \left( \int_{Q(2^k)} w(x)^{-\eta\mu'}\, dx \right)^{\frac{1}{\mu'}} \left( \int_{Q(2^k)} \left| \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} \chi^{(p)}_{Q_{\nu m}}(x) \right|^{p} w(x)\, dx \right)^{\frac{1}{p}}.
\end{aligned}$$

Since $w^{-\eta\mu'} = w^{-\frac{1}{u-1}} \in A^{\rm loc}_{u'}$, we see

$$\int_{Q(2^k)} w(x)^{-\eta\mu'}\, dx \le c\, \exp(N_{p,\eta,u,w}\cdot 2^k) \int_{Q(1)} w(x)^{-\frac{1}{u-1}}\, dx \le c\, \exp(N_{p,\eta,u,w}\cdot 2^k),$$

where $c$ and $N_{p,\eta,u,w}$ depend on $p,\eta,u$ and the $A^{\rm loc}_u$-constant of $w$. Recall that $N$ is at our disposal. Thus, we finally obtain

$$\sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |\lambda_{\nu m} a_{\nu m}(x) \phi_{\nu m}(x)|\, dx \le c\, 2^{-\nu\left(s-\frac{n}{\eta}+n+L+1\right)} q_N(\phi) \left\| \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \,:\, L^w_p \right\|.$$

Now by the assumption, $L$ is sufficiently large:

$$s - \frac{n}{\eta} + n + L + 1 > \sigma_p + un - \frac{n}{\eta} = \sigma_p - n\left(\frac{1}{p} - 1\right) \ge 0.$$

Thus, we are in the position of adding these inequalities over $\nu \in \mathbb{N}_0$:

$$\sum_{\nu=0}^{\infty} \left| \int_{\mathbb{R}^n} \left( \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x) \right) \phi(x)\, dx \right| \le c\, q_N(\phi)\, \|\lambda \,:\, b^w_{p,q}\|. \tag{9}$$

This proves

$$\lim_{P\to\infty} \sum_{\nu=0}^{P} \left( \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \right)$$

exists in $\mathcal{S}'_e$. □

**Theorem 3.4.** *Suppose that the parameters $p, q$ and $s$ satisfy*

$$0 < p < \infty,\ 0 < q \le \infty \text{ and } s \in \mathbb{R}.$$

*Assume that $w \in A^{\rm loc}_u$ with $1 \le u < \infty$. Let $K, L \in \mathbb{Z}$ satisfy*

$$K \ge (1+[s])_+ \text{ and } L \ge \max\left(-1, \left[\sigma_{\frac{p}{u}} - s + (u-1)n\right]\right)$$

*when $A^{s,w}_{p,q}$ denotes $B^{s,w}_{p,q}$ and*

$$K \ge (1+[s])_+ \text{ and } L \ge \max\left(-1, \left[\sigma_{\frac{p}{u}\frac{q}{u}} - s + (u-1)n\right]\right)$$

*when $A^{s,w}_{p,q}$ denotes $F^{s,w}_{p,q}$.*

1. *Let $f \in A^{s,w}_{p,q}$. Then there exist $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^n} \in \mathbf{Atom}_0$ and $\lambda \in a^w_{p,q}$ such that*

$$f = \sum_{\nu\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m},$$

*with*

$$\|\lambda \,:\, a^w_{p,q}\| \le c\,\|f \,:\, A^{s,w}_{p,q}\|.$$

*Here the constant $d > 1$ appearing implicitly in $\mathbf{Atom}_0$ is a fixed large number.*

2. *Suppose that $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^n} \in \mathbf{Atom}_0$ and $\lambda \in a^w_{p,q}$. Then the series*

$$f := \sum_{\nu\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}$$

*belongs to $A^{s,w}_{p,q}$ and satisfies the norm estimate*

$$\|f \,:\, A^{s,w}_{p,q}\| \le c\,\|\lambda \,:\, a^w_{p,q}\|.$$

Let us remark that if $w \equiv 1$, then the conditions on $K$ and $L$ are exactly the ones in [**?**].

*Proof of 1.* Denote by $\{\phi_\nu\}_{\nu\in\mathbb{N}_0}$ the family described in Definition **??**. We utilize the idea in [**?**]. Let $\{\psi_\nu\}_{\nu\in\mathbb{N}_0}$ be taken so that

$$\sum_{\nu\in\mathbb{N}_0} \phi_\nu * \psi_\nu = \delta$$

in $\mathcal{S}'_e$ and that $L_{\psi_1} \ge L$. Here $\delta$ means the Dirac delta distribution. Then we have

$$f = \sum_{\nu\in\mathbb{N}_0} \phi_\nu * \psi_\nu * f = \sum_{\nu\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n} \Phi_{\nu m}(f),$$

where $\Phi_{\nu m}(f)(x) := \displaystyle\int_{Q_{\nu m}} \psi_\nu(x-y)\phi_\nu * f(y)\,dy$. If $\nu \in \mathbb{N}$, then it enjoys the same moment condition as $\psi_\nu$ and the order $L$ of the moment condition can be made as large as we wish. Observe also that $\Phi_{\nu m}$ is supported on $d\,Q_{\nu m}$ for some $d > 1$. Denote

$$\lambda_{\nu m} := \sup_{\substack{\alpha\in\mathbb{N}_0{}^n \\ |\alpha|\le K}} 2^{\nu(s-n/p)-\nu|\alpha|}\|\partial^\alpha \Phi_{\nu m}(f) \,:\, C\|. \tag{10}$$

