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# TAME CHARACTERS AND RAMIFICATION OF FINITE FLAT GROUP SCHEMES

SHIN HATTORI

## 1. INTRODUCTION

Let  $K$  be a complete discrete valuation field of mixed characteristic  $(0, p)$  with residue field  $F$  which may be imperfect,  $G_K$  be its absolute Galois group and  $I_K$  be its inertia subgroup. Let  $\mathcal{G}$  be a finite flat group scheme over  $\mathcal{O}_K$ . When  $\mathcal{G}$  is monogenic, that is to say, when the affine algebra of  $\mathcal{G}$  is generated over  $\mathcal{O}_K$  by one element, it is well-known that the tame characters appearing in the  $I_K$ -module  $\mathcal{G}(\bar{K})$  are determined by the slopes of the Newton polygon of a defining equation of  $\mathcal{G}$  ([15, Proposition 10]).

On the other hand, for an elliptic modular form  $f$  of level  $N$  prime to  $p$ , we also have a description of the tame characters of the associated mod  $p$  Galois representation  $\bar{\rho}_f$  ([10, Theorem 2.5, Theorem 2.6], [9, Section 4.3]). This is based on Raynaud's theory of prolongations of finite flat group schemes or the integral  $p$ -adic Hodge theory. However, for an analogous study of the associated mod  $p$  Galois representation of a Hilbert modular form over a totally real number field, we encounter a local field of higher absolute ramification index. In this case, these two theories no longer work well and we need some other techniques.

In this paper, to propose a new approach toward this problem, we generalize the aforementioned result of [15] to the higher dimensional case using the ramification theory of Abbes and Saito ([2], [3]) and determine the tame characters appearing in  $\mathcal{G}(\bar{K})$  in terms of the ramification of  $\mathcal{G}$  without any restriction on the absolute ramification index of  $K$ . Namely, we show the following theorem.

**Theorem 1.1.** *Let  $\mathcal{G}$  be a finite flat group scheme over  $\mathcal{O}_K$ . Write  $\{\mathcal{G}^j\}_{j \in \mathbb{Q}_{>0}}$  for the ramification filtration of  $\mathcal{G}$  in the sense of [2] and [3]. Then the graded piece  $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K})$  is killed by  $p$  and the  $I_K$ -module  $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$  is the direct sum of  $\mathbb{F}_p$ -conjugates of the fundamental character  $\theta_j$  of level  $j$ .*

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In view of the following corollary, we can regard this theorem as a counterpart for finite flat group schemes of the Hasse-Arf theorem in the classical ramification theory.

**Corollary 1.2.** *Let  $L$  be an abelian extension of  $K$ . Suppose that its integer ring  $\mathcal{O}_L$  is a  $\mathcal{G}$ -torsor over  $\mathcal{O}_K$ . Then the denominator of every jump of the upper numbering ramification filtration  $\{\mathrm{Gal}(L/K)^j\}_{j \in \mathbb{Q}_{>0}}$  ([2]) is a power of  $p$ .*

To prove the main theorem, we firstly show that the tubular neighborhood of  $\mathcal{G}$  can be chosen to have a rigid analytic group structure. Passing to the closed fiber, we realize the graded piece of the ramification filtration of  $\mathcal{G}$  as the kernel of an étale isogeny of the additive groups  $\mathbb{G}_a^r$  over  $\bar{F}$ . Then we determine the tame characters by comparing the  $I_K$ -action on the graded piece with the  $\bar{\mathbb{G}}_m$ -action on  $\mathbb{G}_a^r$  given by the multiplication. As an appendix, we also give an explicit formula for the conductor of a Raynaud  $\mathbb{F}$ -vector space scheme over  $\mathcal{O}_K$  ([14]).

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## 2. REVIEW OF THE RAMIFICATION THEORY OF ABBES AND SAITO

Let  $K$  be a complete discrete valuation field with residue field  $F$  which may be imperfect. Set  $\pi = \pi_K$  to be a uniformizer of  $K$ . The separable closure of  $K$  is denoted by  $\bar{K}$  and the absolute Galois group of  $K$  by  $G_K$ . Let  $\mathfrak{m}_{\bar{K}}$  and  $\bar{F}$  be the maximal ideal and the residue field of  $\mathcal{O}_{\bar{K}}$  respectively. In [2] and [3], Abbes and Saito defined the ramification theory of a finite flat  $\mathcal{O}_K$ -algebra of relative complete intersection. In this section, we gather the necessary definitions and briefly recall their theory.

