On the Stokes resolvent equations in locally uniform L^p spaces in exterior domains

Matthias Geissert and Yoshikazu Giga

Abstract. The Stokes resolvent equations are studied in locally uniform L^p spaces where the domain is an exterior of a bounded domain. The unique existence of a solution of the Stokes resolvent equations is proved with a resolvent estimate. In particular, the analyticity of the Stokes semigroup is established. An interesting aspect of locally uniform L^p spaces is that these spaces contain non-decaying functions.

1. Introduction

In this note we consider the Stokes resolvent equations in locally uniform L^p spaces in an exterior domain, which is a complement of the closure of a bounded open set. We shall prove the analyticity of the Stokes semigroup in these spaces. Note that these spaces contain non-decaying functions. Although there is a huge literature for the analyticity of the Stokes semigroup, results are only known for spaces which exclude non-decaying functions if the domain is an exterior domain.

Throughout this note let $p \in (1, \infty)$ and $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an exterior domain with $C^{2+\mu}$-boundary for some $\mu \in (0, 1)$ and let $G = \Omega$ or $G = \mathbb{R}^n$. We consider the Stokes equations

$$\begin{aligned} \lambda u - \Delta u + \nabla \pi = f, &\quad \text{in } G \\ \text{div } u = 0, &\quad \text{in } G \\ u = 0, &\quad \text{on } \partial G \end{aligned} \tag{1}$$

in locally uniform spaces, i.e.

$$L^p_{\text{uloc}}(G) = \{u \in L^p_{\text{loc}}(G) : \|u\|_{L^p_{\text{uloc}}(G)} < \infty\},$$

where

$$\|u\|_{L^p_{\text{uloc}}(G)} = \sup_{x_0 \in \mathbb{Z}^n} \|u\|_{L^p(B(x_0,2)\cap G)}.$$

Note that the choice of radius 2 for the balls is not important. Indeed, any radius r such that $\Omega \subset \bigcup_{i\in\mathbb{N}} B(x_i, r)$ leads to the same spaces $L^p_{\mathrm{uloc}}(G)$. There are even more possibilities to define locally uniform spaces, see [2] and [7].

Our aim is to show that (1) has a unique solution for solenoidal f in locally uniform L^p spaces in exterior domains and establish a resolvent estimate for large λ which yields analyticity of the Stokes semigroup (Theorem 3.1 and Theorem 3.4).

The advantage of locally uniform spaces is that $L^p_{\mathrm{uloc}}(\Omega)$ inherit many properties of the usual $L^p(\Omega)$ spaces but it contains non-decaying functions. In particular, $L^\infty(\Omega) \subset L^p_{\mathrm{uloc}}(\Omega)$.

Since locally uniform spaces coincide with the usual L^p-spaces if the domain is bounded, unbounded domains are of interest only. Unfortunately, we cannot expect the Helmholtz-projection to be bounded since it is unbounded in locally uniform spaces in $\mathbb{R}^n$. Up to now, [7] is the only work that deals with the Navier-Stokes equations in locally uniform spaces. The authors of [7] prove existence and uniqueness of a mild solution to the Navier-Stokes equations in $\mathbb{R}^n$ by using a variant of the Fujita-Kato iteration. In order to do so, they use kernel estimates for the heat-semigroup to show $L^p - L^q$ smoothing estimates. For further development see [8].

In contrast to the case $\mathbb{R}^n$ there are no kernel estimates for exterior domains available. However, we can construct a solution of (1) using the resolvent of the Laplacian in $\mathbb{R}^n$ in locally uniform spaces, see [2], and the solution of the generalized Stokes resolvent problem in $L^p(\Omega)$, see [4]. This is possible since the boundary of Ω is compact and thus $L^p(\partial\Omega) = L^p_{\mathrm{uloc}}(\partial\Omega)$, see the proof of Theorem 3.1 below.