Define

$$a_{\nu m} := \begin{cases} \lambda_{\nu m}{}^{-1}\cdot \Phi_{\nu m} & \text{if } \lambda_{\nu m} \ne 0 \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Then we have from the observation above $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^n} \in \mathbf{Atom}$. Furthermore $f$ is decomposed as

$$f = \sum_{\nu\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}.$$

Let us see the size of coefficients. To do this we majorize the coefficient with the maximal operator which is given by $\phi^*_{\nu,A,B}f(x) = \sup_{y\in\mathbb{R}^n} \dfrac{|\phi_\nu * f(x-y)|}{m_{\nu,A,B}(y)}$.

$$\begin{aligned}
\lambda_{\nu m} &\le \sup_{\substack{\alpha\in\mathbb{N}_0{}^n\\|\alpha|\le K}} \sup_{x\in\mathbb{R}^n} 2^{\nu(s-n/p)-\nu|\alpha|} \int_{Q_{\nu m}} |\partial^\alpha_x [\psi_\nu(x-y)]\, \phi_\nu * f(y)|\, dy \\
&\le c\, 2^{\nu(s-n/p)} \sup_{y\in Q_{\nu m}+Q(2^{-\nu+1})} |\phi_\nu * f(y)| \\
&\le c\, 2^{\nu(s-n/p)} \sup_{|y|\le c\, 2^{-\nu}} |\phi_\nu * f(x-y)| \\
&\le c\, 2^{\nu(s-n/p)} \phi^*_{\nu,A,B}f(x)
\end{aligned}$$

for all $x \in d\,Q_{\nu m}$. As a consequence

$$\sum_{\nu\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\chi^{(p)}_{\nu m}(x) \le c\, 2^{\nu s}\phi^*_{\nu,A,B}f(x).$$

In view of the maximal estimate (**??**), which can be formulated as

$$\left\|2^{\nu s}\phi^*_{\nu,A,B}f \,:\, L^w_p(l_q)\right\| \le c\, \|f \,:\, F^{s,w}_{p,q}\|$$

under our notation, we obtain

$$\|\lambda \,:\, L^w_p(l_q)\| \le c\, \|f \,:\, F^{s,w}_{p,q}\|.$$

Thus, $f$ was decomposed as we wish. □

*Proof of 2.* We deal with the $F$-scale, the proof for the $B$-scale being similar. Suppose that we are given

$$\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^n} \in \textbf{Atom} \text{ and } \lambda = \{\lambda_{\nu m}\}_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^n} \in a^w_{p,q}.$$

Assume that $\nu > k \ge 0$ or $\nu = k = 0$ for the time being and let us estimate $\phi_k * a_{\nu m}$. The same argument as the non-weighted case works to obtain

$$|2^{ks}\phi_k * a_{\nu m}(x)| \le c\, 2^{-(\nu-k)(L+1+s+n)+k\nu/p} \chi_{c_0\, 2^{\nu-k} Q_{\nu m}}(x),$$

for some $c > 0$. For details of this calculation we refer to [**?**]. Keeping this in mind, let us estimate

$$\sum_{m\in\mathbb{Z}^n} |2^{\nu s}\lambda_{\nu m}\phi_k * a_{\nu m}(x)|.$$

We adopt the following notations:

$$\begin{aligned}
Q(x,r) &:= \{y \in \mathbb{R}^n \,:\, \max(|y_1 - x_1|, |y_2 - x_2|, \ldots, |y_n - x_n|) \le r\} \\
B(x,r) &:= \{y \in \mathbb{R}^n \,:\, |x - y| < r\}.
\end{aligned}$$

Let $\eta < \min\left(1, \dfrac{p}{u}, \dfrac{q}{u}\right)$. A trivial inequality $(a+b)^\theta \le a^\theta + b^\theta$ for $0 < \theta \le 1$ gives us

$$\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| \chi_{c\,Q(2^{-\nu}m, 2^{-k})}(x) \le \sum_{\substack{m \in \mathbb{Z}^n \\ x \in B(2^{-\nu}m, c\,2^{-k})}} |\lambda_{\nu m}| \le \left( \sum_{\substack{m \in \mathbb{Z}^n \\ x \in B(2^{-\nu}m, c\,2^{-k})}} |\lambda_{\nu m}|^\eta \right)^{\frac{1}{\eta}}.$$

