1. Introduction

Let $K$ be a complete discrete valuation field of mixed characteristic $(0, p)$ with residue field $F$ which may be imperfect, $G_K$ be its absolute Galois group and $I_K$ be its inertia subgroup. Let $G$ be a finite flat group scheme over $\mathcal{O}_K$. When $G$ is monogenic, that is to say, when the affine algebra of $G$ is generated over $\mathcal{O}_K$ by one element, it is well-known that the tame characters appearing in the $I_K$-module $G(K)$ are determined by the slopes of the Newton polygon of a defining equation of $G$ ([15, Proposition 10]).

On the other hand, for an elliptic modular form $f$ of level $N$ prime to $p$, we also have a description of the tame characters of the associated mod $p$ Galois representation $\bar{\rho}_f$ ([10, Theorem 2.5, Theorem 2.6], [9, Section 4.3]). This is based on Raynaud’s theory of prolongations of finite flat group schemes or the integral $p$-adic Hodge theory. However, for an analogous study of the associated mod $p$ Galois representation of a Hilbert modular form over a totally real number field, we encounter a local field of higher absolute ramification index. In this case, these two theories no longer work well and we need some other techniques.

In this paper, to propose a new approach toward this problem, we generalize the aforementioned result of [15] to the higher dimensional case using the ramification theory of Abbes and Saito ([2], [3]) and determine the tame characters appearing in $G(K)$ in terms of the ramification of $G$ without any restriction on the absolute ramification index of $K$. Namely, we show the following theorem.

**Theorem 1.1.** Let $G$ be a finite flat group scheme over $\mathcal{O}_K$. Write $\{G^i\}_{i \in \mathbb{Q}_{>0}}$ for the ramification filtration of $G$ in the sense of [2] and [3]. Then the graded piece $G^i(K)/G^{i+}(K)$ is killed by $p$ and the $I_K$-module $G^i(K)/G^{i+}(K) \otimes_{\mathbb{F}_p} \mathbb{F}_p$ is the direct sum of $\mathbb{F}_p$-conjugates of the fundamental character $\theta_j$ of level $j$.
In view of the following corollary, we can regard this theorem as a counterpart for finite flat group schemes of the Hasse-Arf theorem in the classical ramification theory.

**Corollary 1.2.** Let $L$ be an abelian extension of $K$. Suppose that its integer ring $\mathcal{O}_L$ is a $\mathcal{G}$-torsor over $\mathcal{O}_K$. Then the denominator of every jump of the upper numbering ramification filtration $\{\text{Gal}(L/K)^j\}_{j \in \mathbb{Q}_{>0}}$ ([2]) is a power of $p$.

To prove the main theorem, we firstly show that the tubular neighborhood of $\mathcal{G}$ can be chosen to have a rigid analytic group structure. Passing to the closed fiber, we realize the graded piece of the ramification filtration of $\mathcal{G}$ as the kernel of an etale isogeny of the additive groups $\mathcal{G}_a^r$ over $\mathbb{F}$. Then we determine the tame characters by comparing the $I_K$-action on the graded piece with the $\mathcal{G}_a^r$-action on $\mathcal{G}_a^r$ given by the multiplication. As an appendix, we also give an explicit formula for the conductor of a Raynaud $\mathbb{F}$-vector space scheme over $\mathcal{O}_K$ ([14]).

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2. **Review of the ramification theory of Abbes and Saito**

Let $K$ be a complete discrete valuation field with residue field $\mathbb{F}$ which may be imperfect. Set $\pi = \pi_K$ to be an uniformizer of $K$. The separable closure of $K$ is denoted by $\overline{K}$ and the absolute Galois group of $K$ by $G_K$. Let $\mathfrak{m}_K$ and $\mathbb{F}$ be the maximal ideal and the residue field of $\mathcal{O}_K$ respectively. In [2] and [3], Abbes and Saito defined the ramification theory of a finite flat $\mathcal{O}_K$-algebra of relative complete intersection. In this section, we gather the necessary definitions and briefly recall their theory.

Let $A$ be a finite flat $\mathcal{O}_K$-algebra and $\mathbb{A}$ be a complete Noetherian semi-local ring (with its topology defined by $\text{rad}(\mathbb{A})$) which is of formally smooth over $\mathcal{O}_K$ and whose quotient ring $\mathbb{A}/\text{rad}(\mathbb{A})$ is of finite type over $\mathbb{F}$. A surjection of $\mathcal{O}_K$-algebras $\mathbb{A} \to A$ is called an embedding if $\mathbb{A}/\text{rad}(\mathbb{A}) \to A/\text{rad}(A)$ is an isomorphism. For an embedding $(\mathbb{A} \to A)$ and $j \in \mathbb{Q}_{>0}$, the $j$-th tubular neighborhood of $\mathcal{O}_K \langle T_1, \ldots, T_r \rangle$ constructed as follows. Write $j = k/l$ with $k, l$ non-negative integers. Put $I = \text{Ker}(\mathbb{A} \to A)$ and $A_{0}^{k,l} = A[[I^l/\pi^k]]$, where $\langle \rangle$ means the $\pi$-adic completion. Then $A_{0}^{k,l}$ is a quotient ring of the Tate algebra $\mathcal{O}_K \langle T_1, \ldots, T_r \rangle$ for some $r$. Its generic fiber $A_{K}^{j} = A_{0}^{k,l} \otimes_{\mathcal{O}_K} K$ is independent of the choice of a representation.
the category of finite \( G \)-variety is geometrically regular ([3, Lemma 1.6]). We put \( F(A) = \text{Hom}_{\mathcal{O}_K}\text{-alg}(A, \mathcal{O}_K) \) and \( F^j(A) = \varprojlim \pi_0(X^j(\mathbb{A} \to A))_K \). Here \( \pi_0(X)_K \) denotes the set of geometric connected components of a \( K \)-affinoid variety \( X \) and the projective limit is taken in the category of embeddings of \( A \). Note that the projective family \( \pi_0(X^j(\mathbb{A} \to A))_K \) is constant ([3, Section 1.2]). These define contravariant functors \( F \) and \( F^j \) from the category of finite flat \( \mathcal{O}_K \)-algebras to the category of finite \( G_K \)-sets. Moreover, there are morphisms of functors \( F \to F^j \) and \( F^{j'} \to F^j \) for \( j' \geq j > 0 \).

By the finiteness theorem of Grauert and Remmert ([8, Theorem 1.3]), there exists a finite separable extension \( L \) of \( K \) such that the geometric closed fiber of the unit disc \( \mathfrak{X}^j(\mathbb{A} \to A)_{\mathcal{O}_L} \) for the suprenum norm in \( X^j(\mathbb{A} \to A)_L = X^j(\mathbb{A} \to A) \times_K L \) is reduced. Then for any finite separable extension \( L' \) of \( L \), the \( \pi_{L'} \)-adic formal scheme \( \mathfrak{X}^j(\mathbb{A} \to A)_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} \mathcal{O}_{L'} \) coincides with the unit disc for the suprenum norm in \( X^j(\mathbb{A} \to A)_{L'} \) and thus is normal. The \( \pi_L \)-adic formal scheme \( \mathfrak{X}^j(\mathbb{A} \to A)_{\mathcal{O}_L} \) is referred as the stable normalized integral model of \( X^j(\mathbb{A} \to A) \) over \( \mathcal{O}_L \) and its geometric closed fiber is denoted by \( \bar{X}^j(\mathbb{A} \to A) \). If \( L/K \) is Galois, the Galois group \( \text{Gal}(L/K) \) acts on it by the functoriality of the unit disc for the suprenum norm. We have the \( G_K \)-equivariant isomorphism \( \pi_0(\bar{X}^j(\mathbb{A} \to A))_F \to \pi_0(X^j(\mathbb{A} \to A))_K \), where the former is the set of geometric connected components of \( \bar{X}^j(\mathbb{A} \to A) \) ([3, Corollary 1.11]).

