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Uniqueness for geometric solutions of implicit second order ordinary differential equations ^{*}

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Abstract

For implicit second order ordinary differential equations, uniqueness for solutions does not hold in general. In this paper, we give a sufficient condition that implicit second order ordinary differential equations have the unique geometric solution around a point.

1 Introduction

An implicit second order ordinary differential equation (or, briefly, an equation) is given by the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0,$$

where F is a smooth function of the independent variable x , of the function y . It is natural to consider $F = 0$ as being defined on a subset in the space of 2-jets of functions of one variable, $F : U \rightarrow \mathbb{R}$ where U is an open subset in $J^2(\mathbb{R}, \mathbb{R})$. Throughout this paper, we assume that 0 is a regular value of F . It follows that the set $F^{-1}(0)$ is a hypersurface in $J^2(\mathbb{R}, \mathbb{R})$. We call $F^{-1}(0)$ the *equation hypersurface*. Let (x, y, p, q) be a local coordinate in $J^2(\mathbb{R}, \mathbb{R})$ and $\xi \subset TJ^2(\mathbb{R}, \mathbb{R})$ be the canonical contact system on $J^2(\mathbb{R}, \mathbb{R})$ described by the vanishing of the 1-forms $\alpha_1 = dy - pdx$ and $\alpha_2 = dp - qdx$. It is worth noting that the field of planes ξ is an *Engel structure* on the 4-dimensional manifold $J^2(\mathbb{R}, \mathbb{R})$.

If $F = 0$ satisfies $F_q = \partial F / \partial q \neq 0$ at z_0 , by the implicit function theorem, we can locally rewrite this equation in the form $q = f(x, y, p)$, where f is a smooth function. This explicit form $q = f(x, y, p)$ is more convenient than the original one, because there exists the classical existence and uniqueness for (smooth) solutions.

^{*}Dedicated to Professor Goo Ishikawa on the occasion of his fiftieth birthday.

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Here a *smooth solution* (or, a *classical solution*) of $F = 0$ around z_0 is a smooth function germ $y = f(x)$ at a point t_0 such that $(t_0, f(t_0), f'(t_0), f''(t_0)) = z_0$ and $F(x, f(x), f'(x), f''(x)) = 0$. In other words, there exists a smooth function germ $f : (\mathbb{R}, t_0) \rightarrow \mathbb{R}$ such that the image of the 2-jet extension, $j^2 f : (\mathbb{R}, t_0) \rightarrow (J^2(\mathbb{R}, \mathbb{R}), z_0)$, is contained in the equation hypersurface. It is easy to check that the map $j^2 f$ satisfies $(j^2 f)^* \alpha_1 = (j^2 f)^* \alpha_2 = 0$ (i.e. $j^2 f$ is an Engel immersion germ).

More generally, a *geometric solution* of $F = 0$ around z_0 is an Engel immersion germ $\gamma : (\mathbb{R}, t_0) \rightarrow (J^2(\mathbb{R}, \mathbb{R}), z_0)$ such that the image of γ is contained in the equation hypersurface, namely, $\gamma' \neq 0$, $\gamma^* \alpha_1 = \gamma^* \alpha_2 = 0$ and $F(\gamma(t)) = 0$.

By definition, the classical solution is also a geometric solution. Hence for explicit equations, there exists the unique geometric solution around z_0 .

For implicit second order ordinary differential equations, however, uniqueness for geometric solutions does not hold in general (cf. Example 4.2). In this paper, we give a sufficient condition that uniqueness for geometric solutions of implicit second order ordinary differential equations hold. The main result is the following.

Theorem 1.1 *Let $F(x, y, p, q) = 0$ be an equation at z_0 . If $F_x + pF_y + qF_p \neq 0$ at z_0 , then there exists the unique geometric solution around z_0 .*

We remark that we gave an existence condition for a complete solution (two-parameter family of geometric solutions) in [1, 8, 9]. As a consequence, if the equation $F = 0$ satisfies $F_x + pF_y + qF_p \neq 0$ at z_0 , then there exists a geometric solution around z_0 (cf. Proposition 2.1). Therefore we may prove that uniqueness for geometric solutions under this condition.