Therefore, it follows that

$$\begin{aligned}
\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| \chi_{c\,Q(2^{-\nu}m, 2^{-k})} &\le c\,2^{\frac{\nu n}{\eta}} \left( \int_{B(x, c_0\,2^{-k})} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}}(y) \right|^\eta dy \right)^{\frac{1}{\eta}} \\
&= c\,2^{\frac{n(\nu-k)}{\eta}} m^{(\eta)}_{B(x, c_0\,2^{-k})} \left( \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}} \right| \right) \\
&\le c\,2^{\frac{n(\nu-k)}{\eta}} M^{(\eta)}_{\le c_0} \left[ \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}} \right](x).
\end{aligned}$$

Inserting this estimate, we are led to

$$\sum_{m \in \mathbb{Z}^n} |2^{ks} \lambda_{\nu m} \phi_k * a_{\nu m}(x)| \le c\,2^{-2\delta_0(\nu-k)} M^{(\eta)}_{\le c_0} \left[ \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi^{(p)}_{\nu m} \right](x) \tag{12}$$

for some $\delta_0 > 0$ if $\nu > k \ge 0$ or $\nu = k = 0$, and some $\eta$ which is slightly less than $\min\left(1, \dfrac{p}{u}, \dfrac{q}{u}\right)$.

Let us turn to the remaining case when $k \ge \nu \ge 0$ with $k \ne 0$, which requires us an elaboration. Let $L \gg 1$. Since we are assuming $k \ge 1$, a suitable choice of $\phi_0$ allows us to assume

$$\begin{aligned}
\phi_k(x) &= 2^{kn} \phi_0(2^k x) - 2^{(k-1)n} \phi_0(2^{k-1} x) \\
\mathcal{F}\phi_0(x) &= 1 + c_L\,|x|^{2L} + c_{L+1}\,|x|^{2L+2} + \dots.
\end{aligned}$$

As a consequence, we can assume $\phi_1 = \Delta^L \rho$ for some $L \in \mathbb{N}_0$ sufficiently large and $\rho \in \mathcal{S}$. Once $\phi_1$ is decomposed as above, the same argument as the non-weighted case again works and we obtain

$$|2^{ks} \phi_k * a_{\nu m}(x)| \le c\,2^{-(k-\nu)(K-s)+\nu n/p} \chi_{c_0\,Q_{\nu m}}(x) \le c\,2^{-2\delta_0(k-\nu)+\nu n/p} \chi_{c_0\,Q_{\nu m}}(x) \tag{13}$$

for some $\delta_0 > 0$. We remark that the constants $\delta_0$ in (**??**) and (**??**) can be assumed identical if we replace them by smaller numbers if necessary. Therefore, going through a similar argument, we obtain

$$\sum_{m \in \mathbb{Z}^n} |2^{ks} \lambda_{\nu m} \phi_k * a_{\nu m}(x)| \le c\,2^{-2\delta_0(\nu-k)} M^{(\eta)}_{\le c_0} \left[ \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi^{(p)}_{\nu m} \right](x)$$

for $k \ge \nu \ge 0$ with $k \ne 1$.

In view of this estimate, an argument similar to the non-weighted case works for the non-weighted case to obtain

$$\left\| \sum_{\nu\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \,:\, F^{s,w}_{p,q} \right\| \le c \left\| M^{(\eta)}_{\le c_0} \left[ \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} \chi^{(p)}_{\nu m} \right] \,:\, L^w_p(l_q) \right\| \le c\, \|\lambda \,:\, f^w_{p,q}\|.$$

This is the desired result. $\square$

## Atomic decomposition for the weighted Hardy space

Bui considered the weighted local Hardy spaces in [**?**] for $w \in A_\infty$. Let us present the definition of $h^w_p$ with $w \in A^{\mathrm{loc}}_\infty$. For $f \in \mathcal{S}'_e$, we set

$$\phi_0^+ f(x) := \sup_{j\in\mathbb{N}} (\phi_0)_j * f(x),$$

where $(\phi_0)_j(x) := 2^{jn}\phi_0(2^j x)$ for $j \in \mathbb{N}$. We remark that $\phi_j$ and $(\phi_0)_j$ are different. Let $0 < p < \infty$. The weighted local Hardy space $h^w_p$ is a set of all $f \in \mathcal{S}'_e$ for which the seminorm

$$\|f \,:\, h^w_p\| := \|\phi_0^+ f \,:\, L_p\|$$

is finite. Rychkov proved the following equivalence with the class of the weight $w$ extended to the $A^{\mathrm{loc}}_\infty$-class. We can say that the original function space $h^w_p$, which was studied by Bui, is obtained by letting $w \in A_\infty$.