Suppose that \( A \) is of relative complete intersection over \( \mathcal{O}_K \) and \( A \otimes_{\mathcal{O}_K} \mathbb{K} \) is etale over \( K \). Then the natural map \( F(A) \to F^j(A) \) is surjective. The family \( \{ F(A) \to F^j(A) \}_{j \in \mathbb{Q}_{>0}} \) is separated, exhaustive and its jumps are rational ([2, Proposition 6.4]). The conductor of \( A \) is defined to be \( c(A) = \inf \{ j \in \mathbb{Q}_{>0} | F(A) \to F^j(A) \text{ is an isomorphism} \} \).

If \( B \) is the affine algebra of a finite flat group scheme \( \mathcal{G} \) over \( \mathcal{O}_K \) which is generically etale, then \( B \) is of relative complete intersection (for example, [5, Proposition 2.2.2]) and the theory above can all be applied to \( B \). By the functoriality, \( F^j(B) \) is endowed with a \( G_K \)-module structure ([1, Lemme 2.1]) and the natural map \( \mathcal{G}(\mathbb{K}) = F(B) \to F^j(B) \) is a \( G_K \)-homomorphism. Let \( \mathcal{G}^j \) denote the schematic closure ([14]) in \( \mathcal{G} \) of the kernel of this homomorphism. It is called the \( j \)-th ramification filtration of \( \mathcal{G} \). We refer \( c(B) \) as the conductor of \( \mathcal{G} \), which is denoted also by \( c(\mathcal{G}) \). We put \( \mathcal{G}^{j+}(\mathbb{K}) = \bigcup_{j \geq 0} \mathcal{G}^j(\mathbb{K}) \) and define \( \mathcal{G}^{j+} \) to be the schematic closure of \( \mathcal{G}^{j+}(\mathbb{K}) \) in \( \mathcal{G} \).
3. GROUP STRUCTURE ON THE TUBULAR NEIGHBORHOOD OF A FINITE FLAT GROUP SCHEME

Let $K$ denote a complete discrete valuation field of mixed characteristic $(0,p)$ whose residue field $F$ may be imperfect and $v_K$ the valuation of $K$ extended to $\bar{K}$ which is normalized as $v_K(\pi) = 1$. Let $\mathcal{G} = \text{Spec}(B)$ be a connected finite flat group scheme over $\mathcal{O}_K$. We define a formal resolution of $\mathcal{G}$ to be a closed immersion $\mathcal{G} \to \Gamma$ of (profinite) formal group schemes over $\mathcal{O}_K$, where $\Gamma = \text{Spf}(\mathbb{B})$ is connected and smooth. Such an immersion can be constructed as follows. By a theorem of Raynaud ([4, Théorème 3.1.1]), we can find an abelian scheme $A$ over $\mathcal{O}_K$ and a closed immersion of group schemes $G \to A$. Taking the formal completion of $A$ along the zero section, we get a formal resolution of $G$. We refer the relative dimension of $\Gamma$ over $\mathcal{O}_K$ as the dimension of a formal resolution ($G \to \Gamma$). We define a morphism of formal resolutions to be a pair of group homomorphisms $(f, f')$ which makes the following diagram commutative.

$$
\begin{array}{ccc}
\mathcal{G} & \longrightarrow & \Gamma \\
f \downarrow & & \downarrow f' \\
\mathcal{G}' & \longrightarrow & \Gamma'
\end{array}
$$

Note that a formal resolution of $G$ is also an embedding of $B$ in the sense of Section 2. We say $(f, f')$ is finite flat if this is finite flat as a map of embeddings ([3]). Consider the $j$-th tubular neighborhood $X^j(B \to B)$ of the embedding $(B \to B)$, which we also write as $X^j(\mathcal{G} \to \Gamma)$. The following lemma, whose first part is implicit in [1], enables us to introduce a group structure on this affinoid variety.

**Lemma 3.1.** Let $(A \to A) \text{ and } (B \to B)$ be embeddings of finite flat $\mathcal{O}_K$-algebras. Put $C = A \otimes_{\mathcal{O}_K} B$ and $C = A \otimes_{\mathcal{O}_K} B$. Then the surjection $C \to C$ is also an embedding and we have a canonical isomorphism $X^j(C \to C) \to X^j(A \to A) \times_{\mathcal{O}_K} X^j(B \to B)$. Moreover, this isomorphism extends to a canonical isomorphism between their stable normalized integral models $X^j(C \to C)_{\mathcal{O}_K} \to X^j(A \to A)_{\mathcal{O}_K} \times_{\mathcal{O}_K} X^j(B \to B)_{\mathcal{O}_K}$.

**Proof.** By the functoriality of a tubular neighborhood, we have an affinoid map $\Phi : X^j(C \to C) \to X^j(A \to A) \times_{\mathcal{O}_K} X^j(B \to B)$. To see that $\Phi$ is an isomorphism, we may replace $K$ with a finite separable extension and suppose that $A$ and $B$ are local, $j$ is an integer and $A/\text{rad}(A) = B/\text{rad}(B) = F$. Then we may identify $A$ with $\mathcal{O}_K[[T_1, \ldots, T_r]]$ and $B$ with $\mathcal{O}_K[[T'_1, \ldots, T'_{r'}]]$ for some $r$ and $r'$. Let $J = (f_1, \ldots, f_s)$ (resp. $J = (g_1, \ldots, g_{s'})$) be the kernel of the surjection $A =
$\mathcal{O}_K[[T_1, \ldots, T_r]] \to A$ (resp. $B = \mathcal{O}_K[[T'_1, \ldots, T'_{r'}]] \to B$). Then the affinoid algebras of $X^j(A \to A)$, $X^j(B \to B)$ and $X^j(C \to C)$ are equal to $K\langle T_1, \ldots, T_r \rangle (f_1/\pi^j, \ldots, f_s/\pi^j)$, $K\langle T'_1, \ldots, T'_{r'} \rangle (g_1/\pi^j, \ldots, g_{s'}/\pi^j)$ and $K\langle T_1, \ldots, T_r, T'_1, \ldots, T'_{r'} \rangle (f_1/\pi^j, \ldots, f_s/\pi^j, g_1/\pi^j, \ldots, g_{s'}/\pi^j)$ respectively. This shows that $\Phi$ is an isomorphism.

Let $L$ be a finite extension of $K$ where the stable normalized integral models of $X^j(A \to A)$, $X^j(B \to B)$ and $X^j(C \to C)$ are defined. Set $\mathcal{A}_{0,L}^j$ and $\mathcal{A}_{K,L}^j$ as in Section 2. Let $\mathcal{A}_{O_L}^j$ denote the unit disc in $\mathcal{A}_{0,L}^j = \mathcal{A}_{K,L}^j \otimes_K L$ for the supremum norm. Define $\mathcal{B}_{0,L}^j$, $\mathcal{C}_{0,L}^j$, $\mathcal{B}_{K,L}^j$, $\mathcal{C}_{K,L}^j$, $\mathcal{B}_{O_L,L}^j$ and $\mathcal{C}_{O_L,L}^j$ similarly for $B$ and $C$. From the proof of [8, Theorem 1.3], there exists a continuous surjection $\alpha : L\langle T_1, \ldots, T_r \rangle \to \mathcal{A}_{L}^j$ such that $\|\alpha\|_{\sup} = \|\alpha\|_\infty$, where $\|\alpha\|_\infty$ is the residue norm induced by $\alpha$. We also have a surjection $\beta : L\langle U_1, \ldots, U_{r'} \rangle \to \mathcal{B}_{L}^j$ with the same property for $B$. Consider the surjection $\alpha \otimes \beta : L\langle T_1, \ldots, T_r \rangle \otimes_L L\langle U_1, \ldots, U_{r'} \rangle \to \mathcal{A}_{L}^j \otimes_L \mathcal{B}_{L}^j = \mathcal{C}_{L}^j$. The unit disc in $\mathcal{A}_{L}^j \otimes_L \mathcal{B}_{L}^j$ for the residue norm induced by $\alpha \otimes \beta$ is $\mathcal{A}_{O_L,L}^j \otimes_{\pi_L} \mathcal{B}_{O_L,L}^j$, where $\otimes_{\pi_L}$ denotes the $\pi_L$-adic complete tensor product over $\mathcal{O}_L$. Its geometric closed fiber $(\mathcal{A}_{O_L,L}^j \otimes_{\pi_L} \mathcal{B}_{O_L,L}^j)(\bar{F})$ is reduced. By [8, Proposition 1.1], we see that the stable normalized integral model $\mathcal{C}_{O_L,L}^j$ is equal to $\mathcal{A}_{O_L,L}^j \otimes_{\pi_L} \mathcal{B}_{O_L,L}^j$.

