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1. Introduction

Let $K$ be a complete discrete valuation field of mixed characteristic $(0,p)$ with residue field $F$ which may be imperfect, $G_K$ be its absolute Galois group and $I_K$ be its inertia subgroup. Let $G$ be a finite flat group scheme over $\mathcal{O}_K$. When $G$ is monogenic, that is to say, when the affine algebra of $G$ is generated over $\mathcal{O}_K$ by one element, it is well-known that the tame characters appearing in the $I_K$-module $G(K)$ are determined by the slopes of the Newton polygon of a defining equation of $G$ ([15, Proposition 10]).

On the other hand, for an elliptic modular form $f$ of level $N$ prime to $p$, we also have a description of the tame characters of the associated mod $p$ Galois representation $\tilde{\rho}_f$ ([10, Theorem 2.5, Theorem 2.6], [9, Section 4.3]). This is based on Raynaud’s theory of prolongations of finite flat group schemes or the integral $p$-adic Hodge theory. However, for an analogous study of the associated mod $p$ Galois representation of a Hilbert modular form over a totally real number field, we encounter a local field of higher absolute ramification index. In this case, these two theories no longer work well and we need some other techniques.

In this paper, to propose a new approach toward this problem, we generalize the aforementioned result of [15] to the higher dimensional case using the ramification theory of Abbes and Saito ([2], [3]) and determine the tame characters appearing in $G(K)$ in terms of the ramification of $G$ without any restriction on the absolute ramification index of $K$. Namely, we show the following theorem.

**Theorem 1.1.** Let $G$ be a finite flat group scheme over $\mathcal{O}_K$. Write $\{G^j\}_{j \in \mathbb{Q}_{>0}}$ for the ramification filtration of $G$ in the sense of [2] and [3]. Then the graded piece $G^j(K)/G^{j+}(K)$ is killed by $p$ and the $I_K$-module $G^j(K)/G^{j+}(K) \otimes_{\mathbb{F}_p} \mathbb{F}_p$ is the direct sum of $\mathbb{F}_p$-conjugates of the fundamental character $\theta_j$ of level $j$.

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In view of the following corollary, we can regard this theorem as a counterpart for finite flat group schemes of the Hasse-Arf theorem in the classical ramification theory.

**Corollary 1.2.** Let $L$ be an abelian extension of $K$. Suppose that its integer ring $\mathcal{O}_L$ is a $\mathcal{G}$-torsor over $\mathcal{O}_K$. Then the denominator of every jump of the upper numbering ramification filtration $\{\text{Gal}(L/K)^j\}_{j \in \mathbb{Q}_{>0}}$ ([2]) is a power of $p$.

To prove the main theorem, we firstly show that the tubular neighborhood of $\mathcal{G}$ can be chosen to have a rigid analytic group structure. Passing to the closed fiber, we realize the graded piece of the ramification filtration of $\mathcal{G}$ as the kernel of an etale isogeny of the additive groups $\mathcal{G}^a_r$ over $\mathcal{F}$.

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**2. Review of the ramification theory of Abbes and Saito**

Let $K$ be a complete discrete valuation field with residue field $F$ which may be imperfect. Set $\pi = \pi_K$ to be an uniformizer of $K$. The separable closure of $K$ is denoted by $\overline{K}$ and the absolute Galois group of $K$ by $G_K$. Let $\mathfrak{m}_K$ and $\mathfrak{F}$ be the maximal ideal and the residue field of $\mathcal{O}_K$ respectively. In [2] and [3], Abbes and Saito defined the ramification theory of a finite flat $\mathcal{O}_K$-algebra of relative complete intersection. In this section, we gather the necessary definitions and briefly recall their theory.

Let $A$ be a finite flat $\mathcal{O}_K$-algebra and $\mathfrak{A}$ be a complete Noetherian semi-local ring (with its topology defined by $\text{rad}(\mathfrak{A})$) which is of formally smooth over $\mathcal{O}_K$ and whose quotient ring $\mathfrak{A}/\text{rad}(\mathfrak{A})$ is of finite type over $F$. A surjection of $\mathcal{O}_K$-algebras $\mathfrak{A} \rightarrow A$ is called an embedding if $\mathfrak{A}/\text{rad}(\mathfrak{A}) \rightarrow A/\text{rad}(A)$ is an isomorphism. For an embedding $(\mathfrak{A} \rightarrow A)$ and $j \in \mathbb{Q}_{>0}$, the $j$-th tubular neighborhood of $(\mathfrak{A} \rightarrow A)$ is the $K$-affinoid variety $X^j(\mathfrak{A} \rightarrow A)$ constructed as follows. Write $j = k/l$ with $k, l$ non-negative integers. Put $I = \text{Ker}(\mathfrak{A} \rightarrow A)$ and $\mathcal{A}^{k,l}_0 = \mathfrak{A}[I^l/\pi^k]^\wedge$, where $\wedge$ means the $\pi$-adic completion. Then $\mathcal{A}^{k,l}_0$ is a quotient ring of the Tate algebra $\mathcal{O}_K\langle T_1, \ldots, T_r \rangle$ for some $r$. Its generic fiber $\mathcal{A}^{k,l}_K = \mathcal{A}^{k,l}_0 \otimes_{\mathcal{O}_K} K$ is independent of the choice of a representation
the category of finite variety is geometrically regular ([3, Lemma 1.6]).

We put $F(A) = \text{Hom}_{\mathcal{O}_K-\text{alg}}(A, \mathcal{O}_K)$ and $F^j(A) = \varprojlim \pi_0(X^j(\mathbb{A} \to A))_K$. Here $\pi_0(X)_K$ denotes the set of geometric connected components of a $K$-affinoid variety $X$ and the projective limit is taken in the category of embeddings of $A$. Note that the projective family $\pi_0(X^j(\mathbb{A} \to A))_K$ is constant ([3, Section 1.2]). These define contravariant functors $F$ and $F^j$ from the category of finite flat $\mathcal{O}_K$-algebras to the category of finite $G_K$-sets. Moreover, there are morphisms of functors $F \to F^j$ and $F^j \to F^j$ for $j' \geq j > 0$.

By the finiteness theorem of Grauert and Remmert ([8, Theorem 1.3]), there exists a finite separable extension $L$ of $K$ such that the geometric closed fiber of the unit disc $\mathcal{X}^j(\mathbb{A} \to A)_{O_L}$ for the supremum norm in $X^j(\mathbb{A} \to A)_L = X^j(\mathbb{A} \to A) \times_K L$ is reduced. Then for any finite separable extension $L'$ of $L$, the $\pi_{L'}$-adic formal scheme $\mathcal{X}^j(\mathbb{A} \to A)_{O_L} \otimes_{O_L} O_{L'}$ coincides with the unit disc for the supremum norm in $X^j(\mathbb{A} \to A)_{L'}$ and thus is normal. The $\pi_{L'}$-adic formal scheme $\mathcal{X}^j(\mathbb{A} \to A)_{O_L}$ is referred as the stable normalized integral model of $X^j(\mathbb{A} \to A)$ over $O_L$ and its geometric closed fiber is denoted by $\mathcal{X}^j(\mathbb{A} \to A)$. If $L/K$ is Galois, the Galois group $\text{Gal}(L/K)$ acts on it by the functoriality of the unit disc for the supremum norm. We have the $G_K$-equivariant isomorphism $\pi_0(\mathcal{X}^j(\mathbb{A} \to A))_F \to \pi_0(X^j(\mathbb{A} \to A))_K$, where the former is the set of geometric connected components of $\mathcal{X}^j(\mathbb{A} \to A)$ ([3, Corollary 1.11]).

