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ON A LINEARIZED OPERATOR OF THE EQUATION FOR BURGERS VORTICES

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ABSTRACT. Burgers vortices have been used as a model which expresses concentrated vorticity fields in turbulence. Several numerical results indicate that as the vortex Reynolds number is increasing, the associated Burgers vortex becomes more radially symmetric and has simpler structures. The purpose of this paper is to give a mathematical explanation for these numerical observations by studying a linearized operator of the equation for Burgers vortices. Based on the previous works by Th. Gallay and C. E. Wayne, we obtain some estimates and spectrum behavior of the linearized operator at large Reynolds numbers which are compatible with the numerical observations.

1. INTRODUCTION

In this paper, we are interested in Burgers vortices, which give stationary solutions to the three dimensional Navier-Stokes equations with a background straining flow. More precisely, we consider a viscous incompressible fluid in \mathbb{R}^3 whose velocity field takes the following form:

$$(1.1) \quad U(x_1, x_2, x_3) = (\gamma_1 x_1, \gamma_2 x_2, \gamma_3 x_3) + (u_1(x_1, x_2), u_2(x_1, x_2), 0),$$

where $\gamma_1 = -(1 + \lambda)/2$, $\gamma_2 = -(1 - \lambda)/2$, $\gamma_3 = 1$ for $\lambda \in [0, 1)$. Here, the velocity field $(\gamma_1 x_1, \gamma_2 x_2, \gamma_3 x_3)$ expresses a given straining flow with an asymmetric parameter λ , and $u = (u_1(x_1, x_2), u_2(x_1, x_2), 0)$ is an unknown two dimensional perturbation velocity field with $\partial_1 u_1 + \partial_2 u_2 = 0$. The associated vorticity field $\Omega = \nabla \times U$ is given by

$$(1.2) \quad \Omega(x_1, x_2, x_3) = (0, 0, \omega(x_1, x_2)), \quad \omega = \partial_1 u_2 - \partial_2 u_1.$$

We assume that U solves the three dimensional viscous incompressible Navier-Stokes equations. Then ω satisfies the equation:

$$(1.3) \quad -\mathcal{L}\omega + (u, \nabla)\omega = \lambda \mathcal{M}\omega, \quad x \in \mathbb{R}^2,$$

where

$$(1.4) \quad \mathcal{L} = \Delta + \frac{x}{2} \cdot \nabla + 1, \quad \mathcal{M} = \frac{1}{2}(x_1 \partial_1 - x_2 \partial_2).$$

The velocity field u is obtained by the vorticity ω via the Biot-Savart law:

$$(1.5) \quad u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(y) dy =: K * \omega, \quad x^\perp = (-x_2, x_1).$$

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The case $\lambda = 0$ is called axisymmetric. Let G be the two dimensional Gauss kernel:

$$(1.6) \quad G(x) = \frac{1}{4\pi} e^{-\frac{|x|^2}{4}}.$$

In this case, we can check that the function αG solves the equation (1.3) for any $\alpha \in \mathbb{R}$. This family of solutions is called axisymmetric Burgers vortices; see J. M. Burgers [1]. The velocity field $v^G := K * G$ is explicitly represented by

$$(1.7) \quad v^G(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2} (1 - e^{-\frac{|x|^2}{4}}).$$

The parameter α represents the circulation of the velocity field at infinity, and the value $|\alpha|$ is called the (vortex) Reynolds number. The stability of the axisymmetric Burgers vortices was firstly discussed by Y. Giga and T. Kambe [8] for small $|\alpha|$ (see also the related work by A. Carpio [2], and Y. Giga and M.-H. Giga [7]), and this smallness assumption was removed by Th. Gallay and C. E. Wayne [4].

Non-axisymmetric Burgers vortices (1.3) have been studied numerically by many authors; see A. C. Robinson and P. G. Saffman [14], S. Kida and K. Ohkitani [9], H. K. Moffat, S. Kida, and K. Ohkitani [11], A. Prochazka and D. I. Pullin [12], and A. Prochazka and D. I. Pullin [13]. Their results imply that Burgers vortices exist when λ/R is sufficiently small at least for $\lambda \in [0, 1)$, where R represents the vortex Reynolds number (in our case, R coincides with $|\alpha|$). In [5], Th. Gallay and C. E. Wayne gave the rigorous proof for the existence and stability of Burgers vortices for any $\alpha \in \mathbb{R}$ when λ is sufficiently small. However, if asymmetric parameter λ is not small, the equation (1.3) has not yet studied enough. As far as the author knows, the only (rigorous) result is obtained by Th. Gallay and C. E. Wayne [6], in which they showed the existence and stability of Burgers vortices for $\lambda \in [0, 1)$ if α is sufficiently small depending on λ (they also discuss the three dimensional stability of Burgers vortices in [6]). Roughly speaking, when λ is not small, the term $\lambda \mathcal{M}\omega$ breaks the symmetricity of the structure of the equation (1.3) and can lead to the slow spatial decay in x_2 direction. This causes technical difficulties in controlling the nonlinear term $(u, \nabla)\omega$ when the Reynolds number is not small.

On the other hand, numerical results indicate that as the Reynolds number is increasing, the solution of (1.3) becomes more radially symmetric and obtains better stability for any λ ; [14], [9], [13]. These observations suggest that Burgers vortices have simpler structures for high Reynolds numbers. There seems to be no rigorous explanation for this numerical observation, even when λ is small.

The purpose of this paper is to reveal this mechanism, based on the results by Th. Gallay and C. E. Wayne [4] and [5], in terms of the analysis of some linear operators. To describe our main results precisely, we introduce the function spaces and these operators. They are also effectively used in [4], [5]. In order to make it clear the correspondence with their results, we shall use the same notations as in [4], [5].

Let X, Y be the complex Hilbert spaces

$$(1.8) \quad X = \{w \in L^2(\mathbb{R}^2) \mid G^{-\frac{1}{2}}w \in L^2(\mathbb{R}^2), \int_{\mathbb{R}^2} w dx = 0,$$

$$\langle w_1, w_2 \rangle_X = \int_{\mathbb{R}^2} G^{-1}(x) w_1(x) \overline{w_2(x)} dx\},$$

$$(1.9) \quad Y = \{w \in X \mid \partial_i w \in X, i = 1, 2,$$

$$\langle w_1, w_2 \rangle_Y = \int_{\mathbb{R}^2} G^{-1}(x) (w_1(x) \overline{w_2(x)} + \nabla w_1(x) \nabla \overline{w_2(x)}) dx\}.$$

Let us consider the solution of (1.3) with circulation $\alpha = \int_{\mathbb{R}^2} \omega(x) dx$. Expanding the equation (1.3) around the axisymmetric Burgers vortex αG , we obtain the following equation for $w = \omega - \alpha G$:

$$(1.10) \quad (\mathcal{L} - \alpha\Lambda)w = (v, \nabla)w - \lambda\mathcal{M}(w + \alpha G),$$

$$(1.11) \quad v = K * w,$$

where Λ is the integro-differential operator defined on Y :

$$(1.12) \quad \Lambda w = \Lambda^a w + \Lambda^b w$$

where

$$(1.13) \quad \Lambda^a w = (v^G, \nabla)w, \quad \Lambda^b w = (v, \nabla)G.$$

The domain of \mathcal{L} is defined by

$$(1.14) \quad D(\mathcal{L}) = \{w \in X \mid \Delta w + \frac{x}{2} \cdot \nabla w \in X\}.$$

Several important properties for these operators \mathcal{L} , Λ , and $\mathcal{L} - \alpha\Lambda$ are obtained in [5] (see also [4]):

Lemma 1.1 ([5]).

- (1-1) \mathcal{L} is self-adjoint in X and $-\mathcal{L} \geq \frac{1}{2}$.
- (1-2) $(-\mathcal{L})^{-\frac{1}{2}}$ is compact in X and bounded from X into Y .
- (2-1) Λ is bounded from Y into X .
- (2-2) Λ is skew-symmetric: for any $w_1, w_2 \in Y$, we have $\langle \Lambda w_1, w_2 \rangle_X + \langle w_1, \Lambda w_2 \rangle_X = 0$.
- (3) $(\mathcal{L} - \alpha\Lambda)^{-1}$ is compact in X and bounded from X into Y . Moreover, its operator norm is bounded uniformly in α .