In Section 2, we give basic notions for implicit second order ordinary differential equations. In Section 3, we prove Theorem 1.1. In Section 4, we give examples of implicit second order ordinary differential equations. In the Appendix, we consider implicit first order ordinary differential equations and give the corresponding result of Theorem 1.1 by using the Legendre transformation.

All map germs and manifolds considered here are differential of class C^∞ .

2 Basic notions

In this section we give basic notions for implicit second order ordinary differential equations. First we define the notion of complete solution (cf. [1, 3, 8, 9]). By the definition of parametrized version for smoothness of the solutions (i.e. smooth solutions), a *smooth complete solution* of $F = 0$ at z_0 is defined to be a two-parameter family of smooth function germs $y = f(t, r, s)$ such that

$$F \left(t, f(t, r, s), \frac{\partial f}{\partial t}(t, r, s), \frac{\partial^2 f}{\partial t^2}(t, r, s) \right) = 0$$

and the map germ $j_1^2 f : (\mathbb{R} \times \mathbb{R}^2, (t_0, r_0, s_0)) \rightarrow (F^{-1}(0), z_0)$ defined by

$$j_1^2 f(t, r, s) = \left(t, f(t, r, s), \frac{\partial f}{\partial t}(t, r, s), \frac{\partial^2 f}{\partial t^2}(t, r, s) \right)$$

is an immersion. It follows that the equation hypersurface is foliated by a two-parameter family of classical solutions like as explicit equations.

On the other hand, we consider the corresponding definition of parametrized version for geometric solutions. Let $\Gamma : (\mathbb{R} \times \mathbb{R}^2, (t_0, r_0, s_0)) \rightarrow (F^{-1}(0), z_0)$ be a two-parameter family of geometric solutions of $F = 0$. We call Γ a *complete solution* at z_0 if

$$\text{rank} \begin{pmatrix} \partial x/\partial t & \partial y/\partial t & \partial p/\partial t & \partial q/\partial t \\ \partial x/\partial r & \partial y/\partial r & \partial p/\partial r & \partial q/\partial r \\ \partial x/\partial s & \partial y/\partial s & \partial p/\partial s & \partial q/\partial s \end{pmatrix} (t_0, r_0, s_0) = 3,$$

where $\Gamma(t, r, s) = (x(t, r, s), y(t, r, s), p(t, r, s), q(t, r, s))$. This condition means that Γ is an immersion germ, that is, the equation hypersurface is foliated by a two-parameter family of geometric solutions. We say that an equation $F = 0$ is *completely integrable* at z_0 if there exists a complete solution of $F = 0$ at z_0 .

Furthermore, an equation $F = 0$ is of *second order Clairaut type* (for short, *Clairaut type*) at z_0 if there exist smooth function germs $A, B : (J^2(\mathbb{R}, \mathbb{R}), z_0) \rightarrow \mathbb{R}$ such that

$$F_x + p \cdot F_y + q \cdot F_p = A \cdot F + B \cdot F_q,$$

and of *first order type* at z_0 if there exist smooth function germs $A', B' : (J^2(\mathbb{R}, \mathbb{R}), z_0) \rightarrow \mathbb{R}$ such that

$$F_q = A' \cdot F + B' \cdot (F_x + p \cdot F_y + q \cdot F_p).$$

In [9], we gave an existence condition for the complete solution of implicit second order ordinary differential equations.

Proposition 2.1 ([9, Proposition 2.2]) *Let $F(x, y, p, q) = 0$ be an equation at z_0 . $F = 0$ is completely integrable at z_0 if and only if $F = 0$ is of Clairaut type, or of first order type at z_0 .*

If $F = 0$ satisfies $F_x + pF_y + qF_p \neq 0$ at z_0 , then $F = 0$ is of first order type at z_0 . By Proposition 2.1, there exists a complete solution at z_0 and hence a geometric solution around z_0 .