\[ \square \]

**Corollary 3.2.** Let $(G \to \Gamma)$ be a formal resolution of $G$. Then the group structure of $\Gamma$ induces a rigid $K$-analytic group structure on the tubular neighborhood $X^j(G \to \Gamma)$. This group structure also extends to $X^j(G \to \Gamma)_{\mathcal{O}_K}$ (resp. $X^j(G \to \Gamma)$) and endows it with a $\pi$-adic formal group structure over $\mathcal{O}_K$ (resp. an algebraic group structure over $\bar{F}$).

Moreover, for a morphism of formal resolutions $(G \to \Gamma) \to (G' \to \Gamma')$, the induced affinoid map $X^j(G \to \Gamma) \to X^j(G' \to \Gamma')$ is a homomorphism of rigid $K$-analytic groups. This homomorphism also extends to their stable normalized integral models as a homomorphism $\mathcal{X}^j(G \to \Gamma)_{\mathcal{O}_K} \to \mathcal{X}^j(G' \to \Gamma')_{\mathcal{O}_K}$ of $\pi$-adic formal group schemes and to their geometric closed fibers as a homomorphism $X^j(G \to \Gamma) \to X^j(G' \to \Gamma')$ of algebraic groups over $\bar{F}$.

Let $G = \text{Spec}(B)$ be a connected finite flat group scheme over $\mathcal{O}_K$ and $(G \to \Gamma = \text{Spec}(B))$ be a formal resolution of dimension $r$. Set $\text{Spec}(A) = \Gamma/G$ and regard the zero section $\text{Spec}(\mathcal{O}_K) \to \text{Spec}(A)$ as a formal resolution of the trivial group. Then we have a finite flat map
of formal resolutions

\[ \mathcal{G} \longrightarrow \text{Spf}(\mathbb{B}) \]

\[ \text{Spec}(\mathcal{O}_K) \longrightarrow \text{Spf}(\mathbb{A}) \]

By Corollary 3.2 and [3, Lemma 1.8], we get a finite flat map of rigid \( K \)-analytic groups \( f^j : X^j_G = X^j(\mathcal{G} \to \Gamma) \to D^{r,j} = X^j(\text{Spec}(\mathcal{O}_K) \to \text{Spf}(\mathbb{A})) \), where \( D^{r,j} \) denotes the \( r \)-dimensional polydisc \( \{ (z_1, \ldots, z_r) \in \mathcal{O}_K^r \mid v_K(z_i) \geq j \text{ for any } i \} \). We call this the affinoid homomorphism associated to a formal resolution \( (\mathcal{G} \to \Gamma) \). Write \( B^j_K \) and \( A^j_K \) for the \( K \)-affinoid algebras of \( X^j_G \) and \( D^{r,j} \) respectively. The stable normalized integral model over \( \mathcal{O}_L \) of \( X^j_G \) (resp. \( D^{r,j} \)) is denoted by \( X^j_G; \mathcal{O}_L \) (resp. \( D^{r,j}; \mathcal{O}_L \)). Note that the algebraic group \( X^j_G \) is reduced, hence smooth by [16, Theorem 11.6].

**Lemma 3.3.** The affinoid homomorphism \( f^j : X^j_G \to D^{r,j} \) is etale for any \( j > 0 \). Moreover, for \( j > c(\mathcal{G}) \), there exists a finite extension \( K' / K \) such that \( X^j_{G,K'} \) is isomorphic to the disjoint sum of finitely many copies of \( D^{r,j}_{K'} \).

**Proof.** We have \( \Omega^1_{B^j_K/A^j_K} = B^j_K \otimes_B \hat{\Omega}_{B/A} \). It is enough to show that \( \hat{\Omega}_{B/A} \) is a torsion \( \mathcal{O}_K \)-module. Let \( J_A \) and \( J_B \) be the augmentation ideals of \( A \) and \( B \) respectively. Set \( I = \text{Ker}(B \to B) \). Then \( \hat{\Omega}_{B/A} = \text{Coker}(B \otimes_A \hat{\Omega}_{A/O_K} \to \hat{\Omega}_{B/O_K}) \) is equal to \( B \otimes_{\mathcal{O}_K} \text{Coker}(\text{Cot}(A) \to \text{Cot}(B)) = B \otimes_{\mathcal{O}_K} J_B / (I + J_B^2) = B \otimes_{\mathcal{O}_K} \text{Cot}(B) \). This shows the first assertion. For the second assertion, take a finite extension \( K' / K \) where the geometric connected components of \( X^j_G \) are defined. By assumption, each of the connected components of \( X^j_{G,K'} \) is a finite etale cover of \( D^{r,j}_{K'} \) whose degree is one. Thus this is isomorphic to \( D^{r,j}_{K'} \).

Take a finite extension \( L \) of \( K \) where the stable normalized integral models of \( X^j_G \) and \( D^{r,j} \) are defined. The generic fiber \( \mathcal{G}_L = \mathcal{G} \times_{\mathcal{O}_K} L \) can be regarded as a rigid \( L \)-analytic subgroup of \( X^j_{G,L} \) defined by an ideal \( J_L \) of \( B^j_L \). Put \( J = J_L \cap B^j_{\mathcal{O}_L} \) and \( B^j = B^j_{\mathcal{O}_L} / J \). The latter is a subring of \( B_L = B \otimes_K L \). Since we have a commutative diagram with
surjective horizontal arrows and injective vertical arrows

\[
\begin{array}{ccc}
B^k_0 \otimes_{O_K} O_L & \longrightarrow & B \otimes_{O_K} O_L \\
\downarrow & & \downarrow \\
\mathcal{B}^j_{O_L} & \longrightarrow & \hat{\mathcal{B}}^j_{O_L},
\end{array}
\]

we see that \( \hat{\mathcal{B}}^j_{O_L} \) is integral over \( B \otimes_{O_K} O_L \). Thus the \( O_K \)-algebra \( \hat{\mathcal{B}}^j_{O_L} \) is finite flat. Set \( \mathcal{H}^j_{O_L} = \text{Spec}(\hat{\mathcal{B}}^j_{O_L}) \). This can be regarded as a closed \( \pi_L \)-adic formal subscheme of \( \mathcal{X}^j_{G,O_L} \).

**Lemma 3.4.** The group structure of \( \mathcal{G}_L \) extends to \( \mathcal{H}^j_{O_L} \). The group scheme \( \mathcal{H}^j_{O_L} \) is a closed \( \pi_L \)-adic formal subgroup scheme of \( \mathcal{X}^j_{G,O_L} \).

**Proof.** Put \( \mathcal{K} = \text{Ker}(\mathcal{B}^j_{O_L} \otimes_{O_L} \mathcal{B}^j_{O_L} \rightarrow \hat{\mathcal{B}}^j_{O_L} \otimes_{O_L} \hat{\mathcal{B}}^j_{O_L}) \) and \( \mathcal{K}_L = \mathcal{K} \otimes_{O_L} L \). Let \( \mu \) be the coproduct of \( \mathcal{B}^j_{O_L} \). We must show \( \mu(J) \subseteq \mathcal{K} \). This follows from the commutative diagram below whose rows are exact and vertical arrows are injective.