Suppose that $A$ is of relative complete intersection over $\mathcal{O}_K$ and $A \otimes_{\mathcal{O}_K} K$ is etale over $K$. Then the natural map $F(A) \to F^j(A)$ is surjective. The family $\{F(A) \to F^j(A)\}_{j \in \mathbb{Q}_{>0}}$ is separated, exhaustive and its jumps are rational ([2, Proposition 6.4]). The conductor of $A$ is defined to be $c(A) = \inf \{ j \in \mathbb{Q}_{>0} | F(A) \to F^j(A) \text{ is an isomorphism} \}$. If $B$ is the affine algebra of a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$ which is generically etale, then $B$ is of relative complete intersection (for example, [5, Proposition 2.2.2]) and the theory above can all be applied to $B$. By the functoriality, $F^j(B)$ is endowed with a $G_K$-module structure ([1, Lemme 2.1]) and the natural map $\mathcal{G}(K) = F(B) \to F^j(B)$ is a $G_K$-homomorphism. Let $\mathcal{G}^j$ denote the schematic closure ([14]) in $\mathcal{G}$ of the kernel of this homomorphism. It is called the $j$-th ramification filtration of $\mathcal{G}$. We refer $c(B)$ as the conductor of $\mathcal{G}$, which is denoted also by $c(\mathcal{G})$. We put $\mathcal{G}^{j+}(K) = \cup_{j' > j} \mathcal{G}^{j'}(K)$ and define $\mathcal{G}^{j+}$ to be the schematic closure of $\mathcal{G}^{j+}(K)$ in $\mathcal{G}$. 

...
3. GROUP STRUCTURE ON THE TUBULAR NEIGHBORHOOD OF A FINITE FLAT GROUP SCHEME

Let $K$ denote a complete discrete valuation field of mixed characteristic $(0, p)$ whose residue field $F$ may be imperfect and $v_K$ the valuation of $K$ extended to $K$ which is normalized as $v_K(\pi) = 1$. Let $\mathcal{G} = \text{Spec}(B)$ be a connected finite flat group scheme over $\mathcal{O}_K$. We define a formal resolution of $\mathcal{G}$ to be a closed immersion $\mathcal{G} \to \Gamma$ of (profinite) formal group schemes over $\mathcal{O}_K$, where $\Gamma = \text{Spf}(B)$ is connected and smooth. Such an immersion can be constructed as follows. By a theorem of Raynaud ([4, Théorème 3.1.1]), we can find an abelian scheme $A$ over $\mathcal{O}_K$ and a closed immersion of group schemes $G \to A$. Taking the formal completion of $A$ along the zero section, we get a formal resolution of $G$. We refer the relative dimension of $\overline{G}$ over $\mathcal{O}_K$ as the dimension of a formal resolution $(G \to \Gamma)$. We define a morphism of formal resolutions to be a pair of group homomorphisms $(f, \overline{f})$ which makes the following diagram commutative.

\[
\begin{array}{ccc}
G & \longrightarrow & \Gamma \\
\downarrow f & & \downarrow \overline{f} \\
G' & \longrightarrow & \Gamma'
\end{array}
\]

Note that a formal resolution of $G$ is also an embedding of $B$ in the sense of Section 2. We say $(f, \overline{f})$ is finite flat if this is finite flat as a map of embeddings ([3]). Consider the $j$-th tubular neighborhood $X^j(B \to B)$ of the embedding $(B \to B)$, which we also write as $X^j(\mathcal{G} \to \Gamma)$. The following lemma, whose first part is implicit in [1], enables us to introduce a group structure on this affinoid variety.

**Lemma 3.1.** Let $(\mathbb{A} \to A)$ and $(\mathbb{B} \to B)$ be embeddings of finite flat $\mathcal{O}_K$-algebras. Put $C = \mathbb{A} \otimes_{\mathcal{O}_K} \mathbb{B}$ and $C = A \otimes_{\mathcal{O}_K} B$. Then the surjection $\mathbb{C} \to C$ is also an embedding and we have a canonical isomorphism $X^j(\mathbb{C} \to C) \to X^j(\mathbb{A} \to A) \times_K X^j(\mathbb{B} \to B)$. Moreover, this isomorphism extends to a canonical isomorphism between their stable normalized integral models $\mathcal{X}^j(\mathbb{C} \to C)_{\mathcal{O}_K} \to \mathcal{X}^j(\mathbb{A} \to A)_{\mathcal{O}_K} \times_{\mathcal{O}_K} \mathcal{X}^j(\mathbb{B} \to B)_{\mathcal{O}_K}$.

**Proof.** By the functoriality of a tubular neighborhood, we have an affinoid map $\Phi : X^j(\mathbb{C} \to C) \to X^j(\mathbb{A} \to A) \times_K X^j(\mathbb{B} \to B)$. To see that $\Phi$ is an isomorphism, we may replace $K$ with a finite separable extension and suppose that $A$ and $B$ are local, $j$ is an integer and $A/\text{rad}(A) = B/\text{rad}(B) = F$. Then we may identify $\mathbb{A}$ with $\mathcal{O}_K[[T_1, \ldots, T_j]]$ and $\mathbb{B}$ with $\mathcal{O}_K[[T'_1, \ldots, T'_r]]$ for some $r$ and $r'$. Let $J = (f_1, \ldots, f_s)$ (resp. $J = (g_1, \ldots, g_{s'})$) be the kernel of the surjection $\mathbb{A} = \mathbb{B}$.
Corollary 3.2. Let \((\mathcal{G} \to \Gamma)\) be a formal resolution of \(\mathcal{G}\). Then the group structure of \(\Gamma\) induces a rigid \(K\)-analytic group structure on the tubular neighborhood \(X^j(\mathcal{G} \to \Gamma)\). This group structure also extends to \(X^j(\mathcal{G} \to \Gamma)\), and endows it with a \(\pi\)-adic formal group scheme structure over \(\mathcal{O}_K\) (resp. an algebraic group structure over \(\tilde{F}\)).

Moreover, for a morphism of formal resolutions \((\mathcal{G} \to \Gamma) \rightarrow (\mathcal{G}' \to \Gamma')\), the induced affinoid map \(X^j(\mathcal{G} \to \Gamma) \rightarrow X^j(\mathcal{G}' \to \Gamma')\) is a homomorphism of rigid \(K\)-analytic groups. This homomorphism also extends to their stable normalized integral models as a homomorphism \(X^j(\mathcal{G} \to \Gamma) \rightarrow X^j(\mathcal{G}' \to \Gamma')\) of \(\pi\)-adic formal group schemes and to their geometric closed fibers as a homomorphism \(X^j(\mathcal{G} \to \Gamma) \rightarrow X^j(\mathcal{G}' \to \Gamma')\) of algebraic groups over \(\tilde{F}\).