We call points $z \in F^{-1}(0)$ such that the contact plane ξ_z intersects $T_z F^{-1}(0)$ transversality fails *contact singular points*. We denote the set of such points by $\Sigma_c = \Sigma_c(F)$. It is easy to check that the contact singular set is given by

$$\Sigma_c(F) = \{z \in F^{-1}(0) \mid F_x(z) + pF_y(z) + qF_p(z) = 0, F_q(z) = 0\}.$$

As a corollary of Theorem 1.1, we have the following result.

Corollary 2.2 *Let $F(x, y, p, q) = 0$ be an equation at z_0 . If $z_0 \notin \Sigma_c(F)$, then there exists the unique geometric solution around z_0 .*

The last part in this section, we define a dual equation of the original equation $F = 0$. We consider a transformation of $J^2(\mathbb{R}, \mathbb{R})$. Let (X, Y, P, Q) be an another coordinate of $J^2(\mathbb{R}, \mathbb{R})$ by $X = q, Y = p - qx, P = y - px + (1/2)qx^2, Q = x$. We refer to a smooth mapping $E : J^2(\mathbb{R}, \mathbb{R}) \rightarrow J^2(\mathbb{R}, \mathbb{R})$ given by

$$E(x, y, p, q) = \left(q, p - qx, y - px + \frac{1}{2}qx^2, x \right)$$

as an *Engel-Legendre transformation* (or, briefly, *E-L transformation*) (cf. [8]). By definition, we have $E^{-1}(X, Y, P, Q) = (Q, P + YQ + (1/2)XQ^2, Y + XQ, X)$. If we apply the *E-L transformation* to our equation $F = 0$, we obtain a new equation

$$F^*(X, Y, P, Q) = F \circ E^{-1}(X, Y, P, Q) = F(Q, P + YQ + (1/2)XQ^2, Y + XQ, X) = 0$$

in the coordinate system (X, Y, P, Q) .

If we calculate the partial derivatives at the point (X_0, Y_0, P_0, Q_0) corresponding to (x_0, y_0, p_0, q_0) , we can show the following.

$$\begin{aligned} F_Q^*(X_0, Y_0, P_0, Q_0) &= (F_x + pF_y + qF_p)(x_0, y_0, p_0, q_0), \\ F_P^*(X_0, Y_0, P_0, Q_0) &= F_y(x_0, y_0, p_0, q_0), \\ F_Y^*(X_0, Y_0, P_0, Q_0) &= (xF_y + F_p)(x_0, y_0, p_0, q_0), \\ F_X^*(X_0, Y_0, P_0, Q_0) &= ((1/2)x^2F_y + xF_p + F_q)(x_0, y_0, p_0, q_0). \end{aligned}$$

3 Proof of Theorem 1.1

Let $F = 0$ be an equation at z_0 . Suppose that $F_x + pF_y + qF_p \neq 0$ at z_0 . By Proposition 2.1, there exists a geometric solution around z_0 .

First we may assume that $F_x + pF_y + qF_p \equiv 1$ around z_0 . If $\gamma(t) = (x(t), y(t), p(t), q(t))$ is a geometric solution of $F = 0$ around z_0 , then $q'(t) \neq 0$. Hence we can reparametrize $\gamma(t)$ as $(x(t), y(t), p(t), t)$.

Let $\gamma(t) = (x(t), y(t), p(t), t)$ and $\tilde{\gamma}(t) = (\tilde{x}(t), \tilde{y}(t), \tilde{p}(t), t)$ be geometric solutions of $F = 0$ around z_0 . Suppose that $\gamma(t_0) = \tilde{\gamma}(t_0) = z_0$. It is enough to show that $\gamma(t) = \tilde{\gamma}(t)$ for $t_0 \leq t \leq t_0 + \varepsilon$, where ε is a positive small number. If we differentiate the equality $F(\gamma(t)) = F(x(t), y(t), p(t), t) = 0$ with respect to t , then we get $x'(t) = -F_q(x(t), y(t), p(t), t)$. By integrating this equality,

$$x(t) = x(t_0) + \int_{t_0}^t -F_q(x(t), y(t), p(t), t) dt.$$

Since $\gamma(t)$ is a geometric solution, namely $y'(t) = p(t)x'(t)$ and $p'(t) = tx'(t)$, we have

$$y(t) = y(t_0) + \int_{t_0}^t p(t)x'(t) dt, \quad p(t) = p(t_0) + \int_{t_0}^t tx'(t) dt.$$