Let  $A$  be a finite flat  $\mathcal{O}_K$ -algebra and  $\mathbb{A}$  be a complete Noetherian semi-local ring (with its topology defined by  $\mathrm{rad}(\mathbb{A})$ ) which is of formally smooth over  $\mathcal{O}_K$  and whose quotient ring  $\mathbb{A}/\mathrm{rad}(\mathbb{A})$  is of finite type over  $F$ . A surjection of  $\mathcal{O}_K$ -algebras  $\mathbb{A} \rightarrow A$  is called an embedding if  $\mathbb{A}/\mathrm{rad}(\mathbb{A}) \rightarrow A/\mathrm{rad}(A)$  is an isomorphism. For an embedding  $(\mathbb{A} \rightarrow A)$  and  $j \in \mathbb{Q}_{>0}$ , the  $j$ -th tubular neighborhood of  $(\mathbb{A} \rightarrow A)$  is the  $K$ -affinoid variety  $X^j(\mathbb{A} \rightarrow A)$  constructed as follows. Write  $j = k/l$  with  $k, l$  non-negative integers. Put  $I = \mathrm{Ker}(\mathbb{A} \rightarrow A)$  and  $\mathcal{A}_0^{k,l} = \mathbb{A}[I^l/\pi^k]^\wedge$ , where  $\wedge$  means the  $\pi$ -adic completion. Then  $\mathcal{A}_0^{k,l}$  is a quotient ring of the Tate algebra  $\mathcal{O}_K\langle T_1, \dots, T_r \rangle$  for some  $r$ . Its generic fiber  $\mathcal{A}_K^j = \mathcal{A}_0^{k,l} \otimes_{\mathcal{O}_K} K$  is independent of the choice of a representation

$j = k/l$  ([3, Lemma 1.4]) and set  $X^j(\mathbb{A} \rightarrow A) = \mathrm{Sp}(\mathcal{A}_K^j)$ . This affinoid variety is geometrically regular ([3, Lemma 1.6]).

We put  $F(A) = \mathrm{Hom}_{\mathcal{O}_K\text{-alg.}}(A, \mathcal{O}_{\bar{K}})$  and  $F^j(A) = \varprojlim \pi_0(X^j(\mathbb{A} \rightarrow A))_{\bar{K}}$ . Here  $\pi_0(X)_{\bar{K}}$  denotes the set of geometric connected components of a  $K$ -affinoid variety  $X$  and the projective limit is taken in the category of embeddings of  $A$ . Note that the projective family  $\pi_0(X^j(\mathbb{A} \rightarrow A))_{\bar{K}}$  is constant ([3, Section 1.2]). These define contravariant functors  $F$  and  $F^j$  from the category of finite flat  $\mathcal{O}_K$ -algebras to the category of finite  $G_K$ -sets. Moreover, there are morphisms of functors  $F \rightarrow F^j$  and  $F^{j'} \rightarrow F^j$  for  $j' \geq j > 0$ .

By the finiteness theorem of Grauert and Remmert ([8, Theorem 1.3]), there exists a finite separable extension  $L$  of  $K$  such that the geometric closed fiber of the unit disc  $\mathfrak{X}^j(\mathbb{A} \rightarrow A)_{\mathcal{O}_L}$  for the supremum norm in  $X^j(\mathbb{A} \rightarrow A)_L = X^j(\mathbb{A} \rightarrow A) \times_K L$  is reduced. Then for any finite separable extension  $L'$  of  $L$ , the  $\pi_L$ -adic formal scheme  $\mathfrak{X}^j(\mathbb{A} \rightarrow A)_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}$  coincides with the unit disc for the supremum norm in  $X^j(\mathbb{A} \rightarrow A)_{L'}$  and thus is normal. The  $\pi_L$ -adic formal scheme  $\mathfrak{X}^j(\mathbb{A} \rightarrow A)_{\mathcal{O}_L}$  is referred as the stable normalized integral model of  $X^j(\mathbb{A} \rightarrow A)$  over  $\mathcal{O}_L$  and its geometric closed fiber is denoted by  $\bar{X}^j(\mathbb{A} \rightarrow A)$ . If  $L/K$  is Galois, the Galois group  $\mathrm{Gal}(L/K)$  acts on it by the functoriality of the unit disc for the supremum norm. We have the  $G_K$ -equivariant isomorphism  $\pi_0(\bar{X}^j(\mathbb{A} \rightarrow A))_{\bar{F}} \rightarrow \pi_0(X^j(\mathbb{A} \rightarrow A))_{\bar{K}}$ , where the former is the set of geometric connected components of  $\bar{X}^j(\mathbb{A} \rightarrow A)$  ([3, Corollary 1.11]).