Let \(\mathcal{G} = \text{Spec}(B)\) be a connected finite flat group scheme over \(\mathcal{O}_K\) and \((\mathcal{G} \to \Gamma = \text{Spec}(B))\) be a formal resolution of dimension \(r\). Set \(\text{Spf}(\mathcal{A}) = \Gamma/\mathcal{G}\) and regard the zero section \(\text{Spec}(\mathcal{O}_K) \rightarrow \text{Spf}(\mathcal{A})\) as a formal resolution of the trivial group. Then we have a finite flat map.
of formal resolutions

\[
\begin{array}{ccc}
\mathcal{G} & \longrightarrow & \text{Spf} \left( \mathbb{B} \right) \\
\downarrow & & \downarrow \\
\text{Spec} \left( \mathcal{O}_K \right) & \longrightarrow & \text{Spf} \left( \mathbb{A} \right).
\end{array}
\]

By Corollary 3.2 and \cite[Lemma 1.8]{3}, we get a finite flat map of rigid \( K \)-analytic groups \( f^j : X_j^i = X_j^i(\mathcal{G} \to \Gamma) \to D^{r,j} = X_j^i(\text{Spec}(\mathcal{O}_K) \to \text{Spf}(\mathbb{A})) \), where \( D^{r,j} \) denotes the \( r \)-dimensional polydisc \( \{(z_1, \ldots, z_r) \in \mathcal{O}_K^r | v_K(z_i) \geq j \text{ for any } i\} \). We call this the affinoid homomorphism associated to a formal resolution \( (\mathcal{G} \to \Gamma) \). Write \( B_j^I \) and \( A_j^I \) for the \( K \)-affinoid algebras of \( X_j^i \) and \( D^{r,j} \) respectively. The stable normalized integral model over \( \mathcal{O}_L \) of \( X_j^i \) (resp. \( D^{r,j} \)) is denoted by \( X_j^i_{\mathcal{G},\mathcal{O}_L} \) (resp. \( D^{r,j}_{\mathcal{O}_L} \)) and its geometric closed fiber by \( \hat{X}_j^i \) (resp. \( \hat{D}_j^r \)). Note that the algebraic group \( \hat{X}_j^i \) is reduced, hence smooth by \cite[Theorem 11.6]{16}.

**Lemma 3.3.** The affinoid homomorphism \( f^j : X_j^i \to D^{r,j} \) is etale for any \( j > 0 \). Moreover, for \( j > c(\mathcal{G}) \), there exists a finite extension \( K' / K \) such that \( X_j^i_{\mathcal{G},K'} \) is isomorphic to the disjoint sum of finitely many copies of \( D_j^{r,j} \).

**Proof.** We have \( \Omega^1_{B_j^I/\mathcal{O}^I_K} = B_j^I \otimes_B \hat{\Omega}_B/\mathcal{O}^I_K \). It is enough to show that \( \hat{\Omega}_B/\mathcal{O}^I_K \) is a torsion \( \mathcal{O}_K \)-module. Let \( J_A \) and \( J_B \) be the augmentation ideals of \( \mathbb{A} \) and \( \mathbb{B} \) respectively. Set \( I = \text{Ker}(\mathbb{B} \to B) \). Then \( \hat{\Omega}_B/\mathcal{O}^I_K = \text{Coker} \left( \mathbb{B} \otimes_B \hat{\Omega}_B/\mathcal{O}^I_K \to \Omega_B/\mathcal{O}_K \right) \) is equal to \( \mathbb{B} \otimes_B \text{Coker}(\text{Cot}(\mathbb{A}) \to \text{Cot}(\mathbb{B})) = \mathbb{B} \otimes_B \mathcal{O}_K \). This shows the first assertion. For the second assertion, take a finite extension \( K' \) of \( K \) where the geometric connected components of \( X_j^i \) are defined. By assumption, each of the connected components of \( X_j^i_{\mathcal{G},K'} \) is a finite etale cover of \( D_j^{r,j} \) whose degree is one. Thus this is isomorphic to \( D_j^{r,j} \). \hfill \Box

Take a finite extension \( L \) of \( K \) where the stable normalized integral models of \( X_j^i \) and \( D^{r,j} \) are defined. The generic fiber \( \mathcal{G}_L = \mathcal{G} \times_{\mathcal{O}_K} L \) can be regarded as a rigid \( L \)-analytic subgroup of \( X_j^i_{\mathcal{G},L} \) defined by an ideal \( \mathcal{J}_L \) of \( \mathcal{B}_L^i \). Put \( \mathcal{J} = \mathcal{J}_L \cap \mathcal{B}_L^j \) and \( \mathcal{B}_j = \mathcal{B}_L^j / \mathcal{J} \). The latter is a subring of \( B_L = B \otimes_K L \). Since we have a commutative diagram with
surjective horizontal arrows and injective vertical arrows

\[
\begin{array}{ccc}
B_0^j \otimes_{O_K} O_L & \longrightarrow & B \otimes_{O_K} O_L \\
\downarrow & & \downarrow \\
\hat{B}_L^j & \longrightarrow & \hat{B}_L^j
\end{array}
\]

we see that $\hat{B}_L^j$ is integral over $B \otimes_{O_K} O_L$. Thus the $O_K$-algebra $\hat{B}_L^j$ is finite flat. Set $\mathcal{H}'_{O_L} = \text{Spec}(\hat{B}_L^j)$. This can be regarded as a closed $\pi_L$-adic formal subscheme of $X_{\mathcal{G},O_L}$.

**Lemma 3.4.** The group structure of $\mathcal{G}_L$ extends to $\mathcal{H}'_{O_L}$. The group scheme $\mathcal{H}'_{O_L}$ is a closed $\pi_L$-adic formal subgroup scheme of $X_{\mathcal{G},O_L}$.

**Proof.** Put $\mathcal{K} = \text{Ker}(B_0^j \otimes_{\pi_L} \hat{B}_L^j \to \hat{B}_L^j \otimes_{O_L} \hat{B}_L^j)$ and $\mathcal{K}_L = \mathcal{K} \otimes_{O_L} L$. Let $\mu$ be the coproduct of $\hat{B}_L^j$. We must show $\mu(J) \subseteq \mathcal{K}$. This follows from the commutative diagram below whose rows are exact and vertical arrows are injective.

\[
\begin{array}{cccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \hat{B}_L^j \otimes_{\pi_L} \hat{B}_L^j & \longrightarrow & \hat{B}_L^j \otimes_{O_L} \hat{B}_L^j & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}_L & \longrightarrow & \hat{B}_L^j \otimes_{L} \hat{B}_L^j & \longrightarrow & B_L \otimes_L B_L & \longrightarrow & 0
\end{array}
\]

Passing to the generic fiber, we see that the second assertion holds. \(\square\)

**Lemma 3.5.** The associated homomorphism $\tilde{f}^j : X_{\mathcal{G},O_L}^j \to \mathcal{D}_{O_L}^{r,j}$ is finite flat. Moreover, there exists an exact sequence of $\pi_L$-adic formal group schemes

\[
0 \to \mathcal{H}'_{O_L} \to X_{\mathcal{G},O_L}^j \to \mathcal{D}_{O_L}^{r,j} \to 0.
\]

**Proof.** From Lemma 3.3, the associated affinoid map $f^j : X_{\mathcal{G}}^j \to D^{r,j}$ is finite etale. Let $\mathcal{B}_K^j$ and $\mathcal{A}_K^j$ be their affinoid algebras as above. Since $D^{r,j}$ is integral, we see that $f^j$ is surjective and the ring homomorphism $\mathcal{A}_K^j \to \mathcal{B}_K^j$ is injective. Thus we have an injection $\hat{\mathcal{A}}_L^j \to \hat{\mathcal{B}}_L^j$, which is finite by [6, Corollary 6.4.1/6]. Hence $\tilde{f}^j : X_{\mathcal{G}}^j \to D^{r,j}$ is a surjective homomorphism of algebraic groups over $\tilde{F}$. Since $X_{\mathcal{G}}^j$ and $D^{r,j}$ are regular, we see that $f^j$ is faithfully flat by [13, Theorem 23.1]. Since $\hat{\mathcal{A}}_L^j$ and $\hat{\mathcal{B}}_L^j$ is $\pi_L$-torsion free, the map $\tilde{f}^j$ is flat by the local criterion of flatness. Put $\mathcal{H}' = \text{Ker}(\tilde{f}^j)$. This is a closed $\pi_L$-adic formal subgroup scheme of $X_{\mathcal{G},O_L}$ and can be regarded also as a finite flat group scheme over $O_L$. Passing to the generic fiber, we see that $\mathcal{H}'_{O_L}$ is a closed subgroup scheme of $\mathcal{H}'$. Comparing these ranks concludes the lemma.
From Lemma 3.3, we see that for $j > c = c(\mathcal{G})$, the map $\tilde{f}^j$ identifies $\mathfrak{X}_{\mathcal{G}, \mathcal{O}_L}^j$ with the direct sum of finitely many copies of $\mathcal{D}_{\mathcal{O}_L}^{-j}$. More precisely, we have the following.

