TAME CHARACTERS AND RAMIFICATION OF FINITE FLAT GROUP SCHEMES

SHIN HATTORI

1. Introduction

Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ with residue field F which may be imperfect, G_K be its absolute Galois group and I_K be its inertia subgroup. Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$. When $\mathcal{G}$ is monogenic, that is to say, when the affine algebra of $\mathcal{G}$ is generated over $\mathcal{O}_K$ by one element, it is well-known that the tame characters appearing in the I_K-module $\mathcal{G}(\bar{K})$ are determined by the slopes of the Newton polygon of a defining equation of $\mathcal{G}$ ([15, Proposition 10]).

On the other hand, for an elliptic modular form f of level N prime to p, we also have a description of the tame characters of the associated mod p Galois representation $\bar{\rho}_f$ ([10, Theorem 2.5, Theorem 2.6], [9, Section 4.3]). This is based on Raynaud's theory of prolongations of finite flat group schemes or the integral p-adic Hodge theory. However, for an analogous study of the associated mod p Galois representation of a Hilbert modular form over a totally real number field, we encounter a local field of higher absolute ramification index. In this case, these two theories no longer work well and we need some other techniques.

In this paper, to propose a new approach toward this problem, we generalize the aforementioned result of [15] to the higher dimensional case using the ramification theory of Abbes and Saito ([2], [3]) and determine the tame characters appearing in $\mathcal{G}(\bar{K})$ in terms of the ramification of $\mathcal{G}$ without any restriction on the absolute ramification index of K. Namely, we show the following theorem.

Theorem 1.1. *Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$. Write $\{\mathcal{G}^j\}_{j\in\mathbb{Q}_{>0}}$ for the ramification filtration of $\mathcal{G}$ in the sense of* [2] *and* [3]. *Then the graded piece $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K})$ is killed by p and the I_K-module $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K})\otimes_{\mathbb{F}_p}\bar{\mathbb{F}}_p$ is the direct sum of $\mathbb{F}_p$-conjugates of the fundamental character θ_j of level j.*

Date: December 8, 2006.

In view of the following corollary, we can regard this theorem as a counterpart for finite flat group schemes of the Hasse-Arf theorem in the classical ramification theory.

Corollary 1.2. *Let L be an abelian extension of K. Suppose that its integer ring $\mathcal{O}_L$ is a $\mathcal{G}$-torsor over $\mathcal{O}_K$. Then the denominator of every jump of the upper numbering ramification filtration* $\{\mathrm{Gal}(L/K)^j\}_{j\in\mathbb{Q}_{>0}}$ ([2]) *is a power of p.*

To prove the main theorem, we firstly show that the tubular neighborhood of $\mathcal{G}$ can be chosen to have a rigid analytic group structure. Passing to the closed fiber, we realize the graded piece of the ramification filtration of $\mathcal{G}$ as the kernel of an etale isogeny of the additive groups $\bar{\mathbb{G}}_{\mathrm{a}}^r$ over $\bar{F}$. Then we determine the tame characters by comparing the I_K-action on the graded piece with the $\bar{\mathbb{G}}_{\mathrm{m}}$-action on $\bar{\mathbb{G}}_{\mathrm{a}}^r$ given by the multiplication. As an appendix, we also give an explicit formula for the conductor of a Raynaud $\mathbb{F}$-vector space scheme over $\mathcal{O}_K$ ([14]).

Acknowledgements. The author would like to thank Takeshi Saito for his encouragements and valuable discussions. He also wants to thank Ahmed Abbes and Takeshi Tsuji for their helpful comments.

2. Review of the ramification theory of Abbes and Saito

Let K be a complete discrete valuation field with residue field F which may be imperfect. Set $\pi = \pi_K$ to be an uniformizer of K. The separable closure of K is denoted by $\bar{K}$ and the absolute Galois group of K by G_K. Let $\mathfrak{m}_{\bar{K}}$ and $\bar{F}$ be the maximal ideal and the residue field of $\mathcal{O}_{\bar{K}}$ respectively. In [2] and [3], Abbes and Saito defined the ramification theory of a finite flat $\mathcal{O}_K$-algebra of relative complete intersection. In this section, we gather the necessary definitions and briefly recall their theory.

Let A be a finite flat $\mathcal{O}_K$-algebra and $\mathbb{A}$ be a complete Noetherian semi-local ring (with its topology defined by $\mathrm{rad}(\mathbb{A})$) which is of formally smooth over $\mathcal{O}_K$ and whose quotient ring $\mathbb{A}/\mathrm{rad}(\mathbb{A})$ is of finite type over F. A surjection of $\mathcal{O}_K$-algebras $\mathbb{A} \to A$ is called an embedding if $\mathbb{A}/\mathrm{rad}(\mathbb{A}) \to A/\mathrm{rad}(A)$ is an isomorphism. For an embedding $(\mathbb{A} \to A)$ and $j \in \mathbb{Q}_{>0}$, the j-th tubular neighborhood of $(\mathbb{A} \to A)$ is the K-affinoid variety $X^j(\mathbb{A} \to A)$ constructed as follows. Write $j = k/l$ with k, l non-negative integers. Put $I = \mathrm{Ker}(\mathbb{A} \to A)$ and $\mathcal{A}_0^{k,l} = \mathbb{A}[I^l/\pi^k]^\wedge$, where $\wedge$ means the π-adic completion. Then $\mathcal{A}_0^{k,l}$ is a quotient ring of the Tate algebra $\mathcal{O}_K\langle T_1, \ldots, T_r\rangle$ for some r. Its generic fiber $\mathcal{A}_K^j = \mathcal{A}_0^{k,l} \otimes_{\mathcal{O}_K} K$ is independent of the choice of a representation

$j = k/l$ ([3, Lemma 1.4]) and set $X^j(\mathbb{A} \to A) = \mathrm{Sp}(\mathcal{A}^j_K)$. This affinoid variety is geometrically regular ([3, Lemma 1.6]).

We put $F(A) = \mathrm{Hom}_{\mathcal{O}_K\text{-alg.}}(A, \mathcal{O}_{\bar{K}})$ and $F^j(A) = \varprojlim \pi_0(X^j(\mathbb{A} \to A))_{\bar{K}}$. Here $\pi_0(X)_{\bar{K}}$ denotes the set of geometric connected components of a K-affinoid variety X and the projective limit is taken in the category of embeddings of A. Note that the projective family $\pi_0(X^j(\mathbb{A} \to A))_{\bar{K}}$ is constant ([3, Section 1.2]). These define contravariant functors F and F^j from the category of finite flat $\mathcal{O}_K$-algebras to the category of finite G_K-sets. Moreover, there are morphisms of functors $F \to F^j$ and $F^{j'} \to F^j$ for $j' \geq j > 0$.

By the finiteness theorem of Grauert and Remmert ([8, Theorem 1.3]), there exists a finite separable extension L of K such that the geometric closed fiber of the unit disc $\mathfrak{X}^j(\mathbb{A} \to A)_{\mathcal{O}_L}$ for the supremum norm in $X^j(\mathbb{A} \to A)_L = X^j(\mathbb{A} \to A) \times_K L$ is reduced. Then for any finite separable extension L' of L, the π_L-adic formal scheme $\mathfrak{X}^j(\mathbb{A} \to A)_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}$ coincides with the unit disc for the supremum norm in $X^j(\mathbb{A} \to A)_{L'}$ and thus is normal. The π_L-adic formal scheme $\mathfrak{X}^j(\mathbb{A} \to A)_{\mathcal{O}_L}$ is referred as the stable normalized integral model of $X^j(\mathbb{A} \to A)$ over $\mathcal{O}_L$ and its geometric closed fiber is denoted by $\bar{X}^j(\mathbb{A} \to A)$. If L/K is Galois, the Galois group $\mathrm{Gal}(L/K)$ acts on it by the functoriality of the unit disc for the supremum norm. We have the G_K-equivariant isomorphism $\pi_0(\bar{X}^j(\mathbb{A} \to A))_{\bar{F}} \to \pi_0(X^j(\mathbb{A} \to A))_{\bar{K}}$, where the former is the set of geometric connected components of $\bar{X}^j(\mathbb{A} \to A)$ ([3, Corollary 1.11]).