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{B}^j_{O_L} \otimes_{\pi_L} \mathcal{B}^j_{O_L} & \longrightarrow & 
\mathcal{B}^j_{O_L} \otimes_{O_L} \hat{\mathcal{B}}^j_{O_L} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}_L & \longrightarrow & \mathcal{B}^j_{L} \otimes_{L} \mathcal{B}^j_{L} & \longrightarrow & B_L \otimes L B_L & \longrightarrow & 0
\end{array}
\]

Passing to the generic fiber, we see that the second assertion holds. \( \Box \)

**Lemma 3.5.** The associated homomorphism \( \tilde{f}^j : \mathcal{X}^j_{G,O_L} \rightarrow \mathcal{D}^{r,j}_{O_L} \) is finite flat. Moreover, there exists an exact sequence of \( \pi_L \)-adic formal group schemes

\[
(1) \quad 0 \rightarrow \mathcal{H}^j_{O_L} \rightarrow \mathcal{X}^j_{G,O_L} \rightarrow \mathcal{D}^{r,j}_{O_L} \rightarrow 0.
\]

**Proof.** From Lemma 3.3, the associated affinoid map \( f^j : \mathcal{X}^j_G \rightarrow \mathcal{D}^{r,j} \) is finite etale. Let \( \mathcal{B}^j_K \) and \( \mathcal{A}^j_K \) be their affinoid algebras as above. Since \( \mathcal{D}^{r,j} \) is integral, we see that \( f^j \) is surjective and the ring homomorphism \( \mathcal{A}^j_K \rightarrow \mathcal{B}^j_K \) is injective. Thus we have an injection \( \mathcal{A}^j_{O_L} \rightarrow \mathcal{B}^j_{O_L} \), which is finite by [6, Corollary 6.4.1/6]. Hence \( \tilde{f}^j : \mathcal{X}^j_G \rightarrow \mathcal{D}^{r,j} \) is a surjective homomorphism of algebraic groups over \( \bar{F} \). Since \( \mathcal{X}^j_G \) and \( \mathcal{D}^{r,j} \) are regular, we see that \( f^j \) is faithfully flat by [13, Theorem 23.1]. Since \( \mathcal{A}^j_{O_L} \) and \( \mathcal{B}^j_{O_L} \) are \( \pi_L \)-torsion free, the map \( \tilde{f}^j \) is flat by the local criterion of flatness. Put \( \mathcal{H}' = \text{Ker}(\tilde{f}^j) \). This is a closed \( \pi_L \)-adic formal subgroup scheme of \( \mathcal{X}^j_{G,O_L} \) and can be regarded also as a finite flat group scheme over \( O_L \). Passing to the generic fiber, we see that \( \mathcal{H}^j_{O_L} \) is a closed subgroup scheme of \( \mathcal{H}' \). Comparing these ranks concludes the lemma.
From Lemma 3.3, we see that for \( j > c = c(G) \), the map \( \hat{f}^j \) identifies \( X^j_{G,O_L} \) with the direct sum of finitely many copies of \( D^{r,j} \). More precisely, we have the following.

**Lemma 3.6.** Let \( c = c(G) \) be the conductor of \( G \). Then the associated homomorphism \( \hat{f}^j : X^j_{G,O_L} \to D^{r,j} \) is finite etale if and only if \( j \geq c \).

**Proof.** Let \( sp_j : X^j_{G,K} \to X^j_G \) be the specialization map. By Lemma 3.3, we see that \( \hat{f}^c \) is finite etale at \( sp_c(x) \) for any \( x \in G(K) \) as in the proof of [2, Theorem 7.2], using a theorem of Bosch ([7, Lemma 2.1]). We conclude the etaleness of \( \hat{f}^c \) by the existence of the group structure. We have \( \Omega^1_{B_{O_L}/A_{O_L}} \otimes_{O_L} F = 0 \). Since \( B_{O_L} \) is \( \pi_L \)-adic complete and Noetherian, we see that \( \Omega^1_{B_{O_L}/A_{O_L}} = 0 \) and the homomorphism \( \hat{f}^c : X^c_{G,O_L} \to D^{r,c} \) is finite etale.

Let \( \bar{0} \) be the zero section of \( \bar{D}^{r,j} \) and set \( X^j_G = \cup_{j' > j} X^j_G \). Then we have \( (\bar{f}^j)^{-1}(\bar{0}) = sp_j(X^j_G) = sp_j(G(K)) \). If \( \bar{f}^j \) is etale, then \( \sharp(\bar{f}^j)^{-1}(\bar{0}) \) equals the degree of \( \bar{f}^j \), namely \( \sharp(G(K)) \). Thus \( X^j_G \) splits and we have \( j \geq c \).

\[\square\]

4. **Ramification and the \( I_K \)-module structure of a finite flat group scheme**

Consider the right action of \( I_K \) on \( K \) defined by \( \sigma, z = \sigma^{-1}(z) \) for \( \sigma \in I_K \). This action induces a \( K \)-semilinear left action of \( I_K \) on \( X^j_{G,K} = X^j_G \times_K K \), which also extends to an \( O_K \)-semilinear action on its stable normalized integral model \( X^j_{G,O_K} \). Thus we have an \( F \)-linear left action of \( I_K \) on its closed fiber \( X^j_G \). We call this the geometric monodromy action of \( I_K \) and write the action of \( \sigma \in I_K \) as \( \sigma_{\text{geom}} \) (note that here we follow the terminology in [2], and not in [3] where the monodromy action is called arithmetic, since in our case there is no "geometric" action other than the monodromy action). Similarly, we have the geometric monodromy action of \( I_K \) on \( \bar{D}^{r,j} \).

The latter action is described as follows. Let the additive group (resp. multiplicative group) over \( \bar{F} \) be denoted by \( \mathbb{G}_a \) (resp. \( \mathbb{G}_m \)). Consider the left action \( \mathbb{G}_m \times \mathbb{G}_a \to \mathbb{G}_a \) given by the multiplication. Write this action of \( \lambda \in \bar{F}^\times \) as \( [\lambda] \). This action is defined by \( T_i \mapsto \lambda T_i \), where \( \mathbb{G}_a = \text{Spec}(\bar{F}[T_1, \ldots, T_r]) \). For \( j \in \mathbb{Q}_{>0} \), we define the fundamental character \( \theta_j : I_K \to \bar{F}^\times \) to be \( \theta_{p^j} \), where \( k'/l' \) is the prime-to-\( p \)-denominator part
Thus diagram whose rows are exact. Note that, for $j = k/l$ and $l = p^ml_0$ with $(k, l) = 1$ and $p 
mid l_0$, we have $\theta_j = \theta_j^{kp-m}$.

**Lemma 4.1.** The algebraic group $\tilde{D}^{r,j}$ is equal to $\tilde{G}^r_A$. For $\sigma \in I_K$, the geometric monodromy action $\sigma_{\text{geom}}$ on $\tilde{D}^{r,j}$ coincides with the multiplication $[\theta_j(\sigma)]$.

**Proof.** Put $A = \mathcal{O}_K[[T_1, \ldots, T_r]]$ and $j = k/l$ with $(k, l) = 1$. Let $L$ be a finite Galois extension of $K$ containing $\pi^{1/l}$ and $e' = e(L/K)$ be its ramification index over $K$. Then $e'k/l \in \mathbb{Z}$ and the stable normalized integral model of $D^{r,j}$ over $\mathcal{O}_L$ is $\mathcal{O}_L(T_1/(\pi_L)^{e'k/l}, \ldots, T_r/(\pi_L)^{e'k/l}) = \mathcal{O}_L(W_1, \ldots, W_r)$, where $W_i = T_i/(\pi^{1/l})^k$. Set $\mu_A$ to be the coproduct of $A$. We have

$$\mu_A(T_i) = T_i \otimes 1 + 1 \otimes T_i + \text{(higher degree)}$$

and then $\mu_A(W_i) = \mu_A((\pi^{1/l})^k W_i)/(\pi^{1/l})^k$ is equal to

$$W_i \otimes_\pi 1 + 1 \otimes_\pi W_i + (\pi^{1/l})^k \text{(higher degree)}$$

in this $\mathcal{O}_L$-algebra. This shows the first assertion.