The proof of the above lemma will be omitted. We only remark on the operator \mathcal{L} . As stated in [5], the spectrum of \mathcal{L} consists of the eigenvalues $\{-\frac{n}{2} \mid n = 1, 2, \dots\}$ and the associated eigenspace for $-\frac{n}{2}$ is $(n+1)$ dimensional closed subspace spanned by the Hermite functions $\{\partial_1^{\beta_1} \partial_2^{\beta_2} G\}$ with $\beta_1 + \beta_2 = n$.

This paper will be devoted to investigate the above operators in further details. Our main interest is the properties of the operator $\mathcal{L} - \alpha\Lambda$ when the Reynolds number $|\alpha|$ is large. From this point of view, one will find that the definition of Λ by (1.12) is not so useful because the operator Λ defined above is not closed in X (note that $D(\Lambda) = Y$ in the above setting). To avoid this inconvenience, motivated by the analysis of [4], we redefine Λ by using polar coordinates; see Section 2. Then the redefined operator Λ is shown to be a densely-defined closed operator in X and satisfy $\Lambda^* = -\Lambda$. In fact, one can check that this redefined operator is the closure of the original one defined by (1.12) and (1.13); see Remark 2.2.

From the closedness and the property $\Lambda^* = -\Lambda$ for the redefined operator Λ , we obtain the decomposition of $X = \text{Ker } \Lambda \oplus \overline{\text{Ran } \Lambda}$.

We are now in position to state our main results. Our first step is to characterize the kernel of the operator Λ . This is important for our purpose, since the function in $\text{Ker } \Lambda$ cannot be influenced by the value of circulation α with respect to the action of the operator $\mathcal{L} - \alpha\Lambda$.

Let $\mathbb{P}_S X$ be a closed subspace of X consisting of all radially symmetric functions, i.e.,

$$(1.15) \quad \mathbb{P}_S X = \{w \in X \mid w(Rx) = w(x), \text{ a.e. } x \in \mathbb{R}^2, \text{ for all orthogonal matrix } R\}.$$

The following characterization of $\text{Ker } \Lambda$ is essential.

Theorem 1.1 (Characterization of $\text{Ker } \Lambda$).

$$\text{Ker } \Lambda = \mathbb{P}_S X \oplus \{\beta_1 \partial_1 G + \beta_2 \partial_2 G \mid \beta_i \in \mathbb{C}\}.$$

Let $\mathbb{P}_\Lambda = \mathbb{P}_{\text{Ker } \Lambda}$ be the (orthogonal) projection to the kernel of Λ . Let $\mathbb{Q}_\Lambda := 1 - \mathbb{P}_\Lambda$. One of the immediate and important corollaries of the above theorem is that the operator $\mathcal{L} - \alpha\Lambda$ commutes with the projections \mathbb{P}_Λ or \mathbb{Q}_Λ . More precisely, $\text{Ker } \Lambda$ reduces $\mathcal{L} - \alpha\Lambda$, i.e., $\mathbb{P}_\Lambda(\mathcal{L} - \alpha\Lambda) \subset (\mathcal{L} - \alpha\Lambda)\mathbb{P}_\Lambda$; for any $w \in D(\mathcal{L} - \alpha\Lambda) = D(\mathcal{L})$, we have $\mathbb{P}_\Lambda w \in D(\mathcal{L} - \alpha\Lambda)$, and $\mathbb{P}_\Lambda(\mathcal{L} - \alpha\Lambda)w = (\mathcal{L} - \alpha\Lambda)\mathbb{P}_\Lambda w (= \mathcal{L}\mathbb{P}_\Lambda w)$. Indeed, we first remark that the closed subspace $\mathbb{P}_S X$ reduces \mathcal{L} , since $\mathbb{P}_S X$ is spanned by the eigenfunctions of \mathcal{L} such as

$$(1.16) \quad \mathbb{P}_S X = \overline{\text{span} \{(\Delta)^k G \mid k = 1, 2, \dots\}}.$$

Moreover, since $\partial_i G$ is an eigenfunction of \mathcal{L} , the eigenspace $\{\beta_1 \partial_1 G + \beta_2 \partial_2 G \mid \beta_i \in \mathbb{C}\}$ also reduces \mathcal{L} . From these, we find that $\text{Ker } \Lambda$ reduces $\mathcal{L} - \alpha\Lambda$.

This reduction enables us to decompose the operator $\mathcal{L} - \alpha\Lambda$ as follows.

$$(1.17) \quad \mathcal{L} - \alpha\Lambda = \mathcal{L}|_{\text{Ker } \Lambda} \oplus \mathcal{L}_\alpha$$

where

$$\mathcal{L}_\alpha := (\mathcal{L} - \alpha\Lambda)\mathbb{Q}_\Lambda = (\mathcal{L} - \alpha\Lambda)|_{\overline{\text{Ran } \Lambda}} : D(\mathcal{L}_\alpha) \rightarrow \overline{\text{Ran } \Lambda},$$

with its domain $D(\mathcal{L}_\alpha) = D(\mathcal{L}) \cap \overline{\text{Ran } \Lambda}$. Combined with the above characterization, the following theorems will give a well explanation for the numerical observations stated previously.

Theorem 1.2 (Estimates for \mathcal{L}_α^{-1}).

Let \mathcal{L}_α be the operator defined as above. Then we have

$$(1.18) \quad \lim_{|\alpha| \rightarrow \infty} \|\mathcal{L}_\alpha^{-1}\|_{X \rightarrow X} = 0.$$

Theorem 1.3 (Behavior of the spectrum of \mathcal{L}_α).

Let $\sigma(\mathcal{L}_\alpha)$ be the set of the spectrum of the operator \mathcal{L}_α . Then we have

$$(1.19) \quad r_\alpha := \sup_{\mu \in \sigma(\mathcal{L}_\alpha)} \text{Re}(\mu) \rightarrow -\infty, \quad |\alpha| \rightarrow \infty.$$

Here, $\text{Re}(\mu)$ is the real part of μ .

Remark 1.1. Unfortunately, we do not know how fast $\|\mathcal{L}_\alpha^{-1}\|_{X \rightarrow X}$ goes to 0, or r_α goes to $-\infty$ as $|\alpha| \rightarrow \infty$, since our proof is given by a contradiction argument.

Remark 1.2. Let X_1 be the closed subspace of X defined by

$$(1.20) \quad X_1 = \{w \in X \mid \int_{\mathbb{R}^2} x_i w(x) dx = 0, \quad i = 1, 2\}.$$

Thus X_1 is the orthogonal complement of the eigenspace $\{\beta_1 \partial_1 G + \beta_2 \partial_2 G \mid \beta_i \in \mathbb{C}\}$ in X .

Since X_1 is invariant under the equation (1.3), we find that the problem (1.10) in X can be reduced to the problem in X_1 . Then, the operator Λ in X_1 satisfies

$$(1.21) \quad \text{Ker } \Lambda = \mathbb{P}_S X_1,$$

and Theorem 1.2 and Theorem 1.3 hold for the operator $(\mathcal{L} - \alpha\Lambda)|_{(\mathbb{P}_S X_1)^\perp}$ by the relation $\overline{\text{Ran } \Lambda} = (\mathbb{P}_S X_1)^\perp$. Then Theorem 1.2 suggests that the non-radial symmetric parts of Burgers vortices would be smaller compared with the radially symmetric parts in the case of high Reynolds numbers. This is quite compatible with the numerical observations which say that solutions of (1.3) tend to be more circular as the Reynolds number is increasing. Theorem 1.3 suggests that Burgers vortices would have better stabilities in this case. Detailed applications of our results to the original problem (1.3) will be discussed in the forthcoming paper [10].

Let us state the idea of the proofs of the above theorems. The operator Λ is defined by using the Fourier series expansion with respect to the angular variable of polar coordinates, and solutions to non-local linear ordinary differential equations in the radial variable. This ODEs have singularities at the origin which can lead to the delicate dependence of solutions on the potential part of the ODEs. We make use of the property of this potential part in the proofs of the above theorems.