Suppose that  $A$  is of relative complete intersection over  $\mathcal{O}_K$  and  $A \otimes_{\mathcal{O}_K} K$  is etale over  $K$ . Then the natural map  $F(A) \rightarrow F^j(A)$  is surjective. The family  $\{F(A) \rightarrow F^j(A)\}_{j \in \mathbb{Q}_{>0}}$  is separated, exhaustive and its jumps are rational ([2, Proposition 6.4]). The conductor of  $A$  is defined to be  $c(A) = \inf\{j \in \mathbb{Q}_{>0} \mid F(A) \rightarrow F^j(A) \text{ is an isomorphism}\}$ . If  $B$  is the affine algebra of a finite flat group scheme  $\mathcal{G}$  over  $\mathcal{O}_K$  which is generically etale, then  $B$  is of relative complete intersection (for example, [5, Proposition 2.2.2]) and the theory above can all be applied to  $B$ . By the functoriality,  $F^j(B)$  is endowed with a  $G_K$ -module structure ([1, Lemme 2.1]) and the natural map  $\mathcal{G}(\bar{K}) = F(B) \rightarrow F^j(B)$  is a  $G_K$ -homomorphism. Let  $\mathcal{G}^j$  denote the schematic closure ([14]) in  $\mathcal{G}$  of the kernel of this homomorphism. It is called the  $j$ -th ramification filtration of  $\mathcal{G}$ . We refer  $c(B)$  as the conductor of  $\mathcal{G}$ , which is denoted also by  $c(\mathcal{G})$ . We put  $\mathcal{G}^{j+}(\bar{K}) = \cup_{j' > j} \mathcal{G}^{j'}(\bar{K})$  and define  $\mathcal{G}^{j+}$  to be the schematic closure of  $\mathcal{G}^{j+}(\bar{K})$  in  $\mathcal{G}$ .

### 3. GROUP STRUCTURE ON THE TUBULAR NEIGHBORHOOD OF A FINITE FLAT GROUP SCHEME

Let  $K$  denote a complete discrete valuation field of mixed characteristic  $(0, p)$  whose residue field  $F$  may be imperfect and  $v_K$  the valuation of  $K$  extended to  $\bar{K}$  which is normalized as  $v_K(\pi) = 1$ . Let  $\mathcal{G} = \text{Spec}(B)$  be a connected finite flat group scheme over  $\mathcal{O}_K$ . We define a formal resolution of  $\mathcal{G}$  to be a closed immersion  $\mathcal{G} \rightarrow \Gamma$  of (profinite) formal group schemes over  $\mathcal{O}_K$ , where  $\Gamma = \text{Spf}(\mathbb{B})$  is connected and smooth. Such an immersion can be constructed as follows. By a theorem of Raynaud ([4, Théorème 3.1.1]), we can find an abelian scheme  $A$  over  $\mathcal{O}_K$  and a closed immersion of group schemes  $\mathcal{G} \rightarrow A$ . Taking the formal completion of  $A$  along the zero section, we get a formal resolution of  $\mathcal{G}$ . We refer the relative dimension of  $\Gamma$  over  $\mathcal{O}_K$  as the dimension of a formal resolution  $(\mathcal{G} \rightarrow \Gamma)$ . We define a morphism of formal resolutions to be a pair of group homomorphisms  $(f, \mathbf{f})$  which makes the following diagram commutative.

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \Gamma \\ f \downarrow & & \downarrow \mathbf{f} \\ \mathcal{G}' & \longrightarrow & \Gamma' \end{array}$$

Note that a formal resolution of  $\mathcal{G}$  is also an embedding of  $B$  in the sense of Section 2. We say  $(f, \mathbf{f})$  is finite flat if this is finite flat as a map of embeddings ([3]). Consider the  $j$ -th tubular neighborhood  $X^j(\mathbb{B} \rightarrow B)$  of the embedding  $(\mathbb{B} \rightarrow B)$ , which we also write as  $X^j(\mathcal{G} \rightarrow \Gamma)$ . The following lemma, whose first part is implicit in [1], enables us to introduce a group structure on this affinoid variety.

**Lemma 3.1.** *Let  $(\mathbb{A} \rightarrow A)$  and  $(\mathbb{B} \rightarrow B)$  be embeddings of finite flat  $\mathcal{O}_K$ -algebras. Put  $\mathbb{C} = \mathbb{A} \hat{\otimes}_{\mathcal{O}_K} \mathbb{B}$  and  $C = A \otimes_{\mathcal{O}_K} B$ . Then the surjection  $\mathbb{C} \rightarrow C$  is also an embedding and we have a canonical isomorphism  $X^j(\mathbb{C} \rightarrow C) \rightarrow X^j(\mathbb{A} \rightarrow A) \times_K X^j(\mathbb{B} \rightarrow B)$ . Moreover, this isomorphism extends to a canonical isomorphism between their stable normalized integral models  $\mathfrak{X}^j(\mathbb{C} \rightarrow C)_{\mathcal{O}_{\bar{K}}} \rightarrow \mathfrak{X}^j(\mathbb{A} \rightarrow A)_{\mathcal{O}_{\bar{K}}} \times_{\mathcal{O}_{\bar{K}}} \mathfrak{X}^j(\mathbb{B} \rightarrow B)_{\mathcal{O}_{\bar{K}}}$ .*