**Lemma 3.6.** Let $c = c(\mathcal{G})$ be the conductor of $\mathcal{G}$. Then the associated homomorphism $\tilde{f}^j : \mathfrak{X}_{\mathcal{G}, \mathcal{O}_L}^j \to \mathcal{D}_{\mathcal{O}_L}^{-j}$ is finite etale if and only if $j \geq c$.

**Proof.** Let $\text{sp}_j : \mathfrak{X}_{\mathcal{G}}^j \to \tilde{\mathfrak{X}}_{\mathcal{G}}^j$ be the specialization map. By Lemma 3.3, we see that $\tilde{f}^c$ is finite etale at $\text{sp}_c(x)$ for any $x \in \mathcal{G}(\bar{K})$ as in the proof of [2, Theorem 7.2], using a theorem of Bosch ([7, Lemma 2.1]). We conclude the etaleness of $\tilde{f}^c$ by the existence of the group structure. We have $\Omega^1_{\mathcal{B}_{\mathcal{O}_L}/\mathcal{A}_{\mathcal{O}_L}} \otimes_{\mathcal{O}_L} \bar{F} = 0$. Since $\mathcal{B}_{\mathcal{O}_L}$ is $\pi_L$-adic complete and Noetherian, we see that $\Omega^1_{\mathcal{B}_{\mathcal{O}_L}/\mathcal{A}_{\mathcal{O}_L}} = 0$ and the homomorphism $\tilde{f}^c : \mathfrak{X}_{\mathcal{G}, \mathcal{O}_L}^j \to \mathcal{D}_{\mathcal{O}_L}^{c}$ is finite etale.

Let $\bar{0}$ be the zero section of $\tilde{D}^{r-j}$ and set $X^j_{\mathcal{G}} = \cup_{j' > j} X^j_{\mathcal{G}}$. Then we have $(\tilde{f}^j)^{-1}(\bar{0}) = \text{sp}_j(X^j_{\mathcal{G}}) = \text{sp}_j(\mathcal{G}(\bar{K}))$. If $\tilde{f}^j$ is etale, then $\sharp(\tilde{f}^j)^{-1}(\bar{0})$ equals the degree of $\tilde{f}^j$, namely $\sharp\mathcal{G}(\bar{K})$. Thus $X^j_{\mathcal{G}}$ splits and we have $j \geq c$.

4. **Ramification and the $I_K$-module structure of a finite flat group scheme**

Consider the right action of $I_K$ on $\bar{K}$ defined by $\sigma.z = \sigma^{-1}(z)$ for $\sigma \in I_K$. This action induces a $\bar{K}$-semilinear left action of $I_K$ on $X^j_{\mathcal{G}, \mathcal{O}_K} = X^j_{\mathcal{G}} \times_K \bar{K}$, which also extends to an $\mathcal{O}_K$-semilinear action on its stable normalized integral model $X^j_{\mathcal{G}, \mathcal{O}_K}$. Thus we have an $\bar{F}$-linear left action of $I_K$ on its closed fiber $\tilde{X}^j_{\mathcal{G}}$. We call this the geometric monodromy action of $I_K$ and write the action of $\sigma \in I_K$ as $\sigma_{\text{geom}}$ (note that here we follow the terminology in [2], and not in [3] where the monodromy action is called arithmetic, since in our case there is no “geometric” action other than the monodromy action). Similarly, we have the geometric monodromy action of $I_K$ on $\tilde{D}^{r-j}$.

The latter action is described as follows. Let the additive group (resp. multiplicative group) over $\bar{F}$ be denoted by $\mathbb{G}_a$ (resp. $\mathbb{G}_m$). Consider the left action $\mathbb{G}_m \times \mathbb{G}_a \to \mathbb{G}_a$ given by the multiplication. Write this action of $\lambda \in \bar{F}^\times$ as $[\lambda]$. This action is defined by $T_i \mapsto \lambda T_i$, where $\mathbb{G}_a = \text{Spec}(\bar{F}[T_1, \ldots, T_r])$. For $j \in \mathbb{Q}_{>0}$, we define the fundamental character $\theta_j : I_K \to \bar{F}^\times$ to be $\theta_{k'}/l'$, where $k'/l'$ is the prime-to-$p$-denominator part.
Thus the diagram whose rows are exact.

Proof. Put $A = O_K[[T_1, \ldots, T_r]]$ and $j = k/l$ with $(k, l) = 1$. Let $L$ be a finite Galois extension of $K$ containing $\pi^{1/l}$ and $e' = e(L/K)$ be its ramification index over $K$. Then $e'k/l \in \mathbb{Z}$ and the stable normalized integral model of $D^{r,j}$ over $O_L$ is $O_L(T_1/(\pi_L)^{e'k/l}, \ldots, T_r/(\pi_L)^{e'k/l}) = O_L(W_1, \ldots, W_r)$, where $W_i = T_i/(\pi_L)^{k/l}$. Set $\mu_\mathbb{A}$ to be the coproduct of $A$. We have

$$\mu_\mathbb{A}(T_i) = T_i \hat{\otimes} 1 + 1 \hat{\otimes} T_i + \text{(higher degree)}$$

and then $\mu_\mathbb{A}(W_i) = \mu_\mathbb{A}((\pi_L)^{k/l}W_i)/(\pi_L)^{k/l}$ is equal to

$$W_i \hat{\otimes}_\pi 1 + 1 \hat{\otimes}_\pi W_i + (\pi_L)^{k/l}\text{(higher degree)}$$

in this $O_L$-algebra. This shows the first assertion.

On the other hand, we know the geometric monodromy action on $D^{r,j}$ is tame ([2, Lemma 7.7]). Write $l = p^nl_0$ with $p \nmid l_0$. Then for $\sigma \in I_K$, we have $\sigma((\pi_L)^{k/l})(\pi_L)^{k/l} \equiv \theta_l(\sigma)^{k/p^m} \zeta_{p^m}^{N}$ mod $m_K$ with some $N$ and this is equal to $\theta_l(\sigma)$. Thus the action on the affine algebra $F[W_1, \ldots, W_r]$ of $D^{r,j}$ is given by $\sigma^*_\text{geom}(W_i) = \theta_l(\sigma)W_i$. This coincides with $[\theta_l(\sigma)]$. \hfill \square

Next we consider the geometric monodromy action on $X^{j}_G$. Let $\bar{X}^{j,0}_G$ denote the unit component of the algebraic group $\bar{X}^{j}_G$ and $\bar{H}^j$ be the geometric closed fiber of $H^j_{O_L}$. We begin with the following lemma.