Suppose that A is of relative complete intersection over $\mathcal{O}_K$ and $A \otimes_{\mathcal{O}_K} K$ is etale over K. Then the natural map $F(A) \to F^j(A)$ is surjective. The family $\{F(A) \to F^j(A)\}_{j \in \mathbb{Q}_{>0}}$ is separated, exhaustive and its jumps are rational ([2, Proposition 6.4]). The conductor of A is defined to be $c(A) = \inf\{j \in \mathbb{Q}_{>0} | F(A) \to F^j(A) \text{ is an isomorphism}\}$. If B is the affine algebra of a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$ which is generically etale, then B is of relative complete intersection (for example, [5, Proposition 2.2.2]) and the theory above can all be applied to B. By the functoriality, $F^j(B)$ is endowed with a G_K-module structure ([1, Lemme 2.1]) and the natural map $\mathcal{G}(\bar{K}) = F(B) \to F^j(B)$ is a G_K-homomorphism. Let $\mathcal{G}^j$ denote the schematic closure ([14]) in $\mathcal{G}$ of the kernel of this homomorphism. It is called the j-th ramification filtration of $\mathcal{G}$. We refer $c(B)$ as the conductor of $\mathcal{G}$, which is denoted also by $c(\mathcal{G})$. We put $\mathcal{G}^{j+}(\bar{K}) = \cup_{j'>j} \mathcal{G}^{j'}(\bar{K})$ and define $\mathcal{G}^{j+}$ to be the schematic closure of $\mathcal{G}^{j+}(\bar{K})$ in $\mathcal{G}$.

3. Group structure on the tubular neighborhood of a finite flat group scheme

Let K denote a complete discrete valuation field of mixed characteristic $(0,p)$ whose residue field F may be imperfect and v_K the valuation of K extended to $\bar{K}$ which is normalized as $v_K(\pi)=1$. Let $\mathcal{G}=\mathrm{Spec}(B)$ be a connected finite flat group scheme over $\mathcal{O}_K$. We define a formal resolution of $\mathcal{G}$ to be a closed immersion $\mathcal{G}\to\Gamma$ of (profinite) formal group schemes over $\mathcal{O}_K$, where $\Gamma=\mathrm{Spf}(\mathbb{B})$ is connected and smooth. Such an immersion can be constructed as follows. By a theorem of Raynaud ([4, Théorème 3.1.1]), we can find an abelian scheme A over $\mathcal{O}_K$ and a closed immersion of group schemes $\mathcal{G}\to A$. Taking the formal completion of A along the zero section, we get a formal resolution of $\mathcal{G}$. We refer the relative dimension of Γ over $\mathcal{O}_K$ as the dimension of a formal resolution $(\mathcal{G}\to\Gamma)$. We define a morphism of formal resolutions to be a pair of group homomorphisms $(f,\mathbf{f})$ which makes the following diagram commutative.

$$\begin{array}{ccc}\mathcal{G} & \longrightarrow & \Gamma\\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathbf{f}}\\ \mathcal{G}' & \longrightarrow & \Gamma'\end{array}$$

Note that a formal resolution of $\mathcal{G}$ is also an embedding of B in the sense of Section 2. We say $(f,\mathbf{f})$ is finite flat if this is finite flat as a map of embeddings ([3]). Consider the j-th tubular neighborhood $X^j(\mathbb{B}\to B)$ of the embedding $(\mathbb{B}\to B)$, which we also write as $X^j(\mathcal{G}\to\Gamma)$. The following lemma, whose first part is implicit in [1], enables us to introduce a group structure on this affinoid variety.

Lemma 3.1. *Let $(\mathbb{A}\to A)$ and $(\mathbb{B}\to B)$ be embeddings of finite flat $\mathcal{O}_K$-algebras. Put $\mathbb{C}=\mathbb{A}\hat{\otimes}_{\mathcal{O}_K}\mathbb{B}$ and $C=A\otimes_{\mathcal{O}_K}B$. Then the surjection $\mathbb{C}\to C$ is also an embedding and we have a canonical isomorphism $X^j(\mathbb{C}\to C)\to X^j(\mathbb{A}\to A)\times_K X^j(\mathbb{B}\to B)$. Moreover, this isomorphism extends to a canonical isomorphism between their stable normalized integral models $\mathfrak{X}^j(\mathbb{C}\to C)_{\mathcal{O}_{\bar{K}}}\to\mathfrak{X}^j(\mathbb{A}\to A)_{\mathcal{O}_{\bar{K}}}\times_{\mathcal{O}_{\bar{K}}}\mathfrak{X}^j(\mathbb{B}\to B)_{\mathcal{O}_{\bar{K}}}$.*

Proof. By the functoriality of a tubular neighborhood, we have an affinoid map $\Phi: X^j(\mathbb{C}\to C)\to X^j(\mathbb{A}\to A)\times_K X^j(\mathbb{B}\to B)$. To see that Φ is an isomorphism, we may replace K with a finite separable extension and suppose that A and B are local, j is an integer and $A/\mathrm{rad}(A)=B/\mathrm{rad}(B)=F$. Then we may identify $\mathbb{A}$ with $\mathcal{O}_K[[T_1,\dots,T_r]]$ and $\mathbb{B}$ with $\mathcal{O}_K[[T'_1,\dots,T'_{r'}]]$ for some r and r'. Let $I=(f_1,\dots,f_s)$ (*resp.* $J=(g_1,\dots,g_{s'})$) be the kernel of the surjection $\mathbb{A}=$

$\mathcal{O}_K[[T_1, \dots, T_r]] \to A$ (*resp.* $\mathbb{B} = \mathcal{O}_K[[T'_1, \dots, T'_{r'}]] \to B$). Then the affinoid algebras of $X^j(\mathbb{A} \to A)$, $X^j(\mathbb{B} \to B)$ and $X^j(\mathbb{C} \to C)$ are equal to $K\langle T_1, \dots, T_r\rangle\langle f_1/\pi^j, \dots, f_s/\pi^j\rangle$, $K\langle T'_1, \dots, T'_{r'}\rangle\langle g_1/\pi^j, \dots, g_{s'}/\pi^j\rangle$ and $K\langle T_1, \dots, T_r, T'_1, \dots, T'_{r'}\rangle\langle f_1/\pi^j, \dots, f_s/\pi^j, g_1/\pi^j, \dots, g_{s'}/\pi^j\rangle$ respectively. This shows that Φ is an isomorphism.

Let L be a finite extension of K where the stable normalized integral models of $X^j(\mathbb{A} \to A)$, $X^j(\mathbb{B} \to B)$ and $X^j(\mathbb{C} \to C)$ are defined. Set $\mathcal{A}_0^{k,l}$ and $\mathcal{A}_K^j$ as in Section 2. Let $\mathring{\mathcal{A}}^j_{\mathcal{O}_L}$ denote the unit disc in $\mathcal{A}^j_L = \mathcal{A}^j_K\hat{\otimes}_K L$ for the supremum norm. Define $\mathcal{B}_0^{k,l}$, $\mathcal{C}_0^{k,l}$, $\mathcal{B}^j_K$, $\mathcal{C}^j_K$, $\mathring{\mathcal{B}}^j_{\mathcal{O}_L}$ and $\mathring{\mathcal{C}}^j_{\mathcal{O}_L}$ similarly for B and C. From the proof of [8, Theorem 1.3], there exists a continuous surjection $\alpha : L\langle T_1, \dots, T_{r'}\rangle \to \mathcal{A}^j_L$ such that $||_{\sup} = ||_\alpha$, where $||_\alpha$ is the residue norm induced by α. We also have a surjection $\beta : L\langle U_1, \dots, U_{s'}\rangle \to \mathcal{B}^j_L$ with the same property for B. Consider the surjection $\alpha\hat{\otimes}\beta : L\langle T_1, \dots, T_{r'}\rangle\hat{\otimes}_L L\langle U_1, \dots, U_{s'}\rangle \to \mathcal{A}^j_L\hat{\otimes}_L\mathcal{B}^j_L = \mathcal{C}^j_L$. The unit disc in $\mathcal{A}^j_L\hat{\otimes}_L\mathcal{B}^j_L$ for the residue norm induced by $\alpha\hat{\otimes}\beta$ is $\mathring{\mathcal{A}}^j_{\mathcal{O}_L}\hat{\otimes}_{\pi_L}\mathring{\mathcal{B}}^j_{\mathcal{O}_L}$, where $\hat{\otimes}_{\pi_L}$ denotes the π_L-adic complete tensor product over $\mathcal{O}_L$. Its geometric closed fiber $(\mathring{\mathcal{A}}^j_{\mathcal{O}_L}\otimes_{\mathcal{O}_L}\bar{F})\otimes_{\bar{F}}(\mathring{\mathcal{B}}^j_{\mathcal{O}_L}\otimes_{\mathcal{O}_L}\bar{F})$ is reduced. By [8, Proposition 1.1], we see that the stable normalized integral model $\mathring{\mathcal{C}}^j_{\mathcal{O}_L}$ is equal to $\mathring{\mathcal{A}}^j_{\mathcal{O}_L}\hat{\otimes}_{\pi_L}\mathring{\mathcal{B}}^j_{\mathcal{O}_L}$.