The Stokes resolvent problem has not yet been studied much in a space which contains non-decaying functions if G is a domain with non-empty boundary. A few exception is a result by Desch, Hieber and Prüss [3] which established the boundedness and the analyticity of the Stokes semigroup in L^∞ space if the domain is a half space by using an explicit representation of a solution. To show existence and uniqueness of a solution of the Navier-Stokes equations the analyticity of the semigroup is usually not enough so we do not touch this problem in this note.

This work was initiated when the first author visited the Department of Mathematics of Hokkaido University and the Graduate School of Mathematical Sciences of University of Tokyo. Their hospitality is gratefully acknowledged. The work of the first author is partly supported by the Japan program of the German Research Foundation (DFG) and the German Union of Mathematicians (DMV). The work of the second author is partly supported by the Grant-in-Aid for Scientific Research, No. 17654037, No. 18204011, the Japan Society of the Promotion of Science (JSPS) and by COE 'Mathematics of Nonlinear Structures via Singularities' (Hokkaido University) sponsored by JSPS.

2. Preliminaries

Analogous to the homogeneous Sobolev space $\hat{W}^{1,p}(G)$ we define

$$\hat{W}^{1,p}_{\mathrm{uloc}}(G) = \{u \in L^p_{\mathrm{loc}}(\overline{G}) : \nabla u \in L^p_{\mathrm{uloc}}(G)\}.$$

Next, we define the space of solenoidal vector fields.

$$L^p_{\mathrm{ul}\sigma}(G) = \{u \in L^p_{\mathrm{uloc}}(G) : \mathrm{div}\ u = 0,\ u \cdot \nu = 0 \text{ on } \partial G\}.$$

Here, ν denotes the outer normal and the boundary condition $u \cdot \nu = 0$ on ∂G is understood in the sense of the trace theorem based on Gauss' divergence theorem similar as in the L^p-setting. For the convenience of the reader we discuss the differences to the proof for the L^p-setting given in [5, Chapter III.2]. A major difference to the usual L^p-setting is that $C^\infty_c(\overline{\Omega})$ is not dense in

$$H_p(\Omega) := \left\{u \in L^1_{\mathrm{loc}}(\Omega) : \|u\|_{H_p} < \infty\right\},$$

where $\|u\|_{H_p} = \|u\|_{L^p(\Omega)} + \|\mathrm{div}\ u\|_{L^p(\Omega)}$. But it is not difficult to show that $BC^\infty(\overline{\Omega}) = \{u \in C^\infty(\overline{\Omega}) : \partial^\alpha u \text{ is bounded for all } \alpha \in \mathbb{N}^n\}$ is dense in

$$H_{p,\mathrm{uloc}}(\Omega) := \left\{u \in L^1_{\mathrm{loc}}(\Omega) : \|u\|_{H_{p,\mathrm{uloc}}} < \infty\right\},$$

where $\|u\|_{H_{p,\mathrm{uloc}}} = \|u\|_{L^p_{\mathrm{uloc}}(\Omega)} + \|\mathrm{div}\ u\|_{L^p_{\mathrm{uloc}}(\Omega)}$. For $u \in BC^\infty(\overline{\Omega})$ we obtain

$$\int_{\partial\Omega} u\nu\Psi \mathrm{d}x = \int_{\Omega} u\nabla\Psi \mathrm{d}x + \int_{\Omega} \Psi \mathrm{div}\ u \mathrm{d}x, \quad \Psi \in C^\infty_c(\mathbb{R}^n). \tag{2}$$

Obviously, the right hand side does not make sense for all $\Psi \in W^{1,p'}(\Omega)$, where $1/p + 1/p' = 1$. Hence, we have to impose stronger decay properties on Ψ for $|x| \to \infty$ in order to makes sense out of (2). More precisely, let us define

$$L^p_{\mathrm{sum}}(G) = \{u \in L^p_{\mathrm{loc}}(G) : \|u\|_{L^p_{\mathrm{sum}}(G)} < \infty\},$$

where

$$\|u\|_{L^p_{\mathrm{sum}}(G)} = \sum_{x_0 \in \mathbb{Z}^n} \|u\|_{L^p(B(x_0,2)\cap G)}.$$

In contrast to the situation for locally uniform spaces, $C^\infty_c(G)$ is dense in $L^p_{\mathrm{sum}}(G)$. Furthermore, we have $L^p_{\mathrm{sum}}(G) \subsetneq L^p(G) \subsetneq L^p_{\mathrm{uloc}}(G)$.

