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**タイトル**

Tame characters and ramification of finite flat group schemes

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1. Introduction

Let $K$ be a complete discrete valuation field of mixed characteristic $(0, p)$ with residue field $F$ which may be imperfect, $G_K$ be its absolute Galois group and $I_K$ be its inertia subgroup. Let $G$ be a finite flat group scheme over $\mathcal{O}_K$. When $G$ is monogenic, that is to say, when the affine algebra of $G$ is generated over $\mathcal{O}_K$ by one element, it is well-known that the tame characters appearing in the $I_K$-module $G(K)$ are determined by the slopes of the Newton polygon of a defining equation of $G$ ([15, Proposition 10]).

On the other hand, for an elliptic modular form $f$ of level $N$ prime to $p$, we also have a description of the tame characters of the associated mod $p$ Galois representation $\bar{\rho}_f$ ([10, Theorem 2.5, Theorem 2.6], [9, Section 4.3]). This is based on Raynaud’s theory of prolongations of finite flat group schemes or the integral $p$-adic Hodge theory. However, for an analogous study of the associated mod $p$ Galois representation of a Hilbert modular form over a totally real number field, we encounter a local field of higher absolute ramification index. In this case, these two theories no longer work well and we need some other techniques.

In this paper, to propose a new approach toward this problem, we generalize the aforementioned result of [15] to the higher dimensional case using the ramification theory of Abbes and Saito ([2], [3]) and determine the tame characters appearing in $G(K)$ in terms of the ramification of $G$ without any restriction on the absolute ramification index of $K$. Namely, we show the following theorem.

**Theorem 1.1.** Let $G$ be a finite flat group scheme over $\mathcal{O}_K$. Write $\{G^i\}_{j \in \mathbb{Q}_{>0}}$ for the ramification filtration of $G$ in the sense of [2] and [3]. Then the graded piece $G^i(K)/G^{i+}(K)$ is killed by $p$ and the $I_K$-module $G^i(K)/G^{i+}(K) \otimes_{\mathbb{F}_p} \mathbb{F}_p$ is the direct sum of $\mathbb{F}_p$-conjugates of the fundamental character $\theta_j$ of level $j$. 

*Date: December 8, 2006.*
In view of the following corollary, we can regard this theorem as a counterpart for finite flat group schemes of the Hasse-Arf theorem in the classical ramification theory.

**Corollary 1.2.** Let \( L \) be an abelian extension of \( K \). Suppose that its integer ring \( \mathcal{O}_L \) is a \( G \)-torsor over \( \mathcal{O}_K \). Then the denominator of every jump of the upper numbering ramification filtration \( \{ \text{Gal}(L/K)^j \}_{j \in \mathbb{Q}_{>0}} \) ([2]) is a power of \( p \).

To prove the main theorem, we firstly show that the tubular neighborhood of \( G \) can be chosen to have a rigid analytic group structure. Passing to the closed fiber, we realize the graded piece of the ramification filtration of \( G \) as the kernel of an étale isogeny of the additive groups \( G_\mathbb{a}^r \) over \( \mathbb{F} \). Then we determine the tame characters by comparing the \( I_K \)-action on the graded piece with the \( \mathbb{G}_m \)-action on \( G_\mathbb{a}^r \) given by the multiplication. As an appendix, we also give an explicit formula for the conductor of a Raynaud \( \mathbb{F} \)-vector space scheme over \( \mathcal{O}_K \) ([14]).

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2. Review of the ramification theory of Abbes and Saito

Let \( K \) be a complete discrete valuation field with residue field \( F \) which may be imperfect. Set \( \pi = \pi_K \) to be an uniformizer of \( K \). The separable closure of \( K \) is denoted by \( \bar{K} \) and the absolute Galois group of \( K \) by \( G_K \). Let \( \mathfrak{m}_K \) and \( \bar{F} \) be the maximal ideal and the residue field of \( \mathcal{O}_K \) respectively. In [2] and [3], Abbes and Saito defined the ramification theory of a finite flat \( \mathcal{O}_K \)-algebra of relative complete intersection. In this section, we gather the necessary definitions and briefly recall their theory.

Let \( A \) be a finite flat \( \mathcal{O}_K \)-algebra and \( \mathbb{A} \) be a complete Noetherian semi-local ring (with its topology defined by \( \text{rad}(\mathbb{A}) \)) which is of formally smooth over \( \mathcal{O}_K \) and whose quotient ring \( \mathbb{A}/\text{rad}(\mathbb{A}) \) is of finite type over \( \bar{F} \). A surjection of \( \mathcal{O}_K \)-algebras \( \mathbb{A} \to A \) is called an embedding if \( \mathbb{A}/\text{rad}(\mathbb{A}) \to A/\text{rad}(A) \) is an isomorphism. For an embedding \( (\mathbb{A} \to A) \) and \( j \in \mathbb{Q}_{>0} \), the \( j \)-th tubular neighborhood of \( (\mathbb{A} \to A) \) is the \( K \)-affinoid variety \( X^j(\mathbb{A} \to A) \) constructed as follows. Write \( j = k/l \) with \( k, l \) non-negative integers. Put \( I = \text{Ker}(\mathbb{A} \to A) \) and \( \mathcal{A}^{k,l}_0 = \mathbb{A}[I/\pi^k]^\wedge \), where \( \wedge \) means the \( \pi \)-adic completion. Then \( \mathcal{A}^{k,l}_0 \) is a quotient ring of the Tate algebra \( \mathcal{O}_K\langle T_1, \ldots, T_r \rangle \) for some \( r \). Its generic fiber \( \mathcal{A}^j_K = \mathcal{A}^{k,l}_0 \otimes_{\mathcal{O}_K} K \) is independent of the choice of a representation
$j = k/l$ ([3, Lemma 1.4]) and set $X^j(\mathbb{A} \rightarrow A) = \text{Sp}(A^j_\mathbb{K})$. This affinoid variety is geometrically regular ([3, Lemma 1.6]).

We put $F(A) = \text{Hom}_{\mathcal{O}_K\text{-alg.}}(A, \mathcal{O}_K)$ and $F^j(A) = \lim_{\rightarrow} \pi_0(X^j(\mathbb{A} \rightarrow A))_k$. Here $\pi_0(X)_K$ denotes the set of geometric connected components of a $K$-affinoid variety $X$ and the projective limit is taken in the category of embeddings of $A$. Note that the projective family $\pi_0(X^j(\mathbb{A} \rightarrow A))_K$ is constant ([3, Section 1.3]). These define contravariant functors $F$ and $F^j$ from the category of finite flat $\mathcal{O}_K$-algebras to the category of finite $G_K$-sets. Moreover, there are morphisms of functors $F \rightarrow F^j$ and $F^j \rightarrow F^{j'}$ for $j' \geq j > 0$.

By the finiteness theorem of Grauert and Remmert ([8, Theorem 1.3]), there exists a finite separable extension $L/$ of $K$ such that the geometric closed fiber of the unit disc $\mathcal{X}^j(\mathbb{A} \rightarrow A)_{O_L}$ for the supremum norm in $X^j(\mathbb{A} \rightarrow A)_L = X^j(\mathbb{A} \rightarrow A) \times K L$ is reduced. Then for any finite separable extension $L'$ of $L$, the $\pi_{L'}$-adic formal scheme $\mathcal{X}^j(\mathbb{A} \rightarrow A)_{O_L} \otimes_{O_L} O_{L'}$ coincides with the unit disc for the supremum norm in $X^j(\mathbb{A} \rightarrow A)_{L'}$ and thus is normal. The $\pi_{L'}$-adic formal scheme $\mathcal{X}^j(\mathbb{A} \rightarrow A)_{O_L}$ is referred as the stable normalized integral model of $X^j(\mathbb{A} \rightarrow A)$ over $O_L$ and its geometric closed fiber is denoted by $\mathcal{X}^j(\mathbb{A} \rightarrow A).$ If $L/K$ is Galois, the Galois group $\text{Gal}(L/K)$ acts on it by the functoriality of the unit disc for the supremum norm. We have the $G_K$-equivariant isomorphism $\pi_0(\mathcal{X}^j(\mathbb{A} \rightarrow A))_F \rightarrow \pi_0(X^j(\mathbb{A} \rightarrow A))_K$, where the former is the set of geometric connected components of $\mathcal{X}^j(\mathbb{A} \rightarrow A)$ ([3, Corollary 1.11]).