□

Corollary 3.2. *Let $(\mathcal{G} \to \Gamma)$ be a formal resolution of $\mathcal{G}$. Then the group structure of Γ induces a rigid K-analytic group structure on the tubular neighborhood $X^j(\mathcal{G} \to \Gamma)$. This group structure also extends to $\mathfrak{X}^j(\mathcal{G} \to \Gamma)_{\mathcal{O}_{\bar{K}}}$ (resp. $\bar{X}^j(\mathcal{G} \to \Gamma)$) and endows it with a π-adic formal group scheme structure over $\mathcal{O}_{\bar{K}}$ (resp. an algebraic group structure over $\bar{F}$).*

Moreover, for a morphism of formal resolutions $(\mathcal{G} \to \Gamma) \to (\mathcal{G}' \to \Gamma')$, the induced affinoid map $X^j(\mathcal{G} \to \Gamma) \to X^j(\mathcal{G}' \to \Gamma')$ is a homomorphism of rigid K-analytic groups. This homomorphism also extends to their stable normalized integral models as a homomorphism $\mathfrak{X}^j(\mathcal{G} \to \Gamma)_{\mathcal{O}_{\bar{K}}} \to \mathfrak{X}^j(\mathcal{G}' \to \Gamma')_{\mathcal{O}_{\bar{K}}}$ of π-adic formal group schemes and to their geometric closed fibers as a homomorphism $\bar{X}^j(\mathcal{G} \to \Gamma) \to \bar{X}^j(\mathcal{G}' \to \Gamma')$ of algebraic groups over $\bar{F}$.

Let $\mathcal{G} = \mathrm{Spec}(B)$ be a connected finite flat group scheme over $\mathcal{O}_K$ and $(\mathcal{G} \to \Gamma = \mathrm{Spf}(\mathbb{B}))$ be a formal resolution of dimension r. Set $\mathrm{Spf}(\mathbb{A}) = \Gamma/\mathcal{G}$ and regard the zero section $\mathrm{Spec}(\mathcal{O}_K) \to \mathrm{Spf}(\mathbb{A})$ as a formal resolution of the trivial group. Then we have a finite flat map of formal resolutions

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathrm{Spf}(\mathbb{B}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}_K) & \longrightarrow & \mathrm{Spf}(\mathbb{A}). \end{array}$$

By Corollary 3.2 and [3, Lemma 1.8], we get a finite flat map of rigid K-analytic groups $f^j : X^j_{\mathcal{G}} = X^j(\mathcal{G} \to \Gamma) \to D^{r,j} = X^j(\mathrm{Spec}(\mathcal{O}_K) \to \mathrm{Spf}(\mathbb{A}))$, where $D^{r,j}$ denotes the r-dimensional polydisc $\{(z_1, \ldots, z_r) \in \mathcal{O}^r_{\bar{K}} \mid v_K(z_i) \geq j$ for any $i\}$. We call this the affinoid homomorphism associated to a formal resolution $(\mathcal{G} \to \Gamma)$. Write $\mathcal{B}^j_K$ and $\mathcal{A}^j_K$ for the K-affinoid algebras of $X^j_{\mathcal{G}}$ and $D^{r,j}$ respectively. The stable normalized integral model over $\mathcal{O}_L$ of $X^j_{\mathcal{G}}$ (*resp.* $D^{r,j}$) is denoted by $\mathfrak{X}^j_{\mathcal{G},\mathcal{O}_L}$ (*resp.* $\mathfrak{D}^{r,j}_{\mathcal{O}_L}$) and its geometric closed fiber by $\bar{X}^j_{\mathcal{G}}$ (*resp.* $\bar{D}^{r,j}$). Note that the algebraic group $\bar{X}^j_{\mathcal{G}}$ is reduced, hence smooth by [16, Theorem 11.6].

Lemma 3.3. *The affinoid homomorphism $f^j : X^j_{\mathcal{G}} \to D^{r,j}$ is etale for any $j > 0$. Moreover, for $j > c(\mathcal{G})$, there exists a finite extension K'/K such that $X^j_{\mathcal{G},K'}$ is isomorphic to the disjoint sum of finitely many copies of $D^{r,j}_{K'}$.*

Proof. We have $\Omega^1_{\mathcal{B}^j_K/\mathcal{A}^j_K} = \mathcal{B}^j_K \hat{\otimes}_{\mathbb{B}} \hat{\Omega}_{\mathbb{B}/\mathbb{A}}$. It is enough to show that $\hat{\Omega}_{\mathbb{B}/\mathbb{A}}$ is a torsion $\mathcal{O}_K$-module. Let $J_{\mathbb{A}}$ and $J_{\mathbb{B}}$ be the augmentation ideals of $\mathbb{A}$ and $\mathbb{B}$ respectively. Set $I = \mathrm{Ker}(\mathbb{B} \to B)$. Then $\hat{\Omega}_{\mathbb{B}/\mathbb{A}} = \mathrm{Coker}(\mathbb{B} \otimes_{\mathbb{A}} \hat{\Omega}_{\mathbb{A}/\mathcal{O}_K} \to \hat{\Omega}_{\mathbb{B}/\mathcal{O}_K})$ is equal to $\mathbb{B} \otimes_{\mathcal{O}_K} \mathrm{Coker}(\mathrm{Cot}(\mathbb{A}) \to \mathrm{Cot}(\mathbb{B})) = \mathbb{B} \otimes_{\mathcal{O}_K} J_{\mathbb{B}}/(I + J^2_{\mathbb{B}}) = \mathbb{B} \otimes_{\mathcal{O}_K} \mathrm{Cot}(B)$. This shows the first assertion. For the second assertion, take a finite extension K' of K where the geometric connected components of $X^j_{\mathcal{G}}$ are defined. By assumption, each of the connected components of $X^j_{\mathcal{G},K'}$ is a finite etale cover of $D^{r,j}_{K'}$ whose degree is one. Thus this is isomorphic to $D^{r,j}_{K'}$. □

Take a finite extension L of K where the stable normalized integral models of $X^j_{\mathcal{G}}$ and $D^{r,j}$ are defined. The generic fiber $\mathcal{G}_L = \mathcal{G} \times_{\mathcal{O}_K} L$ can be regarded as a rigid L-analytic subgroup of $X^j_{\mathcal{G},L}$ defined by an ideal $\mathcal{J}_L$ of $\mathcal{B}^j_L$. Put $\mathcal{J} = \mathcal{J}_L \cap \mathring{\mathcal{B}}^j_{\mathcal{O}_L}$ and $\mathring{B}^j = \mathring{\mathcal{B}}^j_{\mathcal{O}_L}/\mathcal{J}$. The latter is a subring of $B_L = B \otimes_K L$. Since we have a commutative diagram with surjective horizontal arrows and injective vertical arrows

$$\begin{array}{ccc} \mathcal{B}_0^{k,l} \otimes_{\mathcal{O}_K} \mathcal{O}_L & \longrightarrow & B \otimes_{\mathcal{O}_K} \mathcal{O}_L \\ \downarrow & & \downarrow \\ \mathring{\mathcal{B}}^j_{\mathcal{O}_L} & \longrightarrow & \mathring{B}^j_{\mathcal{O}_L}, \end{array}$$

we see that $\mathring{B}^j_{\mathcal{O}_L}$ is integral over $B \otimes_{\mathcal{O}_K} \mathcal{O}_L$. Thus the $\mathcal{O}_K$-algebra $\mathring{B}^j_{\mathcal{O}_L}$ is finite flat. Set $\mathcal{H}^j_{\mathcal{O}_L} = \mathrm{Spec}(\mathring{B}^j_{\mathcal{O}_L})$. This can be regarded as a closed π_L-adic formal subscheme of $\mathfrak{X}^j_{\mathcal{G},\mathcal{O}_L}$.