It seems to be hard to give direct estimates for \mathcal{L}_α^{-1} or eigenvalues μ of the operator \mathcal{L}_α . So we shall prove Theorem 1.2 and Theorem 1.3 by contradiction arguments. Here we give a rough idea for the proof of Theorem 1.3. Theorem 1.2 is shown by a similar argument. Under the assumption of the uniform bound of r_α (r_α is defined in Theorem 1.3), we have a strong convergent sequence of eigenfunctions of \mathcal{L}_α , and its limit is shown to be a nontrivial eigenfunction for a nonzero eigenvalue of the operator Λ . However, by investigating the ODEs used in the definition of Λ again, we conclude that Λ does not have nonzero eigenvalues. Thus we obtain a contradiction and the theorem is proved. In order to show the nonexistence of nonzero eigenvalues of Λ , we essentially use the fact that any eigenfunction for nonzero eigenvalues (if they exist) must be orthogonal to the function $\partial_i G$ which is the eigenfunction of the eigenvalue 0.

This paper is organized as follows. In Section 2, we redefine the operator Λ . In Section 3, we study the kernel of Λ and prove Theorem 1.1. In Section 4, we prove Theorem 1.2. In Section 5, we prove the spectrum behavior of Theorem 1.3. In the proof of Theorem 1.3, Lemma 5.1 (the non-existence of nonzero eigenvalues of Λ) is crucial.

For not sufficiently small λ , the Gaussian weighted L^2 space X will not be suitable to find solutions of (1.3) because of the slower spatial decay in x_2 direction by the term $\lambda\mathcal{M}\omega$. So we give simple observations of the operators $\mathcal{L} - \alpha\Lambda$ and Λ in the polynomial weighted L^2 spaces in Appendix.

2. REDEFINITION OF THE OPERATOR Λ

In this section, we redefine the operator (1.12) by using polar coordinates; see also the argument in [4, Section 4].

Let $w \in X$ and $n \in \mathbb{Z}$. Let \mathbb{P}_n be the orthogonal projection defined by

$$(2.1) \quad \mathbb{P}_n(w) = \omega_n(r)e^{in\theta},$$

where

$$(2.2) \quad \omega_n(r) = \frac{1}{2\pi} \int_0^{2\pi} w(r \cos \theta, r \sin \theta) e^{-in\theta} d\theta.$$

Then, we see that ω_n belongs to the Hilbert space

$$(2.3) \quad \begin{aligned} Z &= \{f : \mathbb{R}_+ \rightarrow \mathbb{C} \mid \int_0^\infty r g^{-1}(r) |f(r)|^2 dr < \infty\}, \\ &< f_1, f_2 >_Z = \int_0^\infty r g^{-1}(r) f_1(r) \overline{f_2(r)} dr. \end{aligned}$$

Here,

$$(2.4) \quad g(r) = \frac{1}{4\pi} e^{-\frac{r^2}{4}}.$$

Let Λ_n^a, Λ_n^b be the linear operators on Z defined by $\Lambda_0^a = \Lambda_0^b = 0$ and

$$(2.5) \quad \Lambda_n^a(\omega) = in\varphi\omega,$$

$$(2.6) \quad \Lambda_n^b(\omega) = -ing\Omega_n(\omega),$$

where

$$(2.7) \quad \varphi(r) = \frac{1}{2\pi r^2} (1 - e^{-\frac{r^2}{4}}),$$

$$(2.8) \quad \Omega_n(\omega)(r) = \frac{1}{4|n|} \left(\int_0^r \left(\frac{s}{r}\right)^{|n|} s\omega(s) ds + \int_r^\infty \left(\frac{r}{s}\right)^{|n|} s\omega(s) ds \right).$$

Note that $\Omega_n(\omega)$ is a solution of the ODE:

$$(2.9) \quad -\frac{1}{r}(r\Omega_n')' + \frac{n^2}{r^2}\Omega_n - \frac{1}{2}\omega = 0$$

such that Ω_n/r is locally integrable and converges to zero at infinity. We define the operator Λ_n by the relation

$$(2.10) \quad \Lambda_n = \Lambda_n^a + \Lambda_n^b.$$

Now let us recall the result in [4].

Lemma 2.1 ([4, Lemma 4.4]). *Let $\omega \in D(\mathcal{L})$. Then, we have*

$$(2.11) \quad \mathbb{P}_n \Lambda^a w = \Lambda^a \mathbb{P}_n w = \Lambda_n^a(\omega_n) e^{in\theta},$$

$$(2.12) \quad \mathbb{P}_n \Lambda^b w = \Lambda^b \mathbb{P}_n w = \Lambda_n^b(\omega_n) e^{in\theta}.$$

Remark 2.1. *We can easily check that the above assertion holds at least for any $w \in Y$.*

From the above lemma, we have the relation $\mathbb{P}_n \Lambda w = \Lambda \mathbb{P}_n w = \Lambda_n(\omega_n) e^{in\theta}$ at least for $w \in Y$. We also see that Λ_n is a bounded operator in Z . From these facts, we redefine the operator Λ as follows:

$$(2.13) \quad \Lambda w := \sum_{n \in \mathbb{Z}} \Lambda_n(\omega_n) e^{in\theta},$$

and its domain is naturally defined by

$$D(\Lambda) = \left\{ w \in X \mid \sum_{n \in \mathbb{Z}} \|\Lambda_n \omega_n\|_Z^2 < \infty \right\}.$$

Since Λ^b is bounded in X (see [5, Lemma 2.3]), the series $\sum_{n \in \mathbb{Z}} \|\Lambda_n^b(\omega_n)\|_Z^2$ converges for any $w \in X$. Thus the domain $D(\Lambda)$ is the domain of the (re-defined) operator Λ^a , $\Lambda^a w := \sum_{n \in \mathbb{Z}} \Lambda_n^a(\omega_n) e^{in\theta}$, that is,

$$(2.14) \quad D(\Lambda) = \left\{ w \in X \mid \sum_{n \in \mathbb{Z}} n^2 \|\varphi \omega_n\|_Z^2 < \infty \right\}.$$

We remark that the redefined operator Λ defined by (2.13), (2.14) and the original one defined by (1.12) coincide with each other at least on Y . The characterization (2.14) for the domain of (redefined) Λ leads to:

Proposition 2.1. *Let Λ be the operator defined by (2.13). Then,*

- (1) Λ is densely-defined and closed in X ,
- (2) $\Lambda^* = -\Lambda$.

Proof. (1) Since $Y \subset D(\Lambda)$, Λ is densely-defined. For each $m \in \mathbb{N}$, set $\mathcal{P}_m = \sum_{|n| \leq m} \mathbb{P}_n$. Let $x_n \in D(\Lambda)$, $x_n \rightarrow w$ in X and $\Lambda x_n \rightarrow y$ in X . Remark that, since each Λ_n is bounded in Z , the operator $\Lambda \mathcal{P}_m: \Lambda \mathcal{P}_m(w) = \sum_{|n| \leq m} \Lambda_n(\omega_n) e^{in\theta}$ is a bounded operator in X for each m and $\mathcal{P}_m \Lambda \subset \Lambda \mathcal{P}_m$. Then, we see that

$$\begin{aligned} \mathcal{P}_m y &= \mathcal{P}_m \lim_{n \rightarrow \infty} \Lambda x_n = \lim_{n \rightarrow \infty} \mathcal{P}_m \Lambda x_n = \lim_{n \rightarrow \infty} \sum_{|l| \leq m} (\mathbb{P}_l \Lambda)(x_n) \\ &= \sum_{|l| \leq m} (\mathbb{P}_l \Lambda) \left(\lim_{n \rightarrow \infty} x_n \right) = \Lambda \mathcal{P}_m w. \end{aligned}$$

Thus, we have

$$\sum_{|n| \leq m} n^2 \|\varphi \omega_n\|_Z^2 \leq 2(\|\mathcal{P}_m y\|_X^2 + \|\Lambda^b \mathcal{P}_m w\|_X^2) \leq 2(\|y\|_X^2 + C\|w\|_X^2) < \infty.$$

Since m is arbitrary, this shows that $w \in D(\Lambda)$ and $y = \Lambda w$, i.e., Λ is closed in X .