Suppose that $A$ is of relative complete intersection over $\mathcal{O}_K$ and $A \otimes_{\mathcal{O}_K} K$ is etale over $K$. Then the natural map $F(A) \rightarrow F^j(A)$ is surjective. The family $\{F(A) \rightarrow F^j(A)\}_{j \in \mathbb{Q}_{>0}}$ is separated, exhaustive and its jumps are rational ([2, Proposition 6.4]). The conductor of $A$ is defined to be $c(A) = \inf\{j \in \mathbb{Q}_{>0} | F(A) \rightarrow F^j(A)$ is an isomorphism$\}$. If $B$ is the affine algebra of a finite flat group scheme $G$ over $\mathcal{O}_K$ which is generically etale, then $B$ is of relative complete intersection (for example, [5, Proposition 2.2.2]) and the theory above can all be applied to $B$. By the functoriality, $F^j(B)$ is endowed with a $G_K$-module structure ([1, Lemme 2.1]) and the natural map $G(\mathcal{K}) = F(B) \rightarrow F^j(B)$ is a $G_K$-homomorphism. Let $G^j$ denote the schematic closure ([14]) in $G$ of the kernel of this homomorphism. It is called the $j$-th ramification filtration of $G$. We refer $c(B)$ as the conductor of $G$, which is denoted also by $c(G)$. We put $G^{j/+}(\mathcal{K}) = \cup_{j' > j} G^{j'}(\mathcal{K})$ and define $G^{j/+}$ to be the schematic closure of $G^{j/+}(\mathcal{K})$ in $G$. 

TAME CHARACTERS AND RAMIFICATION OF FINITE FLAT GROUP SCHEMES
3. Group structure on the tubular neighborhood of a finite flat group scheme

Let $K$ denote a complete discrete valuation field of mixed characteristic $(0, p)$ whose residue field $F$ may be imperfect and $v_K$ the valuation of $K$ extended to $K$ which is normalized as $v_K(\pi) = 1$. Let $\mathcal{G} = \text{Spec}(B)$ be a connected finite flat group scheme over $\mathcal{O}_K$. We define a formal resolution of $\mathcal{G}$ to be a closed immersion $\mathcal{G} \rightarrow \Gamma$ of (profinite) formal group schemes over $\mathcal{O}_K$, where $\Gamma = \text{Spf}(\mathbb{B})$ is connected and smooth. Such an immersion can be constructed as follows.

By a theorem of Raynaud ([4, Théorème 3.1.1]), we can find an abelian scheme $A$ over $\mathcal{O}_K$ and a closed immersion of group schemes $G \rightarrow A$. Taking the formal completion of $A$ along the zero section, we get a formal resolution of $G$. We refer the relative dimension of $\Gamma$ over $\mathcal{O}_K$ as the dimension of a formal resolution $(G \rightarrow \Gamma)$. We define a morphism of formal resolutions to be a pair of group homomorphisms $(f, \hat{f})$ which makes the following diagram commutative.

\[
\begin{array}{ccc}
G & \longrightarrow & \Gamma \\
\downarrow f & & \downarrow \hat{f} \\
G' & \longrightarrow & \Gamma'
\end{array}
\]

Note that a formal resolution of $G$ is also an embedding of $B$ in the sense of Section 2. We say $(f, \hat{f})$ is finite flat if this is finite flat as a map of embeddings ([3]). Consider the $j$-th tubular neighborhood $X_j(B \rightarrow B)$ of the embedding $(B \rightarrow B)$, which we also write as $X_j(G \rightarrow \Gamma)$. The following lemma, whose first part is implicit in [1], enables us to introduce a group structure on this affinoid variety.

**Lemma 3.1.** Let $(A \rightarrow A)$ and $(B \rightarrow B)$ be embeddings of finite flat $\mathcal{O}_K$-algebras. Put $C = A \otimes_{\mathcal{O}_K} B$ and $C = A \otimes_{\mathcal{O}_K} B$. Then the surjection $C \rightarrow C$ is also an embedding and we have a canonical isomorphism $X_j(C \rightarrow C) \rightarrow X_j(A \rightarrow A) \times_{\mathcal{O}_K} X_j(B \rightarrow B)$. Moreover, this isomorphism extends to a canonical isomorphism between their stable normalized integral models $X_j(C \rightarrow C)_{\mathcal{O}_K} \rightarrow X_j(A \rightarrow A)_{\mathcal{O}_K} \times_{\mathcal{O}_K} X_j(B \rightarrow B)_{\mathcal{O}_K}$.

**Proof.** By the functoriality of a tubular neighborhood, we have an affinoid map $\Phi : X_j(C \rightarrow C) \rightarrow X_j(A \rightarrow A) \times_{\mathcal{O}_K} X_j(B \rightarrow B)$. To see that $\Phi$ is an isomorphism, we may replace $K$ with a finite separable extension and suppose that $A$ and $B$ are local, $j$ is an integer and $A/\text{rad}(A) = B/\text{rad}(B) = F$. Then we may identify $A$ with $\mathcal{O}_K[[T_1, \ldots, T_r]]$ and $B$ with $\mathcal{O}_K[[T'_1, \ldots, T'_{r'}]]$ for some $r$ and $r'$. Let $I = (f_1, \ldots, f_s)$ (resp. $J = (g_1, \ldots, g_{s'})$) be the kernel of the surjection $A = \ldots$
Let \( \mathcal{O}_K[[T_1, \ldots, T_r]] \to A \) (resp. \( \mathbb{B} = \mathcal{O}_K[[T'_1, \ldots, T'_{r'}]] \to B \)). Then the affinoid algebras of \( X^j(\mathcal{A} \to A) \), \( X^j(\mathbb{B} \to B) \) and \( X^j(\mathbb{C} \to C) \) are equal to \( K\langle T_1, \ldots, T_r \rangle\langle f_1/\pi^j, \ldots, f_s/\pi^j \rangle \), \( K\langle T'_1, \ldots, T'_{r'} \rangle\langle g_1/\pi^j, \ldots, g_{s'}/\pi^j \rangle \) and \( K\langle T_1, \ldots, T_r, T'_1, \ldots, T'_{r'} \rangle\langle f_1/\pi^j, \ldots, f_s/\pi^j, g_1/\pi^j, \ldots, g_{s'}/\pi^j \rangle \) respectively. This shows that \( \Phi \) is an isomorphism.