**Lemma 4.2.** If $\psi \in \text{End}(\bar{X}^{j,0}_G)$ induces the zero map on $D^{r,j}$, then $\psi = 0$.

Proof. Put $\bar{H}^j = H^j \cap \bar{X}^{j,0}_G$. This is the kernel of the faithfully flat map $\bar{X}^{j,0}_G \to D^{r,j}$ and by assumption we have the following commutative diagram whose rows are exact.

$$
\begin{array}{cccccc}
0 & \longrightarrow & \bar{H}^j_0 & \longrightarrow & \bar{X}^{j,0}_G & \longrightarrow & D^{r,j} & \longrightarrow & 0 \\
& & \downarrow & & \psi & & 0 & & 0 \\
0 & \longrightarrow & \bar{H}^j_0 & \longrightarrow & \bar{X}^{j,0}_G & \longrightarrow & D^{r,j} & \longrightarrow & 0
\end{array}
$$

Thus $\psi$ factors through $\bar{H}^j_0$. Put $\bar{C} = \text{Im}(\psi)$. Then this is a closed subgroup scheme of $\bar{H}^j_0$ and the map $\bar{X}^{j,0}_G \to \bar{C}$ is faithfully flat. Since $\bar{X}^{j,0}_G$
is regular and connected, we see that \( \bar{C} \) is also regular and connected by [13, Theorem 23.7]. Hence \( \bar{C} = 0 \) and we have \( \psi = 0 \).

**Corollary 4.3.** Let \( \mathcal{G} \) be a connected finite flat group scheme over \( \mathcal{O}_K \). Take a formal resolution \( (\mathcal{G} \rightarrow \Gamma) \) of dimension \( r \). Then the algebraic group \( \bar{X}^j_0 \) is isomorphic to \( \mathcal{G}_\alpha^r \).

**Proof.** By the previous lemma and Lemma 4.1, we see that \( \bar{X}^j_0 \) is killed by \( p \). Hence the assertion follows from [12, Lemma 1.7.1].

**Corollary 4.4.** The geometric monodromy action of \( I_K \) on \( \bar{X}^j_0 \) is tame.

**Proof.** For an element \( \sigma \) of the wild inertia subgroup \( P_K \), the geometric monodromy action \( \sigma_{\text{geom}} \) on \( \bar{D}^r \) is trivial. Applying the lemma to \( \sigma_{\text{geom}} \) shows the assertion.

**Corollary 4.5.** Let \( J \) be a finite cyclic quotient of \( I_K \) through which the tame character \( \theta_j \) factors and \( \tau \) be a generator of \( J \). Let \( F(t) \) denote the minimal polynomial of \( \theta_j(\tau) \in \overline{\mathbb{F}}_p \) over \( \mathbb{F}_p \). Then the geometric monodromy action of \( I_K \) on \( \bar{X}^j_0 \) also factors through \( J \) and the equation \( F(\tau_{\text{geom}}) = 0 \) holds in \( \text{End}(\bar{X}^j_0) \).

**Proof.** The first assertion follows from Lemma 4.2 as the previous corollary. The second assertion also follows from this lemma using Lemma 4.1.

Let \( c = c(\mathcal{G}) \) be the conductor of \( \mathcal{G} \). The lemma below enables us to realize \( \mathcal{G}^c(\bar{K}) \) as a subgroup of \( \bar{X}^c_0 \).

**Lemma 4.6.** The specialization map \( sp_K : \bar{X}^c_{\mathcal{G}, K} \rightarrow \bar{X}^c_0 \) induces an \( I_K \)-equivariant isomorphism \( \mathcal{G}(\bar{K}) \rightarrow \bar{H}^c(\bar{F}) \) and \( \mathcal{G}^c(\bar{K}) \rightarrow \mathcal{H}^c_0(\bar{F}) \). Here we consider on the left-hand side the natural action as the \( K \)-valued points of \( \mathcal{G} \) (resp. \( \mathcal{G}^c \)) and on the right-hand side the restriction of the geometric monodromy action on \( \bar{X}^c_0 \).

**Proof.** By definition, the generic fiber of \( \mathcal{H}^c_{\mathcal{O}_L} \) is equal to \( \mathcal{G}_L \). From the exact sequence (1) and Lemma 3.6, we know that \( \mathcal{H}^c_{\mathcal{O}_L} \) is etale over \( \mathcal{O}_L \) and there is the following exact sequence of algebraic groups over \( \bar{F} \).

\[
0 \rightarrow \bar{H}^c \rightarrow \bar{X}^c_0 \rightarrow \bar{D}^r \rightarrow 0
\]

Thus we have a natural isomorphism \( \mathcal{H}^c_{\mathcal{O}_L}(\bar{K}) \rightarrow \bar{H}^c(\bar{F}) \) and the composite \( \mathcal{G}(\bar{K}) = \mathcal{H}^c_{\mathcal{O}_L}(\bar{K}) \rightarrow \bar{H}^c(\bar{F}) \rightarrow \bar{X}^c_{\mathcal{G}}(\bar{F}) \) coincides with the map
sp_c. From [2, Corollary 4.4], we see that this map sends \( G^c(\bar{K}) \) isomorphically onto \( \mathcal{H}_0^c(F) \).

For \( x \in X^c_0(\bar{K}) \) and \( \sigma \in I_K \), let \( \sigma(x) \) denote the natural action of \( \sigma \) on \( \bar{K} \)-valued points. Then we have \( \sigma_{\text{geom}}(x) \circ \sigma = \sigma(x) \). Taking its specialization shows the \( I_K \)-equivariance.

The following theorem can be regarded as a generalization for a finite flat group scheme over \( O_K \) of the structure theorem of the graded piece of the classical upper numbering ramification filtration.

**Theorem 4.7.** Let \( \mathcal{G} \) be a finite flat group scheme over \( O_K \) and \( j \in \mathbb{Q}_{>0} \). Then the \( G_K \)-module \( \mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K}) \) is tame and killed by \( p \).

**Proof.** Since \( \mathcal{G}^j = (\mathcal{G}^0)^j \), where \( \mathcal{G}^0 \) denotes the unit component of \( \mathcal{G} \), we may assume \( \mathcal{G} \) is connected. Suppose that \( j \) is a jump of the ramification filtration on \( \mathcal{G} \) and consider the quotient \( \mathcal{G} = \mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K}) \). The subgroup \( \mathcal{G}^j(\bar{K}) \subseteq \mathcal{G}(\bar{K}) \) has a non-trivial image in \( (\mathcal{G}/\mathcal{G}^{j+})(\bar{K}) \). By the Herbrand theorem ([1, Lemme 2.10]), the natural map \( \mathcal{G}^t(\bar{K}) \to (\mathcal{G}/\mathcal{G}^{j+})^t(\bar{K}) \) is surjective for any \( t > 0 \). We have \( (\mathcal{G}/\mathcal{G}^{j+})^t = 0 \) for \( t > j \) and \( (\mathcal{G}/\mathcal{G}^{j+})^j \neq 0 \). Thus the ramification filtration on \( \mathcal{G}/\mathcal{G}^{j+} \) jumps at \( j \) and \( (\mathcal{G}/\mathcal{G}^{j+})^j(\bar{K}) = \mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K}) \). Replacing \( \mathcal{G} \) with \( \mathcal{G}/\mathcal{G}^{j+} \), we may assume \( j = c = c(\mathcal{G}) \).

Take a formal resolution \( (\mathcal{G} \to \Gamma) \) of dimension \( r \) and consider its associated affinoid homomorphism \( X^c_0 \to D^{c,c} \). Then the theorem follows from Corollary 4.3, Corollary 4.4, and Lemma 4.6.

From this theorem, we see that the inertia subgroup \( I_K \) acts on \( \mathcal{G}^c(\bar{K})/\mathcal{G}^{c+}(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p \) by the direct sum of tame characters. The theorem below determines these characters up to \( p \)-power exponent.