(2) It is not difficult to see that $(\Lambda \mathcal{P}_m)^* = -\Lambda \mathcal{P}_m$. Indeed, by direct calculation, we can check that $\Lambda_n^* = -\Lambda_n$ in Z , which gives the claim. Then, from the definition of Λ , for $w_1, w_2 \in D(\Lambda)$, we have

$$\begin{aligned} \langle \Lambda w_1, w_2 \rangle_X &= \lim_{m \rightarrow \infty} \langle \mathcal{P}_m \Lambda w_1, w_2 \rangle_X = \lim_{m \rightarrow \infty} \langle \Lambda \mathcal{P}_m w_1, w_2 \rangle_X \\ &= - \lim_{m \rightarrow \infty} \langle w_1, \Lambda \mathcal{P}_m w_2 \rangle_X = - \lim_{m \rightarrow \infty} \langle w_1, \mathcal{P}_m \Lambda w_2 \rangle_X \\ &= - \langle w_1, \Lambda w_2 \rangle_X. \end{aligned}$$

From this relation, we get $D(\Lambda) \subset D(\Lambda^*)$. Take any $w \in D(\Lambda^*)$. Then, for each m ,

$$\mathcal{P}_m \Lambda^* w = -\Lambda \mathcal{P}_m w$$

by $\mathcal{P}_m \Lambda^* = \mathcal{P}_m^* \Lambda^* \subset (\Lambda \mathcal{P}_m)^*$. So we have

$$\begin{aligned} \sum_{|n| \leq m} n^2 \|\varphi \omega_n\|_Z^2 &\leq 2(\|\Lambda \mathcal{P}_m w\|_X^2 + \|\Lambda^b \mathcal{P}_m w\|_X^2) \\ &\leq 2(\|\mathcal{P}_m \Lambda^* w\|_X^2 + C\|w\|_X^2) \\ &\leq 2(\|\Lambda^* w\|_X^2 + C\|w\|_X^2) < \infty. \end{aligned}$$

Hence $w \in D(\Lambda)$. This completes the proof.

From the above proposition, we obtain a simple corollary.

Corollary 2.1. $X = \text{Ker } \Lambda \oplus \overline{\text{Ran } \Lambda}$.

Proof. Since Λ is closed in X , we have $X = \text{Ker } \Lambda \oplus \overline{\text{Ran } \Lambda^*}$. The assertion immediately follows from the fact $\Lambda^* = -\Lambda$.

Remark 2.2. *The operator Λ defined by (2.13) and (2.14) is in fact the closure of the original one defined by (1.12) and (1.13) (we write this original one as Λ' for convenience). Indeed, for any $w \in X$ with $\sum_{n \in \mathbb{Z}} n^2 \|\varphi \omega_n\|_Z^2 < \infty$, there exists a sequence $\{w_k\}_{k \in \mathbb{N}}$ in Y such that $w_k \rightarrow w$ in X as $k \rightarrow \infty$. We may assume that $\|w - w_k\|_X \leq 2^{-k}$. It is not difficult to see that if $f \in Y$, then $\mathbb{P}_n f \in Y$, hence $\mathcal{P}_k f \in Y$ for any $k \in \mathbb{N}$. We claim that $\{\Lambda \mathcal{P}_k w_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence in X , which shows that the redefined Λ is the closure of the original operator Λ' , since $\mathcal{P}_k w_k \in Y$ and $\mathcal{P}_k w_k \rightarrow w$ in X .*

First we note that $\|\Lambda\mathcal{P}_k f\|_X \leq Ck\|f\|_X$ for any $f \in X$. Thus for $k > l$,

$$\begin{aligned} & \|\Lambda\mathcal{P}_k w_k - \Lambda\mathcal{P}_l w_l\|_X \\ & \leq \|\Lambda\mathcal{P}_k(w_k - w)\|_X + \|\Lambda\mathcal{P}_k w - \Lambda\mathcal{P}_l w\|_X + \|\Lambda\mathcal{P}_l(w_l - w)\|_X \\ & \leq Ck2^{-k} + \left(\sum_{|n|=l+1}^k \|\Lambda_n \omega_n\|_Z^2 \right)^{\frac{1}{2}} + Cl2^{-l} \\ & \rightarrow 0, \quad l \rightarrow \infty. \end{aligned}$$

Here we used the definition (2.13) and (2.14). This proves the claim.

3. CHARACTERIZATION OF Ker Λ

In this section, we shall give the proof for Theorem 1.1. First of all, we remark on the following simple facts.

Proposition 3.1. (1) Let $w \in X$. Then, $w \in \mathbb{P}_S X$ if and only if $\omega_n = e^{-in\theta} \mathbb{P}_n w = 0$ for any $n \neq 0$,

(2) For any $\omega \in Z$, the function $\Omega_n = \Omega_n(\omega)$ defined by (2.8) is continuously differentiable in $(0, \infty)$. Moreover, when $|n| \geq 2$, Ω_n/r , Ω'_n are bounded near $r = 0$, and $r^{|n|}\Omega_n$, $r^{|n|+1}\Omega'_n$ are bounded for $r \gg 1$. When $|n| = 1$, $\Omega_n/(r \log r)$, $\Omega'_n/\log r$ are bounded near $r = 0$, and $r\Omega_n$, $r^2\Omega'_n$ are bounded for $r \gg 1$.

Remark 3.1. The properties (2) above give the necessary conditions at the boundary $r = 0$ or $r = \infty$ for solutions Ω_n , which will play important roles in the proofs of the theorems.

Proof. (1) Let $w \in \mathbb{P}_S X$. Since w is radial symmetric, we have $w(r \cos \theta, r \sin \theta) = \tilde{w}(r)$. Then, for $n \neq 0$, we have

$$\omega_n(r) = \frac{1}{2\pi} \int_0^{2\pi} w(r \cos \theta, r \sin \theta) e^{-in\theta} d\theta = \frac{1}{2\pi} \tilde{w}(r) \int_0^{2\pi} e^{-in\theta} d\theta = 0,$$

for almost every r . Conversely, if $\omega_n = e^{-in\theta} \mathbb{P}_n w = 0$ for any $n \neq 0$, we have

$$w = \mathbb{P}_0(w) = \frac{1}{2\pi} \int_0^{2\pi} w(r \cos \theta, r \sin \theta) d\theta = \omega_0,$$

thus, w is radial symmetric.

(2) Since $\omega \in L^2_{loc}(0, \infty)$, from the interior regularities for solutions to the second order linear elliptic equations, we obtain $\Omega_n \in H^2_{loc}(0, \infty)$. Thus, by Sobolev's embedding lemma, we have $\Omega_n \in C^1_{loc}(0, \infty)$. The latter assertion of (2) follows from the representation (5.8) and Hölder's inequalities. Indeed, for each $r > 0$, we have

$$\begin{aligned} |\Omega_n(\omega)(r)| & \leq \frac{1}{4|n|} \left\{ r^{-|n|} \left(\int_0^r s^{2|n|+1} g(s) ds \right)^{\frac{1}{2}} \left(\int_0^\infty s g^{-1}(s) |\omega(s)|^2 ds \right)^{\frac{1}{2}} \right. \\ & \quad \left. + r^{|n|} \left(\int_r^\infty s^{1-2|n|} g(s) ds \right)^{\frac{1}{2}} \left(\int_0^\infty s g^{-1}(s) |\omega(s)|^2 ds \right)^{\frac{1}{2}} \right\} \\ & \leq C \left\{ r^{-|n|} \left(\int_0^r s^{2|n|+1} g(s) ds \right)^{\frac{1}{2}} + r^{|n|} \left(\int_r^\infty s^{1-2|n|} g(s) ds \right)^{\frac{1}{2}} \right\} \|\omega\|_Z. \end{aligned}$$

This proves the assertion for Ω_n . Similarly, the assertion for Ω'_n follows from the representation

$$\Omega'_n(r) = \frac{1}{4} \left\{ -r^{-|n|-1} \int_0^r s^{|n|+1} \omega(s) ds + r^{|n|-1} \int_r^\infty s^{-|n|+1} \omega(s) ds \right\}.$$

We omit the details.