On the other hand, we know the geometric monodromy action on $\tilde{D}^{r,j}$ is tame ([2, Lemma 7.7]). Write $l = p^ml_0$ with $p \nmid l_0$. Then for $\sigma \in I_K$, we have $\sigma((\pi^{1/l})^k)/(\pi^{1/l})^k \equiv \theta_l(\sigma)^{kp-m} \zeta_{p^m}$ mod $m_K$ with some $N$ and this is equal to $\theta_j(\sigma)$. Thus the action on the affine algebra $F[W_1, \ldots, W_r]$ of $\tilde{D}^{r,j}$ is given by $\sigma^\ast_{\text{geom}}(W_i) = \theta_j(\sigma)W_i$. This coincides with $[\theta_j(\sigma)]$. \hfill \Box

Next we consider the geometric monodromy action on $\tilde{X}_G^j$. Let $\tilde{X}_G^{j,0}$ denote the unit component of the algebraic group $\tilde{X}_G^j$ and $\mathcal{H}_G^j$ be the geometric closed fiber of $\mathcal{H}_G^j$. We begin with the following lemma.

**Lemma 4.2.** If $\psi \in \text{End}(\tilde{X}_G^{j,0})$ induces the zero map on $\tilde{D}^{r,j}$, then $\psi = 0$.

**Proof.** Put $\mathcal{H}_G^j = \mathcal{H}_G^j \cap \tilde{X}_G^{j,0}$. This is the kernel of the faithfully flat map $\tilde{X}_G^{j,0} \to \tilde{D}^{r,j}$ and by assumption we have the following commutative diagram whose rows are exact.

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{H}_G^j & \longrightarrow & \tilde{X}_G^{j,0} & \longrightarrow & \tilde{D}^{r,j} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{H}_0^j & \longrightarrow & \tilde{X}_G^{j,0} & \longrightarrow & \tilde{D}^{r,j} & \longrightarrow & 0
\end{array}
$$

Thus $\psi$ factors through $\mathcal{H}_G^j$. Put $\mathcal{C} = \text{Im}(\psi)$. Then this is a closed subgroup scheme of $\mathcal{H}_0^j$ and the map $\tilde{X}_G^{j,0} \to \mathcal{C}$ is faithfully flat. Since $\tilde{X}_G^{j,0}$
is regular and connected, we see that $\tilde{C}$ is also regular and connected by [13, Theorem 23.7]. Hence $\tilde{C} = 0$ and we have $\psi = 0$.

\begin{corollary}
Let $G$ be a connected finite flat group scheme over $\mathcal{O}_K$. Take a formal resolution $(G \to \Gamma)$ of dimension $r$. Then the algebraic group $\tilde{X}_G^{1,0}$ is isomorphic to $\tilde{G}^r$.
\end{corollary}

\begin{proof}
By the previous lemma and Lemma 4.1, we see that $\tilde{X}_G^{1,0}$ is killed by $p$. Hence the assertion follows from [12, Lemma 1.7.1].
\end{proof}

\begin{corollary}
The geometric monodromy action of $I_K$ on $\tilde{X}_G^{1,0}$ is tame.
\end{corollary}

\begin{proof}
For an element $\sigma$ of the wild inertia subgroup $P_K$, the geometric monodromy action $\sigma_{\text{geom}}$ on $\tilde{D}^{r,j}$ is trivial. Applying the lemma to $\sigma_{\text{geom}} - \text{id} \in \text{End}(\tilde{X}_G^{1,0})$ shows the assertion.
\end{proof}

\begin{corollary}
Let $J$ be a finite cyclic quotient of $I_K$ through which the tame character $\theta_j$ factors and $\tau$ be a generator of $J$. Let $F(t)$ denote the minimal polynomial of $\theta_j(\tau) \in \bar{\mathbb{F}}_p$ over $\mathbb{F}_p$. Then the geometric monodromy action of $I_K$ on $\tilde{X}_G^{1,0}$ also factors through $J$ and the equation $F(\tau_{\text{geom}}) = 0$ holds in $\text{End}(\tilde{X}_G^{1,0})$.
\end{corollary}

\begin{proof}
The first assertion follows from Lemma 4.2 as the previous corollary. The second assertion also follows from this lemma using Lemma 4.1.
\end{proof}

Let $c = c(G)$ be the conductor of $G$. The lemma below enables us to realize $G^c(\bar{K})$ as a subgroup of $\tilde{X}_G^{1,0}$.

\begin{lemma}
The specialization map $\text{sp}_c : X_{G,K}^c \to \tilde{X}_G^c$ induces an $I_K$-equivariant isomorphism $G(\bar{K}) \to \mathcal{H}^c(\bar{F})$ and $G^c(\bar{K}) \to \mathcal{H}^c_0(\bar{F})$. Here we consider on the left-hand side the natural action as the $K$-valued points of $G$ (resp. $G^c$) and on the right-hand side the restriction of the geometric monodromy action on $X_G^c$.
\end{lemma}

\begin{proof}
By definition, the generic fiber of $\mathcal{H}^c_{\mathcal{O}_L}$ is equal to $G_L$. From the exact sequence (1) and Lemma 3.6, we know that $\mathcal{H}^c_{\mathcal{O}_L}$ is etale over $\mathcal{O}_L$ and there is the following exact sequence of algebraic groups over $\bar{F}$.

\begin{equation}
0 \to \mathcal{H}^c \to \tilde{X}_G^c \to \tilde{D}^{r,j} \to 0
\end{equation}

Thus we have a natural isomorphism $\mathcal{H}^c_{\mathcal{O}_L}(\bar{K}) \to \mathcal{H}^c(\bar{F})$ and the composite $G(\bar{K}) = \mathcal{H}^c_{\mathcal{O}_L}(\bar{K}) \to \mathcal{H}^c(\bar{F}) \to \tilde{X}_G^c(\bar{F})$ coincides with the map...
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From [2, Corollary 4.4], we see that this map sends $G^c(\bar{K})$ isomorphically onto $H^0_c(F)$. For $x \in X^c_0(\bar{K})$ and $\sigma \in I_K$, let $\sigma(x)$ denote the natural action of $\sigma$ on $\bar{K}$-valued points. Then we have $\sigma_{\text{geom}}(x) \circ \sigma = \sigma(x)$. Taking its specialization shows the $I_K$-equivariance.

The following theorem can be regarded as a generalization for a finite flat group scheme over $\mathcal{O}_K$ of the structure theorem of the graded piece of the classical upper numbering ramification filtration.

**Theorem 4.7.** Let $G$ be a finite flat group scheme over $\mathcal{O}_K$ and $j \in \mathbb{Q}_{>0}$. Then the $G_K$-module $G^j(\bar{K})/G^{j+}(\bar{K})$ is tame and killed by $p$.

**Proof.** Since $G^j = (G^0)^j$, where $G^0$ denotes the unit component of $G$, we may assume $G$ is connected. Suppose that $j$ is a jump of the ramification filtration on $G$ and consider the quotient $G^j(\bar{K}) \subseteq G(\bar{K})$ has a non-trivial image in $(G/G^{j+})(\bar{K})$. By the Herbrand theorem ([1, Lemme 2.10]), the natural map $G^t(\bar{K}) \rightarrow (G/G^{j+})^t(\bar{K})$ is surjective for any $t > 0$. We have $(G/G^{j+})^t = 0$ for $t > j$ and $(G/G^{j+})^j \neq 0$. Thus the ramification filtration on $G/G^{j+}$ jumps at $j$ and $(G/G^{j+})^j(\bar{K}) = G^j(\bar{K})/G^{j+}(\bar{K})$. Replacing $G$ with $G/G^{j+}$, we may assume $j = c = c(G)$.