**Proposition 3.5.** [**?**, Theorem 2.25] *Let $0 < p < \infty$ and $w \in A^{\mathrm{loc}}_\infty$. Then there exists a constant $c > 0$ such that*

$$c^{-1}\|f \,:\, h^w_p\| \le \|f \,:\, F^{0,w}_{p,2}\| \le c\,\|f \,:\, h^w_p\|$$

*for all $f \in \mathcal{S}'_e$.*

**Corollary 3.6.** *Let $w \in A^{\mathrm{loc}}_u$ with $1 \le u < \infty$. Then any element $f$ in $h^w_p$ admits the following decomposition.*

$$f = \sum_{\nu\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m},$$

*where the coefficients $\{\lambda_{\nu m}\}_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^n}$ satisfies*

$$\left\| \left( \sum_{\nu\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^n} 2^{\frac{2n}{p}} |\lambda_{\nu m}|^2 \chi_{Q_{\nu m}} \right)^{\frac{1}{2}} \,:\, L^w_p \right\| \le c\,\|f \,:\, h^w_p\|$$

*and each $a_{\nu m} \in C^K$ is a function satisfying*

$$\mathrm{supp}(a_{0m}) \subset d\,Q_{0m},\ \|\partial^\alpha a_{0m} \,:\, C\| \le 1$$

*for all multiindices $\alpha$ with $|\alpha| \le K$ and*

$$\mathrm{supp}(a_{\nu m}) \subset d\,Q_{\nu m},\ \|\partial^\alpha a_{\nu m} \,:\, C\| \le 2^{\frac{\nu n}{p}+|\alpha|\nu},\ \int_{\mathbb{R}^n} x^\beta a_{\nu m}(x)\,dx = 0$$

*for all $\nu \in \mathbb{N}$ and multiindices $\alpha$, $\beta$ with $|\alpha| \le K$ and $|\beta| \le L$. Here $d > 1$ is a fixed number and the integers $K \in \mathbb{N}$, $L \ge \sigma_{\frac{p}{u}\frac{2}{u}} + (u-1)n$ are chosen as large as we wish.*

## 4 Local means for non-weighted case revisited

Let us now reconsider the local means for the non-weighted case. When we consider the case when $w = 1$, let us abbreviate $w$ to write

$$\|f \,:\, A^{s,w}_{p,q}\| =: \|f \,:\, A^{s}_{p,q}\|,$$

for example. In [?] Triebel has shown the following.

**Theorem 4.1.** *Let $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Then choose $\tau_0, \tau \in \mathcal{S}$ so that*

$$\chi_{B(2)} \le \tau_0 \le \chi_{B(4)} \text{ and } \chi_{B(4)\setminus B(2)} \le \tau \le \chi_{B(8)\setminus B(1)}.$$

*Define $\tau_j(x) := \tau(2^{-j+1}x)$ for $j \in \mathbb{N}$. Given $f \in \mathcal{S}'$, we set*

$$\|f \,:\, B^{s}_{p,q}\|^* := \|2^{js}\mathcal{F}^{-1}(\tau_j \cdot \mathcal{F}f) \,:\, l_q(L_p)\|,\ \|f \,:\, F^{s}_{p,q}\|^* := \|2^{js}\mathcal{F}^{-1}(\tau_j \cdot \mathcal{F}f) \,:\, L_p(l_q)\|.$$

*Let us use $\|f \,:\, A^{s}_{p,q}\|^*$ to denote either $\|f \,:\, B^{s}_{p,q}\|^*$ or $\|f \,:\, F^{s}_{p,q}\|^*$. Then there exists $c > 0$ such that*

$$c^{-1}\|f \,:\, A^{s}_{p,q}\|^* \le \|f \,:\, A^{s}_{p,q}\| \le c\,\|f \,:\, A^{s}_{p,q}\|^*, \tag{14}$$

*whenever $f \in \mathcal{S}'$ satisfies $\|f \,:\, A^{s}_{p,q}\|^* < \infty$.*

If $p, q \ge 1$, the superfluous condition $\|f \,:\, A^{s}_{p,q}\|^* < \infty$ can be dilated. For details we refer to [?]. However, in general it had been essential that we assume $\|f \,:\, A^{s}_{p,q}\|^* < \infty$. This is because of so called "absorbing argument". However, having obtained the atomic decomposition starting from the local means, we are now eliminate this assumption in any case. That is, we have the following.