Lemma 3.4. *The group structure of $\mathcal{G}_L$ extends to $\mathcal{H}^j_{\mathcal{O}_L}$. The group scheme $\mathcal{H}^j_{\mathcal{O}_L}$ is a closed π_L-adic formal subgroup scheme of $\mathfrak{X}^j_{\mathcal{G},\mathcal{O}_L}$.*

Proof. Put $\mathcal{K} = \mathrm{Ker}(\mathring{\mathcal{B}}^j_{\mathcal{O}_L} \hat{\otimes}_{\pi_L} \mathring{\mathcal{B}}^j_{\mathcal{O}_L} \to \mathring{B}^j_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} \mathring{B}^j_{\mathcal{O}_L})$ and $\mathcal{K}_L = \mathcal{K} \otimes_{\mathcal{O}_L} L$. Let μ be the coproduct of $\mathring{\mathcal{B}}^j_{\mathcal{O}_L}$. We must show $\mu(\mathcal{J}) \subseteq \mathcal{K}$. This follows from the commutative diagram below whose rows are exact and vertical allows are injective.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathring{\mathcal{B}}^j_{\mathcal{O}_L} \hat{\otimes}_{\pi_L} \mathring{\mathcal{B}}^j_{\mathcal{O}_L} & \longrightarrow & \mathring{B}^j_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} \mathring{B}^j_{\mathcal{O}_L} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K}_L & \longrightarrow & \mathcal{B}^j_L \hat{\otimes}_L \mathcal{B}^j_L & \longrightarrow & B_L \otimes_L B_L & \longrightarrow & 0 \end{array}$$

Passing to the generic fiber, we see that the second assertion holds. □

Lemma 3.5. *The associated homomorphism $\mathring{f}^j : \mathfrak{X}^j_{\mathcal{G},\mathcal{O}_L} \to \mathfrak{D}^{r,j}_{\mathcal{O}_L}$ is finite flat. Moreover, there exists an exact sequence of π_L-adic formal group schemes*

$$0 \to \mathcal{H}^j_{\mathcal{O}_L} \to \mathfrak{X}^j_{\mathcal{G},\mathcal{O}_L} \to \mathfrak{D}^{r,j}_{\mathcal{O}_L} \to 0. \tag{1}$$

Proof. From Lemma 3.3, the associated affinoid map $f^j : X^j_{\mathcal{G}} \to D^{r,j}$ is finite etale. Let $\mathcal{B}^j_K$ and $\mathcal{A}^j_K$ be their affinoid algebras as above. Since $D^{r,j}$ is integral, we see that f^j is surjective and the ring homomorphism $\mathcal{A}^j_K \to \mathcal{B}^j_K$ is injective. Thus we have an injection $\mathring{\mathcal{A}}^j_{\mathcal{O}_L} \to \mathring{\mathcal{B}}^j_{\mathcal{O}_L}$, which is finite by [6, Corollary 6.4.1/6]. Hence $\bar{f}^j : \bar{X}^j_{\mathcal{G}} \to \bar{D}^{r,j}$ is a surjective homomorphism of algebraic groups over $\bar{F}$. Since $\bar{X}^j_{\mathcal{G}}$ and $\bar{D}^{r,j}$ are regular, we see that $\bar{f}^j$ is faithfully flat by [13, Theorem 23.1]. Since $\mathring{\mathcal{A}}^j_{\mathcal{O}_L}$ and $\mathring{\mathcal{B}}^j_{\mathcal{O}_L}$ is π_L-torsion free, the map $\mathring{f}^j$ is flat by the local criterion of flatness. Put $\mathcal{H}' = \mathrm{Ker}(\mathring{f}^j)$. This is a closed π_L-adic formal subgroup scheme of $\mathfrak{X}^j_{\mathcal{G},\mathcal{O}_L}$ and can be regarded also as a finite flat group scheme over $\mathcal{O}_L$. Passing to the generic fiber, we see that $\mathcal{H}^j_{\mathcal{O}_L}$ is a closed subgroup scheme of $\mathcal{H}'$. Comparing these ranks concludes the lemma.

□

From Lemma 3.3, we see that for $j > c = c(\mathcal{G})$, the map $\mathring{f}^j$ identifies $\mathfrak{X}^j_{\mathcal{G},\mathcal{O}_L}$ with the direct sum of finitely many copies of $\mathfrak{D}^{r,j}_{\mathcal{O}_L}$. More precisely, we have the following.

Lemma 3.6. *Let $c = c(\mathcal{G})$ be the conductor of $\mathcal{G}$. Then the associated homomorphism $\mathring{f}^j : \mathfrak{X}^j_{\mathcal{G},\mathcal{O}_L} \to \mathfrak{D}^{r,j}_{\mathcal{O}_L}$ is finite etale if and only if $j \geq c$.*

Proof. Let $\mathrm{sp}_j : X^j_{\mathcal{G}} \to \bar{X}^j_{\mathcal{G}}$ be the specialization map. By Lemma 3.3, we see that $\bar{f}^c$ is finite etale at $\mathrm{sp}_c(x)$ for any $x \in \mathcal{G}(\bar{K})$ as in the proof of [2, Theorem 7.2], using a theorem of Bosch ([7, Lemma 2.1]). We conclude the etaleness of $\bar{f}^c$ by the existence of the group structure. We have $\Omega^1_{\mathring{\mathcal{B}}^c_{\mathcal{O}_L}/\mathring{\mathcal{A}}^c_{\mathcal{O}_L}} \otimes_{\mathcal{O}_L} \bar{F} = 0$. Since $\mathring{\mathcal{B}}^c_{\mathcal{O}_L}$ is π_L-adic complete and Noetherian, we see that $\Omega^1_{\mathring{\mathcal{B}}^c_{\mathcal{O}_L}/\mathring{\mathcal{A}}^c_{\mathcal{O}_L}} = 0$ and the homomorphism $\mathring{f}^c : \mathfrak{X}^c_{\mathcal{G},\mathcal{O}_L} \to \mathfrak{D}^{r,c}_{\mathcal{O}_L}$ is finite etale.

Let $\bar{0}$ be the zero section of $\bar{D}^{r,j}$ and set $X^{j+}_{\mathcal{G}} = \cup_{j'>j} X^{j'}_{\mathcal{G}}$. Then we have $(\bar{f}^j)^{-1}(\bar{0}) = \mathrm{sp}_j(X^{j+}_{\mathcal{G}}) = \mathrm{sp}_j(\mathcal{G}(\bar{K}))$. If $\bar{f}^j$ is etale, then $\sharp(\bar{f}^j)^{-1}(\bar{0})$ equals the degree of $\bar{f}^j$, namely $\sharp\mathcal{G}(\bar{K})$. Thus $X^{j+}_{\mathcal{G}}$ splits and we have $j \geq c$.

□

4. Ramification and the I_K-module structure of a finite flat group scheme

Consider the right action of I_K on $\bar{K}$ defined by $\sigma.z = \sigma^{-1}(z)$ for $\sigma \in I_K$. This action induces a $\bar{K}$-semilinear left action of I_K on $X^j_{\mathcal{G},\bar{K}} = X^j_{\mathcal{G}} \times_K \bar{K}$, which also extends to an $\mathcal{O}_{\bar{K}}$-semilinear action on its stable normalized integral model $\mathfrak{X}^j_{\mathcal{G},\mathcal{O}_{\bar{K}}}$. Thus we have an $\bar{F}$-linear left action of I_K on its closed fiber $\bar{X}^j_{\mathcal{G}}$. We call this the geometric monodromy action of I_K and write the action of $\sigma \in I_K$ as σ_{geom} (note that here we follow the terminology in [2], and not in [3] where the monodromy action is called arithmetic, since in our case there is no "geometric" action other than the monodromy action). Similarly, we have the geometric monodromy action of I_K on $\bar{D}^{r,j}$.

The latter action is described as follows. Let the additive group (*resp.* multiplicative group) over $\bar{F}$ be denoted by $\bar{\mathbb{G}}_{\mathrm{a}}$ (*resp.* $\bar{\mathbb{G}}_{\mathrm{m}}$). Consider the left action $\bar{\mathbb{G}}_{\mathrm{m}} \times \bar{\mathbb{G}}^r_{\mathrm{a}} \to \bar{\mathbb{G}}^r_{\mathrm{a}}$ given by the multiplication. Write this action of $\lambda \in \bar{F}^\times$ as $[\lambda]$. This action is defined by $T_i \mapsto \lambda T_i$, where $\bar{\mathbb{G}}^r_{\mathrm{a}} = \mathrm{Spec}(\bar{F}[T_1, \ldots, T_r])$. For $j \in \mathbb{Q}_{>0}$, we define the fundamental character $\theta_j : I_K \to \bar{F}^\times$ to be $\theta^{k'}_{l'}$, where k'/l' is the prime-to-p-denominator part of j mod $\mathbb{Z}$ ([15]). In other words, we set $\theta_j(\sigma) = (\sigma(\pi^{1/l'})/\pi^{1/l'})^{k'}$ mod $\mathfrak{m}_{\bar{K}}$. Note that, for $j = k/l$ and $l = p^m l_0$ with $(k,l) = 1$ and $p \nmid l_0$, we have $\theta_j = \theta_{l_0}^{kp^{-m}}$.