Take a formal resolution $(G \rightarrow \Gamma)$ of dimension $r$ and consider its associated affinoid homomorphism $X^c_0 \rightarrow D^{nc}$. Then the theorem follows from Corollary 4.3, Corollary 4.4, and Lemma 4.6.

From this theorem, we see that the inertia subgroup $I_K$ acts on $G^j(\bar{K})/G^{j+}(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ by the direct sum of tame characters. The theorem below determines these characters up to $p$-power exponent.

**Theorem 4.8.** Let $G$ be a finite flat group scheme over $\mathcal{O}_K$ and $j \in \mathbb{Q}_{>0}$. Then $I_K$ acts on $G^j(\bar{K})/G^{j+}(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ by the direct sum of $\mathbb{F}_p$-conjugates of the fundamental character $\theta_j$.

**Proof.** We may assume that $G$ is connected. The same argument as in the proof of Theorem 4.7 using the Herbrand theorem reduces the claim to the case where $j = c = c(G)$. Take a formal resolution $(G \rightarrow \Gamma)$ of dimension $r$. Let $J$ and $\tau$ be as in Corollary 4.5. Then Corollary 4.5 and Lemma 4.6 show that every eigenvalue of the action of $\tau_{\text{geom}}$ on the finite dimensional $\mathbb{F}_p$-vector space $G^c(\bar{K})$ is a conjugate of $\theta_j(\tau)$ over $\mathbb{F}_p$. Since the order of $J$ is prime to $p$, we conclude that $I_K$ acts on $G^c(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ by the direct sum of $\mathbb{F}_p$-conjugates of $\theta_c$. 


Corollary 4.9. Let $G$ be a finite flat group scheme over $\mathcal{O}_K$. Then the order of the image of the homomorphism $I_K \to \text{Aut}(G(\bar{K}))$ is a power of $p$ if and only if every jump $j$ of the ramification filtration $\{G^j\}_{j \in \mathbb{Q}_{>0}}$ is an element of $\mathbb{Z}[1/p]$.

Proof. From Theorem 4.8, we see that the jumps are in $\mathbb{Z}[1/p]$ if and only if the graded pieces $G^j(\bar{K})/G^{j+}(\bar{K})$ are unramified. By Theorem 4.7, this is equivalent to the condition that $\sharp \text{Im}(I_K \to \text{Aut}(G(\bar{K})))$ is a $p$-power.

When $G(\bar{K})$ is unramified and killed by $p$, we have the following reinforcement of this result, which is an easy corollary of the Herbrand theorem. The author does not know if every jump is an integer whenever $G(\bar{K})$ is unramified. If $G$ is monogenic, then we see that this holds true from [11, Theorem 4].

Proposition 4.10. Let $G$ be a finite flat group scheme over $\mathcal{O}_K$ which is killed by $p$. Suppose that the $G_K$-module $G(\bar{K})$ is unramified. Then every jump $j$ of the ramification filtration $\{G^j\}_{j \in \mathbb{Q}_{>0}}$ is an element of $p\mathbb{Z}$.

Proof. We may assume $K = K^{nr}$ and $G_K$ acts trivially on $G(\bar{K})$. There is a quotient $W$ of $G(\bar{K})/G^{j+}(\bar{K})$ where $G^j(\bar{K})$ has a non-trivial image and of rank one over $\mathbb{F}_p$. Taking the schematic closure, $W$ extends to a finite flat group scheme $\mathcal{W}$ over $\mathcal{O}_K$ which is a quotient of $G/G^{j+}$. By the Herbrand theorem, we see that the ramification filtration of $\mathcal{W}$ jumps at $j$. On the other hand, $W$ is a Raynaud $\mathbb{F}_p$-vector space scheme with unramified generic fiber. Thus the proposition follows from [11, Theorem 4].

For the rest of this section, we state some corollaries in the case where $G$ is an $\mathbb{F}$-vector space scheme of rank one or two for a finite extension $\mathbb{F}$ over $\mathbb{F}_p$. In the case of rank one, Theorem 4.8 directly shows the claim below, while this can be shown also as a corollary of a determination of the conductor of a Raynaud $\mathbb{F}$-vector space scheme (Theorem 5.5).

Corollary 4.11. Let $G$ be an $\mathbb{F}$-vector space scheme of rank one over $\mathcal{O}_K$ and $c = c(G)$. Then the $I_K$-action on the $\mathbb{F}$-vector space $G(\bar{K})$ of rank one is given by the character $\theta^p_n$ for some $n$.

In the case of rank two, we have the following.

Corollary 4.12. Let $G$ be a finite flat $\mathbb{F}$-vector space scheme of rank two over $\mathcal{O}_K$ and $c = c(G)$. Then the $I_K$-module $G(\bar{K}) \otimes_{\mathcal{O}_K} \mathbb{F}_p$ contains
the character $\theta^n_c$ for some $n$. If the $G_K$-module $\mathcal{G}(\bar{K})$ is reducible, this holds true for $\mathcal{G}(\bar{K})$ itself.

**Proof.** The first assertion follows easily from Theorem 4.8 and the surjection $\mathcal{G}^c(\bar{K}) \otimes_{\mathbb{F}_p} \mathbb{F}_p \to \mathcal{G}^c(\bar{K}) \otimes_{\mathbb{F}_p} \mathbb{F}_p$. Suppose the $I_K$-module $\mathcal{G}(\bar{K})$ is reducible. When $G^c$ is of rank one, the assertion is clear from Theorem 4.8. If $G^c = G$, then $G^c$ is reducible and the assertion follows also from Theorem 4.8.

The corollary below indicates that the conductor $c(\mathcal{G})$ carries information about not only the tame characters but also their extension structures in the $I_K$-module $\mathcal{G}(\bar{K})$.

**Corollary 4.13.** Consider an exact sequence of finite flat $\mathbb{F}$-vector space schemes over $\mathcal{O}_K$

$$0 \to \mathcal{G}_1 \to \mathcal{G} \to \mathcal{G}_2 \to 0$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are connected of rank one. If $c(\mathcal{G}) = c(\mathcal{G}_2)$, then the $I_K$-module $\mathcal{G}(\bar{K})$ splits.

**Proof.** Put $c = c(\mathcal{G})$. Take a formal resolution $(\mathcal{G} \to \Gamma)$ of dimension $r$ and put $\Gamma_2 = \Gamma/\mathcal{G}_1$. Then we get a finite flat map of formal resolutions

$$\begin{array}{ccc}
\mathcal{G} & \longrightarrow & \Gamma \\
\downarrow & & \downarrow \\
\mathcal{G}_2 & \longrightarrow & \Gamma_2.
\end{array}$$

Therefore we have a finite flat homomorphism of rigid $K$-analytic groups $X^i(\mathcal{G} \to \Gamma) \to X^i(\mathcal{G}_2 \to \Gamma_2)$ by Corollary 3.2. As in the proof of Lemma 3.3, we see that this map is finite etale.