Since $C^\infty_c(\overline{\Omega})$ is dense in $W^{1,p'}_{\mathrm{sum}}(\Omega)$, by Hölder's inequality, (2) is valid for $\varphi \in W^{1,p'}_{\mathrm{sum}}(\Omega)$ with $1/p + 1/p' = 1$. Now, we can proceed as in [5, Chapter III.2] since the trace space of $W^{1,p'}_{\mathrm{sum}}(\Omega)$ is $W^{1-1/p',p'}(\partial\Omega)$.

Lemma 2.1. *Let $1/p + 1/p' = 1$. Then*

$$L^p_{\mathrm{ul}\sigma}(G) = \left\{ f \in L^p_{\mathrm{uloc}}(G) : \int_G f\nabla\varphi \mathrm{d}x = 0 \textit{ for all } \varphi \in W^{1,p'}_{\mathrm{sum}}(G) \right\}. \tag{3}$$

Proof. This easily follows from (2). □

Next, we characterize all $\pi \in \hat{W}^{1,p}_{\mathrm{uloc}}(G)$ satisfying $\nabla\pi \in L^p_{\mathrm{ul}\sigma}(G)$. We start with the case $G = \mathbb{R}^n$.

Lemma 2.2. *Let $\pi \in \hat{W}^{1,p}_{\mathrm{uloc}}(\mathbb{R}^n)$ satisfy $\nabla\pi \in L^p_{\mathrm{ul}\sigma}(\mathbb{R}^n)$. Then $\nabla\pi = K$ for some $K \in \mathbb{C}^n$.*

Proof. We only prove the assertion for $n \geq 3$. The case $n = 2$ follows similarly. Let $\alpha, \beta \in \mathbb{N}_0^n$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$. We set $\Psi = E * \partial^\alpha\varphi$, where E denotes the fundamental solution of the Laplace equation. Then, an explicit calculation for $x \notin \mathrm{supp}\ \varphi$ yields

$$\left|\partial^\beta\Psi(x)\right| = \left|\left((\partial^{\alpha+\beta}E) * \varphi\right)(x)\right| \leq \frac{C(\varphi)}{\mathrm{dist}\ (x, \mathrm{supp}\ \varphi)^{n-2+|\alpha|+|\beta|}}.$$

Moreover, $\Psi \in C^\infty(\mathbb{R}^n)$ and $\Delta\Psi = \partial^\alpha\varphi$.

Since $\nabla\pi \in L^p_{\mathrm{uloc}}(\Omega)$ is harmonic, we have $\nabla\pi \in L^\infty(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$. Hence, $|\pi(x) - \pi(0)| \leq \|\nabla\pi\|_{L^\infty(\mathbb{R}^n)}|x|$, $x \in \mathbb{R}^n$. Therefore, integration by parts yields

$$0 = \int_{\mathbb{R}^n} \nabla\pi\nabla\Psi \mathrm{d}x = -\int_{\mathbb{R}^n} \pi\Delta\Psi \mathrm{d}x = -\int_{\mathbb{R}^n} \pi\partial^\alpha\varphi \mathrm{d}x = \int_{\mathbb{R}^n} \partial^\alpha\pi\varphi \mathrm{d}x$$

provided $|\alpha|$ is large enough. Since $\pi \in \hat{W}^{1,p}_{\mathrm{uloc}}(\mathbb{R}^n)$ by assumption, $\nabla\pi = K$ for some $K \in \mathbb{C}^n$. □

In particular, it follows from the previous lemma that $K \in L^p_{\mathrm{ul}\sigma}(\mathbb{R}^n)$. Hence, $L^p_\sigma(\mathbb{R}^n) \subsetneq L^p_{\mathrm{ul}\sigma}(\mathbb{R}^n)$.