*Proof.* By the functoriality of a tubular neighborhood, we have an affinoid map  $\Phi : X^j(\mathbb{C} \rightarrow C) \rightarrow X^j(\mathbb{A} \rightarrow A) \times_K X^j(\mathbb{B} \rightarrow B)$ . To see that  $\Phi$  is an isomorphism, we may replace  $K$  with a finite separable extension and suppose that  $A$  and  $B$  are local,  $j$  is an integer and  $A/\text{rad}(A) = B/\text{rad}(B) = F$ . Then we may identify  $\mathbb{A}$  with  $\mathcal{O}_K[[T_1, \dots, T_r]]$  and  $\mathbb{B}$  with  $\mathcal{O}_K[[T'_1, \dots, T'_{r'}]]$  for some  $r$  and  $r'$ . Let  $I = (f_1, \dots, f_s)$  (resp.  $J = (g_1, \dots, g_{s'})$ ) be the kernel of the surjection  $\mathbb{A} =$

$\mathcal{O}_K[[T_1, \dots, T_r]] \rightarrow A$  (resp.  $\mathbb{B} = \mathcal{O}_K[[T'_1, \dots, T'_{r'}]] \rightarrow B$ ). Then the affinoid algebras of  $X^j(\mathbb{A} \rightarrow A)$ ,  $X^j(\mathbb{B} \rightarrow B)$  and  $X^j(\mathbb{C} \rightarrow C)$  are equal to  $K\langle T_1, \dots, T_r \rangle \langle f_1/\pi^j, \dots, f_s/\pi^j \rangle$ ,  $K\langle T'_1, \dots, T'_{r'} \rangle \langle g_1/\pi^j, \dots, g_{s'}/\pi^j \rangle$  and  $K\langle T_1, \dots, T_r, T'_1, \dots, T'_{r'} \rangle \langle f_1/\pi^j, \dots, f_s/\pi^j, g_1/\pi^j, \dots, g_{s'}/\pi^j \rangle$  respectively. This shows that  $\Phi$  is an isomorphism.

Let  $L$  be a finite extension of  $K$  where the stable normalized integral models of  $X^j(\mathbb{A} \rightarrow A)$ ,  $X^j(\mathbb{B} \rightarrow B)$  and  $X^j(\mathbb{C} \rightarrow C)$  are defined. Set  $\mathcal{A}_0^{k,l}$  and  $\mathcal{A}_K^j$  as in Section 2. Let  $\mathring{\mathcal{A}}_{\mathcal{O}_L}^j$  denote the unit disc in  $\mathcal{A}_L^j = \mathcal{A}_K^j \hat{\otimes}_K L$  for the supremum norm. Define  $\mathcal{B}_0^{k,l}$ ,  $\mathcal{C}_0^{k,l}$ ,  $\mathcal{B}_K^j$ ,  $\mathcal{C}_K^j$ ,  $\mathring{\mathcal{B}}_{\mathcal{O}_L}^j$  and  $\mathring{\mathcal{C}}_{\mathcal{O}_L}^j$  similarly for  $B$  and  $C$ . From the proof of [8, Theorem 1.3], there exists a continuous surjection  $\alpha : L\langle T_1, \dots, T_{r'} \rangle \rightarrow \mathcal{A}_L^j$  such that  $\|\cdot\|_{\text{sup}} = \|\cdot\|_{\alpha}$ , where  $\|\cdot\|_{\alpha}$  is the residue norm induced by  $\alpha$ . We also have a surjection  $\beta : L\langle U_1, \dots, U_{s'} \rangle \rightarrow \mathcal{B}_L^j$  with the same property for  $B$ . Consider the surjection  $\alpha \hat{\otimes} \beta : L\langle T_1, \dots, T_{r'} \rangle \hat{\otimes}_L L\langle U_1, \dots, U_{s'} \rangle \rightarrow \mathcal{A}_L^j \hat{\otimes}_L \mathcal{B}_L^j = \mathcal{C}_L^j$ . The unit disc in  $\mathcal{A}_L^j \hat{\otimes}_L \mathcal{B}_L^j$  for the residue norm induced by  $\alpha \hat{\otimes} \beta$  is  $\mathring{\mathcal{A}}_{\mathcal{O}_L}^j \hat{\otimes}_{\pi_L} \mathring{\mathcal{B}}_{\mathcal{O}_L}^j$ , where  $\hat{\otimes}_{\pi_L}$  denotes the  $\pi_L$ -adic complete tensor product over  $\mathcal{O}_L$ . Its geometric closed fiber  $(\mathring{\mathcal{A}}_{\mathcal{O}_L}^j \otimes_{\mathcal{O}_L} \bar{F}) \otimes_{\bar{F}} (\mathring{\mathcal{B}}_{\mathcal{O}_L}^j \otimes_{\mathcal{O}_L} \bar{F})$  is reduced. By [8, Proposition 1.1], we see that the stable normalized integral model  $\mathring{\mathcal{C}}_{\mathcal{O}_L}^j$  is equal to  $\mathring{\mathcal{A}}_{\mathcal{O}_L}^j \hat{\otimes}_{\pi_L} \mathring{\mathcal{B}}_{\mathcal{O}_L}^j$ .  $\square$