Suppose $\mathcal{G}^c(\bar{K})$ is of rank one. If $\mathcal{G}^c(\bar{K}) \neq \mathcal{G}_1(\bar{K})$ as an $\mathbb{F}$-subspace of $\mathcal{G}(\bar{K})$, the $I_K$-module $\mathcal{G}(\bar{K})$ splits and the proposition follows. Suppose $\mathcal{G}^c(\bar{K}) = \mathcal{G}_1(\bar{K})$. The affinoid variety $X^c(\mathcal{G} \to \Gamma)$ decomposes to $rF$ components over some finite extension $K'$ of $K$. Each component is a Zariski open and closed subset of $X^c(\mathcal{G} \to \Gamma)_{K'}$. As the map $f : X^c(\mathcal{G} \to \Gamma)_{K'} \to X^c(\mathcal{G}_2 \to \Gamma_2)_{K'}$ is finite etale and $X^c(\mathcal{G}_2 \to \Gamma_2)_{K'}$ is connected, every component $X^{ci}(\mathcal{G} \to \Gamma)_{K'}$ maps surjectively to $X^c(\mathcal{G}_2 \to \Gamma_2)_{K'}$. Take some $g_i \in \mathcal{G}^c(\bar{K}) \cap X^{ci}(\mathcal{G} \to \Gamma)_{K'}$. Using the group structure, we see that $\mathcal{G}(\bar{K}) \cap X^{ci}(\mathcal{G} \to \Gamma)_{K'} = g_i + \mathcal{G}^c(\bar{K}) = g_i + \mathcal{G}_1(\bar{K})$ and $f(\mathcal{G}(\bar{K}) \cap X^{ci}(\mathcal{G} \to \Gamma)_{K'}) = f(g_i)$. However, we have $f^{-1}(\mathcal{G}_2(\bar{K})) = \mathcal{G}(\bar{K})$ and thus $f(\mathcal{G}(\bar{K}) \cap X^{ci}(\mathcal{G} \to \Gamma)_{K'}) = \mathcal{G}_2(\bar{K})$. This is a contradiction. Therefore we may assume $\mathcal{G}^c(\bar{K}) = \mathcal{G}(\bar{K})$. In this case, the proposition follows from Theorem 4.7.
5. Example: Rank One Calculation

In this section, we calculate the conductor of a Raynaud $\mathbb{F}$-vector space scheme over $\mathcal{O}_K$. The point is that, as we can see from the bound of the conductor ([11, Theorem 7]), it is enough to consider the $j$-th tubular neighborhood only for $j \leq pe/(p-1) + \varepsilon$ with sufficiently small $\varepsilon > 0$. For such $j$, we can compute the tubular neighborhood easily by Lemma 5.4 below.

Let $K$ be a complete discrete valuation field of mixed characteristic $(0, p)$. We write $\pi = \pi_K$ for its uniformizer and $e$ for its absolute ramification index. We normalize a valuation $v_K$ of $K$ as $v_K(\pi) = 1$ and extend it to the algebraic closure $\bar{K}$ of $K$. For $a \in \bar{K}$ and $j \in \mathbb{R}$, let $D(a, j)$ denote the closed disc $\{ z \in \mathcal{O}_K \mid v_K(z-a) \geq j \}$. This is the underlying subset of a $K(a)$-affinoid subdomain of the unit disc over $K(a)$.

For integers $0 \leq s_1, \ldots, s_r \leq e$, let $\mathcal{G}(s_1, \ldots, s_r)$ denote the Raynaud $\mathbb{F}_p$-vector space scheme over $\mathcal{O}_K$ defined by the $r$ equations

$T_1 = \pi s_1 T_2, T_2 = \pi s_2 T_3, \ldots, T_r = \pi s_r T_1$ ([14]). We set $j_k = (ps_k + p^2 s_{k-1} + \ldots + p^k s_1 + p^{k+1} s_r + p^{k+2} s_{r-1} + \cdots + p^r s_{k+1})/(p^r - 1)$. Before the calculation of $c(\mathcal{G}(s_1, \ldots, s_r))$, we gather some elementary lemmas.

**Lemma 5.1.** Let $a \in \mathcal{O}_K$ and $s = v_K(a)$. Then the affinoid variety $X^j(K) = \{ x \in \mathcal{O}_K \mid v_K(x^p - a) \geq j \}$ is equal to

$$
\{ D(a^{1/p}, j/p) \cap \prod_{i=0}^{p-1} D(a^{1/p} \zeta_{p}^i, j - e - (p-1)s/p) \mid j \geq s + pe/(p-1) \}.
$$

**Proof.** We have $v_K(x^p - a) = \sum_i v_K(x - a^{1/p} \zeta_{p}^i)$. If $v_K(x - a^{1/p} \zeta_{p}^i) \geq v_K(x - a^{1/p} \zeta_{p}^{i'})$ for any $i' \neq i$, then $v_K(x - a^{1/p} \zeta_{p}^{i'}) \leq v_K(a^{1/p} \zeta_{p}^{i'}(1 - \zeta_{p}^{i'})) = s/p + e/(p-1)$. Thus we have

$$
v_K(x - a^{1/p} \zeta_{p}^i) \geq \sup(j/p, j - (p-1)s/p - e)
$$

and

$$
X^j(K) \subseteq \bigcup_i D(a^{1/p} \zeta_{p}^i, \sup(j/p, j - (p-1)s/p - e)).
$$

Suppose that $j/p \geq j - (p-1)s/p - e$. Then we have $v_K(a^{1/p}(1-\zeta_{p}^i)) = s/p + e/(p-1) \geq j/p$, $D(a^{1/p}, j/p) = D(a^{1/p} \zeta_{p}^i, j/p)$ for any $i$. Thus

$$
X^j(K) = D(a^{1/p}, j/p).
$$

When $j/p < j - (p-1)s/p - e$, we have $v_K(a^{1/p}(1-\zeta_{p}^i)) = s/p + e/(p-1) < j - (p-1)s/p - e$. This means that if $w \in D(a^{1/p} \zeta_{p}^{i'}, j - (p-1)s/p - e)$ for some $i$, then $v_K(w - a^{1/p} \zeta_{p}^{i'}) < j - (p-1)s/p - e$ for any other $i'$. 


Thus the discs \( D(a^{1/p}c_i^{j}, j - (p - 1)s/p - e) \) are disjoint and
\[
X^j(\bar{K}) = \prod_i D(a^{1/p}c_i^{j}, j - (p - 1)s/p - e).
\]

These are equalities of the underlying sets of affinoid subdomains of the unit disc over \( K(a^{1/p}, \zeta_p) \). By the universality of an affinoid subdomain, this extends to an isomorphism of affinoid varieties.

\[\square\]

We can prove the following lemma just in the same way.

**Lemma 5.2.** The affinoid variety \( \{x \in \mathcal{O}_K \mid v_K(x^{p^r} - ax) \geq j\} \) is equal to
\[
\begin{cases}
D(0, j/p^r) & \text{if } j \leq p^r v(a)/(p^r - 1), \\
\prod_{i=0}^{r-1} D(\sigma_i, j - v(a)) & \text{if } j > p^r v(a)/(p^r - 1),
\end{cases}
\]
where \( \sigma_i \)'s are the roots of \( X^{p^r} = ax \).

**Lemma 5.3.** For \( g_1(Y_1, \ldots, Y_d), g_2(Y_1, \ldots, Y_d) \in K[Y_1, \ldots, Y_d] \) and \( j_1 \geq j_2 \), the affinoid variety \( \{(x, y_1, \ldots, y_d) \in \mathcal{O}_K \times \mathcal{O}_K^d \mid v_K(x - g_1(y_1, \ldots, y_d)) \geq j_1, v_K(x - g_2(y_1, \ldots, y_d)) \geq j_2\} \) is equal to \( \{(x, y_1, \ldots, y_d) \in \mathcal{O}_K \times \mathcal{O}_K^d \mid v_K(x - g_1(y_1, \ldots, y_d)) \geq j_1, v_K(g_1(y_1, \ldots, y_d) - g_2(y_1, \ldots, y_d)) \geq j_2\} \).