Lemma 4.1. *The algebraic group $\bar{D}^{r,j}$ is equal to $\bar{\mathbb{G}}_{\mathrm{a}}^r$. For $\sigma \in I_K$, the geometric monodromy action σ_{geom} on $\bar{D}^{r,j}$ coincides with the multiplication $[\theta_j(\sigma)]$.*

Proof. Put $\mathbb{A} = \mathcal{O}_K[[T_1, \dots, T_r]]$ and $j = k/l$ with $(k,l) = 1$. Let L be a finite Galois extension of K containing $\pi^{1/l}$ and $e' = e(L/K)$ be its ramification index over K. Then $e'k/l \in \mathbb{Z}$ and the stable normalized integral model of $D^{r,j}$ over $\mathcal{O}_L$ is $\mathcal{O}_L\langle T_1/(\pi_L)^{e'k/l}, \dots, T_r/(\pi_L)^{e'k/l}\rangle = \mathcal{O}_L\langle W_1, \dots, W_r\rangle$, where $W_i = T_i/(\pi^{1/l})^k$. Set $\mu_{\mathbb{A}}$ to be the coproduct of $\mathbb{A}$. We have

$$\mu_{\mathbb{A}}(T_i) = T_i\hat{\otimes}1 + 1\hat{\otimes}T_i + (\text{higher degree})$$

and then $\mu_{\mathbb{A}}(W_i) = \mu_{\mathbb{A}}((\pi^{1/l})^k W_i)/(\pi^{1/l})^k$ is equal to

$$W_i\hat{\otimes}_\pi 1 + 1\hat{\otimes}_\pi W_i + (\pi^{1/l})^k(\text{higher degree})$$

in this $\mathcal{O}_L$-algebra. This shows the first assertion.

On the other hand, we know the geometric monodromy action on $\bar{D}^{r,j}$ is tame ([2, Lemma 7.7]). Write $l = p^m l_0$ with $p \nmid l_0$. Then for $\sigma \in I_K$, we have $\sigma((\pi^{1/l})^k)/(\pi^{1/l})^k \equiv \theta_{l_0}(\sigma)^{kp^{-m}}\zeta_{p^m}^N$ mod $\mathfrak{m}_{\bar{K}}$ with some N and this is equal to $\theta_j(\sigma)$. Thus the action on the affine algebra $\bar{F}[W_1, \dots, W_r]$ of $\bar{D}^{r,j}$ is given by $\sigma_{\mathrm{geom}}^*(W_i) = \theta_j(\sigma)W_i$. This coincides with $[\theta_j(\sigma)]$. □

Next we consider the geometric monodromy action on $\bar{X}_{\mathcal{G}}^j$. Let $\bar{X}_{\mathcal{G}}^{j,0}$ denote the unit component of the algebraic group $\bar{X}_{\mathcal{G}}^j$ and $\bar{\mathcal{H}}^j$ be the geometric closed fiber of $\mathcal{H}_{\mathcal{O}_L}^j$. We begin with the following lemma.

Lemma 4.2. *If $\psi \in \mathrm{End}(\bar{X}_{\mathcal{G}}^{j,0})$ induces the zero map on $\bar{D}^{r,j}$, then $\psi = 0$.*

Proof. Put $\bar{\mathcal{H}}_0^j = \bar{\mathcal{H}}^j \cap \bar{X}_{\mathcal{G}}^{j,0}$. This is the kernel of the faithfully flat map $\bar{X}_{\mathcal{G}}^{j,0} \to \bar{D}^{r,j}$ and by assumption we have the following commutative diagram whose rows are exact.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bar{\mathcal{H}}_0^j & \longrightarrow & \bar{X}_{\mathcal{G}}^{j,0} & \longrightarrow & \bar{D}^{r,j} & \longrightarrow & 0 \\
 & & \downarrow & & \psi\downarrow & & 0\downarrow & & \\
0 & \longrightarrow & \bar{\mathcal{H}}_0^j & \longrightarrow & \bar{X}_{\mathcal{G}}^{j,0} & \longrightarrow & \bar{D}^{r,j} & \longrightarrow & 0
\end{array}$$

Thus ψ factors through $\bar{\mathcal{H}}_0^j$. Put $\bar{C} = \mathrm{Im}(\psi)$. Then this is a closed subgroup scheme of $\bar{\mathcal{H}}_0^j$ and the map $\bar{X}_{\mathcal{G}}^{j,0} \to \bar{C}$ is faithfully flat. Since $\bar{X}_{\mathcal{G}}^{j,0}$

is regular and connected, we see that $\bar{C}$ is also regular and connected by [13, Theorem 23.7]. Hence $\bar{C} = 0$ and we have $\psi = 0$. □

Corollary 4.3. *Let $\mathcal{G}$ be a connected finite flat group scheme over $\mathcal{O}_K$. Take a formal resolution $(\mathcal{G} \to \Gamma)$ of dimension r. Then the algebraic group $\bar{X}_{\mathcal{G}}^{j,0}$ is isomorphic to $\bar{\mathbb{G}}_{\mathrm{a}}^r$.*

Proof. By the previous lemma and Lemma 4.1, we see that $\bar{X}_{\mathcal{G}}^{j,0}$ is killed by p. Hence the assertion follows from [12, Lemma 1.7.1]. □

Corollary 4.4. *The geometric monodromy action of I_K on $\bar{X}_{\mathcal{G}}^{j,0}$ is tame.*

Proof. For an element σ of the wild inertia subgroup P_K, the geometric monodromy action σ_{geom} on $\bar{D}^{r,j}$ is trivial. Applying the lemma to $\sigma_{\mathrm{geom}} - \mathrm{id} \in \mathrm{End}(\bar{X}_{\mathcal{G}}^{j,0})$ shows the assertion. □

Corollary 4.5. *Let J be a finite cyclic quotient of I_K through which the tame character θ_j factors and τ be a generator of J. Let $F(t)$ denote the minimal polynomial of $\theta_j(\tau) \in \bar{\mathbb{F}}_p$ over $\mathbb{F}_p$. Then the geometric monodromy action of I_K on $\bar{X}_{\mathcal{G}}^{j,0}$ also factors through J and the equation $F(\tau_{\mathrm{geom}}) = 0$ holds in $\mathrm{End}(\bar{X}_{\mathcal{G}}^{j,0})$.*

Proof. The first assertion follows from Lemma 4.2 as the previous corollary. The second assertion also follows from this lemma using Lemma 4.1. □

Let $c = c(\mathcal{G})$ be the conductor of $\mathcal{G}$. The lemma below enables us to realize $\mathcal{G}^c(\bar{K})$ as a subgroup of $\bar{X}_{\mathcal{G}}^{c,0}$.

Lemma 4.6. *The specialization map $\mathrm{sp}_c : X_{\mathcal{G},\bar{K}}^c \to \bar{X}_{\mathcal{G}}^c$ induces an I_K-equivariant isomorphism $\mathcal{G}(\bar{K}) \to \bar{\mathcal{H}}^c(\bar{F})$ and $\mathcal{G}^c(\bar{K}) \to \bar{\mathcal{H}}_0^c(\bar{F})$. Here we consider on the left-hand side the natural action as the $\bar{K}$-valued points of $\mathcal{G}$ (resp. $\mathcal{G}^c$) and on the right-hand side the restriction of the geometric monodromy action on $\bar{X}_{\mathcal{G}}^c$.*

Proof. By definition, the generic fiber of $\mathcal{H}_{\mathcal{O}_L}^c$ is equal to $\mathcal{G}_L$. From the exact sequence (1) and Lemma 3.6, we know that $\mathcal{H}_{\mathcal{O}_L}^c$ is etale over $\mathcal{O}_L$ and there is the following exact sequence of algebraic groups over $\bar{F}$.

$$0 \to \bar{\mathcal{H}}^c \to \bar{X}_{\mathcal{G}}^c \to \bar{D}^{r,j} \to 0 \tag{2}$$

Thus we have a natural isomorphism $\mathcal{H}_{\mathcal{O}_L}^c(\bar{K}) \to \bar{\mathcal{H}}^c(\bar{F})$ and the composite $\mathcal{G}(\bar{K}) = \mathcal{H}_{\mathcal{O}_L}^c(\bar{K}) \to \bar{\mathcal{H}}^c(\bar{F}) \to \bar{X}_{\mathcal{G}}^c(\bar{F})$ coincides with the map

sp_c. From [2, Corollary 4.4], we see that this map sends $\mathcal{G}^c(\bar{K})$ isomorphically onto $\bar{\mathcal{H}}_0^c(\bar{F})$.

For $x \in X_{\mathcal{G}}^c(\bar{K})$ and $\sigma \in I_K$, let $\sigma(x)$ denote the natural action of σ on $\bar{K}$-valued points. Then we have $\sigma_{\mathrm{geom}}(x) \circ \sigma = \sigma(x)$. Taking its specialization shows the I_K-equivariance.

□

The following theorem can be regarded as a generalization for a finite flat group scheme over $\mathcal{O}_K$ of the structure theorem of the graded piece of the classical upper numbering ramification filtration.