It follows that

$$x(t) - \tilde{x}(t) = \int_{t_0}^t (-F_q(x(t), y(t), p(t), t) + F_q(\tilde{x}(t), \tilde{y}(t), \tilde{p}(t), t)) dt, \quad (1)$$

$$\begin{aligned} y(t) - \tilde{y}(t) &= \int_{t_0}^t (p(t)x'(t) - \tilde{p}(t)\tilde{x}'(t)) dt \\ &= \int_{t_0}^t p(t)(x'(t) - \tilde{x}'(t)) dt + \int_{t_0}^t \tilde{x}'(t)(p(t) - \tilde{p}(t)) dt \end{aligned} \quad (2)$$

and

$$\begin{aligned} p(t) - \tilde{p}(t) &= \int_{t_0}^t t(x'(t) - \tilde{x}'(t)) dt \\ &= \int_{t_0}^t t(-F_q(x(t), y(t), p(t), t) + F_q(\tilde{x}(t), \tilde{y}(t), \tilde{p}(t), t)) dt. \end{aligned} \quad (3)$$

Since $F(x, y, p, q)$ is a smooth mapping, there exists some number K such that

$$| -F_q(x(t), y(t), p(t), t) + F_q(\tilde{x}(t), \tilde{y}(t), \tilde{p}(t), t) | \leq K(|x(t) - \tilde{x}(t)| + |y(t) - \tilde{y}(t)| + |p(t) - \tilde{p}(t)|),$$

where $t_0 \leq t \leq t_0 + \varepsilon$. Moreover since $\gamma(t)$ and $\tilde{\gamma}(t)$ are smooth mappings, we put

$$a = \max\{|t_0|, |t_0 + \varepsilon|\}, \quad b = \max_{t_0 \leq t \leq t_0 + \varepsilon} \{p(t)\}, \quad c = \max_{t_0 \leq t \leq t_0 + \varepsilon} \{\tilde{x}'(t)\}.$$

By (1),(2) and (3),

$$\begin{aligned} & |x(t) - \tilde{x}(t)| + |y(t) - \tilde{y}(t)| + |p(t) - \tilde{p}(t)| \\ & \leq \int_{t_0}^t | -F_q(x(t), y(t), p(t), t) + F_q(\tilde{x}(t), \tilde{y}(t), \tilde{p}(t), t) | dt \\ & \quad + \int_{t_0}^t |p(t)| \cdot | -F_q(x(t), y(t), p(t), t) + F_q(\tilde{x}(t), \tilde{y}(t), \tilde{p}(t), t) | dt \\ & \quad + \int_{t_0}^t \left(|\tilde{x}'(t)| \cdot \int_{t_0}^t |t| \cdot | -F_q(x(t), y(t), p(t), t) + F_q(\tilde{x}(t), \tilde{y}(t), \tilde{p}(t), t) | dt \right) dt \\ & \quad + \int_{t_0}^t |t| \cdot | -F_q(x(t), y(t), p(t), t) + F_q(\tilde{x}(t), \tilde{y}(t), \tilde{p}(t), t) | dt \\ & \leq (1 + a + b)K \int_{t_0}^t (|x(t) - \tilde{x}(t)| + |y(t) - \tilde{y}(t)| + |p(t) - \tilde{p}(t)|) dt \\ & \quad + acK \int_{t_0}^t \left(\int_{t_0}^t (|x(t) - \tilde{x}(t)| + |y(t) - \tilde{y}(t)| + |p(t) - \tilde{p}(t)|) dt \right) dt. \end{aligned} \quad (4)$$

If we put

$$L = \max\{1 + a + b, ac\}, \quad M = \max_{t_0 \leq t \leq t_0 + \varepsilon} \{|x(t) - \tilde{x}(t)| + |y(t) - \tilde{y}(t)| + |p(t) - \tilde{p}(t)|\},$$

then it follows from (4) that

$$|x(t) - \tilde{x}(t)| + |y(t) - \tilde{y}(t)| + |p(t) - \tilde{p}(t)| \leq LKM \left((t - t_0) + \frac{1}{2}(t - t_0)^2 \right). \quad (5)$$