Let \( L \) be a finite extension of \( K \) where the stable normalized integral models of \( X^j(\mathcal{A} \to A) \), \( X^j(\mathbb{B} \to B) \) and \( X^j(\mathbb{C} \to C) \) are defined. Set \( \mathcal{A}^j_0 \) and \( \mathcal{A}^j_K \) as in Section 2. Let \( \mathcal{A}^j_{\mathcal{O}_L} \) denote the unit disc in \( \mathcal{A}^j_L = \mathcal{A}^j_K \otimes_K L \) for the supremum norm. Define \( \mathcal{B}_0^j \), \( \mathcal{C}_0^j \), \( \mathcal{B}_K^j \), \( \mathcal{C}_K^j \), \( \mathcal{B}_{\mathcal{O}_L}^j \) and \( \mathcal{C}_{\mathcal{O}_L}^j \) similarly for \( B \) and \( C \). From the proof of [8, Theorem 1.3], there exists a continuous surjection \( \alpha : L\langle T_1, \ldots, T_r \rangle \to \mathcal{A}^j_L \) such that \( \|\alpha\| = \|\alpha\|_\alpha \), where \( \|\cdot\| \) is the residue norm induced by \( \alpha \). We also have a surjection \( \beta : L\langle U_1, \ldots, U_{s'} \rangle \to \mathcal{B}^j_L \) with the same property for \( B \). Consider the surjection \( \alpha \otimes \beta : L\langle T_1, \ldots, T_r \rangle \otimes_L L\langle U_1, \ldots, U_{s'} \rangle \to \mathcal{A}^j_L \otimes_L \mathcal{B}^j_L = \mathcal{C}^j_L \). The unit disc in \( \mathcal{A}^j_L \otimes_L \mathcal{B}^j_L \) for the residue norm induced by \( \alpha \otimes \beta \) is \( \mathcal{A}^j_{\mathcal{O}_L} \otimes_{\pi_L} \mathcal{B}^j_{\mathcal{O}_L} \), where \( \otimes_{\pi_L} \) denotes the \( \pi_L \)-adic complete tensor product over \( \mathcal{O}_L \). Its geometric closed fiber \( (\mathcal{A}^j_{\mathcal{O}_L} \otimes_{\pi_L} \mathcal{F}) \otimes_{\mathcal{F}} (\mathcal{B}^j_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} \mathcal{F}) \) is reduced. By [8, Proposition 1.1], we see that the stable normalized integral model \( \mathcal{C}^j_{\mathcal{O}_L} \) is equal to \( \mathcal{A}^j_{\mathcal{O}_L} \otimes_{\pi_L} \mathcal{B}^j_{\mathcal{O}_L} \).

Corollary 3.2. Let \( (\mathcal{G} \to \Gamma) \) be a formal resolution of \( \mathcal{G} \). Then the group structure of \( \Gamma \) induces a rigid \( K \)-analytic group structure on the tubular neighborhood \( X^j(\mathcal{G} \to \Gamma) \). This group structure also extends to \( X^j(\mathcal{G} \to \Gamma)_{\mathcal{O}_K} \) (resp. \( X^j(\mathcal{G} \to \Gamma) \)) and endows it with a \( \pi \)-adic formal group scheme structure over \( \mathcal{O}_K \) (resp. an algebraic group structure over \( \mathcal{F} \)).

Moreover, for a morphism of formal resolutions \( (\mathcal{G} \to \Gamma) \to (\mathcal{G}' \to \Gamma') \), the induced affinoid map \( X^j(\mathcal{G} \to \Gamma) \to X^j(\mathcal{G}' \to \Gamma') \) is a homomorphism of rigid \( K \)-analytic groups. This homomorphism also extends to their stable normalized integral models as a homomorphism \( X^j(\mathcal{G} \to \Gamma)_{\mathcal{O}_K} \to X^j(\mathcal{G}' \to \Gamma')_{\mathcal{O}_K} \) of \( \pi \)-adic formal group schemes and to their geometric closed fibers as a homomorphism \( X^j(\mathcal{G} \to \Gamma) \to X^j(\mathcal{G}' \to \Gamma') \) of algebraic groups over \( \mathcal{F} \).

Let \( \mathcal{G} = \text{Spec}(B) \) be a connected finite flat group scheme over \( \mathcal{O}_K \) and \( (\mathcal{G} \to \Gamma = \text{Spec}(\mathbb{B})) \) be a formal resolution of dimension \( r \). Set \( \text{Spf}(\mathcal{A}) = \Gamma/\mathcal{G} \) and regard the zero section \( \text{Spec}(\mathcal{O}_K) \to \text{Spf}(\mathcal{A}) \) as a formal resolution of the trivial group. Then we have a finite flat map
of formal resolutions

\[ \mathcal{G} \longrightarrow \text{Spf}(\mathbb{B}) \]

\[ \text{Spec}(\mathcal{O}_K) \longrightarrow \text{Spf}(\mathbb{A}). \]

By Corollary 3.2 and [3, Lemma 1.8], we get a finite flat map of rigid \( K \)-analytic groups \( f^j : X^j_G = X^j(\mathcal{G} \to \Gamma) \to D^{r,j} = X^j(\text{Spec}(\mathcal{O}_K) \to \text{Spf}(\mathbb{A})) \), where \( D^{r,j} \) denotes the \( r \)-dimensional polydisc \( \{(z_1, \ldots, z_r) \in \mathcal{O}_K^r \mid v_K(z_i) \geq j \text{ for any } i\} \). We call this the affinoid homomorphism associated to a formal resolution \((\mathcal{G} \to \Gamma)\). Write \( B^j_K \) and \( A^j_K \) for the \( K \)-affinoid algebras of \( X^j_G \) and \( D^{r,j} \) respectively. The stable normalized integral model over \( \mathcal{O}_L \) of \( X^j_G \) (resp. \( D^{r,j} \)) is denoted by \( X^j_G;\mathcal{O}_L \) (resp. \( D^{r,j};\mathcal{O}_{D^j} \)) and its geometric closed fiber by \( \bar{X}^j_G \) (resp. \( \bar{D}^{r,j} \)). Note that the algebraic group \( \bar{X}^j_G \) is reduced, hence smooth by [16, Theorem 11.6].

**Lemma 3.3.** The affinoid homomorphism \( f^j : X^j_G \to D^{r,j} \) is etale for any \( j > 0 \). Moreover, for \( j > c(\mathcal{G}) \), there exists a finite extension \( K'/K \) such that \( X^j_G;K' \) is isomorphic to the disjoint sum of finitely many copies of \( D^{r,j}_{K'} \).

**Proof.** We have \( \Omega^1_{B^j_K/A^j_K} = B^j_A \otimes_{\mathbb{B}} \hat{\Omega}_{\mathbb{B}/A} \). It is enough to show that \( \hat{\Omega}_{\mathbb{B}/A} \) is a torsion \( \mathcal{O}_K \)-module. Let \( J_A \) and \( J_B \) be the augmentation ideals of \( A \) and \( \mathbb{B} \) respectively. Set \( I = \text{Ker}(\mathbb{B} \to B) \). Then \( \hat{\Omega}_{\mathbb{B}/A} = \text{Coker}(\mathbb{B} \otimes_{\mathbb{A}} \hat{\Omega}_{\mathbb{A}/\mathcal{O}_K} \to \hat{\Omega}_{\mathbb{B}/\mathcal{O}_K}) \) is equal to \( \mathbb{B} \otimes_{\mathcal{O}_K} \text{Coker}(\text{Cot}(A) \to \text{Cot}(\mathbb{B})) = \mathbb{B} \otimes_{\mathcal{O}_K} (I + J_B^2) = \mathbb{B} \otimes_{\mathcal{O}_K} \text{Cot}(B) \). This shows the first assertion. For the second assertion, take a finite extension \( K' \) of \( K \) where the geometric connected components of \( X^j_G \) are defined. By assumption, each of the connected components of \( X^j_G;K' \) is a finite etale cover of \( D^{r,j}_{K'} \) whose degree is one. Thus this is isomorphic to \( D^{r,j}_{K'} \).

Take a finite extension \( L \) of \( K \) where the stable normalized integral models of \( X^j_G \) and \( D^{r,j} \) are defined. The generic fiber \( \mathcal{G}_L = \mathcal{G} \times_{\mathcal{O}_K} L \) can be regarded as a rigid \( L \)-analytic subgroup of \( X^j_G;L \) defined by an ideal \( J_L \) of \( \mathcal{O}_L \). Put \( J = J_L \cap \mathcal{O}_L \) and \( \mathcal{B} = \mathcal{B}_{\mathcal{O}_L}/J \). The latter is a subring of \( B_L = B \otimes_K L \). Since we have a commutative diagram with
surjective horizontal arrows and injective vertical arrows

\[
\begin{array}{ccc}
B_0^{k,l} \otimes_{\mathcal{O}_K} \mathcal{O}_L & \longrightarrow & B \otimes_{\mathcal{O}_K} \mathcal{O}_L \\
\downarrow & & \downarrow \\
\mathcal{B}^j_{\mathcal{O}_L} & \longrightarrow & \mathcal{B}^j_{\mathcal{O}_L},
\end{array}
\]

we see that \(\mathcal{B}^j_{\mathcal{O}_L}\) is integral over \(B \otimes_{\mathcal{O}_K} \mathcal{O}_L\). Thus the \(\mathcal{O}_K\)-algebra \(\mathcal{B}^j_{\mathcal{O}_L}\) is finite \(\pi_L\)-algebra.