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1.

The relation $\mathbb{P}_S X \subset \text{Ker } \Lambda$ is obvious from the definition of Λ and the above proposition. Let $w = \partial_i G$. Then, we have

$$(3.1) \quad \Lambda w = (K * G, \nabla) \partial_i G + (K * \partial_i G, \nabla) G = \partial_i (K * G, \nabla) G = 0.$$

This proves the relation $\mathbb{P}_S X \oplus \{\beta_1 \partial_1 G + \beta_2 \partial_2 G \mid \beta_i \in \mathbb{C}\} \subset \text{Ker } \Lambda$.

Let $w \in \text{Ker } \Lambda$. Then, by the definition of Λ (2.13), we have $0 = \Lambda_n \omega_n = in(\varphi \omega_n - g \Omega_n(\omega_n))$ for all n . It suffices to consider the case $n \neq 0$. First, we consider the case $|n| \geq 2$. Since $\varphi(r) = \frac{1}{2\pi r^2} (1 - e^{-\frac{r^2}{4}})$ is a strictly positive and decreasing function, we can write $\omega_n = \frac{g}{\varphi} \Omega_n(\omega_n)$. Thus, by (2.9), $\Omega_n = \Omega_n(\omega_n)$ solves the ODE

$$(3.2) \quad -\frac{1}{r} (r \Omega_n')' + \left(\frac{n^2}{r^2} - \frac{g}{2\varphi} \right) \Omega_n = 0.$$

Multiplying both sides by $r \overline{\Omega_n}$ and integrating over $(1/R, R)$ for $R \gg 1$, we have

$$\int_{\frac{1}{R}}^R -(r \Omega_n')' \overline{\Omega_n} dr + \int_{\frac{1}{R}}^R r \left(\frac{n^2}{r^2} - \frac{g}{2\varphi} \right) |\Omega_n|^2 dr = 0.$$

Then, by integration by parts, it follows

$$\int_{\frac{1}{R}}^R r |\Omega_n'|^2(r) dr + \int_{\frac{1}{R}}^R r \left(\frac{n^2}{r^2} - \frac{g}{2\varphi} \right) |\Omega_n|^2(r) dr = R \Omega_n'(R) \overline{\Omega_n}(R) - \frac{1}{R} \Omega_n' \left(\frac{1}{R} \right) \overline{\Omega_n} \left(\frac{1}{R} \right).$$

Tending R to infinity, by the decay properties Proposition 3.1 (2), we have

$$(3.3) \quad \int_0^\infty r |\Omega_n'|^2(r) dr + \int_0^\infty r \left(\frac{n^2}{r^2} - \frac{g}{2\varphi} \right) |\Omega_n|^2(r) dr = 0.$$

However, if $|n| \geq 2$, we see that the function φ_n defined by

$$(3.4) \quad \varphi_n(r) = \frac{n^2}{r^2} - \frac{g}{2\varphi}$$

is strictly positive for any r . This gives $\Omega_n \equiv 0$, that is, $\omega_n \equiv 0$ when $|n| \geq 2$.

Next we consider the case $|n| = 1$. In this case, the sign of the function φ_1 defined by (3.4) changes at some $r > 0$, which breaks the "uniqueness" of the ODE (3.2). Indeed, $\Omega_1(rg) = r\varphi$ solves the ODE under our decay and integrability conditions of Proposition 3.1 (2). Remark that this solution corresponds with the case $w \in \{\beta_1 \partial_1 G + \beta_2 \partial_2 G\}$ in Theorem 1.1. So we have to show that there is no solution of (3.2) for $|n| = 1$ which is linearly independent of $\Omega_1(rg)$ and satisfies the conditions of Proposition 3.1 (2).

Let Ω_1 be any solution of (3.2) satisfying the conditions of Proposition 3.1 (2). It is not difficult to see that the Wronskian $W(r) = \Omega_1(\Omega_1(rg))' - \Omega_1(rg)\Omega_1'$ satisfies $W(r) = \frac{r}{s} W(s)$ for any $r > s > 0$. Then, by taking $s \rightarrow 0$ (or $r \rightarrow \infty$), from the conditions of Proposition 3.1 (2), we see that $W(r) \equiv 0$. This implies that Ω_1 cannot be linearly independent of $\Omega_1(rg)$. By the relation $\omega_1 = \frac{g}{2\varphi} \Omega_1(\omega_1)$, the above argument shows that $\omega_1(r) = Crg(r)$ for some constant C , which completes the proof of Theorem 1.1.

4. ESTIMATES FOR THE OPERATOR \mathcal{L}_α^{-1}

In this section, we prove Theorem 1.2 by a contradiction argument. We assume that

$$(4.1) \quad \limsup_{|\alpha| \rightarrow \infty} \|\mathcal{L}_\alpha^{-1}\|_{X \rightarrow X} = \delta > 0.$$

Then we have a sequence $\{\alpha_i\}_{i \in \mathbb{N}}$ and a sequence of functions $\{f_i\}_{i \in \mathbb{N}}$ in $\overline{\text{Ran } \Lambda}$ with $\|f_i\|_X = 1$ such that $|\alpha_i| \rightarrow \infty$ and $\|\mathcal{L}_{\alpha_i}^{-1} f_i\|_X \geq \frac{\delta}{2} > 0$ for any i . We set $g_i = \mathcal{L}_{\alpha_i}^{-1} f_i$. Then the sequence $\{g_i\}_{i \in \mathbb{N}}$ is compact in X . Indeed, we have

$$\begin{aligned} \langle f_i, g_i \rangle_X &= \langle \mathcal{L}_{\alpha_i} g_i, g_i \rangle_X = - \langle -\mathcal{L} g_i, g_i \rangle_X - \alpha_i \langle \Lambda g_i, g_i \rangle_X \\ &= -\|(-\mathcal{L})^{\frac{1}{2}} g_i\|_X^2, \end{aligned}$$

thus we have

$$\|(-\mathcal{L})^{\frac{1}{2}} g_i\|_X \leq 4\|f_i\|_X = 4.$$

Here we used the skew-symmetry of Λ and the property $-\mathcal{L} \geq \frac{1}{2}$. Since $(-\mathcal{L})^{-\frac{1}{2}}$ is a compact operator in X (see Lemma 1.1 (1-1)), $\{g_i\}_{i \in \mathbb{N}}$ is compact in X . We take a strongly convergent subsequence of $\{g_i\}_{i \in \mathbb{N}}$ (again we write this subsequence as $\{g_i\}_{i \in \mathbb{N}}$). Let $g_\infty \in \overline{\text{Ran } \Lambda}$ be the limit function. We note that $g_\infty \neq 0$. Then we have for any $h \in D(\mathcal{L})$,

$$\begin{aligned} \langle \Lambda g_\infty, h \rangle_X &= - \langle g_\infty, \Lambda h \rangle_X = - \lim_{i \rightarrow \infty} \langle g_i, \Lambda h \rangle_X \\ &= - \lim_{i \rightarrow \infty} \frac{1}{\alpha_i} \langle g_i, \mathcal{L}_{-\alpha_i} h \rangle_X \\ &= - \lim_{i \rightarrow \infty} \frac{1}{\alpha_i} \langle \mathcal{L}_{\alpha_i} g_i, h \rangle_X \\ &= - \lim_{i \rightarrow \infty} \frac{1}{\alpha_i} \langle f_i, h \rangle_X \\ &= 0. \end{aligned}$$

By the density argument, this shows that $\Lambda g_\infty = 0$, i.e., $g_\infty \in \text{Ker } \Lambda$. However, since $g_\infty \in \overline{\text{Ran } \Lambda}$, we conclude $g_\infty = 0$, which leads to a contradiction. Now the proof of Theorem 1.2 is completed.

5. BEHAVIOR OF THE SPECTRUM OF \mathcal{L}_α

In this section, we shall give the proof of Theorem 1.3. We first note the following proposition, which is essentially obtained in [4] and [5].