**Corollary 3.2.** *Let  $(\mathcal{G} \rightarrow \Gamma)$  be a formal resolution of  $\mathcal{G}$ . Then the group structure of  $\Gamma$  induces a rigid  $K$ -analytic group structure on the tubular neighborhood  $X^j(\mathcal{G} \rightarrow \Gamma)$ . This group structure also extends to  $\mathfrak{X}^j(\mathcal{G} \rightarrow \Gamma)_{\mathcal{O}_{\bar{K}}}$  (resp.  $\bar{X}^j(\mathcal{G} \rightarrow \Gamma)$ ) and endows it with a  $\pi$ -adic formal group scheme structure over  $\mathcal{O}_{\bar{K}}$  (resp. an algebraic group structure over  $\bar{F}$ ).*

*Moreover, for a morphism of formal resolutions  $(\mathcal{G} \rightarrow \Gamma) \rightarrow (\mathcal{G}' \rightarrow \Gamma')$ , the induced affinoid map  $X^j(\mathcal{G} \rightarrow \Gamma) \rightarrow X^j(\mathcal{G}' \rightarrow \Gamma')$  is a homomorphism of rigid  $K$ -analytic groups. This homomorphism also extends to their stable normalized integral models as a homomorphism  $\mathfrak{X}^j(\mathcal{G} \rightarrow \Gamma)_{\mathcal{O}_{\bar{K}}} \rightarrow \mathfrak{X}^j(\mathcal{G}' \rightarrow \Gamma')_{\mathcal{O}_{\bar{K}}}$  of  $\pi$ -adic formal group schemes and to their geometric closed fibers as a homomorphism  $\bar{X}^j(\mathcal{G} \rightarrow \Gamma) \rightarrow \bar{X}^j(\mathcal{G}' \rightarrow \Gamma')$  of algebraic groups over  $\bar{F}$ .*

Let  $\mathcal{G} = \text{Spec}(B)$  be a connected finite flat group scheme over  $\mathcal{O}_K$  and  $(\mathcal{G} \rightarrow \Gamma = \text{Spf}(\mathbb{B}))$  be a formal resolution of dimension  $r$ . Set  $\text{Spf}(\mathbb{A}) = \Gamma/\mathcal{G}$  and regard the zero section  $\text{Spec}(\mathcal{O}_K) \rightarrow \text{Spf}(\mathbb{A})$  as a formal resolution of the trivial group. Then we have a finite flat map

of formal resolutions

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathrm{Spf}(\mathbb{B}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}_K) & \longrightarrow & \mathrm{Spf}(\mathbb{A}). \end{array}$$

By Corollary 3.2 and [3, Lemma 1.8], we get a finite flat map of rigid  $K$ -analytic groups  $f^j : X_{\mathcal{G}}^j = X^j(\mathcal{G} \rightarrow \Gamma) \rightarrow D^{r,j} = X^j(\mathrm{Spec}(\mathcal{O}_K) \rightarrow \mathrm{Spf}(\mathbb{A}))$ , where  $D^{r,j}$  denotes the  $r$ -dimensional polydisc  $\{(z_1, \dots, z_r) \in \mathcal{O}_{\bar{K}}^r \mid v_K(z_i) \geq j \text{ for any } i\}$ . We call this the affinoid homomorphism associated to a formal resolution  $(\mathcal{G} \rightarrow \Gamma)$ . Write  $\mathcal{B}_K^j$  and  $\mathcal{A}_K^j$  for the  $K$ -affinoid algebras of  $X_{\mathcal{G}}^j$  and  $D^{r,j}$  respectively. The stable normalized integral model over  $\mathcal{O}_L$  of  $X_{\mathcal{G}}^j$  (resp.  $D^{r,j}$ ) is denoted by  $\mathfrak{X}_{\mathcal{G}, \mathcal{O}_L}^j$  (resp.  $\mathfrak{D}_{\mathcal{O}_L}^{r,j}$ ) and its geometric closed fiber by  $\bar{X}_{\mathcal{G}}^j$  (resp.  $\bar{D}^{r,j}$ ). Note that the algebraic group  $\bar{X}_{\mathcal{G}}^j$  is reduced, hence smooth by [16, Theorem 11.6].