**Lemma 3.4.** The group structure of \(\mathcal{G}_L\) extends to \(\mathcal{H}^j_{\mathcal{O}_L}\). The group scheme \(\mathcal{H}^j_{\mathcal{O}_L}\) is a closed \(\pi_L\)-adic formal subgroup scheme of \(\mathcal{X}^j_{\mathcal{G},\mathcal{O}_L}\).

**Proof.** Put \(\mathcal{K} = \text{Ker}(B^j_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} B^j_{\mathcal{O}_L} \to B^j_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} B^j_{\mathcal{O}_L})\) and \(\mathcal{K}_L = \mathcal{K} \otimes_{\mathcal{O}_L} L\). Let \(\mu\) be the coproduct of \(B^j_{\mathcal{O}_L}\). We must show \(\mu(J) \subseteq \mathcal{K}\). This follows from the commutative diagram below whose rows are exact and vertical allows are injective.

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{B}^j_{\mathcal{O}_L} \otimes_{\pi_L} \mathcal{B}^j_{\mathcal{O}_L} & \longrightarrow & \mathcal{B}^j_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} \mathcal{B}^j_{\mathcal{O}_L} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{K}_L & \longrightarrow & \mathcal{B}^j_L \otimes_L \mathcal{B}^j_L & \longrightarrow & \mathcal{B}_L \otimes_L \mathcal{B}_L & \longrightarrow & 0
\end{array}
\]

Passing to the generic fiber, we see that the second assertion holds. \(\square\)

**Lemma 3.5.** The associated homomorphism \(\tilde{f}^j : \mathcal{X}^j_{\mathcal{G},\mathcal{O}_L} \to \mathcal{D}^{r,j}_{\mathcal{O}_L}\) is finite flat. Moreover, there exists an exact sequence of \(\pi_L\)-adic formal group schemes

\[(1) \quad 0 \to \mathcal{H}^j_{\mathcal{O}_L} \to \mathcal{X}^j_{\mathcal{G},\mathcal{O}_L} \to \mathcal{D}^{r,j}_{\mathcal{O}_L} \to 0.
\]

**Proof.** From Lemma 3.3, the associated affinoid map \(f^j : \mathcal{X}^j_{\mathcal{G}} \to \mathcal{D}^{r,j}_{\mathcal{O}_L}\) is finite etale. Let \(\mathcal{B}^j_{\mathcal{K}}\) and \(\mathcal{A}^j_{\mathcal{K}}\) be their affinoid algebras as above. Since \(\mathcal{D}^{r,j}_{\mathcal{O}_L}\) is integral, we see that \(f^j\) is surjective and the ring homomorphism \(\mathcal{A}^j_{\mathcal{K}} \to \mathcal{B}^j_{\mathcal{K}}\) is injective. Thus we have an injection \(\mathcal{A}^j_{\mathcal{O}_L} \to \mathcal{B}^j_{\mathcal{O}_L}\), which is finite by [6, Corollary 6.4.1/6]. Hence \(\tilde{f}^j : \mathcal{X}^j_{\mathcal{G}} \to \mathcal{D}^{r,j}_{\mathcal{O}_L}\) is a surjective homomorphism of algebraic groups over \(\mathcal{F}\). Since \(\mathcal{X}^j_{\mathcal{G}}\) and \(\mathcal{D}^{r,j}_{\mathcal{O}_L}\) are regular, we see that \(\tilde{f}^j\) is faithfully flat by [13, Theorem 23.1]. Since \(\mathcal{A}^j_{\mathcal{O}_L}\) and \(\mathcal{B}^j_{\mathcal{O}_L}\) are \(\pi_L\)-torsion free, the map \(\tilde{f}^j\) is flat by the local criterion of flatness. Put \(\mathcal{H} = \text{Ker}(\tilde{f}^j)\). This is a closed \(\pi_L\)-adic formal subgroup scheme of \(\mathcal{X}^j_{\mathcal{G},\mathcal{O}_L}\) and can be regarded also as a finite flat group scheme over \(\mathcal{O}_L\). Passing to the generic fiber, we see that \(\mathcal{H}^j_{\mathcal{O}_L}\) is a closed subgroup scheme of \(\mathcal{H}'\). Comparing these ranks concludes the lemma.
From Lemma 3.3, we see that for \( j > c = c(\mathcal{G}) \), the map \( \hat{f}^j \) identifies \( X_{\mathcal{G}, \mathcal{O}_L}^j \) with the direct sum of finitely many copies of \( \mathfrak{D}^r_{\mathcal{O}_L} \). More precisely, we have the following.

**Lemma 3.6.** Let \( c = c(\mathcal{G}) \) be the conductor of \( \mathcal{G} \). Then the associated homomorphism \( \hat{f}^j : X_{\mathcal{G}, \mathcal{O}_L}^j \to \mathfrak{D}^r_{\mathcal{O}_L} \) is finite etale if and only if \( j \geq c \).

**Proof.** Let \( \text{sp}_j : X_{\mathcal{G}}^j \to \tilde{X}_{\mathcal{G}}^j \) be the specialization map. By Lemma 3.3, we see that \( \hat{f}^c \) is finite etale at \( \text{sp}_c(x) \) for any \( x \in \mathcal{G}(\overline{K}) \) as in the proof of [2, Theorem 7.2], using a theorem of Bosch ([7, Lemma 2.1]). We conclude the etaleness of \( \hat{f}^c \) by the existence of the group structure. We have \( \Omega^1_{\mathfrak{B}_{\mathcal{O}_L}/\mathfrak{A}_{\mathcal{O}_L}} \otimes_{\mathcal{O}_L} \overline{F} = 0 \). Since \( \mathfrak{B}_{\mathcal{O}_L} \) is \( \pi_L \)-adic complete and Noetherian, we see that \( \Omega^1_{\mathfrak{B}_{\mathcal{O}_L}/\mathfrak{A}_{\mathcal{O}_L}} = 0 \) and the homomorphism \( \hat{f}^c : X_{\mathcal{G}, \mathcal{O}_L} \to \mathfrak{D}^r_{\mathcal{O}_L} \) is finite etale.

Let \( \tilde{0} \) be the zero section of \( \tilde{D}^r \) and set \( X_{\mathcal{G}}^{j+} = \cup_{j' > j} X_{\mathcal{G}}^{j'} \). Then we have \( (\hat{f}^j)^{-1}(\tilde{0}) = \text{sp}_j(X_{\mathcal{G}}^{j+}) = \text{sp}_j(\mathcal{G}(\overline{K})) \). If \( \hat{f}^j \) is etale, then \( \#(\hat{f}^j)^{-1}(\tilde{0}) \) equals the degree of \( \hat{f}^j \), namely \( \sharp \mathcal{G}(\overline{K}) \). Thus \( X_{\mathcal{G}}^{j+} \) splits and we have \( j \geq c \).

\( \square \)

4. Ramification and the \( I_K \)-module structure of a finite flat group scheme

Consider the right action of \( I_K \) on \( \overline{K} \) defined by \( \sigma.z = \sigma^{-1}(z) \) for \( \sigma \in I_K \). This action induces a \( K \)-semilinear left action of \( I_K \) on \( X_{\mathcal{G}, K}^j = X_{\mathcal{G}}^j \times_K \overline{K} \), which also extends to an \( \mathcal{O}_K \)-semilinear action on its stable normalized integral model \( X_{\mathcal{G}, \mathcal{O}_K}^j \). Thus we have an \( \overline{F} \)-linear left action of \( I_K \) on its closed fiber \( \tilde{X}_{\mathcal{G}}^j \). We call this the geometric monodromy action of \( I_K \) and write the action of \( \sigma \in I_K \) as \( \sigma_{\text{geom}} \) (note that here we follow the terminology in [2], and not in [3] where the monodromy action is called arithmetic, since in our case there is no "geometric" action other than the monodromy action). Similarly, we have the geometric monodromy action of \( I_K \) on \( \tilde{D}^r \).