Proposition 5.1. (1) *For any $\alpha \in \mathbb{R}$, the spectrum of $\mathcal{L} - \alpha\Lambda$ consists of eigenvalues with finite multiplicities.*

(2) *For all $\alpha \in \mathbb{R}$, $\sigma(\mathcal{L} - \alpha\Lambda) \subset \{\mu \mid \text{Re}(\mu) \leq -\frac{1}{2}\}$.*

Proof. (1) Since the resolvent $(\mathcal{L} - \alpha\Lambda)^{-1}$ is compact by Lemma 1.1 (3), the assertion follows.

(2) This is obtained in [4, Proposition 4.1]. But we recall their argument for convenience to reader. From (1) above, it suffices to consider the eigenvalue of $\mathcal{L} - \alpha\Lambda$. Let w be a nonzero eigenfunction of the eigenvalue μ . Then,

$$\mu \langle w, w \rangle_X = \langle \mathcal{L} w, w \rangle_X - \alpha \langle \Lambda w, w \rangle_X.$$

By the skew-symmetry of Λ , we see that

$$\text{Re}(\mu) \langle w, w \rangle_X = \langle \mathcal{L} w, w \rangle_X \leq -\frac{1}{2} \langle w, w \rangle_X.$$

Here, we used the fact $\mathcal{L} \leq -\frac{1}{2}$; see Lemma 1.1 (1-1). This proves the proposition.

As stated in Section 1, Theorem 1.1 leads to the decomposition of the operator $\mathcal{L} - \alpha\Lambda$ such as $\mathcal{L} - \alpha\Lambda = \mathcal{L}|_{\text{Ker } \Lambda} \oplus \mathcal{L}_\alpha$, where \mathcal{L}_α is the restriction of $\mathcal{L} - \alpha\Lambda$ to the

closed subspace $\overline{\text{Ran } \Lambda}$. Obviously, the operator \mathcal{L}_α also satisfies the same properties in the above proposition. Especially, for each α , there exists an eigenvalue μ_α which attains the maximum of $\{\text{Re}(\mu) \mid \mu \in \sigma(\mathcal{L}_\alpha)\}$, i.e., $r_\alpha = \text{Re}(\mu_\alpha)$. From (2) of the above proposition, r_α is negative for any α . Thus, it suffices to show that $|r_\alpha|$ goes to infinity as $|\alpha| \rightarrow \infty$. We shall show this by a contradiction argument.

We assume that there exists a sequence $\{\alpha_i\}_{i=1}^\infty$ such that

$$(5.1) \quad \sup_i |r_{\alpha_i}| < \infty$$

and $|\alpha_i| \rightarrow \infty$ as $i \rightarrow \infty$. Let w_{α_i} be a normalized eigenfunction of \mathcal{L}_{α_i} associated with the eigenvalue μ_{α_i} such that $r_{\alpha_i} = \text{Re}(\mu_{\alpha_i})$. Note that w_{α_i} belongs to $D(\mathcal{L}) \cap \overline{\text{Ran } \Lambda}$. Moreover, we have

$$(5.2) \quad \|(-\mathcal{L})^{\frac{1}{2}} w_{\alpha_i}\|_X^2 = \langle -\mathcal{L} w_{\alpha_i}, w_{\alpha_i} \rangle_X = -\text{Re}(\mu_{\alpha_i}) \|w_{\alpha_i}\|_X^2 = |r_{\alpha_i}| < \infty.$$

Thus, by the assumption (5.1), $(-\mathcal{L})^{\frac{1}{2}} w_{\alpha_i}$ is uniformly bounded. Since $(-\mathcal{L})^{-\frac{1}{2}}$ is compact in X , there exists a convergent subsequence of $\{w_{\alpha_i}\}_i$. We write this sequence as $\{w_{\alpha_i}\}_i$ again and its limit as w_∞ . Remark that $w_\infty \in \overline{\text{Ran } \Lambda}$, $\|w_\infty\|_X = 1$ and especially w_∞ is not identically zero.

Now we consider the equation which w_{α_i} satisfies in "weak" form. Given $f \in D(\mathcal{L})$, we have

$$(5.3) \quad \alpha_i^{-1} \langle w_{\alpha_i}, \mathcal{L} f \rangle_X + \langle w_{\alpha_i}, \Lambda f \rangle_X = \frac{\mu_{\alpha_i}}{\alpha_i} \langle w_{\alpha_i}, f \rangle_X.$$

We claim that $\frac{\text{Im}(\mu_{\alpha_i})}{\alpha_i}$ converges to a nonzero number in \mathbb{R} .

Indeed, by the equation (5.3), we have

$$(5.4) \quad \langle w_{\alpha_i}, \Lambda w_{\alpha_i} \rangle_X = i \frac{\text{Im}(\mu_{\alpha_i})}{\alpha_i}.$$

We used the facts that Λ is skew-symmetric and w_{α_i} is normalized. Since w_{α_i} converges in X and is uniformly bounded in Y , we see that the left hand side of (5.3) converges to the value $\langle w_\infty, \Lambda w_\infty \rangle_X$. Thus, $\lim_{i \rightarrow \infty} \frac{\text{Im}(\mu_{\alpha_i})}{\alpha_i}$ exists. If $\lim_{i \rightarrow \infty} \frac{\text{Im}(\mu_{\alpha_i})}{\alpha_i} = 0$, then by taking the limit in the equation (5.3), we have $\langle w_\infty, \Lambda f \rangle_X = 0$ for any $f \in D(\mathcal{L})$. By the density argument and the fact that $w_\infty \in \overline{D(\Lambda)}$, we have $\Lambda w_\infty = 0$, i.e., $w \in \text{Ker } \Lambda$. However, since we also have $w \in \overline{\text{Ran } \Lambda}$, we conclude $w_\infty \equiv 0$, which contradicts the fact $\|w_\infty\|_X = 1$.

Let $\mu_\infty = \lim_{i \rightarrow \infty} \frac{\text{Im}(\mu_{\alpha_i})}{\alpha_i} \neq 0$. Then, again by the equation (5.3) and taking the limit, we have $\langle \Lambda w_\infty, f \rangle_X = -\langle w_\infty, \Lambda f \rangle_X = -i\mu_\infty \langle w_\infty, f \rangle_X$ for any $f \in D(\mathcal{L})$, which leads to

$$(5.5) \quad \Lambda w_\infty = -i\mu_\infty w_\infty.$$

That is, $-i\mu_\infty$ is a nonzero eigenvalue of Λ and w_∞ is a nonzero eigenfunction of $-i\mu_\infty$ in X . Now Theorem 1.3 is a direct consequence of the following lemma.

Lemma 5.1. *The operator Λ has no eigenvalues except for zero.*

Proof. By the skew-symmetry of Λ , the eigenvalue takes the form $i\mu$, for a real number μ . We consider solutions to the equations

$$(5.6) \quad \Lambda w_\infty = -i\mu_\infty w_\infty.$$

Without loss of generality, we may assume that $\mu > 0$. Let w be a solution of (5.6) in X . Then, by our definition of Λ , we have for each nonzero integer n ,

$$(5.7) \quad in(\varphi\omega_n - g\Omega_n(\omega_n)) = i\mu\omega_n,$$

$$(5.8) \quad \Omega_n(\omega_n)(r) = \frac{1}{4|n|} \left(\int_0^r \left(\frac{s}{r}\right)^{|n|} s\omega_n(s) ds + \int_r^\infty \left(\frac{r}{s}\right)^{|n|} s\omega_n(s) ds \right),$$

or,

$$(5.9) \quad -\frac{1}{r}(r\Omega_n')' + \frac{n^2}{r^2}\Omega_n - \frac{1}{2}\omega_n = 0, \quad r > 0.$$

Our goal is to conclude $\omega_n \equiv 0$ for each n . Note that $\Omega_n = \Omega_n(\omega_n)$ satisfies the conditions in Proposition 3.1 (2).

Let $n \leq -2$.