**Theorem 4.8.** Let \( \mathcal{G} \) be a finite flat group scheme over \( O_K \) and \( j \in \mathbb{Q}_{>0} \). Then \( I_K \) acts on \( \mathcal{G}^c(\bar{K})/\mathcal{G}^{c+}(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p \) by the direct sum of \( \mathbb{F}_p \)-conjugates of the fundamental character \( \theta_j \).

**Proof.** We may assume that \( \mathcal{G} \) is connected. The same argument as in the proof of Theorem 4.7 using the Herbrand theorem reduces the claim to the case where \( j = c = c(\mathcal{G}) \). Take a formal resolution \( (\mathcal{G} \to \Gamma) \) of dimension \( r \). Let \( J \) and \( \tau \) be as in Corollary 4.5. Then Corollary 4.5 and Lemma 4.6 show that every eigenvalue of the action of \( \tau_{\text{geom}} \) on the finite dimensional \( \mathbb{F}_p \)-vector space \( \mathcal{G}^c(\bar{K}) \) is a conjugate of \( \theta_j(\tau) \) over \( \mathbb{F}_p \). Since the order of \( J \) is prime to \( p \), we conclude that \( I_K \) acts on \( \mathcal{G}^c(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p \) by the direct sum of \( \mathbb{F}_p \)-conjugates of \( \theta_c \).
Corollary 4.9. Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$. Then the order of the image of the homomorphism $I_K \to \text{Aut}(\mathcal{G}(\bar{K}))$ is a power of $p$ if and only if every jump $j$ of the ramification filtration $\{\mathcal{G}^j\}_{j \in \mathbb{Q}_{>0}}$ is an element of $\mathbb{Z}[1/p]$.

**Proof.** From Theorem 4.8, we see that the jumps are in $\mathbb{Z}[1/p]$ if and only if the graded pieces $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K})$ are unramified. By Theorem 4.7, this is equivalent to the condition that $\sharp \text{Im}(I_K \to \text{Aut}(\mathcal{G}(\bar{K})))$ is a $p$-power.

When $\mathcal{G}(\bar{K})$ is unramified and killed by $p$, we have the following reinforcement of this result, which is an easy corollary of the Herbrand theorem. The author does not know if every jump is an integer whenever $\mathcal{G}(\bar{K})$ is unramified. If $\mathcal{G}$ is monogenic, then we see that this holds true from [11, Theorem 4].

**Proposition 4.10.** Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$ which is killed by $p$. Suppose that the $G_K$-module $\mathcal{G}(\bar{K})$ is unramified. Then every jump $j$ of the ramification filtration $\{\mathcal{G}^j\}_{j \in \mathbb{Q}_{>0}}$ is an element of $p\mathbb{Z}$.

**Proof.** We may assume $K = K^{nr}$ and $G_K$ acts trivially on $\mathcal{G}(\bar{K})$. There is a quotient $W$ of $\mathcal{G}(\bar{K})/\mathcal{G}^{j+}(\bar{K})$ where $\mathcal{G}^j(\bar{K})$ has a non-trivial image and of rank one over $\mathbb{F}_p$. Taking the schematic closure, $W$ extends to a finite flat group scheme $\mathcal{W}$ over $\mathcal{O}_K$ which is a quotient of $\mathcal{G}/\mathcal{G}^{j+}$. By the Herbrand theorem, we see that the ramification filtration of $\mathcal{W}$ jumps at $j$. On the other hand, $\mathcal{W}$ is a Raynaud $\mathbb{F}_p$-vector space scheme with unramified generic fiber. Thus the proposition follows from [11, Theorem 4].

For the rest of this section, we state some corollaries in the case where $\mathcal{G}$ is an $\mathbb{F}$-vector space scheme of rank one or two for a finite extension $\mathbb{F}$ over $\mathbb{F}_p$. In the case of rank one, Theorem 4.8 directly shows the claim below, while this can be shown also as a corollary of a determination of the conductor of a Raynaud $\mathbb{F}$-vector space scheme (Theorem 5.5).

**Corollary 4.11.** Let $\mathcal{G}$ be an $\mathbb{F}$-vector space scheme of rank one over $\mathcal{O}_K$ and $c = c(\mathcal{G})$. Then the $I_K$-action on the $\mathbb{F}$-vector space $\mathcal{G}(\bar{K})$ of rank one is given by the character $\theta \bar{c}^n$ for some $n$.

In the case of rank two, we have the following.

**Corollary 4.12.** Let $\mathcal{G}$ be a finite flat $\mathbb{F}$-vector space scheme of rank two over $\mathcal{O}_K$ and $c = c(\mathcal{G})$. Then the $I_K$-module $\mathcal{G}(\bar{K}) \otimes_{\mathbb{F}} \bar{\mathbb{F}}_p$ contains
the character $\theta^n$ for some $n$. If the $G_K$-module $G(\bar{K})$ is reducible, this holds true for $G(\bar{K})$ itself.

**Proof.** The first assertion follows easily from Theorem 4.8 and the surjection $G^c(\bar{K}) \otimes_{\mathbb{F}_p} \mathbb{F}_p \to G^c(\bar{K}) \otimes_{\mathbb{F}_p} \mathbb{F}_p$. Suppose the $I_K$-module $G(\bar{K})$ is reducible. When $G^c$ is of rank one, the assertion is clear from Theorem 4.8. If $G^c = G$, then $G^c$ is reducible and the assertion follows also from Theorem 4.8.

The corollary below indicates that the conductor $\text{c}(G)$ carries information about not only the tame characters but also their extension structures in the $I_K$-module $G(\bar{K})$.

**Corollary 4.13.** Consider an exact sequence of finite flat $\mathbb{F}$-vector space schemes over $\mathcal{O}_K$

$$0 \to G_1 \to G \to G_2 \to 0$$

where $G_1$ and $G_2$ are connected of rank one. If $c(G) = c(G_2)$, then the $I_K$-module $G(\bar{K})$ splits.

**Proof.** Put $c = c(G)$. Take a formal resolution $(G \to \Gamma)$ of dimension $r$ and put $\Gamma_2 = \Gamma/G_1$. Then we get a finite flat map of formal resolutions

$$G \longrightarrow \Gamma$$

$$\downarrow \quad \quad \quad \downarrow$$

$$G_2 \longrightarrow \Gamma_2.$$ 

Therefore we have a finite flat homomorphism of rigid $K$-analytic groups $X_j(G \to \Gamma) \to X_j(G_2 \to \Gamma_2)$ by Corollary 3.2. As in the proof of Lemma 3.3, we see that this map is finite etale.

Suppose $G^c(\bar{K})$ is of rank one. If $G^c(\bar{K}) \neq G_1(\bar{K})$ as an $\mathbb{F}$-subspace of $G(\bar{K})$, the $I_K$-module $G(\bar{K})$ splits and the proposition follows. Suppose $G^c(\bar{K}) = G_1(\bar{K})$. The affinoid variety $X^c(G \to \Gamma)$ decomposes to $\#F$ components over some finite extension $K'$ of $K$. Each component is a Zariski open and closed subset of $X^c(G \to \Gamma)_{K'}$. As the map $f : X^c(G \to \Gamma)_{K'} \to X^c(G_2 \to \Gamma_2)_{K'}$ is finite etale and $X^c(G_2 \to \Gamma_2)_{K'}$ is connected, every component $X^c(G \to \Gamma)_{K'}$ maps surjectively to $X^c(G_2 \to \Gamma_2)_{K'}$. Take some $g_i \in G(\bar{K}) \cap X^c(G \to \Gamma)_{K'}$. Using the group structure, we see that $G(\bar{K}) \cap X^c(G \to \Gamma)_{K'} = g_i + G^c(\bar{K}) = g_i + G_1(\bar{K})$ and $f(G(\bar{K}) \cap X^c(G \to \Gamma)_{K'}) = f(g_i)$. However, we have $f^{-1}(G_2(\bar{K})) = G(\bar{K})$ and thus $f(G(\bar{K}) \cap X^c(G \to \Gamma)_{K'}) = G_2(\bar{K})$. This is a contradiction. Therefore we may assume $G^c(\bar{K}) = G(\bar{K})$. In this case, the proposition follows from Theorem 4.7.