Inductively, by put (5) into (4), we can get

$$|x(t) - \tilde{x}(t)| + |y(t) - \tilde{y}(t)| + |p(t) - \tilde{p}(t)| \leq L^n K^n M \sum_{k=0}^n \frac{1}{(n+k)!} \binom{n}{k} (t - t_0)^{n+k},$$

for each n . Since

$$\sum_{k=0}^n \frac{1}{(n+k)!} \binom{n}{k} (t - t_0)^{n+k} \leq \sum_{k=0}^n \frac{1}{n!} \binom{n}{k} (t - t_0)^{n+k} = \frac{(t - t_0)^n}{n!} (1 + t - t_0)^n,$$

the inequality

$$|x(t) - \tilde{x}(t)| + |y(t) - \tilde{y}(t)| + |p(t) - \tilde{p}(t)| \leq \frac{1}{n!} (t - t_0)^n (1 + t - t_0)^n L^n K^n M$$

holds, where $t_0 \leq t \leq t_0 + \varepsilon$. Hence we get $|x(t) - \tilde{x}(t)| + |y(t) - \tilde{y}(t)| + |p(t) - \tilde{p}(t)| = 0$. It concludes that $\gamma(t) = \tilde{\gamma}(t)$ for $t_0 \leq t \leq t_0 + \varepsilon$. This completes the proof of Theorem 1.1 \square

Remark 3.1 We can consider another proof of Theorem 1.1. We shall take small number ε which satisfies $0 < \varepsilon < -1 + \sqrt{1 + (2/KL)}$. By (5), we have $M \leq LKM(\varepsilon + (1/2)\varepsilon^2)$. This implies that $M = 0$. It concludes that $\gamma(t) = \tilde{\gamma}(t)$ for $t_0 \leq t \leq t_0 + \varepsilon$.

Remark 3.2 Since the E-L transformation is not an Engel diffeomorphism, we can not prove by the same method of those of first order case, see Appendix.

4 Examples

We give two examples for completely integrable second order ordinary differential equations. One is satisfied the condition of Theorem 1.1 and the other is not.

Example 4.1 Let $F(x, y, p, q) = x - q^n = 0$ where $n > 1$. Then $F_x + pF_y + qF_p = 1$ and $\Sigma_c(F) = \emptyset$. In this case, the complete solution $\Gamma : \mathbb{R} \times \mathbb{R}^2 \rightarrow F^{-1}(0)$ is given by

$$\Gamma(t, r, s) = \left(t^n, \frac{n^2}{(2n+1)(n+1)}t^{2n+1} + rt^n + s, \frac{n}{n+1}t^{n+1} + r, t \right).$$

By Theorem 1.1 or Corollary 2.2, there exists the unique geometric solution of $F = 0$ at each point which is given by the complete solution.

Example 4.2 Let $F(x, y, p, q) = p - q^n = 0$ where $n > 1$. Then $F_x + pF_y + qF_p = q$ and $F_q = -nq^{n-1}$. Hence $F = 0$ is a first order type equation and $\Sigma_c(F) = \{(x, y, 0, 0)\}$. The complete solution $\Gamma : \mathbb{R} \times \mathbb{R}^2 \rightarrow F^{-1}(0)$ is given by

$$\Gamma(t, r, s) = \left(\frac{n}{n-1}t^{n-1} + r, \frac{n}{2n-1}t^{2n-1} + s, t^n, t \right).$$

In this case, we have the complete singular solution $\Phi : \mathbb{R} \times \mathbb{R} \rightarrow \Sigma_c(F) \subset F^{-1}(0)$; $\Phi(t, a) = (t, a, 0, 0)$. The definition and properties of complete singular solutions please refer to [1, 9]. For example, at least there is two geometric solutions of $F = 0$ at the origin.

5 Appendix: First order ordinary differential equations

For implicit first order ordinary differential equations, we can prove the corresponding result of Theorem 1.1 by using the Legendre transformation.