In this case, we can apply the same argument as in Section 2. Indeed, from (5.7), we have

$$(5.10) \quad \omega_n = \frac{g}{\varphi - \frac{\mu}{n}} \Omega_n.$$

Then, multiplying $r\overline{\Omega_n}$ and integrating over $(0, \infty)$, we have

$$(5.11) \quad \int_0^\infty r|\Omega_n'|^2 dr + \int_0^\infty r\left(\frac{n^2}{r^2} - \frac{g}{2(\varphi - \frac{\mu}{n})}\right)|\Omega_n|^2 dr = 0.$$

Here, we used the integration by parts. From the fact $\mu > 0$ and $n \leq -2$, we have

$$(5.12) \quad \frac{n^2}{r^2} - \frac{g}{2(\varphi - \frac{\mu}{n})} \geq \frac{n^2}{r^2} - \frac{g}{2\varphi} > 0, \quad r > 0.$$

This derives $\Omega_n \equiv 0$, that is, $\omega_n \equiv 0$ by (5.10).

Let $n = -1$.

In this case, in general, we cannot conclude $\frac{n^2}{r^2} - \frac{g}{2(\varphi - \frac{\mu}{n})}$ is positive. So we use the different argument. Recall that $rg e^{-i\theta}$ is an eigenfunction for zero eigenvalue of Λ . Thus the function rg is orthogonal to $\overline{\omega_{-1}}$ in the Hilbert space Z . Hence we have

$$(5.13) \quad 0 = \int_0^\infty rg^{-1}(r)rg(r)\omega_{-1}(r)dr = \int_0^\infty r^2\omega_{-1}(r)dr.$$

Now the representation (5.8) is useful.

$$\begin{aligned} \Omega_{-1}(r) &= \frac{1}{4} \left(\int_0^r \frac{s^2}{r} \omega_{-1}(s) ds + r \int_r^\infty \omega_{-1}(s) ds \right) \\ &= \frac{1}{4} \left(-\frac{1}{r} \int_r^\infty s^2 \omega_{-1}(s) ds + r \int_r^\infty \omega_{-1}(s) ds \right) \\ &= \frac{1}{4} \left(-\frac{1}{r} \int_r^\infty s^2 \frac{g(s)}{\varphi(s) + \mu} \Omega_{-1}(s) ds + r \int_r^\infty \frac{g(s)}{\varphi(s) + \mu} \Omega_{-1}(s) ds \right). \end{aligned}$$

Here, we used (5.13) and (5.10).

Let $R > 0$. Then, for any $r \geq R$, we have, from the above equation,

$$\begin{aligned} &|\Omega_{-1}(r)| \\ &\leq \frac{1}{4} \left(\sup_{r \geq R} \frac{1}{r} \int_r^\infty s^2 \frac{g(s)}{\varphi(s) + \mu} ds + \sup_{r \geq R} r \int_r^\infty \frac{g(s)}{\varphi(s) + \mu} ds \right) \sup_{r \geq R} |\Omega_{-1}(r)|, \end{aligned}$$

thus,

$$\sup_{r \geq R} |\Omega_{-1}(r)| \leq \epsilon(R) \sup_{r \geq R} |\Omega_{-1}(r)|,$$

where $\epsilon(R)$ is a constant such that $\epsilon(R) \rightarrow 0$ as $R \rightarrow \infty$. This shows that there exists an $R > 1$ such that $\Omega_{-1}(r) \equiv 0$ for any $r \geq R$. Then we can conclude that $\Omega_{-1}(r)$ is identically zero for any $r > 0$. Indeed, we just solve the ODE (5.9) with $\omega_{-1} = \frac{g}{\varphi+\mu}\Omega_{-1}$ for initial data $\Omega_{-1}(R) = \Omega'_{-1}(R) = 0$. Since we have no singularities for the coefficients of (5.9) around $R > 0$, we can apply the usual uniqueness results of the ODEs to our case, which gives $\Omega_{-1} \equiv 0$.

Finally, we consider the case $n \geq 1$. In this case, the function $\psi_n(r) = \varphi(r) - \frac{\mu}{n}$ plays an important role. Note that

$$(5.14) \quad \psi'_n(r) = \varphi'(r) < 0, \text{ for any } r > 0.$$

Suppose that $\psi_n(0) \leq 0$. Then we have $\psi_n(r) < 0$ for any $r > 0$. Hence the function $\frac{n^2}{r^2} - \frac{g}{2\psi_n}$ is always positive. This shows that $\Omega_n \equiv 0$ by the equality (5.11), that is, $\omega_n \equiv 0$ by (5.10).

Let $\psi_n(0) > 0$. Then, since $\psi'_n(r) < 0$ and $\lim_{r \rightarrow \infty} \varphi(r) = 0$, there is a (unique) strictly positive number r_n such that $\psi_n(r_n) = 0$. We recall that $\omega_n \in L^2_{loc}(0, \infty)$. Since $\Omega_n \in C^1_{loc}(0, \infty)$, we also have $\omega_n \in C(I \setminus \{r_n\})$ for each compact interval $I \subset (0, \infty)$ by the relation (5.10). This implies that

$$(5.15) \quad |\Omega_n(r_n)| = \liminf_{r \rightarrow r_n} |\Omega_n(r)| = \liminf_{r \rightarrow r_n} |\psi_n(r)\omega_n(r)| = 0.$$

Let $\epsilon > 0$. Multiplying $r\overline{\Omega_n}$ to (5.9) and integrating over $(r_n + \epsilon, \infty)$, we have

$$\int_{r_n+\epsilon}^{\infty} r|\Omega'_n(r)|^2 dr + \int_{r_n+\epsilon}^{\infty} r\left(\frac{n^2}{r^2} - \frac{g}{2\psi_n}\right)|\Omega_n(r)|^2 dr = -(r_n + \epsilon)\Omega'_n(r_n + \epsilon)\Omega_n(r_n + \epsilon).$$

Tending ϵ to zero, we obtain

$$(5.16) \quad \int_{r_n}^{\infty} r|\Omega'_n(r)|^2 dr + \int_{r_n}^{\infty} r\left(\frac{n^2}{r^2} - \frac{g}{2\psi_n}\right)|\Omega_n(r)|^2 dr = 0.$$

Since $\frac{n^2}{r^2} - \frac{g}{2\psi_n} > 0$ for $r > r_n$, we conclude $\Omega_n(r) = 0$ for any $r > r_n$. Since we have already known that $\Omega_n \in C^1_{loc}(0, \infty)$, we also get $\Omega'_n(r_n) = 0$. Then, we can reduce the equation (5.9) to

$$(5.17) \quad -\frac{1}{r}(r\Omega'_n)' + \frac{n^2}{r^2}\Omega_n = \frac{g}{2\psi_n}\Omega_n, \quad 0 < r < r_n,$$

with initial data $\Omega_n(r_n) = \Omega'_n(r_n) = 0$. The advantage of the above reduction is that the coefficients of the equation has only (apparent) singularity ψ_n^{-1} near initial time $r_n > 0$. We rewrite the inhomogeneous term $\frac{g}{2\psi_n}\Omega_n$ as $\frac{g}{2\varphi'}\tilde{\Omega}'_n$ where

$$(5.18) \quad \tilde{\varphi}'(r) = \int_0^1 (\varphi')(\tau r + (1-\tau)r_n) d\tau,$$

$$(5.19) \quad \tilde{\Omega}'_n(r) = \int_0^1 (\Omega'_n)(\tau r + (1-\tau)r_n) d\tau.$$

Note that since $\varphi'(r) < 0$ for any $r > 0$, our new "coefficient" $\frac{g}{2\varphi'}$ is bounded near $r_n > 0$. Then, by the variation-of-constants formula (see [3, Theorem 6.4]), Ω_n is represented by

$$(5.20) \quad \Omega_n(r) = \frac{1}{2|n|} \left(-r^{|n|} \int_{r_n}^r s^{-|n|+1} b(s) ds + r^{-|n|} \int_{r_n}^r s^{|n|+1} b(s) ds \right),$$

where

$$(5.21) \quad b(r) = \frac{g}{2\varphi'} \int_0^1 (\Omega'_n)(\tau r + (1-\tau)r_n) d\tau.$$

Thus,

$$(5.22) \quad \Omega'_n(r) = -\frac{1}{2} \left(r^{|n|-1} \int_{r_n}^r s^{-|n|+1} b(s) ds + r^{-|n|-1} \int_{r_n}^r s^{|n|+1} b(s) ds \right).$$