$\square$
5. Example: Rank One Calculation

In this section, we calculate the conductor of a Raynaud $\mathbb{F}$-vector space scheme over $\mathcal{O}_K$. The point is that, as we can see from the bound of the conductor ([11, Theorem 7]), it is enough to consider the $j$-th tubular neighborhood only for $j \leq pe/(p-1) + \varepsilon$ with sufficiently small $\varepsilon > 0$. For such $j$, we can compute the tubular neighborhood easily by Lemma 5.4 below.

Let $K$ be a complete discrete valuation field of mixed characteristic $(0, p)$. We write $\pi = \pi_K$ for its uniformizer and $e$ for its absolute ramification index. We normalize a valuation $v_K$ of $K$ as $v_K(\pi) = 1$ and extend it to the algebraic closure $\bar{K}$ of $K$. For $a \in \bar{K}$ and $j \in \mathbb{R}$, let $D(a, j)$ denote the closed disc $\{z \in \mathcal{O}_K \mid v_K(z-a) \geq j\}$. This is the underlying subset of a $K(a)$-affinoid subdomain of the unit disc over $K(a)$.

For integers $0 \leq s_1, \ldots, s_r \leq e$, let $\mathcal{G}(s_1, \ldots, s_r)$ denote the Raynaud $\mathbb{F}_p$-vector space scheme over $\mathcal{O}_K$ defined by the $r$ equations $T_i^p = \pi^{s_i} T_{i-1}, T_r^p = \pi^{s_r} T_r$ ([14]). We set $j_k = (ps_k + p^2 s_k - 1 + \cdots + p^k s_1 + p^{k+1} s_r + p^{k+2} s_{r-1} + \cdots + p^r s_k + 1)/(p^r - 1)$. Before the calculation of $c(\mathcal{G}(s_1, \ldots, s_r))$, we gather some elementary lemmas.

**Lemma 5.1.** Let $a \in \mathcal{O}_K$ and $s = v_K(a)$. Then the affinoid variety $X^j(K) = \{x \in \mathcal{O}_K \mid v_K(x^p - a) \geq j\}$ is equal to

$$\left\{ \begin{array}{ll}
D(a^{1/p}, j/p) & \text{if } j \leq s + pe/(p-1), \\
\prod_{i=0}^{p-1} D(a^{1/p^{i+1}}, j - e - (p-1)s/p) & \text{if } j > s + pe/(p-1).
\end{array} \right.$$  

**Proof.** We have $v_K(x^p - a) = \sum_i v_K(x - a^{1/p^{i+1}})$. If $v_K(x - a^{1/p^{i+1}}) \geq v_K(x - a^{1/p^{i'}})$ for any $i' \neq i$, then $v_K(x - a^{1/p^{i+1}}) \leq v_K(a^{1/p^{i'}}(1 - \zeta_{p^{i'-i}})) = s/p + e/(p-1)$. Thus we have $v_K(x - a^{1/p^{i+1}}) \geq \sup(j/p, j - (p-1)s/p - e)$ and $X^j(K) \subseteq \bigcup_i D(a^{1/p^{i+1}}, j/p, j - (p-1)s/p - e)$.

Suppose that $j/p \geq j - (p-1)s/p - e$. Then we have $v_K(a^{1/p}(1-\zeta_{p^i})) = s/p + e/(p-1) \geq j/p, D(a^{1/p}, j/p) = D(a^{1/p^{i+1}}, j/p)$ for any $i$ and thus $X^j(K) = D(a^{1/p}, j/p)$.

When $j/p < j - (p-1)s/p - e$, we have $v_K(a^{1/p}(1-\zeta_{p^i})) = s/p + e/(p-1) < j - (p-1)s/p - e$. This means that if $w \in D(a^{1/p^{i+1}}, j - (p-1)s/p - e)$ for some $i$, then $v_K(w - a^{1/p^{i'}}) < j - (p-1)s/p - e$ for any other $i'$.
Thus the discs $D(a_1^j/\mathbb{C}^p_j, j - (p - 1)s/p - e)$ are disjoint and
\[
X^j(\overline{K}) = \bigsqcap_i D(a_1^j/\mathbb{C}^p_j, j - (p - 1)s/p - e).
\]

These are equalities of the underlying sets of affinoid subdomains of the unit disc over $K(a_1^p, \zeta_p)$. By the universality of an affinoid subdomain, this extends to an isomorphism of affinoid varieties.

\[\square\]

We can prove the following lemma just in the same way.

\begin{lemma}
The affinoid variety $\{ x \in \mathcal{O}_K \mid v_K(x^{p^r} - ax) \geq j \}$ is equal to
\[
\begin{cases}
D(0, j/p^r) & \text{if } j \leq p^r v(a)/(p^r - 1), \\
\bigsqcap_{i=0}^{r-1} D(\sigma_i, j - v(a)) & \text{if } j > p^r v(a)/(p^r - 1),
\end{cases}
\]
where $\sigma_i$’s are the roots of $X^{p^r} = aX$.
\end{lemma}

\begin{lemma}
For $g_1(Y_1, \ldots, Y_d), g_2(Y_1, \ldots, Y_d) \in K[Y_1, \ldots, Y_d]$ and $j_1 \geq j_2$, the affinoid variety $\{(x, y_1, \ldots, y_d) \in \mathcal{O}_K \times \mathcal{O}_K^d \mid v_K(x - g_1(y_1, \ldots, y_d)) \geq j_1, v_K(x - g_2(y_1, \ldots, y_d)) \geq j_2 \}$ is equal to $\{(x, y_1, \ldots, y_d) \in \mathcal{O}_K \times \mathcal{O}_K^d \mid v_K(x - g_1(y_1, \ldots, y_d)) \geq j_1, v_K(g_1(y_1, \ldots, y_d) - g_2(y_1, \ldots, y_d)) \geq j_2 \}$.
\end{lemma}

\textbf{Proof.} For fixed $(x, y)$, these two conditions are equivalent. The universality of an affinoid subdomain proves the lemma.

\[\square\]

\begin{lemma}
Let $a \in \mathcal{O}_K$ and $s = v_K(a)$. If $j \leq pe/(p - 1) + s$, then the affinoid variety $X^j(\overline{K}) = \{(x, y) \in \mathcal{O}_K \times \mathcal{O}_K \mid v_K(x^{p^e} - ay^{p^e}) \geq j \}$ is equal to $\{(x, y) \in \mathcal{O}_K \times \mathcal{O}_K \mid v_K(x - a^{1/p}y^{p^{e-1}}) \geq j/p \}$.
\end{lemma}