Lemma 2.3. *Let $\pi \in \hat{W}^{1,p}_{\mathrm{uloc}}(\Omega)$ satisfy $\nabla\pi \in L^p_{\mathrm{ul}\sigma}(\Omega)$. Then $\pi = p_K + Kx$ for some $K \in \mathbb{C}^n$ and $p_k \in \hat{W}^{1,p}(\Omega)$, where p_K is uniquely determined. In particular, if $\pi \in \hat{W}^{1,p}(\Omega)$ then $\nabla\pi \equiv 0$.*

Proof. Let $\tilde{\pi}$ denote a smooth extension of π to $\mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} \nabla\tilde{\pi}\nabla\Psi \mathrm{d}x = \int_{\Omega} \nabla\pi\nabla\Psi \mathrm{d}x + \int_{\Omega^c} \nabla\tilde{\pi}\nabla\Psi \mathrm{d}x = \int_{\Omega^c} f\nabla\Psi \mathrm{d}x, \quad \Psi \in C_c^\infty(\mathbb{R}^n),$$

where $f = \nabla\tilde{\pi}|_{\Omega^c}$. Then the solution $\hat{\pi}$ of $\Delta\hat{\pi} = \mathrm{div}\ f$ in $\mathbb{R}^n$ satisfies $\hat{\pi} \in \hat{W}^{1,p}(\mathbb{R}^n)$. Since

$$\int_{\mathbb{R}^n} \nabla(\tilde{\pi} - \hat{\pi})\nabla\Psi \mathrm{d}x = 0, \quad \Psi \in C_c^\infty(\mathbb{R}^n),$$

and $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}_{\mathrm{sum}}(\mathbb{R}^n)$, by Lemma 2.2, there exists $K \in \mathbb{C}^n$ with $\nabla(\tilde{\pi} - \hat{\pi}) = K$. Hence, $\nabla\pi = \nabla\hat{\pi}|_\Omega + K$. □

3. The Stokes Operator in L^p_{uloc} Spaces in Exterior Domains

In this section we present our main results for the Stokes operator in locally uniform spaces in exterior domains. We define $\Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \theta\}$. Here and in the following, we always assume $\theta \in (0, \pi)$.

Theorem 3.1. *Fix $\gamma > 0$ and let $\lambda \in \Sigma_\theta$ with $|\lambda| \geq \gamma$. Then, for $f \in L^p_{\text{ul}\sigma}(\Omega)$ there exists $u \in W^{2,p}_{\text{uloc}}(\Omega) \cap L^p_{\text{ul}\sigma}(\Omega)$ and $p \in \hat{W}^{1,p}(\Omega)$ satisfying (1) with $G = \Omega$. Moreover, there exists $C > 0$, independent of u, p, f and λ, such that*

$$\lambda \|u\|_{L^p_{\text{uloc}}(\Omega)} + \|u\|_{W^{2,p}_{\text{uloc}}(\Omega)} + \|\nabla p\|_{L^p(\Omega)} \leq C \|f\|_{L^p_{\text{ul}\sigma}(\Omega)}. \tag{4}$$

Proof. Let $\tilde{f}$ denote the extension of f by 0. By [2, Proposition 2.1 and Theorem 2.1] there exists a solution u_1 to

$$\lambda u_1 - \Delta u_1 = \tilde{f}, \quad \text{in } \mathbb{R}^n,$$

satisfying

$$\|u_1\|_{W^{2,p}_{\text{uloc}}(\mathbb{R}^n)} + |\lambda| \|u_1\|_{L^p_{\text{uloc}}(\mathbb{R}^n)} \leq C_1 \|\tilde{f}\|_{L^p_{\text{uloc}}(\mathbb{R}^n)} = C_1 \|f\|_{L^p_{\text{uloc}}(\Omega)}, \tag{5}$$

where $C_1 > 0$ is independent of f. Furthermore, we have div $u_1 = 0$. However, the boundary conditions are not fulfilled since u_1 is a solution in the whole space only.