Let $0 < \delta < r_n$ and $r \in [r_n - \delta, r_n)$. Then, by (5.20),

$$\begin{aligned} |\Omega'_n(r)| &\leq M \left| \int_{r_n}^r |b(s)| ds \right| \\ &\leq M\delta \sup_{r_n - \delta \leq r < r_n} \int_0^1 |(\Omega'_n)(\tau r + (1 - \tau)r_n)| d\tau \\ &\leq M\delta \sup_{r_n - \delta \leq r < r_n} |\Omega'_n(r)|, \end{aligned}$$

for some constant M . Hence, we obtain

$$\sup_{r_n - \delta \leq r < r_n} |\Omega'_n(r)| \leq M\delta \sup_{r_n - \delta \leq r < r_n} |\Omega'_n(r)|,$$

which shows that $\Omega'_n \equiv 0$ on $[r_n - \delta, r_n]$ for sufficiently small δ . Thus, we obtain $\Omega_n \equiv 0$ on $[r_n - \delta, r_n]$. This is enough for the conclusion $\Omega_n \equiv 0$ in the whole interval $(0, r_n]$, because the coefficients of the equation (5.17) has no singularities around new initial time $r_n - \delta$ and we can apply usual uniqueness results of the ODEs. Thus, we obtain $\Omega_n \equiv 0$, i.e., $\omega_n \equiv 0$ in $(0, \infty)$ when $n \geq 1$. The proof of the lemma is now completed.

6. APPENDIX

If we consider the equations (1.3) for not sufficiently small $\lambda > 0$, the Gaussian weighted L^2 space X will not be suitable to find solutions. Indeed, in [6], Th. Gallay and C. E. Wayne constructed solutions in the polynomial weighted L^2 space near the function G_λ , where

$$(6.1) \quad G_\lambda(x) = \frac{\sqrt{1 - \lambda^2}}{4\pi} e^{-\frac{1+\lambda}{4}x_1^2 - \frac{1-\lambda}{4}x_2^2}.$$

Clearly, the function G_λ does not belong to X if $\lambda \geq \frac{1}{2}$, so it seems that X is not suitable to solve (1.3) in this case.

In this section we consider the operator $\mathcal{L} - \alpha\Lambda$ in the polynomial weighted L^2 spaces defined by

$$(6.2) \quad \begin{aligned} L^2(m) &= \{f \in L^2(\mathbb{R}^2) \mid (1 + |x|^2)^{\frac{m}{2}} f \in L^2(\mathbb{R}^2), \\ &\quad \langle f_1, f_2 \rangle_{L^2(m)} = \int_{\mathbb{R}^2} (1 + |x|^2)^m f_1(x) \overline{f_2(x)} dx \} \end{aligned}$$

for $m > 0$. We also define the closed subspace $L_0^2(m)$ as

$$(6.3) \quad L_0^2(m) = \{f \in L^2(m) \mid \int_{\mathbb{R}^2} f(x) dx = 0\}.$$

As in [4, Section 4], we consider the associated function spaces

$$(6.4) \quad \begin{aligned} Z(m) &= \{f : \mathbb{R}_+ \rightarrow \mathbb{C} \mid \int_0^\infty r(1 + r^2)^m |f(r)|^2 dr < \infty, \\ &\quad \langle f_1, f_2 \rangle_{Z(m)} = \int_0^\infty r(1 + r^2)^m f_1(r) \overline{f_2(r)} dr \}. \end{aligned}$$

Then the operator Λ defined by (2.13) with its domain

$$(6.5) \quad D(\Lambda) = \{w \in L_0^2(m) \mid \sum_{n \in \mathbb{Z}} n^2 \|\varphi \omega_n\|_{Z(m)}^2 < \infty\}$$

is again a closed operator in $L_0^2(m)$.

We consider the case $m > 1$. Then Proposition 3.1 (2) becomes:

Proposition 6.1. *Let $m > 1$. Then for any $\omega \in Z(m)$, the function $\Omega_n = \Omega_n(\omega)$ defined by (2.8) is continuously differentiable in $(0, \infty)$. Moreover, when $|n| \geq 2$, $\Omega_n/r, \Omega'_n$ are bounded near $r = 0$, and $\Omega_n, r\Omega'_n$ are bounded for $r \gg 1$. When $|n| = 1$, $\Omega_n/(r \log r), \Omega'_n/\log r$ are bounded near $r = 0$, and $\Omega_n, r\Omega'_n$ are bounded for $r \gg 1$. More precisely, we have*

$$(6.6) \quad \lim_{r \rightarrow \infty} \Omega_n(r) = \lim_{r \rightarrow \infty} r\Omega'_n(r) = 0,$$

for any $|n| \geq 1$.

The proof of this proposition is given by the same argument as in Proposition 3.1 (2). So we omit the details.

It is not difficult to see that by Proposition 6.1, the argument of the proof for Theorem 1.1 is still available for the case $L_0^2(m)$ when $m > 1$. That is, we have

Theorem 6.1. *Let $m > 1$. Then*

$$(6.7) \quad \text{Ker } \Lambda = \mathbb{P}_S L_0^2(m) \oplus \{\beta_1 \partial_1 G + \beta_2 \partial_2 G \mid \beta_i \in \mathbb{C}\}$$

where

$$(6.8) \quad \mathbb{P}_S L_0^2(m) = \{f \in L_0^2(m) \mid f(Rx) = f(x) \text{ a.a } x \text{ for all orthogonal matrix } R\}.$$

We recall that some eigenfunctions for the operator $\mathcal{L} - \alpha\Lambda$ in $L^2(m)$ have the Gaussian decay properties; see [4, Lemma 4.5]. Thus we have the following theorem as a corollary of Theorem 1.3. Let \mathcal{L}_α be the restriction of $\mathcal{L} - \alpha\Lambda$ on $(\text{Ker } \Lambda)^\perp$ with its domain $D(\mathcal{L}_\alpha) = D(\mathcal{L}) \cap (\text{Ker } \Lambda)^\perp$. Here $D(\mathcal{L}) = \{f \in L_0^2(m) \mid \Delta f + \frac{x}{2} \cdot \nabla f \in L_0^2(m)\}$.

Theorem 6.2. *Let $m > 1$. Let $\sigma(\mathcal{L}_\alpha)$ be the set of spectrum of \mathcal{L}_α . Then there exists a positive number $R = R(m)$ such that for any α with $|\alpha| \geq R$, we have*

$$(6.9) \quad r_\alpha = \sup_{\mu \in \sigma(\mathcal{L}_\alpha)} \text{Re } (\mu) \leq \frac{1-m}{2}$$

Proof. By [4, Section 4], the spectrum μ of the operator $\mathcal{L} - \alpha\Lambda$ (in $L^2(m)$) with $\text{Re } \mu > \frac{1-m}{2}$ is an eigenvalue of $\mathcal{L} - \alpha\Lambda$. Moreover, by [4, Lemma 4.5], eigenfunctions of the operator $\mathcal{L} - \alpha\Lambda$ in $L_0^2(m)$ for eigenvalues μ with $\text{Re } \mu > \frac{1-m}{2}$ must belong to the Gaussian weighted L^2 space X . Hence the assertion of the theorem is an immediate consequence of Theorem 1.3.

Remark 6.1. *Let $m > 2$. If we consider the problem in the space $L_1^2(m)$ defined by*

$$(6.10) \quad L_1^2(m) = \{f \in L_0^2(m) \mid \int_{\mathbb{R}^2} x_i f(x) dx = 0, i = 1, 2\},$$

then we have

$$(6.11) \quad \text{Ker } \Lambda = \mathbb{P}_S L_1^2(m)$$

where

$$(6.12) \quad \mathbb{P}_S L_1^2(m) = \{f \in L_1^2(m) \mid f(Rx) = f(x) \text{ a.a } x \text{ for all orthogonal matrix } R\}.$$

Moreover, for α with $|\alpha| \geq R$ (R is given by Theorem 6.2), we have

$$(6.13) \quad r_\alpha = \sup_{\mu \in \sigma(\mathcal{L}_\alpha)} \text{Re } (\mu) \leq \frac{1-m}{2},$$

where \mathcal{L}_α is the restriction of $\mathcal{L} - \alpha\Lambda$ on $(\mathbb{P}_S L_1^2(m))^\perp$ in this case.

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