\textbf{Proof.} Lemma 5.1 shows that the fiber of the second projection $X^j(\overline{K}) \to \mathcal{O}_K$ at $y$ is equal to
\[
\begin{cases}
D(a_1^{1/p}y^{p^{e-1}}, j/p) & \text{if } j \leq s + p^{e-1}v_K(y) + pe/(p - 1), \\
\bigsqcap_{i=0}^{e-1} D(a_1^{1/p}y^{p^{e-1}}, j - e - (p - 1)(s + p^{e-1}v_K(y))/p) & \text{otherwise}.
\end{cases}
\]
Thus we have $X^j(\overline{K}) = \{(x, y) \in \mathcal{O}_K \times \mathcal{O}_K \mid v_K(x - a^{1/p}y^{p^{e-1}}) \geq j/p \}$ for $j \leq pe/(p - 1) + s$. This is the underlying set of a $K(a_1^p)$-affinoid variety. Again this equality extends to an isomorphism over $K(a_1^p)$.

\[\square\]

Now we proceed to the proof of the main theorem of this section.

\begin{theorem}
$c(G(s_1, \ldots, s_r)) = \sup_k j_k$.
\end{theorem}
Proof. We may assume that $j_r$ is the supremum of $j_k$’s. If $j_r = 0$, then $\mathcal{G}(s_1, \ldots, s_r)$ is etale and $c(\mathcal{G}(s_1, \ldots, s_r)) = 0$. Thus we may assume $j_r > 0$. Consider the homomorphism of $\mathcal{O}_K$-algebras

$$ A = \mathcal{O}_K[T_1, \ldots, T_r]/(T_1^p - \pi^{s_1}T_2, \ldots, T_r^p - \pi^{s_r}T_1) \to $$

$$ B = \mathcal{O}_K[W, T_2, \ldots, T_r]/(W^p - \pi^{s_1}T_2, T_2^p - \pi^{s_2}T_3, \ldots, $$

$$ T_{r-1}^p - \pi^{s_{r-1}}T_r, T_r^p - \pi^{s_r}W^{p^{r-1}}), $$

defined by $T_1 \mapsto W^{p^{r-1}}$. This induces a surjection of $K$-affinoid varieties

$$ X^j_B(\bar{K}) \ni (w, t_2, \ldots, t_r) \mapsto (w^{p^{r-1}}, t_2, \ldots, t_r) \in X^{j}_A(\bar{K}), $$

where

$$ X^j_A(\bar{K}) = \{(t_1, \ldots, t_r) \in \mathcal{O}^r_K \mid v_K(t_1^p - \pi^{s_1}t_2) \geq j, \ldots, $$

$$ v_K(t_{r-1}^p - \pi^{s_{r-1}}t_r) \geq j, v_K(t_r^p - \pi^{s_r}t_s) \geq j \} $$

and

$$ X^j_B(\bar{K}) = \{(w, t_2, \ldots, t_r) \in \mathcal{O}^r_K \mid v_K(w^p - \pi^{s_1}t_2) \geq j, $$

$$ v_K(t_2^p - \pi^{s_2}t_3) \geq j, \ldots, v_K(t_r^p - \pi^{s_r}w^{p^{r-1}}) \geq j \}. $$

These are affinoid subdomains of the $r$-dimensional unit polydisc over $K$. We calculate a jump of $\{F^j(B)\}_{j \in \mathbb{Q}_{>0}}$ at first.

**Lemma 5.6.** If $j_r < pe/(p - 1)$, then the first jump of $\{F^j(B)\}_{j \in \mathbb{Q}_{>0}}$ occurs at $j = j_r$ and $\sharp F^{j_r}(B) = p^r$.

Note that the base change from $K$ to a finite extension $L$ multiplies $s_i$’s, $j_i$’s and $e$ by the ramification index of $L/K$. Thus, to prove Lemma 5.6 and Theorem 5.5, we may assume that $p^{r-1}$ divides $s_i$’s and $e$.

**Proof.** Consider the $K$-affinoid variety $X^j_B$ for $j \leq pe/(p - 1)$. Then the iterative use of Lemma 5.4 and Lemma 5.3 shows that the affinoid variety $X^j_B(\bar{K})$ is equal to

$$ \{v_K(w^p - \pi^{(s_r+ps_{r-1}+\ldots+p^{r-1}s_1)/p^{r-1}}w) \geq pl_1(j), v_K(t_2 - g_2(w)) \geq u_2, $$

$$ v_K(t_3 - g_3(t_2, w)) \geq u_3, \ldots, v_K(t_r - g_r(t_{r-1}, w)) \geq u_r \}, $$

where $l_i(j), g_i(t_{i-1}, w), g_2(w)$ and $u_i$ are defined as follows;

- $l_r(j) = j/p$,
- $l_{i-1}(j) = \inf(j, l_i(j) + s_{i-1})/p$,
- $g_i(t_{i-1}, w) = t_{i-1}^p/\pi^{s_{i-1}}$ and $u_i = j - s_{i-1}$ if $j \geq l_i(j) + s_{i-1}$,
- $g_i(t_{i-1}, w) = \pi^{s_r+ps_{r-1}+\ldots+p^{r-1}s_1/p^{r-1}}w^{p^{r-2}}$ and $u_i = l_i(j)$ if $j < l_i(j) + s_{i-1}$. 

g_2(w) = g_2(w^{p^r-1}, w).

Note that l_1(j) is a strictly monotone increasing function of j. This affinoid variety is isomorphic to the product of the affinoid variety \( \{ w \in \mathcal{O}_K \mid v(w^{p^r} - \pi(s_r + ps_{r-1} + \ldots + s_1)/p^{r-1}w) \geq pl_1(j) \} \) and discs. Therefore, from Lemma 5.2, we see that the first jump of \( \{ F^j(B) \}_{j \in \mathbb{Q}_{>0}} \) occurs at j such that \( pl_1(j) = j_r \), provided this j satisfies \( 0 < j < pe/(p-1) \). Moreover, then we have \( F^j(B) = p^r \). Thus the following lemma and the strict monotonicity of \( l_1 \) terminate the proof of Lemma 5.6.

**Lemma 5.7.** \( l_1(j_r) = j_r/p \).

**Proof.** Suppose that there is k such that \( l_k(j_r) = j_r/p \) and \( j_r \geq l_{k'}(j_r) + s_{k'} \) for any \( 1 < k' \leq k \). Then we have \( l_1(j_r) = \inf(j_r, (j_r + ps_{k-1} + p^2s_{k-2} + \ldots + p^{k-1}s_1)/p^{k-1})/p \) and the assumption \( j_{k-1} \leq j_r \) implies \( l_1(j_r) = j_r/p \).

On the other hand, let \( s = (s_r + ps_{r-1} + \ldots + p^{r-1}s_1)/p^{r-1} \) and \( \sigma_0, \ldots, \sigma_{p^r-1} \) be the roots of the equation \( X^{p^r} - \pi^sX = 0 \). Then we see that the images by \( w \mapsto w^{p^r-1} \) of the discs \( D(\sigma_i, pl_1(j) - s) \) are disjoint for \( j > j_r \). Hence the surjection \( \pi_0(X^1_{B}(\bar{K})) \to \pi_0(X^1_A(\bar{K})) \) is bijective for \( 0 < j \leq pe/(p-1) \) and the first (and the last) jump of \( \{ F^j(A) \}_{j \in \mathbb{Q}_{>0}} \) also occurs at \( j_r \), provided \( j_r < pe/(p-1) \).

When \( j_r = pe/(p-1) \), we see that \( s_k = e > 0 \) for any k. Thus we can use Lemma 5.4 for \( j < pe/(p-1) + \varepsilon \) with sufficiently small \( \varepsilon > 0 \). Then, by the same reasoning as above, we conclude that \( c(A) = pe/(p-1) \).

**References**


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