The latter action is described as follows. Let the additive group (resp. multiplicative group) over \( \overline{F} \) be denoted by \( \mathbb{G}_a \) (resp. \( \mathbb{G}_m \)). Consider the left action \( \mathbb{G}_m \times \mathbb{G}_a^s \to \mathbb{G}_a^s \) given by the multiplication. Write this action of \( \lambda \in \overline{F}^\times \) as \( [\lambda] \). This action is defined by \( T_i \mapsto \lambda T_i \), where \( \mathbb{G}_a^s = \text{Spec}(\overline{F}[T_1, \ldots, T_r]) \). For \( j \in \mathbb{Q}_{>0} \), we define the fundamental character \( \theta_j : I_K \to \overline{F}^\times \) to be \( \theta_k^{k'} \), where \( k'/l' \) is the prime-to-\( p \)-denominator part.
Thus diagram whose rows are exact. \[ X \] ramification index over group scheme of \( K \).

**Proof.** Put Lemma 4.2. If geometric closed fiber of \( H \) denote the unit component of the algebraic group \( A \) is tame (\([2, \text{Lemma } 7.7]\)). Write \( F \) with \( \sigma = \theta_j(\sigma) \). Then for \( \sigma \in I_K \), we have \( \sigma((\pi^{1/l})^k/\pi^{1/l})^k \equiv \theta_j(\sigma)^{k_p} \mod m_K \) with some \( N \) and this is equal to \( \theta_j(\sigma) \). Thus the action on the affine algebra \( \mathcal{O}_L \) of \( D^{r,j} \) is given by \( \sigma^{\text{geom}}(W_i) = \theta_j(\sigma)W_i \). This coincides with \( [\theta_j(\sigma)] \).

Next we consider the geometric monodromy action on \( \mathcal{X}_{G}^j \). Let \( \mathcal{X}_{G}^{j,0} \) denote the unit component of the algebraic group \( \mathcal{X}_{G}^j \) and \( \mathcal{H}^j \) be the geometric closed fiber of \( \mathcal{H}^j \). We begin with the following lemma.

**Lemma 4.2.** If \( \psi \in \text{End}(\mathcal{X}_{G}^{j,0}) \) induces the zero map on \( D^{r,j} \), then \( \psi = 0 \).

**Proof.** Put \( \mathcal{H}^j = \mathcal{H}^j \cap \mathcal{X}_{G}^{j,0} \). This is the kernel of the faithfully flat map \( \mathcal{X}_{G}^{j,0} \to D^{r,j} \) and by assumption we have the following commutative diagram whose rows are exact.

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{H}^j_0 & \longrightarrow & \mathcal{X}_{G}^{j,0} & \longrightarrow & D^{r,j} & \longrightarrow & 0 \\
& & \downarrow \mathcal{H}^j & & \downarrow \psi & & \downarrow 0 & & \\
0 & \longrightarrow & \mathcal{H}^j_0 & \longrightarrow & \mathcal{X}_{G}^{j,0} & \longrightarrow & D^{r,j} & \longrightarrow & 0
\end{array}
\]

Thus \( \psi \) factors through \( \mathcal{H}^j_0 \). Put \( \mathcal{C} = \text{Im}(\psi) \). Then this is a closed subgroup scheme of \( \mathcal{H}^j_0 \) and the map \( \mathcal{X}_{G}^{j,0} \to \mathcal{C} \) is faithfully flat. Since \( \mathcal{X}_{G}^{j,0} \)
is regular and connected, we see that $\tilde{C}$ is also regular and connected by [13, Theorem 23.7]. Hence $\bar{C} = 0$ and we have $\psi = 0$.

**Corollary 4.3.** Let $\mathcal{G}$ be a connected finite flat group scheme over $\mathcal{O}_K$. Take a formal resolution $(\mathcal{G} \to \Gamma)$ of dimension $r$. Then the algebraic group $\bar{X}_G^{j,0}$ is isomorphic to $\bar{\mathcal{G}}^*$. 

**Proof.** By the previous lemma and Lemma 4.1, we see that $\bar{X}_G^{j,0}$ is killed by $p$. Hence the assertion follows from [12, Lemma 1.7.1].

**Corollary 4.4.** The geometric monodromy action of $I_K$ on $\bar{X}_G^{j,0}$ is tame.

**Proof.** For an element $\sigma$ of the wild inertia subgroup $P_K$, the geometric monodromy action $\sigma_{\text{geom}}$ on $D^{r,j}$ is trivial. Applying the lemma to $\sigma_{\text{geom}} - \text{id} \in \text{End}(\bar{X}_G^{j,0})$ shows the assertion.

**Corollary 4.5.** Let $J$ be a finite cyclic quotient of $I_K$ through which the tame character $\theta_j$ factors and $\tau$ be a generator of $J$. Let $F(t)$ denote the minimal polynomial of $\theta_j(\tau) \in \bar{F}$ over $\mathbb{F}_p$. Then the geometric monodromy action of $I_K$ on $\bar{X}_G^{j,0}$ also factors through $J$ and the equation $F(\tau_{\text{geom}}) = 0$ holds in $\text{End}(\bar{X}_G^{j,0})$.

**Proof.** The first assertion follows from Lemma 4.2 as the previous corollary. The second assertion also follows from this lemma using Lemma 4.1.

Let $c = c(\mathcal{G})$ be the conductor of $\mathcal{G}$. The lemma below enables us to realize $G^c(\overline{K})$ as a subgroup of $\bar{X}_G^{c,0}$.

**Lemma 4.6.** The specialization map $sp_e : X_{G,K}^c \to \bar{X}_G^c$ induces an $I_K$-equivariant isomorphism $\mathcal{G}(\overline{K}) \to \mathcal{H}^e(\overline{F})$ and $\mathcal{G}^c(\overline{K}) \to \mathcal{H}^c_0(\overline{F})$. Here we consider on the left-hand side the natural action as the $K$-valued points of $\mathcal{G}$ (resp. $\mathcal{G}^c$) and on the right-hand side the restriction of the geometric monodromy action on $X_G^c$.

**Proof.** By definition, the generic fiber of $\mathcal{H}_{\mathcal{O}_L}^c$ is equal to $\mathcal{G}_L$. From the exact sequence (1) and Lemma 3.6, we know that $\mathcal{H}_{\mathcal{O}_L}^c$ is etale over $\mathcal{O}_L$ and there is the following exact sequence of algebraic groups over $\overline{F}$.

\begin{equation}
0 \to \mathcal{H}^e \to X_G^c \to D^{r,j} \to 0
\end{equation}

Thus we have a natural isomorphism $\mathcal{H}_{\mathcal{O}_L}^c(\overline{K}) \to \mathcal{H}^e(\overline{F})$ and the composite $\mathcal{G}(\overline{K}) = \mathcal{H}_{\mathcal{O}_L}^c(\overline{K}) \to \mathcal{H}^e(\overline{F}) \to X_G^c(\overline{F})$ coincides with the map.
sp_c. From [2, Corollary 4.4], we see that this map sends \( G^c(\bar{K}) \) isomorphically onto \( H^c_0(F) \).

For \( x \in X^c_0(\bar{K}) \) and \( \sigma \in I_K \), let \( \sigma(x) \) denote the natural action of \( \sigma \) on \( \bar{K} \)-valued points. Then we have \( \sigma_{\text{geom}}(x) \circ \sigma = \sigma(x) \). Taking its specialization shows the \( I_K \)-equivariance.