Since Ω^c is compact, $u_1|_{\Omega^c} \in W^{2,p}(\Omega)$. Let E denote a strong 2-extension operator for Ω^c (see [1, Thm. 5.22]) and set $u_2 = Eu_1$. We then have $u_2 = u_1$ in Ω^c, and there exist $C_2, C_3 > 0$, independent of u_1, such that

$$\|u_2\|_{W^{s,p}(\mathbb{R}^n)} \leq C_2 \|u_1\|_{W^{s,p}(\Omega^c)} \leq C_2 C_3 \|u_1\|_{W^{s,p}_{\text{uloc}}(\mathbb{R}^n)}, \quad s = 0, 1, 2. \tag{6}$$

By [4, Thm. 2.1], there exists $u_3 \in W^{2,p}(\Omega)$, $p_3 \in \hat{W}^{1,p}(\Omega)$ such that

$$\begin{aligned} \lambda u_3 - \Delta u_3 + \nabla p_3 &= \lambda u_2 - \Delta u_2, && \text{in } \Omega, \\ \text{div } u_3 &= \text{div } u_2, && \text{in } \Omega, \\ u_3 &= 0, && \text{on } \Omega. \end{aligned}$$

Moreover, it follows from (5), (6) and [4, Thm. 2.1] that

$$\begin{aligned} |\lambda| \|u_3\|_{L^p(\Omega)} + \|\nabla^2 u_3\|_{L^p(\Omega)} + \|\nabla p\|_{L^p(\Omega)} &\leq C_4 \left(\|u_2\|_{W^{2,p}(\Omega)} + |\lambda| \|u_2\|_{L^p(\mathbb{R}^n)} \right) \\ &\leq C_1 C_2 C_3 C_4 \|f\|_{L^p_{\text{uloc}}(\Omega)}, \end{aligned}$$

where C_4 is independent of u_2 but it may depend on γ. Finally, we set $u := u_1 - u_2 + u_3$ and $p := p_3$. Then (u, p) satisfies (4) and

$$\begin{aligned} \lambda u - \Delta u + \nabla p &= f \text{ in } \Omega, \\ \text{div } u &= 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

The proof is complete. □

Next, we investigate uniqueness of solutions to (1). Again, we start with the case $G = \mathbb{R}^n$.

Lemma 3.2. *Let* $p \in (1,\infty)$, $\lambda \in \Sigma_\theta \cup \{0\}$. *Assume that* $u \in W^{2,p}_{\mathrm{uloc}}(\mathbb{R}^n)$ *and* $\pi \in \hat{W}^{1,p}_{\mathrm{uloc}}(\mathbb{R}^n)$ *satisfy* (1) *with* $f \equiv 0$ *and* $G = \mathbb{R}^n$. *Then* $\pi = \lambda K x$ *and* $u = K$ *for some* $K \in \mathbb{C}^n$.

Proof. Multiplying (1) by $\nabla\Psi$, where $\Psi \in W^{1,p'}_{\mathrm{sum}}(\mathbb{R}^n)$, and integrating by parts, we obtain

$$\int_{\mathbb{R}^n} \nabla\pi\nabla\Psi \mathrm{d}x = 0.$$

Hence, by Lemma 2.2, $\nabla\pi = K$ for some $K \in \mathbb{C}^n$. Obviously, $\tilde{u} := K/\lambda$ and $\pi = Kx$ is a solution of (1) for $\lambda \neq 0$. Since the solution is unique by [2, Proposition 2.1] the lemma follows for $\lambda \neq 0$. The case $\lambda = 0$ follows by standard arguments using the fact that ∇u is harmonic. □