**Lemma 3.3.** *The affinoid homomorphism  $f^j : X_{\mathcal{G}}^j \rightarrow D^{r,j}$  is etale for any  $j > 0$ . Moreover, for  $j > c(\mathcal{G})$ , there exists a finite extension  $K'/K$  such that  $X_{\mathcal{G}, K'}^j$  is isomorphic to the disjoint sum of finitely many copies of  $D_{K'}^{r,j}$ .*

*Proof.* We have  $\Omega_{\mathcal{B}_K^j/\mathcal{A}_K^j}^1 = \mathcal{B}_K^j \hat{\otimes}_{\mathbb{B}} \hat{\Omega}_{\mathbb{B}/\mathbb{A}}$ . It is enough to show that  $\hat{\Omega}_{\mathbb{B}/\mathbb{A}}$  is a torsion  $\mathcal{O}_K$ -module. Let  $J_{\mathbb{A}}$  and  $J_{\mathbb{B}}$  be the augmentation ideals of  $\mathbb{A}$  and  $\mathbb{B}$  respectively. Set  $I = \mathrm{Ker}(\mathbb{B} \rightarrow \mathbb{A})$ . Then  $\hat{\Omega}_{\mathbb{B}/\mathbb{A}} = \mathrm{Coker}(\mathbb{B} \otimes_{\mathbb{A}} \hat{\Omega}_{\mathbb{A}/\mathcal{O}_K} \rightarrow \hat{\Omega}_{\mathbb{B}/\mathcal{O}_K})$  is equal to  $\mathbb{B} \otimes_{\mathcal{O}_K} \mathrm{Coker}(\mathrm{Cot}(\mathbb{A}) \rightarrow \mathrm{Cot}(\mathbb{B})) = \mathbb{B} \otimes_{\mathcal{O}_K} J_{\mathbb{B}}/(I + J_{\mathbb{B}}^2) = \mathbb{B} \otimes_{\mathcal{O}_K} \mathrm{Cot}(B)$ . This shows the first assertion. For the second assertion, take a finite extension  $K'$  of  $K$  where the geometric connected components of  $X_{\mathcal{G}}^j$  are defined. By assumption, each of the connected components of  $X_{\mathcal{G}, K'}^j$  is a finite etale cover of  $D_{K'}^{r,j}$  whose degree is one. Thus this is isomorphic to  $D_{K'}^{r,j}$ .  $\square$

Take a finite extension  $L$  of  $K$  where the stable normalized integral models of  $X_{\mathcal{G}}^j$  and  $D^{r,j}$  are defined. The generic fiber  $\mathcal{G}_L = \mathcal{G} \times_{\mathcal{O}_K} L$  can be regarded as a rigid  $L$ -analytic subgroup of  $X_{\mathcal{G}, L}^j$  defined by an ideal  $\mathcal{J}_L$  of  $\mathcal{B}_L^j$ . Put  $\mathcal{J} = \mathcal{J}_L \cap \hat{\mathcal{B}}_{\mathcal{O}_L}^j$  and  $\hat{B}^j = \hat{\mathcal{B}}_{\mathcal{O}_L}^j/\mathcal{J}$ . The latter is a subring of  $B_L = B \otimes_K L$ . Since we have a commutative diagram with

surjective horizontal arrows and injective vertical arrows

$$\begin{array}{ccc} \mathcal{B}_0^{k,l} \otimes_{\mathcal{O}_K} \mathcal{O}_L & \longrightarrow & B \otimes_{\mathcal{O}_K} \mathcal{O}_L \\ \downarrow & & \downarrow \\ \mathring{\mathcal{B}}_{\mathcal{O}_L}^j & \longrightarrow & \mathring{B}_{\mathcal{O}_L}^j, \end{array}$$

we see that  $\mathring{B}_{\mathcal{O}_L}^j$  is integral over  $B \otimes_{\mathcal{O}_K} \mathcal{O}_L$ . Thus the  $\mathcal{O}_K$ -algebra  $\mathring{B}_{\mathcal{O}_L}^j$  is finite flat. Set  $\mathcal{H}_{\mathcal{O}_L}^j = \text{Spec}(\mathring{B}_{\mathcal{O}_L}^j)$ . This can be regarded as a closed  $\pi_L$ -adic formal subscheme of  $\mathfrak{X}_{\mathcal{G}, \mathcal{O}_L}^j$ .