Theorem 4.7. *Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$ and $j \in \mathbb{Q}_{>0}$. Then the G_K-module $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K})$ is tame and killed by p.*

Proof. Since $\mathcal{G}^j = (\mathcal{G}^0)^j$, where $\mathcal{G}^0$ denotes the unit component of $\mathcal{G}$, we may assume $\mathcal{G}$ is connected. Suppose that j is a jump of the ramification filtration on $\mathcal{G}$ and consider the quotient $\mathcal{G}/\mathcal{G}^{j+}$. The subgroup $\mathcal{G}^j(\bar{K}) \subseteq \mathcal{G}(\bar{K})$ has a non-trivial image in $(\mathcal{G}/\mathcal{G}^{j+})(\bar{K})$. By the Herbrand theorem ([1, Lemme 2.10]), the natural map $\mathcal{G}^t(\bar{K}) \to (\mathcal{G}/\mathcal{G}^{j+})^t(\bar{K})$ is surjective for any $t > 0$. We have $(\mathcal{G}/\mathcal{G}^{j+})^t = 0$ for $t > j$ and $(\mathcal{G}/\mathcal{G}^{j+})^j \neq 0$. Thus the ramification filtration on $\mathcal{G}/\mathcal{G}^{j+}$ jumps at j and $(\mathcal{G}/\mathcal{G}^{j+})^j(\bar{K}) = \mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K})$. Replacing $\mathcal{G}$ with $\mathcal{G}/\mathcal{G}^{j+}$, we may assume $j = c = c(\mathcal{G})$.

Take a formal resolution $(\mathcal{G} \to \Gamma)$ of dimension r and consider its associated affinoid homomorphism $X_{\mathcal{G}}^c \to D^{r,c}$. Then the theorem follows from Corollary 4.3, Corollary 4.4, and Lemma 4.6.

□

From this theorem, we see that the inertia subgroup I_K acts on $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ by the direct sum of tame characters. The theorem below determines these characters up to p-power exponent.

Theorem 4.8. *Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$ and $j \in \mathbb{Q}_{>0}$. Then I_K acts on $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ by the direct sum of $\mathbb{F}_p$-conjugates of the fundamental character θ_j.*

Proof. We may assume that $\mathcal{G}$ is connected. The same argument as in the proof of Theorem 4.7 using the Herbrand theorem reduces the claim to the case where $j = c = c(\mathcal{G})$. Take a formal resolution $(\mathcal{G} \to \Gamma)$ of dimension r. Let J and τ be as in Corollary 4.5. Then Corollary 4.5 and Lemma 4.6 show that every eigenvalue of the action of τ_{geom} on the finite dimensional $\mathbb{F}_p$-vector space $\mathcal{G}^c(\bar{K})$ is a conjugate of $\theta_j(\tau)$ over $\mathbb{F}_p$. Since the order of J is prime to p, we conclude that I_K acts on $\mathcal{G}^c(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ by the direct sum of $\mathbb{F}_p$-conjugates of θ_c.

□

Corollary 4.9. *Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$. Then the order of the image of the homomorphism $I_K \to \mathrm{Aut}(\mathcal{G}(\bar{K}))$ is a power of p if and only if every jump j of the ramification filtration $\{\mathcal{G}^j\}_{j\in\mathbb{Q}_{>0}}$ is an element of $\mathbb{Z}[1/p]$.*

Proof. From Theorem 4.8, we see that the jumps are in $\mathbb{Z}[1/p]$ if and only if the graded pieces $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K})$ are unramified. By Theorem 4.7, this is equivalent to the condition that $\sharp\mathrm{Im}(I_K \to \mathrm{Aut}(\mathcal{G}(\bar{K})))$ is a p-power.

□

When $\mathcal{G}(\bar{K})$ is unramified and killed by p, we have the following reinforcement of this result, which is an easy corollary of the Herbrand theorem. The author does not know if every jump is an integer whenever $\mathcal{G}(\bar{K})$ is unramified. If $\mathcal{G}$ is monogenic, then we see that this holds true from [11, Theorem 4].

Proposition 4.10. *Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$ which is killed by p. Suppose that the G_K-module $\mathcal{G}(\bar{K})$ is unramified. Then every jump j of the ramification filtration $\{\mathcal{G}^j\}_{j\in\mathbb{Q}_{>0}}$ is an element of $p\mathbb{Z}$.*

Proof. We may assume $K = K^{\mathrm{nr}}$ and G_K acts trivially on $\mathcal{G}(\bar{K})$. There is a quotient W of $\mathcal{G}(\bar{K})/\mathcal{G}^{j+}(\bar{K})$ where $\mathcal{G}^j(\bar{K})$ has a non-trivial image and of rank one over $\mathbb{F}_p$. Taking the schematic closure, W extends to a finite flat group scheme $\mathcal{W}$ over $\mathcal{O}_K$ which is a quotient of $\mathcal{G}/\mathcal{G}^{j+}$. By the Herbrand theorem, we see that the ramification filtration of $\mathcal{W}$ jumps at j. On the other hand, $\mathcal{W}$ is a Raynaud $\mathbb{F}_p$-vector space scheme with unramified generic fiber. Thus the proposition follows from [11, Theorem 4].

□

For the rest of this section, we state some corollaries in the case where $\mathcal{G}$ is an $\mathbb{F}$-vector space scheme of rank one or two for a finite extension $\mathbb{F}$ over $\mathbb{F}_p$. In the case of rank one, Theorem 4.8 directly shows the claim below, while this can be shown also as a corollary of a determination of the conductor of a Raynaud $\mathbb{F}$-vector space scheme (Theorem 5.5).

Corollary 4.11. *Let $\mathcal{G}$ be an $\mathbb{F}$-vector space scheme of rank one over $\mathcal{O}_K$ and $c = c(\mathcal{G})$. Then the I_K-action on the $\mathbb{F}$-vector space $\mathcal{G}(\bar{K})$ of rank one is given by the character $\theta_c^{p^n}$ for some n.*

In the case of rank two, we have the following.

Corollary 4.12. *Let $\mathcal{G}$ be a finite flat $\mathbb{F}$-vector space scheme of rank two over $\mathcal{O}_K$ and $c = c(\mathcal{G})$. Then the I_K-module $\mathcal{G}(\bar{K}) \otimes_{\mathbb{F}} \bar{\mathbb{F}}_p$ contains* *the character $\theta_c^{p^n}$ for some n. If the G_K-module $\mathcal{G}(\bar{K})$ is reducible, this holds true for $\mathcal{G}(\bar{K})$ itself.*

Proof. The first assertion follows easily from Theorem 4.8 and the surjection $\mathcal{G}^c(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p \to \mathcal{G}^c(\bar{K}) \otimes_{\mathbb{F}} \bar{\mathbb{F}}_p$. Suppose the I_K-module $\mathcal{G}(\bar{K})$ is reducible. When $\mathcal{G}^c$ is of rank one, the assertion is clear from Theorem 4.8. If $\mathcal{G}^c = \mathcal{G}$, then $\mathcal{G}^c$ is reducible and the assertion follows also from Theorem 4.8. □

The corollary below indicates that the conductor $c(\mathcal{G})$ carries information about not only the tame characters but also their extension structures in the I_K-module $\mathcal{G}(\bar{K})$.

Corollary 4.13. *Consider an exact sequence of finite flat $\mathbb{F}$-vector space schemes over $\mathcal{O}_K$*

$$0 \to \mathcal{G}_1 \to \mathcal{G} \to \mathcal{G}_2 \to 0$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are connected of rank one. If $c(\mathcal{G}) = c(\mathcal{G}_2)$, then the I_K-module $\mathcal{G}(\bar{K})$ splits.

Proof. Put $c = c(\mathcal{G})$. Take a formal resolution $(\mathcal{G} \to \Gamma)$ of dimension r and put $\Gamma_2 = \Gamma/\mathcal{G}_1$. Then we get a finite flat map of formal resolutions

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \mathcal{G}_2 & \longrightarrow & \Gamma_2. \end{array}$$

Therefore we have a finite flat homomorphism of rigid K-analytic groups $X^j(\mathcal{G} \to \Gamma) \to X^j(\mathcal{G}_2 \to \Gamma_2)$ by Corollary 3.2. As in the proof of Lemma 3.3, we see that this map is finite etale.