**Theorem 4.2.** *Keep to the same notation as above. If $f \in \mathcal{S}'_e$ satisfies $\|f \,:\, A^{s}_{p,q}\| < \infty$, then we have $\|f \,:\, A^{s}_{p,q}\|^* < \infty$ and the norm equivalence (**??**) as well as $f$ extends to a continuous functional in $\mathcal{S}$.*

*Proof.* Starting from the assumption $\|f \,:\, A^{s}_{p,q}\| < \infty$, we are able to obtain the atomic decomposition described in this paper. However, the atomic decomposition is exactly the one in [?] for the non-weighted case. Therefore, from the atomic decomposition in [?] we conclude $f \in \mathcal{S}'$ and $\|f \,:\, A^{s}_{p,q}\|^* < \infty$. $\square$

## 5 Applications

Now we investigate the key properties of the function spaces $A^{s,w}_{p,q}$. We shall deal with the multiplier inequality and the diffeomorphic property. To do this, it is helpful to use the lifting property. In [?] Rychkov proved the following.

**Proposition 5.1.** *Let $w \in A_\infty^{\mathrm{loc}}$. Suppose that the parameters $p, q$ and $s$ satisfy*

$$0 < p < \infty,\ 0 < q \leq \infty \text{ and } s \in \mathbb{R}.$$

*Then there exists $t_0 = t_0(w, n) > 0$ depending only on the dimension $n$ and the weight $w$ such that*

$$\mathcal{L}_t^a : f \in A_{p,q}^{s,w} \mapsto (1 - t^2\Delta)^{\frac{a}{2}} f \in A_{p,q}^{s-a,w}$$

*is an isomorphism whenever $t < t_0\, p$.*

It is not so hard to see

**Proposition 5.2.** *Keep to the same assumption on the parameters $p, q, s$ and the weight $w$ as Proposition* **??**. *Then*

$$\partial^\alpha : A_{p,q}^{s,w} \to A_{p,q}^{s-|\alpha|,w}$$

*is a continuous linear operator.*

*Proof.* Let $f \in A_{p,q}^{s,w}$. Then by Theorem **??** we are in the position of forming the atomic decomposition of $f$:

$$f = \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}.$$

Here the coefficients $\{\lambda_{\nu m}\}_{\nu \in \mathbb{N}_0,\, m \in \mathbb{Z}^n}$ are subject to some condition, to which we do not allude for the sake of simplicity. And the collection of functions $\{a_{\nu m}\}_{\nu \in \mathbb{N}_0,\, m \in \mathbb{Z}^n}$ belongs to $\mathbf{Atom}_0$. Note that the smoothness of $a$ is obtained as much as we wish. Therefore, we obtain $\{\partial^\alpha a_{\nu m}\}_{\nu \in \mathbb{N}_0,\, m \in \mathbb{Z}^n} \in \mathbf{Atom}_0$ and

$$\partial^\alpha f = \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \partial^\alpha a_{\nu m}$$

is again an atomic decomposition for $\partial^\alpha f$ in $A_{p,q}^{s-|\alpha|,w}$ and we obtain the desired result. □

## 5.1 Pointwise multiplier

Let us denote by $C^M$ the Banach space of $C^M$-functions $f$ for which the norm

$$\|f \,:\, C^M\| := \sum_{\substack{\alpha \in {\mathbb{N}_0}^n \\ |\alpha| \leq M}} \|\partial^\alpha f \,:\, C^M\| \tag{15}$$

is finite.

**Theorem 5.3.** *Suppose that $w \in A_\infty^{\mathrm{loc}}$ and that the parameters $p, q$ and $s$ satisfy*

$$0 < p < \infty,\ 0 < q \leq \infty \text{ and } s \in \mathbb{R}.$$

*Then there exist constants $c > 0$ and $M > 0$ depending only on the parameters $p, q, s$ and the weight $w$ such that we can define the pointwise multiplication $h \cdot f$ naturally and*

$$\|h \cdot f \,:\, A^{s,w}_{p,q}\| \le c\,\|h \,:\, C^M\| \cdot \|f \,:\, A^{s,w}_{p,q}\| \tag{16}$$

*for all $h \in C^M$ and $f \in A^{s,w}_{p,q}$.*

*Proof.* Assume that $w \in A^{\mathrm{loc}}_u$ with $1 \le u < \infty$. First we begin with the regular case. That is, let us assume

$$s > \sigma_{\frac{p}{u}\frac{q}{u}} + (u-1)n.$$

Then we remark that there is no need to postulate the moment condition on atoms.