The following theorem can be regarded as a generalization for a finite flat group scheme over \( O_K \) of the structure theorem of the graded piece of the classical upper numbering ramification filtration.

**Theorem 4.7.** Let \( \mathcal{G} \) be a finite flat group scheme over \( O_K \) and \( j \in \mathbb{Q}_{>0} \). Then the \( G_K \)-module \( G^j(\bar{K})/G^{j+}(\bar{K}) \) is tame and killed by \( p \).

**Proof.** Since \( G^j = (G^0)^j \), where \( G^0 \) denotes the unit component of \( G \), we may assume \( \mathcal{G} \) is connected. Suppose that \( j \) is a jump of the ramification filtration on \( \mathcal{G} \) and consider the quotient \( \mathcal{G} = G^j(K) \). The subgroup \( G^j(K) \subseteq G(K) \) has a non-trivial image in \( (G/G^{j+})(K) \). By the Herbrand theorem ([1, Lemme 2.10]), the natural map \( G^j(K) \to (G/G^{j+})(K) \) is surjective for any \( t > 0 \). We have \( (G/G^{j+})^t = 0 \) for \( t > j \) and \( (G/G^{j+})^j \neq 0 \). Thus the ramification filtration on \( G/G^{j+} \) jumps at \( j \) and \( (G/G^{j+})^j(K) = G^j(K)/G^{j+}(K) \). Replacing \( \mathcal{G} \) with \( G/G^{j+} \), we may assume \( j = c = c(\mathcal{G}) \).

Take a formal resolution \( (\mathcal{G} \to \Gamma) \) of dimension \( r \) and consider its associated affinoid homomorphism \( X^c_0 \to D^{\infty} \). Then the theorem follows from Corollary 4.3, Corollary 4.4, and Lemma 4.6.

From this theorem, we see that the inertia subgroup \( I_K \) acts on \( G^j(K)/G^{j+}(K) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p} \) by the direct sum of tame characters. The theorem below determines these characters up to \( p \)-power exponent.

**Theorem 4.8.** Let \( \mathcal{G} \) be a finite flat group scheme over \( O_K \) and \( j \in \mathbb{Q}_{>0} \). Then \( I_K \) acts on \( G^j(K)/G^{j+}(K) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p} \) by the direct sum of \( \mathbb{F}_p \)-conjugates of the fundamental character \( \theta_j \).

**Proof.** We may assume that \( \mathcal{G} \) is connected. The same argument as in the proof of Theorem 4.7 using the Herbrand theorem reduces the claim to the case where \( j = c = c(\mathcal{G}) \). Take a formal resolution \( (\mathcal{G} \to \Gamma) \) of dimension \( r \). Let \( J \) and \( \tau \) be as in Corollary 4.5. Then Corollary 4.5 and Lemma 4.6 show that every eigenvalue of the action of \( \tau_{\text{geom}} \) on the finite dimensional \( \mathbb{F}_p \)-vector space \( G^c(\bar{K}) \) is a conjugate of \( \theta_j(\tau) \) over \( \mathbb{F}_p \). Since the order of \( J \) is prime to \( p \), we conclude that \( I_K \) acts on \( G^c(\bar{K}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p} \) by the direct sum of \( \mathbb{F}_p \)-conjugates of \( \theta_c \).
Corollary 4.9. Let $G$ be a finite flat group scheme over $\mathcal{O}_K$. Then the order of the image of the homomorphism $I_K \rightarrow \text{Aut}(G(\overline{K}))$ is a power of $p$ if and only if every jump $j$ of the ramification filtration $\{G^j\}_{j \in \mathbb{Q}_{>0}}$ is an element of $\mathbb{Z}[1/p]$.

Proof. From Theorem 4.8, we see that the jumps are in $\mathbb{Z}[1/p]$ if and only if the graded pieces $G^j(\overline{K})/G^{j+}(\overline{K})$ are unramified. By Theorem 4.7, this is equivalent to the condition that $\sharp \text{Im}(I_K \rightarrow \text{Aut}(G(\overline{K})))$ is a $p$-power.

When $G(\overline{K})$ is unramified and killed by $p$, we have the following re-inforcement of this result, which is an easy corollary of the Herbrand theorem. The author does not know if every jump is an integer whenever $G(\overline{K})$ is unramified. If $G$ is monogenic, then we see that this holds true from [11, Theorem 4].

Proposition 4.10. Let $G$ be a finite flat group scheme over $\mathcal{O}_K$ which is killed by $p$. Suppose that the $G_K$-module $G(\overline{K})$ is unramified. Then every jump $j$ of the ramification filtration $\{G^j\}_{j \in \mathbb{Q}_{>0}}$ is an element of $p\mathbb{Z}$.

Proof. We may assume $K = K^{ur}$ and $G_K$ acts trivially on $G(\overline{K})$. There is a quotient $W$ of $G(\overline{K})/G^{j+}(\overline{K})$ where $G^j(\overline{K})$ has a non-trivial image and of rank one over $\mathbb{F}_p$. Taking the schematic closure, $W$ extends to a finite flat group scheme $W$ over $\mathcal{O}_K$ which is a quotient of $G/G^{j+}$. By the Herbrand theorem, we see that the ramification filtration of $W$ jumps at $j$. On the other hand, $W$ is a Raynaud $\mathbb{F}_p$-vector space scheme with unramified generic fiber. Thus the proposition follows from [11, Theorem 4].

For the rest of this section, we state some corollaries in the case where $G$ is an $\mathbb{F}$-vector space scheme of rank one or two for a finite extension $\mathbb{F}$ over $\mathbb{F}_p$. In the case of rank one, Theorem 4.8 directly shows the claim below, while this can be shown also as a corollary of a determination of the conductor of a Raynaud $\mathbb{F}$-vector space scheme (Theorem 5.5).

Corollary 4.11. Let $G$ be an $\mathbb{F}$-vector space scheme of rank one over $\mathcal{O}_K$ and $c = c(G)$. Then the $I_K$-action on the $\mathbb{F}$-vector space $G(\overline{K})$ of rank one is given by the character $\theta_{c^n}$ for some $n$.

In the case of rank two, we have the following.

Corollary 4.12. Let $G$ be a finite flat $\mathbb{F}$-vector space scheme of rank two over $\mathcal{O}_K$ and $c = c(G)$. Then the $I_K$-module $G(\overline{K}) \otimes_{\mathbb{F}} \mathbb{F}_p$ contains
the character $\theta_n^m$ for some $n$. If the $G_K$-module $G(\bar{K})$ is reducible, this holds true for $G(\bar{K})$ itself.

**Proof.** The first assertion follows easily from Theorem 4.8 and the surjection $G_c(\bar{K}) \otimes_{\mathbb{F}_p} \mathbb{F}_p \to G_c(\bar{K}) \otimes_{\mathbb{F}_p} \mathbb{F}_p$. Suppose the $I_K$-module $G(\bar{K})$ is reducible. When $G^c$ is of rank one, the assertion is clear from Theorem 4.8. If $G^c = G$, then $G^c$ is reducible and the assertion follows also from Theorem 4.8.

The corollary below indicates that the conductor $c(G)$ carries information about not only the tame characters but also their extension structures in the $I_K$-module $G(\bar{K})$.

**Corollary 4.13.** Consider an exact sequence of finite flat $\mathbb{F}$-vector space schemes over $\mathcal{O}_K$

$$0 \to G_1 \to G \to G_2 \to 0$$

where $G_1$ and $G_2$ are connected of rank one. If $c(G) = c(G_2)$, then the $I_K$-module $G(\bar{K})$ splits.

**Proof.** Put $c = c(G)$. Take a formal resolution $(G \to \Gamma)$ of dimension $r$ and put $\Gamma_2 = \Gamma/G_1$. Then we get a finite flat map of formal resolutions

$$
\begin{array}{ccc}
G & \longrightarrow & \Gamma \\
\downarrow & & \downarrow \\
G_2 & \longrightarrow & \Gamma_2.
\end{array}
$$

Therefore we have a finite flat homomorphism of rigid $K$-analytic groups $X^i(G \to \Gamma) \to X^i(G_2 \to \Gamma_2)$ by Corollary 3.2. As in the proof of Lemma 3.3, we see that this map is finite etale.