Suppose $\mathcal{G}^c(\bar{K})$ is of rank one. If $\mathcal{G}^c(\bar{K}) \neq \mathcal{G}_1(\bar{K})$ as an $\mathbb{F}$-subspace of $\mathcal{G}(\bar{K})$, the I_K-module $\mathcal{G}(\bar{K})$ splits and the proposition follows. Suppose $\mathcal{G}^c(\bar{K}) = \mathcal{G}_1(\bar{K})$. The affinoid variety $X^c(\mathcal{G} \to \Gamma)$ decomposes to $\sharp\mathbb{F}$ components over some finite extension K' of K. Each component is a Zariski open and closed subset of $X^c(\mathcal{G} \to \Gamma)_{K'}$. As the map $f : X^c(\mathcal{G} \to \Gamma)_{K'} \to X^c(\mathcal{G}_2 \to \Gamma_2)_{K'}$ is finite etale and $X^c(\mathcal{G}_2 \to \Gamma_2)_{K'}$ is connected, every component $X^{c,i}(\mathcal{G} \to \Gamma)_{K'}$ maps surjectively to $X^c(\mathcal{G}_2 \to \Gamma_2)_{K'}$. Take some $g_i \in \mathcal{G}(\bar{K}) \cap X^{c,i}(\mathcal{G} \to \Gamma)_{K'}$. Using the group structure, we see that $\mathcal{G}(\bar{K}) \cap X^{c,i}(\mathcal{G} \to \Gamma)_{K'} = g_i + \mathcal{G}^c(\bar{K}) = g_i + \mathcal{G}_1(\bar{K})$ and $f(\mathcal{G}(\bar{K}) \cap X^{c,i}(\mathcal{G} \to \Gamma)_{K'}) = f(g_i)$. However, we have $f^{-1}(\mathcal{G}_2(\bar{K})) = \mathcal{G}(\bar{K})$ and thus $f(\mathcal{G}(\bar{K}) \cap X^{c,i}(\mathcal{G} \to \Gamma)_{K'}) = \mathcal{G}_2(\bar{K})$. This is a contradiction. Therefore we may assume $\mathcal{G}^c(\bar{K}) = \mathcal{G}(\bar{K})$. In this case, the proposition follows from Theorem 4.7. □

5. Example: rank one calculation

In this section, we calculate the conductor of a Raynaud $\mathbb{F}$-vector space scheme over $\mathcal{O}_K$. The point is that, as we can see from the bound of the conductor ([11, Theorem 7]), it is enough to consider the j-th tubular neighborhood only for $j \leq pe/(p-1)+\varepsilon$ with sufficiently small $\varepsilon > 0$. For such j, we can compute the tubular neighborhood easily by Lemma 5.4 below.

Let K be a complete discrete valuation field of mixed characteristic $(0,p)$. We write $\pi = \pi_K$ for its uniformizer and e for its absolute ramification index. We normalize a valuation v_K of K as $v_K(\pi) = 1$ and extend it to the algebraic closure $\bar{K}$ of K. For $a \in \bar{K}$ and $j \in \mathbb{R}$, let $D(a,j)$ denote the closed disc $\{z \in \mathcal{O}_{\bar{K}} \mid v_K(z-a) \geq j\}$. This is the underlying subset of a $K(a)$-affinoid subdomain of the unit disc over $K(a)$.

For integers $0 \leq s_1, \ldots, s_r \leq e$, let $\mathcal{G}(s_1, \ldots, s_r)$ denote the Raynaud $\mathbb{F}_{p^r}$-vector space scheme over $\mathcal{O}_K$ defined by the r equations $T_1^p = \pi^{s_1} T_2, T_2^p = \pi^{s_1} T_3, \ldots, T_r^p = \pi^{s_r} T_1$ ([14]). We set $j_k = (ps_k + p^2 s_{k-1} + \ldots + p^k s_1 + p^{k+1} s_r + p^{k+2} s_{r-1} + \cdots + p^r s_{k+1})/(p^r - 1)$. Before the calculation of $c(\mathcal{G}(s_1, \ldots, s_r))$, we gather some elementary lemmas.

Lemma 5.1. *Let $a \in \mathcal{O}_K$ and $s = v_K(a)$. Then the affinoid variety $X^j(\bar{K}) = \{x \in \mathcal{O}_{\bar{K}} \mid v_K(x^p - a) \geq j\}$ is equal to*

$$\begin{cases} D(a^{1/p}, j/p) & \text{if } j \leq s + pe/(p-1), \\ \coprod_{i=0}^{p-1} D(a^{1/p}\zeta_p^i, j - e - (p-1)s/p) & \text{if } j > s + pe/(p-1). \end{cases}$$

Proof. We have $v_K(x^p - a) = \sum_i v_K(x - a^{1/p}\zeta_p^i)$. If $v_K(x - a^{1/p}\zeta_p^i) \geq v_K(x - a^{1/p}\zeta_p^{i'})$ for any $i' \neq i$, then $v_K(x - a^{1/p}\zeta_p^{i'}) \leq v_K(a^{1/p}\zeta_p^{i'}(1 - \zeta_p^{i-i'})) = s/p + e/(p-1)$. Thus we have $v_K(x - a^{1/p}\zeta_p^i) \geq \sup(j/p, j - (p-1)s/p - e)$ and

$$X^j(\bar{K}) \subseteq \bigcup_i D(a^{1/p}\zeta_p^i, \sup(j/p, j-(p-1)s/p-e)).$$

Suppose that $j/p \geq j-(p-1)s/p-e$. Then we have $v_K(a^{1/p}(1-\zeta_p^i)) = s/p + e/(p-1) \geq j/p$, $D(a^{1/p}, j/p) = D(a^{1/p}\zeta_p^i, j/p)$ for any i and thus

$$X^j(\bar{K}) = D(a^{1/p}, j/p).$$

When $j/p < j-(p-1)s/p-e$, we have $v_K(a^{1/p}(1-\zeta_p^i)) = s/p+e/(p-1) < j-(p-1)s/p-e$. This means that if $w \in D(a^{1/p}\zeta_p^i, j-(p-1)s/p-e)$ for some i, then $v_K(w - a^{1/p}\zeta_p^{i'}) < j-(p-1)s/p-e$ for any other i'.

Thus the discs $D(a^{1/p}\zeta_p^i, j-(p-1)s/p-e)$ are disjoint and

$$X^j(\bar{K}) = \coprod_i D(a^{1/p}\zeta_p^i, j-(p-1)s/p-e).$$

These are equalities of the underlying sets of affinoid subdomains of the unit disc over $K(a^{1/p}, \zeta_p)$. By the universality of an affinoid subdomain, this extends to an isomorphism of affinoid varieties.

□

We can prove the following lemma just in the same way.

Lemma 5.2. *The affinoid variety* $\{x \in \mathcal{O}_{\bar{K}} \mid v_K(x^{p^r} - ax) \geq j\}$ *is equal to*

$$\begin{cases} D(0, j/p^r) & \text{if } j \leq p^r v(a)/(p^r-1), \\ \coprod_{i=0}^{p^r-1} D(\sigma_i, j-v(a)) & \text{if } j > p^r v(a)/(p^r-1), \end{cases}$$

where σ_i*'s are the roots of* $X^{p^r} = aX$.

Lemma 5.3. *For* $g_1(Y_1, \ldots, Y_d), g_2(Y_1, \ldots, Y_d) \in K[Y_1, \ldots, Y_d]$ *and* $j_1 \geq j_2$*, the affinoid variety* $\{(x, y_1, \ldots, y_d) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}}^d \mid v_K(x - g_1(y_1, \ldots, y_d)) \geq j_1, v_K(x - g_2(y_1, \ldots, y_d)) \geq j_2\}$ *is equal to* $\{(x, y_1, \ldots, y_d) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}}^d \mid v_K(x - g_1(y_1, \ldots, y_d)) \geq j_1, v_K(g_1(y_1, \ldots, y_d) - g_2(y_1, \ldots, y_d)) \geq j_2\}$.

Proof. For fixed (x, y), these two conditions are equivalent. The universality of an affinoid subdomain proves the lemma.

□

Lemma 5.4. *Let* $a \in \mathcal{O}_K$ *and* $s = v_K(a)$*. If* $j \leq pe/(p-1) + s$*, then the affinoid variety* $X^j(\bar{K}) = \{(x, y) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}} \mid v_K(x^p - ay^{p^n}) \geq j\}$ *is equal to* $\{(x, y) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}} \mid v_K(x - a^{1/p}y^{p^{n-1}}) \geq j/p\}$.

Proof. Lemma 5.1 shows that the fiber of the second projection $X^j(\bar{K}) \to \mathcal{O}_{\bar{K}}$ at y is equal to

$$\begin{cases} D(a^{1/p}y^{p^{n-1}}, j/p) & \text{if } j \leq s + p^{n-1}v_K(y) + pe/(p-1), \\ \coprod_{i=0}^{p-1} D(a^{1/p}\zeta_p^i y^{p^{n-1}}, j - e - (p-1)(s + p^{n-1}v_K(y))/p) & \text{otherwise.} \end{cases}$$

Thus we have $X^j(\bar{K}) = \{(x, y) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}} \mid v_K(x - a^{1/p}y^{p^{n-1}}) \geq j/p\}$ for $j \leq pe/(p-1) + s$. This is the underlying set of a $K(a^{1/p})$-affinoid variety. Again this equality extends to an isomorphism over $K(a^{1/p})$.

□

Now we proceed to the proof of the main theorem of this section.

Theorem 5.5. $c(\mathcal{G}(s_1, \ldots, s_r)) = \sup_k j_k$.