**Lemma 3.4.** *The group structure of  $\mathcal{G}_L$  extends to  $\mathcal{H}_{\mathcal{O}_L}^j$ . The group scheme  $\mathcal{H}_{\mathcal{O}_L}^j$  is a closed  $\pi_L$ -adic formal subgroup scheme of  $\mathfrak{X}_{\mathcal{G}, \mathcal{O}_L}^j$ .*

*Proof.* Put  $\mathcal{K} = \text{Ker}(\mathring{\mathcal{B}}_{\mathcal{O}_L}^j \hat{\otimes}_{\pi_L} \mathring{\mathcal{B}}_{\mathcal{O}_L}^j \rightarrow \mathring{B}_{\mathcal{O}_L}^j \otimes_{\mathcal{O}_L} \mathring{B}_{\mathcal{O}_L}^j)$  and  $\mathcal{K}_L = \mathcal{K} \otimes_{\mathcal{O}_L} L$ . Let  $\mu$  be the coproduct of  $\mathring{\mathcal{B}}_{\mathcal{O}_L}^j$ . We must show  $\mu(\mathcal{J}) \subseteq \mathcal{K}$ . This follows from the commutative diagram below whose rows are exact and vertical arrows are injective.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathring{\mathcal{B}}_{\mathcal{O}_L}^j \hat{\otimes}_{\pi_L} \mathring{\mathcal{B}}_{\mathcal{O}_L}^j & \longrightarrow & \mathring{B}_{\mathcal{O}_L}^j \otimes_{\mathcal{O}_L} \mathring{B}_{\mathcal{O}_L}^j \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{K}_L & \longrightarrow & \mathcal{B}_L^j \hat{\otimes}_L \mathcal{B}_L^j & \longrightarrow & B_L \otimes_L B_L \longrightarrow 0 \end{array}$$

Passing to the generic fiber, we see that the second assertion holds.  $\square$

**Lemma 3.5.** *The associated homomorphism  $f^j : \mathfrak{X}_{\mathcal{G}, \mathcal{O}_L}^j \rightarrow \mathfrak{D}_{\mathcal{O}_L}^{r,j}$  is finite flat. Moreover, there exists an exact sequence of  $\pi_L$ -adic formal group schemes*

$$(1) \quad 0 \rightarrow \mathcal{H}_{\mathcal{O}_L}^j \rightarrow \mathfrak{X}_{\mathcal{G}, \mathcal{O}_L}^j \rightarrow \mathfrak{D}_{\mathcal{O}_L}^{r,j} \rightarrow 0.$$

*Proof.* From Lemma 3.3, the associated affinoid map  $f^j : X_{\mathcal{G}}^j \rightarrow D^{r,j}$  is finite etale. Let  $\mathcal{B}_K^j$  and  $\mathcal{A}_K^j$  be their affinoid algebras as above. Since  $D^{r,j}$  is integral, we see that  $f^j$  is surjective and the ring homomorphism  $\mathcal{A}_K^j \rightarrow \mathcal{B}_K^j$  is injective. Thus we have an injection  $\mathring{\mathcal{A}}_{\mathcal{O}_L}^j \rightarrow \mathring{\mathcal{B}}_{\mathcal{O}_L}^j$ , which is finite by [6, Corollary 6.4.1/6]. Hence  $\bar{f}^j : \bar{X}_{\mathcal{G}}^j \rightarrow \bar{D}^{r,j}$  is a surjective homomorphism of algebraic groups over  $\bar{F}$ . Since  $\bar{X}_{\mathcal{G}}^j$  and  $\bar{D}^{r,j}$  are regular, we see that  $\bar{f}^j$  is faithfully flat by [13, Theorem 23.1]. Since  $\mathring{\mathcal{A}}_{\mathcal{O}_L}^j$  and  $\mathring{\mathcal{B}}_{\mathcal{O}_L}^j$  is  $\pi_L$ -torsion free, the map  $f^j$  is flat by the local criterion of flatness. Put  $\mathcal{H}' = \text{Ker}(f^j)$ . This is a closed  $\pi_L$ -adic formal subgroup scheme of  $\mathfrak{X}_{\mathcal{G}, \mathcal{O}_L}^j$  and can be regarded also as a finite flat group scheme over  $\mathcal{O}_L$ . Passing to the generic fiber, we see that  $\mathcal{H}_{\mathcal{O}_L}^j$  is a closed subgroup scheme of  $\mathcal{H}'$ . Comparing these ranks concludes the lemma.



□

From Lemma 3.3, we see that for  $j > c = c(\mathcal{G})$ , the map  $\mathring{f}^j$  identifies  $\mathfrak{X}_{\mathcal{G}, \mathcal{O}_L}^j$  with the direct sum of finitely many copies of  $\mathfrak{D}_{\mathcal{O}_L}^{r,j}$ . More precisely, we have the following.