Suppose $G^c(\bar{K})$ is of rank one. If $G^c(\bar{K}) \neq G_1(\bar{K})$ as an $\mathbb{F}$-subspace of $G(\bar{K})$, the $I_K$-module $G(\bar{K})$ splits and the proposition follows. Suppose $G^c(\bar{K}) = G_1(\bar{K})$. The affinoid variety $X^c(G \to \Gamma)$ decomposes to $\sharp\mathbb{F}$ components over some finite extension $K'$ of $K$. Each component is a Zariski open and closed subset of $X^c(G \to \Gamma)_{K'}$. As the map $f : X^c(G \to \Gamma)_{K'} \to X^c(G_2 \to \Gamma_2)_{K'}$ is finite etale and $X^c(G_2 \to \Gamma_2)_{K'}$ is connected, every component $X^{ci}(G \to \Gamma)_{K'}$ maps surjectively to $X^c(G_2 \to \Gamma_2)_{K'}$. Take some $g_i \in G(\bar{K}) \cap X^{ci}(G \to \Gamma)_{K'}$. Using the group structure, we see that $G(\bar{K}) \cap X^{ci}(G \to \Gamma)_{K'} = g_i + G^c(\bar{K}) = g_i + G_1(\bar{K})$ and $f(G(\bar{K}) \cap X^{ci}(G \to \Gamma)_{K'}) = f(g_i)$. However, we have $f^{-1}(G_2(\bar{K})) = G(\bar{K})$ and thus $f(G(\bar{K}) \cap X^{ci}(G \to \Gamma)_{K'}) = G_2(\bar{K})$. This is a contradiction. Therefore we may assume $G^c(\bar{K}) = G(\bar{K})$. In this case, the proposition follows from Theorem 4.7.

$\square$
5. Example: rank one calculation

In this section, we calculate the conductor of a Raynaud $\mathbb{F}$-vector space scheme over $\mathcal{O}_K$. The point is that, as we can see from the bound of the conductor ([11, Theorem 7]), it is enough to consider the $j$-th tubular neighborhood only for $j \leq pe/(p-1) + \varepsilon$ with sufficiently small $\varepsilon > 0$. For such $j$, we can compute the tubular neighborhood easily by Lemma 5.4 below.

Let $K$ be a complete discrete valuation field of mixed characteristic $(0, p)$. We write $\pi = \pi_K$ for its uniformizer and $e$ for its absolute ramification index. We normalize a valuation $v_K$ of $K$ as $v_K(\pi) = 1$ and extend it to the algebraic closure $\bar{K}$ of $K$. For $a \in \bar{K}$ and $j \in \mathbb{R}$, let $D(a, j)$ denote the closed disc $\{z \in \mathcal{O}_K \mid v_K(z-a) \geq j\}$. This is the underlying subset of a $K(a)$-affinoid subdomain of the unit disc over $K(a)$.

For integers $0 \leq s_1, \ldots, s_r \leq e$, let $\mathcal{G}(s_1, \ldots, s_r)$ denote the Raynaud $\mathbb{F}_p$-vector space scheme over $\mathcal{O}_K$ defined by the $r$ equations $T_1^e = \pi^{s_1}T_2, T_2^e = \pi^{s_2}T_3, \ldots, T_r^e = \pi^{s_r}T_1$ ([14]). We set $j_k = (ps_k + p^2s_{k-1} + \ldots + p^ks_1 + p^{k+1}s_r + p^{k+2}s_{r-1} + \cdots + p^ks_{k+1})/(p^k - 1)$. Before the calculation of $c(\mathcal{G}(s_1, \ldots, s_r))$, we gather some elementary lemmas.

**Lemma 5.1.** Let $a \in \mathcal{O}_K$ and $s = v_K(a)$. Then the affinoid variety $X^j(K) = \{x \in \mathcal{O}_K \mid v_K(x^p - a) \geq j\}$ is equal to

$$\left\{ \begin{array}{ll} D(a^{1/p}, j/p) & \text{if } j \leq s + pe/(p-1), \\
\prod_{i=0}^{p-1} D(a^{1/p}c_{p^i}, j - e - (p-1)s/p) & \text{if } j > s + pe/(p-1). \end{array} \right.$$ 

**Proof.** We have $v_K(x^p - a) = \sum_i v_K(x - a^{1/p}c_{p^i})$. If $v_K(x - a^{1/p}c_{p^i}) \geq v_K(x - a^{1/p}c_{p^i})$ for any $i' \neq i$, then $v_K(x - a^{1/p}c_{p^i}) \leq v_K(a^{1/p}c_{p^i}(1 - c_{p^i}^{-1}))) = s/p + e/(p-1)$. Thus we have $v_K(x - a^{1/p}c_{p^i}) \geq \sup(j/p, j - (p-1)s/p - e)$ and

$$X^j(K) \subseteq \bigcup_i D(a^{1/p}c_{p^i}, \sup(j/p, j - (p-1)s/p - e)).$$

Suppose that $j/p \geq j - (p-1)s/p - e$. Then we have $v_K(a^{1/p}(1 - c_{p^i})) = s/p + e/(p-1) \geq j/p$, $D(a^{1/p}, j/p) = D(a^{1/p}c_{p^i}, j/p)$ for any $i$ and thus

$$X^j(K) = D(a^{1/p}, j/p).$$

When $j/p < j - (p-1)s/p - e$, we have $v_K(a^{1/p}(1 - c_{p^i})) = s/p + e/(p-1) < j - (p-1)s/p - e$. This means that if $w \in D(a^{1/p}c_{p^i}, j - (p-1)s/p - e)$ for some $i$, then $v_K(w - a^{1/p}c_{p^i}) < j - (p-1)s/p - e$ for any other $i'$. 

Thus the discs $D(a^{1/p^i}j, (p-1)s/p - e)$ are disjoint and
\[X^j(K) = \prod_i D(a^{1/p^i}j, (p-1)s/p - e).\]

These are equalities of the underlying sets of affinoid subdomains of the unit disc over $K(a^{1/p}, \zeta_p)$. By the universality of an affinoid subdomain, this extends to an isomorphism of affinoid varieties.

We can prove the following lemma just in the same way.

**Lemma 5.2.** The affinoid variety $\{x \in O_K \mid v_K(x^{p^r} - ax) \geq j\}$ is equal to
\[
\begin{cases}
D(0, j/p^r) & \text{if } j \leq p^r v(a)/(p - 1), \\
\prod_{i=0}^{r-1} D(\sigma_i, j - v(a)) & \text{if } j > p^r v(a)/(p - 1),
\end{cases}
\]
where $\sigma_i$'s are the roots of $X^{p^r} = ax$.

**Lemma 5.3.** For $g_1(Y_1, \ldots, Y_d), g_2(Y_1, \ldots, Y_d) \in K[Y_1, \ldots, Y_d]$ and $j_1 \geq j_2$, the affinoid variety $\{(x, y_1, \ldots, y_d) \in \O_K \times \O_K \mid v_K(x - g_1(y_1, \ldots, y_d)) \geq j_1, v_K(x - g_2(y_1, \ldots, y_d)) \geq j_2\}$ is equal to $\{(x, y_1, \ldots, y_d) \in \O_K \times \O_K \mid v_K(x - g_1(y_1, \ldots, y_d)) \geq j_1, v_K(g_1(y_1, \ldots, y_d) - g_2(y_1, \ldots, y_d)) \geq j_2\}$.

**Proof.** For fixed $(x, y)$, these two conditions are equivalent. The universality of an affinoid subdomain proves the lemma.