Lemma 3.3. *Let* $p \in (1,\infty)$, $\lambda \in \Sigma_\theta$ *and let* $u \in W^{2,p}_{\mathrm{uloc}}(\Omega)$ *and* $\pi \in \hat{W}^{1,p}_{\mathrm{uloc}}(\Omega)$ *satisfy* (1) *with* $f = 0$ *and* $G = \Omega$. *Then* $u = u_K + K$ *and* $\pi = \pi_K + \lambda K x$ *with some* $K \in \mathbb{C}^n$, $u_K \in W^{2,p}(\Omega)$ *and* $\pi_K \in \hat{W}^{1,p}(\Omega)$. *In particular, if* $\pi \in \hat{W}^{1,p}(\Omega)$, *then* $u = 0$, $\nabla\pi = 0$.

Proof. We follow the ideas of the proof of [9, Theorem 1.2]. Let $\tilde{u}$, $\tilde{\pi}$ be a (smooth) extension to $\mathbb{R}^n$. Then $\tilde{u}$ and $\tilde{\pi}$ solve

$$\begin{aligned} \lambda\tilde{u} - \Delta\tilde{u} + \nabla\tilde{\pi} &= \tilde{f}, \quad \text{in } \mathbb{R}^n \\ \mathrm{div}\ \tilde{u} &= \tilde{g}, \quad \text{in } \mathbb{R}^n \end{aligned}$$

where $\tilde{g} := \mathrm{div}\ \tilde{u}$ and $\tilde{f} = \lambda\tilde{u} - \Delta\tilde{u} + \nabla\tilde{\pi}$. Note that $\tilde{g}$ and $\tilde{f}$ are compactly supported. Hence, $\tilde{g} \in W^{1,p}(\mathbb{R}^n)$ and $\tilde{f} \in L^p(\Omega)$. Taking divergence, we obtain

$$\Delta\tilde{\pi} = \mathrm{div}\ \tilde{f} - \lambda\tilde{g} - \Delta\tilde{g} = \mathrm{div}\ \tilde{f} - \lambda\mathrm{div}\ \tilde{u} - \Delta\tilde{g}. \tag{7}$$

We set $\hat{\pi} = E * (\mathrm{div}\ \tilde{f} - \lambda\mathrm{div}\ \tilde{u}) + \tilde{g}$, where E denotes the fundamental solution of the Laplace equation. It then follows that $\hat{\pi} \in \hat{W}^{1,p}(\mathbb{R}^n)$. Moreover, $\hat{\pi}$ satisfies (7). Hence,

$$\hat{u} := (\lambda - \Delta)^{-1}(\tilde{f} - \nabla\hat{\pi}) \in W^{2,p}(\mathbb{R}^n) \cap L^p_\sigma(\mathbb{R}^n)$$

and $\hat{\pi}$ satisfies (1) with $G = \mathbb{R}^n$ and $f = 0$. Therefore, Lemma 3.2 yields $\hat{u} - \tilde{u} = K$ and $\hat{\pi} - \tilde{\pi} = \lambda K x$ for some $K \in \mathbb{R}^n$. In particular, $u = K - \hat{u}$ and $\pi = \hat{\pi} - \lambda K x$. If $\pi \in \hat{W}^{1,p}(\Omega)$, then K must be zero so that $u \in W^{2,p}(\Omega)$ and $\pi \in \hat{W}^{1,p}(\Omega)$. By uniqueness results in $L^p(\Omega)$ (see [6], [4]), we have $u = 0$ and $\nabla\pi = 0$. □