Let $f \in A^{s,w}_{p,q}$. Then we can decompose $f$ as follows:

$$f = \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m},$$

where $\{a_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{Atom}_0$ and the collection of coefficients $\lambda := \{\lambda_{\nu m}\}_{\nu \in \mathbb{N}_0,\, m \in \mathbb{Z}^n}$ satisfies

$$\|\lambda \,:\, a^w_{p,q}\| \le c\,\|f \,:\, A^{s,w}_{p,q}\|. \tag{17}$$

As we have remarked, we do not have to pose the functions on the moment condition. Therefore the decomposition

$$h \cdot f = \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} (h \cdot a_{\nu m})$$

satisfies $\{h \cdot a_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{Atom}$ as well as the estimates of coefficients (**??**). It follows from the Leibnitz rule that there exists a constant $c > 0$ independent of the functions $f, h$ such that

$$\{c\,\|h \,:\, C^M\|^{-1} h \cdot a_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{Atom}_0,$$

once we exclude the trivial case when $h \equiv 0$. Therefore, we obtain

$$\|h \cdot f \,:\, A^{s,w}_{p,q}\| \le c\,\|h \,:\, C^M\| \cdot \|\lambda \,:\, a^w_{p,q}\| \le c\,\|h \,:\, C^M\| \cdot \|f \,:\, A^{s,w}_{p,q}\|.$$

Thus, the theorem is proved when $s$ is sufficiently large, say $s > \sigma_{\frac{p}{u}\frac{q}{u}} + (u-1)n$.

Now we pass to the general case. To do this, fix the parameters $p, q$ and the weight $w$. We set

$$S = S_{p,q,w} := \{s \in \mathbb{R} \,:\, (\mathbf{??}) \text{ holds for } s\}.$$

Assuming that $w \in A^{\mathrm{loc}}_u$, in the paragraph above, we have shown that

$$\left(\sigma_{\frac{p}{u}\frac{q}{u}} + (u-1)n, \infty\right) \subset S.$$

Therefore, to complete the proof, it suffices to show that

$$s, s+1 \in S \Longrightarrow s-1 \in S.$$

We utilize Proposition **??** and fix the constant $t_0 = t_0(w,n)$ of Proposition **??**. Let $f \in A^{s-1,w}_{p,q}$. Then set $g = (1-t^2\Delta)^{-1} f$ with $t = \dfrac{t_0 p}{2}$. Then we have by virtue of the aforementioned proposition that

$$\|g \,:\, A^{s+1,w}_{p,q}\| \le c\,\|f \,:\, A^{s-1,w}_{p,q}\|. \tag{18}$$

Furthermore, we have

$$h \cdot f = h \cdot (1-t^2\Delta) g = (1-t^2\Delta)(h \cdot g) + 2t^2 \sum_{j=1}^{n} \partial_j h \cdot \partial_j g + t^2 \Delta h \cdot g.$$

Therefore, we obtain

$$\begin{aligned}
&\|h \cdot f \,:\, A^{s-1,w}_{p,q}\| \\
&\le c \left( \|(1-t^2\Delta)(h \cdot g) \,:\, A^{s-1,w}_{p,q}\| + \sum_{j=1}^{n} \|\partial_j h \cdot \partial_j g \,:\, A^{s-1,w}_{p,q}\| + \|\Delta h \cdot g \,:\, A^{s-1,w}_{p,q}\| \right) \\
&\le c \left( \|h \cdot g \,:\, A^{s+1,w}_{p,q}\| + \sum_{j=1}^{n} \|\partial_j h \cdot \partial_j g \,:\, A^{s,w}_{p,q}\| + \|\Delta h \cdot g \,:\, A^{s+1,w}_{p,q}\| \right).
\end{aligned}$$

Since we are assuming $s, s+1 \in S$, we obtain, together with Proposition **??**,

$$\begin{aligned}
\|h \cdot g \,:\, A^{s+1,w}_{p,q}\| &\le c\,\|h \,:\, C^{M'}\| \cdot \|g \,:\, A^{s+1,w}_{p,q}\|, \\
\|\partial_j h \cdot \partial_j g \,:\, A^{s,w}_{p,q}\| &\le c\,\|h \,:\, C^{M'}\| \cdot \|\partial_j g \,:\, A^{s+1,w}_{p,q}\| \le c\,\|h \,:\, C^{M'}\| \cdot \|g \,:\, A^{s+1,w}_{p,q}\| \\
&\text{and} \\
\|\Delta h \cdot g \,:\, A^{s+1,w}_{p,q}\| &\le c\,\|h \,:\, C^{M'}\| \cdot \|g \,:\, A^{s+1,w}_{p,q}\| \le c\,\|h \,:\, C^{M'}\| \cdot \|g \,:\, A^{s+1,w}_{p,q}\|.
\end{aligned}$$

Here $M' \in \mathbb{N}$ is a large integer.

Inserting these estimates and (**??**), we obtain

$$\|h \cdot f \,:\, A^{s-1,w}_{p,q}\| \le c\,\|h \,:\, C^{M'}\| \cdot \|g \,:\, A^{s+1,w}_{p,q}\| \le c\,\|h \,:\, C^{M'}\| \cdot \|f \,:\, A^{s-1,w}_{p,q}\|,$$

which establishes $s-1 \in S$. □

### 5.2 Diffeomorphic property

Finally in this paper we consider the diffeomorphic property of the function space $A^{s,w}_{p,q}$ : We take up the composite of the elements in $A^{s,w}_{p,q}$ and a diffeomorphism on $\mathbb{R}^n$. To deal with the Sobolev-like spaces, we need to postulate some conditions on the diffeomorphisms we are going to take up.