**Proof.** For fixed \( (x, y) \), these two conditions are equivalent. The universality of an affinoid subdomain proves the lemma.

\[\square\]

**Lemma 5.4.** Let \( a \in \mathcal{O}_K \) and \( s = v_K(a) \). If \( j \leq pe/(p - 1) + s \), then the affinoid variety \( X^j(\bar{K}) = \{(x, y) \in \mathcal{O}_K \times \mathcal{O}_K \mid v_K(x^{p^r} - ay^{p^r}) \geq j\} \) is equal to \( \{(x, y) \in \mathcal{O}_K \times \mathcal{O}_K \mid v_K(x - a^{1/p}y^{p^{n-1}}) \geq j/p\} \).

**Proof.** Lemma 5.1 shows that the fiber of the second projection \( X^j(\bar{K}) \to \mathcal{O}_K \) at \( y \) is equal to
\[
\begin{cases}
D(a^{1/p}y^{p^{n-1}}, j/p) & \text{if } j \leq s + p^{n-1}v_K(y) + pe/(p - 1), \\
\prod_{i=0}^{p-1} D(a^{1/p}c_i^{j}y^{p^{n-1}}, j - e - (p - 1)(s + p^{n-1}v_K(y))/p) & \text{otherwise}.
\end{cases}
\]

Thus we have \( X^j(\bar{K}) = \{(x, y) \in \mathcal{O}_K \times \mathcal{O}_K \mid v_K(x - a^{1/p}y^{p^{n-1}}) \geq j/p\} \) for \( j \leq pe/(p - 1) + s \). This is the underlying set of a \( K(a^{1/p}) \)-affinoid variety. Again this equality extends to an isomorphism over \( \bar{K}(a^{1/p}) \).

\[\square\]

Now we proceed to the proof of the main theorem of this section.

**Theorem 5.5.** \( c(\mathcal{G}(s_1, \ldots, s_r)) = \sup_k j_k \).
Lemma 5.6. If $X$ is a variety, then $p$ divides $s_i$, $s_1$ for $i < j$, where $j$ is the supremum of $j_i$'s. If $j = 0$, then $G(s_1, \ldots, s_r)$ is etale and $c(G(s_1, \ldots, s_r)) = 0$. Thus we may assume $j > 0$. Consider the homomorphism of $O_K$-algebras

\[
A = O_K[T_1, \ldots, T_r]/(T_1^p - \pi s_1 T_2, \ldots, T_r^p - \pi s_r T_1) \rightarrow \quad B = O_K[W, T_2, \ldots, T_r]/(W^p - \pi s_1 T_2, T_2^p - \pi s_2 T_3, \ldots, T_{r-1}^p - \pi s_{r-1} T_r, T_r^p - \pi s_r W^{p-1}),
\]

defined by $T_1 \mapsto W^{p-1}$. This induces a surjection of $K$-affinoid varieties

\[
X_B^r(K) \ni (w, t_2, \ldots, t_r) \mapsto (w^{p-1}, t_2, \ldots, t_r) \in X_A^r(K),
\]

where

\[
X_A^r(K) = \{(t_1, \ldots, t_r) \in O_K^r \mid v_K(t_1^p - \pi s_1 t_2) \geq j, \ldots, v_K(t_{r-1}^p - \pi s_{r-1} t_r) \geq j, v_K(t_r^p - \pi s_r) \geq j \}
\]

and

\[
X_B^r(K) = \{(w, t_2, \ldots, t_r) \in O_K^r \mid v_K(w^p - \pi s_1 t_2) \geq j, v_K(t_2^p - \pi s_2 t_3) \geq j, \ldots, v_K(t_r^p - \pi s_r w^{p-1}) \geq j \}.
\]

These are affinoid subdomains of the $r$-dimensional unit polydisc over $K$. We calculate a jump of $\{F^j(B)\}_j \in \mathbb{Q}_{>0}$ at first.

**Lemma 5.6.** If $j \leq j_i$, then the first jump of $\{F^j(B)\}_j \in \mathbb{Q}_{>0}$ occurs at $j = j_i$ and $\# F^{j_i}(B) = p^j$.

Note that the base change from $K$ to a finite extension $L$ multiplies $s_i$, $j_i$ and $e$ by the ramification index of $L/K$. Thus, to prove Lemma 5.6 and Theorem 5.5, we may assume that $p^{r-1}$ divides $s_i$ and $e$.

**Proof.** Consider the $K$-affinoid variety $X_B^r(K)$ for $j \leq j_i$. Then the iterative use of Lemma 5.4 and Lemma 5.3 shows that the affinoid variety $X_B^r(K)$ is equal to

\[
\{v_K(w^p - \pi^{s_1+p s_{r-1}+\ldots+p^{r-1} s_1}/p^{r-1} w) \geq p l_1(j), v_K(t_2 - g_2(w)) \geq u_2, v_K(t_3 - g_3(t_2, w)) \geq u_3, \ldots, v_K(t_r - g_r(t_{r-1}, w)) \geq u_r \},
\]

where $l_i(j), g_i(t_{i-1}, w), g_2(w)$ and $u_i$ are defined as follows:

- $l_j(j) = j/p$,
- $l_{i-1}(j) = \text{inf}(j, l_i(j) + s_{i-1})/p$,
- $g_i(t_{i-1}, w) = p_{i-1}^i / p s_{i-1}$ and $u_i = j - s_{i-1}$ if $j \geq l_i(j) + s_{i-1}$,
- $g_i(t_{i-1}, w) = \pi^{s_1+p s_{r-1}+\ldots+p^{r-1} s_1}/p^{r+1}$ and $u_i = l_i(j)$ if $j < l_i(j) + s_{i-1}$,
Note that $l_1(j)$ is a strictly monotone increasing function of $j$. This affinoid variety is isomorphic to the product of the affinoid variety $\{ w \in \mathcal{O}_F | v(w^{p^r} - \pi(s_r + ps_{r-1} + \ldots + p^{r-1}s_1))/p^{r-1}w) \geq pl_1(j) \}$ and discs. Therefore, from Lemma 5.2, we see that the first jump of $\{ F^j(B) \}_{j \in \mathbb{Q}_{>0}}$ occurs at $j$ such that $pl_1(j) = j_r$, provided this $j$ satisfies $0 < j < pe/(p-1)$. Moreover, then we have $\sharp F^j(B) = p^r$. Thus the following lemma and the strict monotonicity of $l_1$ terminate the proof of Lemma 5.6.

**Lemma 5.7.** $l_1(j_r) = j_r/p$.

**Proof.** Suppose that there is $k$ such that $l_k(j_r) = j_r/p$ and $j_r \geq l_k'(j_r) + s_k$ for any $1 < k' \leq k$. Then we have $l_1(j_r) = \inf(j_r, (j_r + ps_{k-1} + p^2s_{k-2} + \ldots + p^{k-1}s_1))/p^{k-1})/p$ and the assumption $j_{k-1} \leq j_r$ implies $l_1(j_r) = j_r/p$.

On the other hand, let $s = (s_r + ps_{r-1} + \ldots + p^{r-1}s_1)/p^{r-1}$ and $\sigma_0, \ldots, \sigma_{p^r-1}$ be the roots of the equation $X^{p^r} - \pi X = 0$. Then we see that the images by $w \mapsto w^{p^r-1}$ of the discs $D(\sigma_i, pl_1(j) - s)$ are disjoint for $j > j_r$. Hence the surjection $\pi_0(X^p_{\mathbb{Q}}) \to \pi_0(X_{\mathbb{Q}})$ is bijective for $0 < j \leq pe/(p-1)$ and the first (and the last) jump of $\{ F^j(A) \}_{j \in \mathbb{Q}_{>0}}$ also occurs at $j_r$, provided $j_r < pe/(p-1)$.

When $j_r = pe/(p-1)$, we see that $s_k = e > 0$ for any $k$. Thus we can use Lemma 5.4 for $j < pe/(p-1)+\varepsilon$ with sufficiently small $\varepsilon > 0$. Then, by the same reasoning as above, we conclude that $c(A) = pe/(p-1)$.

**References**


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