Our existence and uniqueness result yields the analyticity of the Stokes semigroup in locally uniform L^p spaces. Let $R(\lambda)f$ denote the solution u of (1) in Theorem 3.1. The estimate (4) implies that $R(\lambda)$ is a bounded linear operator from $L^p_{\mathrm{ul}\sigma}(\Omega)$ to $W^{2,p}_{\mathrm{uloc}}(\Omega)$ for $\lambda \in \Sigma = \mathbb{C}\backslash(-\infty, 0]$. We define a closed linear operator in $L^p_{\mathrm{ul}\sigma}(\Omega)$ by

$$A := \lambda I - R(\lambda)^{-1}$$

whose domain equals the range of $R(\lambda)$ where $\lambda \in \Sigma$. We call this operator the Stokes operator in $L^p_{\mathrm{ul}\sigma}(\Omega)$. Apparently, the definition depends on λ. However, we easily obtain from (1) the 'resolvent identity'

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu) = (\mu - \lambda)R(\mu)R(\lambda)$$

by observing that the difference $w = R(\lambda)f - R(\mu)f$ solves

$$\begin{aligned} (\lambda - \Delta)w + \nabla q &= (\mu - \lambda)R(\mu)f && \text{in } G \\ \operatorname{div} w &= 0 && \text{in } G \\ w &= 0 && \text{on } \partial G \end{aligned}$$

with some $q \in \hat{W}^{1,p}(\Omega)$. The resolvent identity implies that the definition of the operator A is independent of $\lambda \in \Sigma$. Now, Theorem 3.1 yields the analyticity of the semigroup generated by A.

Theorem 3.4. *The operator $-A$ generates an analytic semigroup e^{-tA} in $L^p_{\mathrm{ul}\sigma}(\Omega)$.*

Remark 3.5. *The estimate* (4) *in Theorem* 3.1 *is not enough to claim that e^{-tA} is a bounded analytic semigroup since* (4) *is not uniform near $\lambda = 0$. Moreover, e^{-tA} is not expected to be a C_0-semigroup since the domain is not dense in $L^p_{\mathrm{ul}\sigma}(\Omega)$ and it is not C_0 even for $G = \mathbb{R}^n$.*

References

[1] Robert A. Adams. *Sobolev Spaces.* Academic Press, New York-London, 1975.

[2] J. M. Arrieta, A. Rodriguez-Bernal, J. W. Cholewa, and T. Dlotko. Linear parabolic equations in locally uniform spaces. *Math. Models Methods Appl. Sci.*, **14(2)**,253–293, 2004.

[3] W. Desch, M. Hieber and J. Prüss, L^p-theory of the Stokes equation in a half space. *J. Evol. Equ. 1* , **1**, (2001), 115-142.

[4] R. Farwig and H. Sohr. Generalized resolvent estimates for the Stokes system in bounded and unbounded domains. *J. Math. Soc. Japan*, 46(4):607–643, 1994.

[5] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I*, Springer-Verlag, New York, 1994.

[6] Y. Giga and H. Sohr, On the Stokes operator in exterior domains. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* , **36**, (1989), 103-130.

[7] Y. Maekawa and Y. Terasawa. The Navier-Stokes equations with initial data in uniformly local L^p spaces. *Differential Integral Equations*, **19(4)**, 369–400, 2006.

[8] Y. Maekawa, Y. Terasawa and T. Yoneda, Navier-Stokes equations in the amalgam spaces $(L^p,\ \ell^q)$. In preparation.

[9] W. Varnhorn. Solutions in Lebesgue spaces to the stationary Stokes equations in unbounded open sets of $\mathbf{R}^n$. *Adv. Math. Sci. Appl.*, **4(1)**,191–208, 1994.

Matthias Geissert
Schlossgartenstr. 7
FB Mathematik
TU Darmstadt
64289 Darmstadt
Germany
e-mail: geissert@mathematik.tu-darmstadt.de

Yoshikazu Giga
Graduate School of Mathematical Sciences
University of Tokyo
Komaba, 3-8-1, Meguro
Tokyo 153-8914
Japan
e-mail: labgiga@ms.u-tokyo.ac.jp