**Definition 5.4.** A homeomorphism $\psi$ on $\mathbb{R}^n$ is said to be a regular $C^N$-diffeomorphism, if the following conditions below are fulfilled.

1. $\psi \in C^N$, where $C^N$ is a Banach space normed by (**??**).

2. Denote by $J\psi$ the Jacobian matrix. Then $\psi$ satisfies

$$\inf_{x \in \mathbb{R}^n} |\det(J\psi)(x)| > 0.$$

Let $\psi$ be a regular $C^N$-diffeomorphism. Then observe that $\psi^{-1}$ is a regular $C^N$-diffeomorphism as well. Denote by $C^N_c$ the set of all $C^N$-functions with compact support. Then define

$$B_\psi : f \in C^N_c \to f \circ \psi \in C^N_c.$$

Then we see $B_\psi$ is an isomorphism on $C^N_c$.

**Theorem 5.5.** *Suppose that the parameters $p, q$ and $s$ satisfy*

$$0 < p < \infty,\ 0 < q \leq \infty \text{ and } s \in \mathbb{R}$$

*and let $w \in A^{\rm loc}_\infty$. Assume that $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is a regular $C^N$-diffeomorphism. If $N$ is sufficiently large, the mapping $B_\psi$, defined initially on $C^N_c$, can be extended to an isomorphism on $A^{s,w}_{p,q}$.*

*Proof.* Let $w \in A^{\rm loc}_u$ with $1 \leq u < \infty$. By symmetry we have only to prove

$$\|f \circ \psi \,:\, A^{s,w}_{p,q}\| \leq c\,\|f \,:\, A^{s,w}_{p,q}\|.$$

Assume first that $s$ is large enough, which allows us to disregard the moment condition. Let $f \in A^{s,w}_{p,q}$ and form its atomic decomposition:

$$f = \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m},$$

where $\lambda = \{\lambda_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in a^w_{p,q}$ and $\{a_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{Atom}_0$.

First of all, let us see that the sum

$$F := \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \cdot a_{\nu m} \circ \psi$$

converges in $\mathcal{S}'_e$ and belongs to $A^{s,w}_{p,q}$. To do this, we observe that there exists a constant $c > 0$ such that

$$c^{-1}|x - x'| \leq |\psi(x) - \psi(x')| \leq c\,|x - x'| \tag{19}$$

for all $x, x' \in \mathbb{R}^n$. Therefore, the distorted dyadic cubes $\{\psi(d\,Q_{\nu m})\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ are nicely separated. A geometric observation using (**??**) shows that there exist $I \in \mathbb{N}$ and $D > 1$ independent of $\nu \in \mathbb{N}_0$ with the following property:

1. For each $\nu \in \mathbb{N}_0$, $\mathbb{Z}^n$ is partitioned into a disjoint union of $I$ subsets:

$$\mathbb{Z}^n = M^\nu_1 \coprod M^\nu_2 \coprod \cdots \coprod M^\nu_I.$$

2. For each $\nu \in \mathbb{N}_0$ and $i = 1, 2, \ldots, I$, there exist an injection $\alpha_i^\nu : M_i^\nu \to \mathbb{Z}^n$ such that
$$\psi(d\,Q_{\nu m}) \subset D\,Q_{\nu\,\alpha_i^\nu(m)}$$
for all $m \in M_i^\nu$, where $D$ is a new constant that depends only on $d$ and $\psi$.

Let $i$ and $\nu$ be fixed for the time being. Set $\overline{M}_i^\nu := \alpha_i^\nu(M_i^\nu)$. Let us regard each $\alpha_i^\nu$ as a bijection from $M_i^\nu$ to $\overline{M}_i^\nu$ with inverse $\beta_i^\nu$. For each $\overline{m} \in \overline{M}_i^\nu$, we define

$$\overline{\lambda}^i_{\nu\overline{m}} := \lambda_{\nu\,\beta_i^\nu(\overline{m})} \quad \text{and} \quad \overline{a}^i_{\nu\overline{m}} := a_{\nu\,\beta_i^\nu(\overline{m})} \circ \psi.$$

If $\overline{m} \in \mathbb{Z}^n \setminus \overline{M}_i^\nu$, then define

$$\overline{\lambda}^i_{\nu\overline{m}} := 0 \quad \text{and} \quad \overline{a}^i_{\nu\overline{m}} := 0.$$

With this preparation in mind, we decompose the sum according to the partition above.