**Lemma 5.4.** Let $a \in O_K$ and $s = v_K(a)$. If $j \leq pe/(p - 1) + s$, then the affinoid variety $X^j(K) = \{(x, y) \in \O_K \times \O_K \mid v_K(x^{p^r} - ay^{p^r}) \geq j\}$ is equal to $\{(x, y) \in \O_K \times \O_K \mid v_K(x - a^{1/p^r}y^{p^r - 1}) \geq j/p\}$.

**Proof.** Lemma 5.1 shows that the fiber of the second projection $X^j(K) \to \O_K$ at $y$ is equal to
\[
\begin{cases}
D(a^{1/p^r}y^{p^r - 1}, j/p) & \text{if } j \leq s + p^{n-1}v_K(y) + pe/(p - 1), \\
\prod_{i=0}^{p-1} D(a^{1/p^i}y^{p^i - 1}, j - e - (p - 1)(s + p^{n-1}v_K(y))/p) & \text{otherwise}.
\end{cases}
\]
Thus we have $X^j(K) = \{(x, y) \in \O_K \times \O_K \mid v_K(x - a^{1/p^r}y^{p^r - 1}) \geq j/p\}$ for $j \leq pe/(p - 1) + s$. This is the underlying set of a $K(a^{1/p})$-affinoid variety. Again this equality extends to an isomorphism over $K(a^{1/p})$.

Now we proceed to the proof of the main theorem of this section.

**Theorem 5.5.** $c(G(s_1, \ldots, s_r)) = \sup_k j_k$. 
Proof. We may assume that \( j_r \) is the supremum of \( j_k \)'s. If \( j_r = 0 \), then \( \mathcal{G}(s_1, \ldots, s_r) \) is etale and \( c(\mathcal{G}(s_1, \ldots, s_r)) = 0 \). Thus we may assume \( j_r > 0 \). Consider the homomorphism of \( \mathcal{O}_K \)-algebras

\[
A = \mathcal{O}_K[T_1, \ldots, T_r]/(T_1^p - \pi s_1 T_2, \ldots, T_r^p - \pi s_r T_1) \\
B = \mathcal{O}_K[W, T_2, \ldots, T_r]/(W^p - \pi s_1 T_2, T_2^p - \pi s_2 T_3, \ldots, \]

\[
T_{r-1}^p - \pi s_{r-1} T_r, T_r^p - \pi s_r W^{p^{r-1}})
\]

defined by \( T_1 \mapsto W^{p^{r-1}} \). This induces a surjection of \( K \)-affinoid varieties

\[
X_B^j(\bar{K}) \ni (w, t_2, \ldots, t_r) \mapsto (w^{p^{r-1}}, t_2, \ldots, t_r) \in X_A^j(\bar{K}),
\]

where

\[
X_A^j(\bar{K}) = \{(t_1, \ldots, t_r) \in \mathcal{O}_K^r \mid v_K(t_1^p - \pi s_1 t_2) \geq j, \ldots, v_K(t_{r-1}^p - \pi s_{r-1} t_r) \geq j, v_K(t_r^p - \pi s_r t_3) \geq j\}
\]

and

\[
X_B^j(\bar{K}) = \{(w, t_2, \ldots, t_r) \in \mathcal{O}_K^r \mid v_K(w^p - \pi s_1 t_2) \geq j, v_K(t_2^p - \pi s_2 t_3) \geq j, \ldots, v_K(t_r^p - \pi s_r w^{p^{r-1}}) \geq j\}.
\]

These are affinoid subdomains of the \( r \)-dimensional unit polydisc over \( K \). We calculate a jump of \( \{F^j(B)\}_{j \in \mathbb{Q}_{>0}} \) at first.

**Lemma 5.6.** If \( j_r < pe/(p - 1) \), then the first jump of \( \{F^j(B)\}_{j \in \mathbb{Q}_{>0}} \) occurs at \( j = j_r \) and \( F^{j_r}(B) = p^r \).

Note that the base change from \( K \) to a finite extension \( L \) multiplies \( s_i \)'s, \( j_i \)'s and \( e \) by the ramification index of \( L/K \). Thus, to prove Lemma 5.6 and Theorem 5.5, we may assume that \( p^{r-1} \) divides \( s_i \)'s and \( e \).

**Proof.** Consider the \( K \)-affinoid variety \( X_B^j \) for \( j \leq pe/(p - 1) \). Then the iterative use of Lemma 5.4 and Lemma 5.3 shows that the affinoid variety \( X_B^j(\bar{K}) \) is equal to

\[
\{v_K(w^p - \pi^{(s_r + \ldots + s_1)/p^{r-1}} w) \geq pl_1(j), v_K(t_2 - g_2(w)) \geq u_2, v_K(t_3 - g_3(t_2, w)) \geq u_3, \ldots, v_K(t_r - g_r(t_{r-1}, w)) \geq u_r\},
\]

where \( l_i(j), g_i(t_{i-1}, w), g_2(w) \) and \( u_i \) are defined as follows;

- \( l_r(j) = j/p \),
- \( l_{i-1}(j) = \inf(j, l_i(j) + s_{i-1})/p \),
- \( g_i(t_{i-1}, w) = t_{i-1}^p/\pi^{s_{i-1}} \) and \( u_i = j - s_{i-1} \) if \( j \geq l_i(j) + s_{i-1} \),
- \( g_i(t_{i-1}, w) = \pi^{s_r + \ldots + s_1}/p^{r-i+1} q^{2p^{r-1}} \) and \( u_i = l_i(j) \) if \( j < l_i(j) + s_{i-1} \),
Note that \( l_1(j) \) is a strictly monotone increasing function of \( j \). This affinoid variety is isomorphic to the product of the affinoid variety \( \{ w \in \mathcal{O}_k \mid v(w^p - \pi(s_r + ps_{r-1} + \ldots + p^{r-1}s_1)/p^{r-1}w) \geq pl_1(j) \} \) and discs. Therefore, from Lemma 5.2, we see that the first jump of \( \{ F^j(B) \}_{j \in \mathbb{Q}_{>0}} \) occurs at \( j \) such that \( pl_1(j) = j_r \), provided this \( j \) satisfies \( 0 < j < pe/(p-1) \). Moreover, then we have \( F^j(B) = p^r \). Thus the following lemma and the strict monotonicity of \( l_1 \) terminate the proof of Lemma 5.6.

**Lemma 5.7.** \( l_1(j_r) = j_r/p \).

**Proof.** Suppose that there is \( k \) such that \( l_k(j_r) = j_r/p \) and \( j_r \geq l_{k'}(j_r) + s_{k'} \) for any \( 1 < k' \leq k \). Then we have \( l_1(j_r) = \inf(j_r, (j_r + ps_{k-1} + p^2s_{k-2} + \ldots + p^{k-1}s_1)/p^{k-1})/p \) and the assumption \( j_{k-1} \leq j_r \) implies \( l_1(j_r) = j_r/p \).

On the other hand, let \( s = (s_r + ps_{r-1} + \ldots + p^{r-1}s_1)/p^{r-1} \) and \( \sigma_0, \ldots, \sigma_{p-1} \) be the roots of the equation \( X^p - \pi^*X = 0 \). Then we see that the images by \( w \mapsto w^p - 1 \) of the discs \( D(\sigma_i, pl_1(j) - s) \) are disjoint for \( j > j_r \). Hence the surjection \( \pi_0(X_B^s(\overline{K})) \to \pi_0(X^s_{A}(\overline{K})) \) is bijective for \( 0 < j \leq pe/(p-1) \) and the first (and the last) jump of \( \{ F^j(A) \}_{j \in \mathbb{Q}_{>0}} \) also occurs at \( j_r \), provided \( j_r < pe/(p-1) \).

When \( j_r = pe/(p-1) \), we see that \( s_k = e > 0 \) for any \( k \). Thus we can use Lemma 5.4 for \( j < pe/(p-1) + \varepsilon \) with sufficiently small \( \varepsilon > 0 \). Then, by the same reasoning as above, we conclude that \( c(A) = pe/(p-1) \).

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**References**


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