Proof. We may assume that j_r is the supremum of j_k's. If $j_r = 0$, then $\mathcal{G}(s_1, \dots, s_r)$ is etale and $c(\mathcal{G}(s_1, \dots, s_r)) = 0$. Thus we may assume $j_r > 0$. Consider the homomorphism of $\mathcal{O}_K$-algebras

$$\begin{aligned} A = \mathcal{O}_K[T_1, \dots, T_r]/({T_1}^p - \pi^{s_1}T_2, \dots, {T_r}^p - \pi^{s_r}T_1) \to \\ B = \mathcal{O}_K[W, T_2, \dots, T_r]/(W^{p^r} - \pi^{s_1}T_2, {T_2}^p - \pi^{s_2}T_3, \dots, \\ {T_{r-1}}^p - \pi^{s_{r-1}}T_r, {T_r}^p - \pi^{s_r}W^{p^{r-1}}), \end{aligned}$$

defined by $T_1 \mapsto W^{p^{r-1}}$. This induces a surjection of K-affinoid varieties

$$X_B^j(\bar{K}) \ni (w, t_2, \dots, t_r) \mapsto (w^{p^{r-1}}, t_2, \dots, t_r) \in X_A^j(\bar{K}),$$

where

$$\begin{aligned} X_A^j(\bar{K}) = \{(t_1, \dots, t_r) \in \mathcal{O}_{\bar{K}}^r \mid v_K({t_1}^p - \pi^{s_1}t_2) \geq j, \dots, \\ v_K({t_{r-1}}^p - \pi^{s_{r-1}}t_r) \geq j, v_K({t_r}^p - \pi^{s_r}t_1) \geq j\} \end{aligned}$$

and

$$\begin{aligned} X_B^j(\bar{K}) = \{(w, t_2, \dots, t_r) \in \mathcal{O}_{\bar{K}}^r \mid v_K(w^{p^r} - \pi^{s_1}t_2) \geq j, \\ v_K({t_2}^p - \pi^{s_2}t_3) \geq j, \dots, v_K({t_r}^p - \pi^{s_r}w^{p^{r-1}}) \geq j\}. \end{aligned}$$

These are affinoid subdomains of the r-dimensional unit polydisc over K. We calculate a jump of $\{F^j(B)\}_{j \in \mathbb{Q}_{>0}}$ at first.

Lemma 5.6. *If $j_r < pe/(p-1)$, then the first jump of $\{F^j(B)\}_{j \in \mathbb{Q}_{>0}}$ occurs at $j = j_r$ and $\sharp F^{j_r}(B) = p^r$.*

Note that the base change from K to a finite extension L multiplies s_i's, j_i's and e by the ramification index of L/K. Thus, to prove Lemma 5.6 and Theorem 5.5, we may assume that p^{r-1} divides s_i's and e.

Proof. Consider the K-affinoid variety X_B^j for $j \leq pe/(p-1)$. Then the iterative use of Lemma 5.4 and Lemma 5.3 shows that the affinoid variety $X_B^j(\bar{K})$ is equal to

$$\begin{aligned} \{v_K(w^{p^r} - \pi^{(s_r + ps_{r-1} + \dots + p^{r-1}s_1)/p^{r-1}}w) \geq pl_1(j),\ v_K(t_2 - g_2(w)) \geq u_2, \\ v_K(t_3 - g_3(t_2, w)) \geq u_3, \dots,\ v_K(t_r - g_r(t_{r-1}, w)) \geq u_r\}, \end{aligned}$$

where $l_i(j)$, $g_i(t_{i-1}, w)$, $g_2(w)$ and u_i are defined as follows;

- $l_r(j) = j/p$,
- $l_{i-1}(j) = \inf(j, l_i(j) + s_{i-1})/p$,
- $g_i(t_{i-1}, w) = t_{i-1}^p/\pi^{s_{i-1}}$ and $u_i = j - s_{i-1}$ if $j \geq l_i(j) + s_{i-1}$,
- $g_i(t_{i-1}, w) = \pi^{s_r + ps_{r-1} + \dots + p^{r-i}s_i/p^{r-i+1}}w^{p^{i-2}}$ and $u_i = l_i(j)$ if $j < l_i(j) + s_{i-1}$,

- $g_2(w) = g_2(w^{p^{r-1}}, w)$.

Note that $l_i(j)$ is a strictly monotone increasing function of j. This affinoid variety is isomorphic to the product of the affinoid variety $\{w \in \mathcal{O}_{\bar{K}} \mid v(w^{p^r} - \pi^{(s_r+ps_{r-1}+\ldots+p^{r-1}s_1)/p^{r-1}}w) \geq pl_1(j)\}$ and discs. Therefore, from Lemma 5.2, we see that the first jump of $\{F^j(B)\}_{j\in\mathbb{Q}_{>0}}$ occurs at j such that $pl_1(j) = j_r$, provided this j satisfies $0 < j < pe/(p-1)$. Moreover, then we have $\sharp F^j(B) = p^r$. Thus the following lemma and the strict monotonicity of l_1 terminate the proof of Lemma 5.6. □

Lemma 5.7. $l_1(j_r) = j_r/p$.

Proof. Suppose that there is k such that $l_k(j_r) = j_r/p$ and $j_r \geq l_{k'}(j_r) + s_{k'}$ for any $1 < k' \leq k$. Then we have $l_1(j_r) = \inf(j_r, (j_r + ps_{k-1} + p^2 s_{k-2} + \ldots + p^{k-1}s_1)/p^{k-1})/p$ and the assumption $j_{k-1} \leq j_r$ implies $l_1(j_r) = j_r/p$.

□

On the other hand, let $s = (s_r + ps_{r-1} + \ldots + p^{r-1}s_1)/p^{r-1}$ and $\sigma_0, \ldots, \sigma_{p^r-1}$ be the roots of the equation $X^{p^r} - \pi^s X = 0$. Then we see that the images by $w \mapsto w^{p^{r-1}}$ of the discs $D(\sigma_i, pl_1(j) - s)$ are disjoint for $j > j_r$. Hence the surjection $\pi_0(X_B^j(\bar{K})) \to \pi_0(X_A^j(\bar{K}))$ is bijective for $0 < j \leq pe/(p-1)$ and the first (and the last) jump of $\{F^j(A)\}_{j\in\mathbb{Q}_{>0}}$ also occurs at j_r, provided $j_r < pe/(p-1)$.

When $j_r = pe/(p-1)$, we see that $s_k = e > 0$ for any k. Thus we can use Lemma 5.4 for $j < pe/(p-1)+\varepsilon$ with sufficiently small $\varepsilon > 0$. Then, by the same reasoning as above, we conclude that $c(A) = pe/(p-1)$.

□

References

[1] A. Abbes and A. Mokrane: *Sous-groupes canoniques et cycles évanescents p-adiques pour les variétés abéliennes*, Publ. Math. IHES **99** (2004), 117-162

[2] A. Abbes and T. Saito: *Ramification of local fields with imperfect residue fields I*, Amer. J. Math. **124** (2002), 879-920

[3] A. Abbes and T. Saito: *Ramification of local fields with imperfect residue fields II*, Documenta Math. Extra volume: Kazuya Kato's Fiftieth Birthday (2003), 5-72

[4] P. Berthelot, L. Breen and W. Messing: *Théorie de Dieudonné Cristalline II*, Lecture Note in Math. **930**

[5] C. Breuil: *Groupes p-divisibles, groupes finis et modules filtrés*, Ann. of Math. (2) **152** (2000), 489-549

[6] S. Bosch, U. Güntzer and R. Remmert: *Non-Archimedean Analysis*, Springer-Verlag, 1984

[7] S. Bosch and W. Lütkebohmert: *Stable reduction and uniformization of abelian varieties I*, Math. Ann. **270** (1985), 349-379

[8] S. Bosch, W. Lütkebohmert and M. Raynaud: *Formal and rigid geometry IV. The reduced fiber theorem*, Invent. Math. **119** (1995), 361-398
[9] C. Breuil and A. Mézard: *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbb{Z}_p)$ *et de* $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ *en* $l = p$, Duke Math. J. **115** (2002), 205-310
[10] B. Edixhoven: *The weight in Serre's conjectures on modular forms*, Invent. Math. **109** (1992), 563-594
[11] S. Hattori: *Ramification of a finite flat group scheme over a local field*, J. of Number Theory **118** (2006), 145-154
[12] T. Kambayashi, M. Miyanishi and M. Takeuchi: *Unipotent algebraic groups*, Lecture Note in Math. **414**
[13] H. Matsumura: *Commutative ring theory*, Cambridge university press, 1986
[14] M. Raynaud: *Schémas en groupes de type* $(p, ..., p)$, Bull. Soc. Math. France **102** (1974), 241-280
[15] J.-P. Serre: *Propriétés galoisiennes des points d'ordre fini des courbes elliptiques*, Invent. Math. **15** (1972), 259-331
[16] W. C. Waterhouse: *Introduction to Affine Group Schemes*, Springer-Verlag, 1979

E-mail address: `shin-h@ms.u-tokyo.ac.jp`