Theorem 5.1 ([2, 7]) *Let $G(x, y, p) = 0$ be an implicit first order ordinary differential equation at z_0 . If $G_x + pG_y \neq 0$ at z_0 , then there exists the unique geometric solution around z_0 .*

First we quickly review some notions for implicit first order ordinary differential equations (cf. [5, 6, 10]). We consider an implicit first order ordinary differential equation

$$G\left(x, y, \frac{dy}{dx}\right) = 0,$$

where G is a smooth function defined on an open subset U in $J^1(\mathbb{R}, \mathbb{R})$ such that $\text{grad } G \neq 0$ at any point $z_0 = (x_0, y_0, p_0)$ in U . Let (x, y, p) be a local coordinate of $J^1(\mathbb{R}, \mathbb{R})$ and $K \subset$

$TJ^1(\mathbb{R}, \mathbb{R})$ be the canonical contact structure on $J^1(\mathbb{R}, \mathbb{R})$. It is well-known that locally the contact structure is given by the set of the kernel of the 1-form $\theta = dy - pdx$.

A *smooth solution* (or, a *classical solution*) of $G = 0$ around z_0 is a smooth function germ $y = g(x)$ at a point t_0 such that $(t_0, g(t_0), g'(t_0)) = z_0$ and $G(x, g(x), g'(x)) = 0$. In other words, there exists a smooth function germ $g : (\mathbb{R}, t_0) \rightarrow \mathbb{R}$ such that $j^1g : (\mathbb{R}, t_0) \rightarrow (G^{-1}(0), z_0)$. It is easy to check that the map j^1g is a Legendrian immersion germ (i.e. $(j^1g)^*\theta = 0$).

More generally, a *geometric solution* of $G = 0$ around z_0 is a Legendrian immersion germ $\gamma : (\mathbb{R}, t_0) \rightarrow (G^{-1}(0), z_0)$, namely, $\gamma' \neq 0$, $\gamma^*\theta = 0$ and $G(\gamma(t)) = 0$ for each $t \in (\mathbb{R}, t_0)$.

A notion of Legendre transformation can be used to set up a dual relationship between equations. We adopt another coordinate system (X, Y, P) of $J^1(\mathbb{R}, \mathbb{R})$ by $X = p, Y = xp - y, P = x$. We refer to a smooth mapping $L : J^1(\mathbb{R}, \mathbb{R}) \rightarrow J^1(\mathbb{R}, \mathbb{R})$ given by

$$L(x, y, p) = (p, xp - y, x)$$

as a *Legendre transformation* (cf. [2, 4, 6, 7]). By definition, we have $L^{-1}(X, Y, P) = (P, XP - Y, X)$. If we apply the Legendre transformation to our equation $G = 0$, we obtain a new equation

$$G^*(X, Y, P) = G \circ L^{-1}(X, Y, P) = G(P, XP - Y, X) = 0$$

in the coordinate system (X, Y, P) . If we calculate the partial derivatives at the point (X_0, Y_0, P_0) corresponding to (x_0, y_0, p_0) , we can show the following.

$$\begin{aligned} G_X^*(X_0, Y_0, P_0) &= (G_p + xG_y)(x_0, y_0, p_0), \\ G_Y^*(X_0, Y_0, P_0) &= -G_y(x_0, y_0, p_0), \\ G_P^*(X_0, Y_0, P_0) &= (G_x + pG_y)(x_0, y_0, p_0). \end{aligned} \tag{6}$$

Proof of Theorem 5.1. Suppose that $\gamma : (\mathbb{R}, t_0) \rightarrow (G^{-1}(0), z_0)$ is a geometric solution. Since the Legendre transformation L is a contact diffeomorphism (contactomorphism), $L \circ \gamma : (\mathbb{R}, t_0) \rightarrow (G^{*-1}(0), L(z_0))$ is also a geometric solution and vice versa. By (6), if $G = 0$ satisfies $G_x + pG_y \neq 0$ at z_0 , then $G^* = 0$ satisfies $G_P^* \neq 0$ at $L(z_0)$ so that there exists the unique geometric solution of $G^* = 0$ around $L(z_0)$. Since $G = G^* \circ L$, we have the unique geometric solution of $G = 0$ around z_0 . \square

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