$$\begin{aligned}
\sum_{\nu\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\cdot a_{\nu m}\circ\psi &= \sum_{i=1}^{I}\left(\sum_{\nu\in\mathbb{N}_0}\sum_{m\in M_i} \lambda_{\nu m}\cdot a_{\nu m}\circ\psi\right)\\
&= \sum_{i=1}^{I}\left(\sum_{\nu\in\mathbb{N}_0}\sum_{\overline{m}\in\overline{M}_i^\nu} \overline{\lambda}^i_{\nu\overline{m}}\overline{a}^i_{\nu\overline{m}}\right)\\
&= \sum_{i=1}^{I}\left(\sum_{\nu\in\mathbb{N}_0}\sum_{\overline{m}\in\mathbb{Z}^n} \overline{\lambda}^i_{\nu\overline{m}}\overline{a}^i_{\nu\overline{m}}\right).
\end{aligned}$$

Note that $\{\overline{a}^i_{\nu\overline{m}}\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n} \in \mathbf{Atom}$ for each $i = 1, 2, \ldots, I$ because there is no need to take into account the moment condition. Therefore, each summand in the parenthesis belongs to $A^{s,w}_{p,q}$. Thus, we see that

$$F := \sum_{\nu\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\cdot a_{\nu m}\circ\psi \tag{20}$$

belongs to $A^{s,w}_{p,q}$.

Once we prove that $F$ is not dependent on the choice of the atomic decomposition of $f$, it follows from the estimate of the coefficients that $f \in A^{s,w}_{p,q} \mapsto F \in A^{s,w}_{p,q}$ is continuous. To prove that $F$ is independent of the choice of the atomic decomposition we may assume that $p, q$ are finite and hence the convergence

$$\sum_{\nu\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}$$

takes place in the topology of $A^{s,w}_{p,q}$. This is because of the embedding

$$A^{s+\epsilon,w}_{p,\infty} \subset A^{s,w}_{p,q}$$

and we are assuming $s$ is sufficiently large. In this case we can use

$$\langle F, \eta\rangle = \langle f, \eta\circ\psi^{-1}\cdot|\det(J\psi)|^{-1}\rangle,$$

where we used the fact that $f$ can be extended continuously to the functional on $C^N$. Note that (**??**) converges in the topology of $A^{s,w}_{p,q}$. Therefore $C_c^N$ is dense in $A^{s,w}_{p,q}$. As a result the composition mapping is extended to a bounded linear operator provided $s$ is sufficiently large.

Now let us pass to the general case. To do this we set

$$S = S_{p,q,w} := \{s \in \mathbb{R} \,:\, \text{Theorem } \textbf{??} \text{ holds for } s\}$$

with the parameters $p, q$ and the weight $w$ fixed.

What we have struggled to prove can be summarized as follows:

$$[\alpha, \infty) \subset S$$

for some $\alpha \gg 0$.

Let $f \in A^{s,w}_{p,q}$ with $s < \alpha$. Then by Proposition **??** we obtain

$$h = (1 - t^2\,\Delta)^{-2L} f \in A^{s+2L,w}_{p,q},\ \|h \,:\, A^{s+2L,w}_{p,q}\| \leq c\,\|f \,:\, A^{s,w}_{p,q}\|,$$

where $L$ is a constant larger than $\dfrac{\alpha - s}{2L}$ and $t$ is a small parameter as before. Then

$$f = (1 - t^2\,\Delta)h = (1 - t^2\,\Delta)(h \circ \psi \circ \psi^{-1}) = \sum_{\substack{\alpha \in \mathbb{N}_0 \\ |\alpha| \leq 2L}} \tilde{\phi}_\alpha \cdot \partial^\alpha (h \circ \psi) \circ \psi^{-1}.$$

Here $\tilde{\phi}_\alpha$ belongs to $C^{M'}$ that depends only on $\psi$ and $\alpha$. Therefore, we obtain

$$f \circ \psi = \sum_{\substack{\alpha \in \mathbb{N}_0 \\ |\alpha| \leq 2L}} \tilde{\phi}_\alpha \cdot \partial^\alpha (h \circ \psi) \circ \psi^{-1}.$$

If we invoke Proposition **??** and Theorem **??**, we obtain

$$\|f \circ \psi \,:\, A^{s,w}_{p,q}\| \leq c \sum_{\substack{\alpha \in \mathbb{N}_0 \\ |\alpha| \leq 2L}} \|\tilde{\phi}_\alpha \cdot \partial^\alpha (h \circ \psi) \,:\, A^{s,w}_{p,q}\| \leq c\,\|h \circ \psi \,:\, A^{s+2L,w}_{p,q}\|.$$

Now that $s + 2L \in S$ and hence we have

$$\|h \circ \psi \,:\, A^{s+2L,w}_{p,q}\| \leq c\,\|h \,:\, A^{s+2L,w}_{p,q}\|.$$

Putting together these observations, we obtain $s \in S$. □

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