**Lemma 3.6.** *Let  $c = c(\mathcal{G})$  be the conductor of  $\mathcal{G}$ . Then the associated homomorphism  $\mathring{f}^j : \mathfrak{X}_{\mathcal{G}, \mathcal{O}_L}^j \rightarrow \mathfrak{D}_{\mathcal{O}_L}^{r,j}$  is finite etale if and only if  $j \geq c$ .*

*Proof.* Let  $\text{sp}_j : X_{\mathcal{G}}^j \rightarrow \bar{X}_{\mathcal{G}}^j$  be the specialization map. By Lemma 3.3, we see that  $\bar{f}^c$  is finite etale at  $\text{sp}_c(x)$  for any  $x \in \mathcal{G}(\bar{K})$  as in the proof of [2, Theorem 7.2], using a theorem of Bosch ([7, Lemma 2.1]). We conclude the etaleness of  $\bar{f}^c$  by the existence of the group structure. We have  $\Omega_{\mathring{\mathcal{B}}_{\mathcal{O}_L}^c / \mathring{\mathcal{A}}_{\mathcal{O}_L}^c}^1 \otimes_{\mathcal{O}_L} \bar{F} = 0$ . Since  $\mathring{\mathcal{B}}_{\mathcal{O}_L}^c$  is  $\pi_L$ -adic complete and Noetherian, we see that  $\Omega_{\mathring{\mathcal{B}}_{\mathcal{O}_L}^c / \mathring{\mathcal{A}}_{\mathcal{O}_L}^c}^1 = 0$  and the homomorphism  $\mathring{f}^c : \mathfrak{X}_{\mathcal{G}, \mathcal{O}_L}^c \rightarrow \mathfrak{D}_{\mathcal{O}_L}^{r,c}$  is finite etale.

Let  $\bar{0}$  be the zero section of  $\bar{D}^{r,j}$  and set  $X_{\mathcal{G}}^{j+} = \cup_{j' > j} X_{\mathcal{G}}^{j'}$ . Then we have  $(\bar{f}^j)^{-1}(\bar{0}) = \text{sp}_j(X_{\mathcal{G}}^{j+}) = \text{sp}_j(\mathcal{G}(\bar{K}))$ . If  $\bar{f}^j$  is etale, then  $\sharp(\bar{f}^j)^{-1}(\bar{0})$  equals the degree of  $\bar{f}^j$ , namely  $\sharp\mathcal{G}(\bar{K})$ . Thus  $X_{\mathcal{G}}^{j+}$  splits and we have  $j \geq c$ .

□

#### 4. RAMIFICATION AND THE $I_K$ -MODULE STRUCTURE OF A FINITE FLAT GROUP SCHEME

Consider the right action of  $I_K$  on  $\bar{K}$  defined by  $\sigma.z = \sigma^{-1}(z)$  for  $\sigma \in I_K$ . This action induces a  $\bar{K}$ -semilinear left action of  $I_K$  on  $X_{\mathcal{G}, \bar{K}}^j = X_{\mathcal{G}}^j \times_K \bar{K}$ , which also extends to an  $\mathcal{O}_{\bar{K}}$ -semilinear action on its stable normalized integral model  $\mathfrak{X}_{\mathcal{G}, \mathcal{O}_{\bar{K}}}^j$ . Thus we have an  $\bar{F}$ -linear left action of  $I_K$  on its closed fiber  $\bar{X}_{\mathcal{G}}^j$ . We call this the geometric monodromy action of  $I_K$  and write the action of  $\sigma \in I_K$  as  $\sigma_{\text{geom}}$  (note that here we follow the terminology in [2], and not in [3] where the monodromy action is called arithmetic, since in our case there is no “geometric” action other than the monodromy action). Similarly, we have the geometric monodromy action of  $I_K$  on  $\bar{D}^{r,j}$ .

The latter action is described as follows. Let the additive group (*resp.* multiplicative group) over  $\bar{F}$  be denoted by  $\bar{\mathbb{G}}_a$  (*resp.*  $\bar{\mathbb{G}}_m$ ). Consider the left action  $\bar{\mathbb{G}}_m \times \bar{\mathbb{G}}_a^r \rightarrow \bar{\mathbb{G}}_a^r$  given by the multiplication. Write this action of  $\lambda \in \bar{F}^\times$  as  $[\lambda]$ . This action is defined by  $T_i \mapsto \lambda T_i$ , where  $\bar{\mathbb{G}}_a^r = \text{Spec}(\bar{F}[T_1, \dots, T_r])$ . For  $j \in \mathbb{Q}_{>0}$ , we define the fundamental character  $\theta_j : I_K \rightarrow \bar{F}^\times$  to be  $\theta_{l'}^{k'}$ , where  $k'/l'$  is the prime-to- $p$ -denominator part

of  $j \bmod \mathbb{Z}$  ([15]). In other words, we set  $\theta_j(\sigma) = (\sigma(\pi^{1/l'})/\pi^{1/l'})^{k'} \bmod \mathfrak{m}_{\bar{K}}$ . Note that, for  $j = k/l$  and  $l = p^m l_0$  with  $(k, l) = 1$  and  $p \nmid l_0$ , we have  $\theta_j = \theta_{l_0}^{kp^{-m}}$ .

**Lemma 4.1.** *The algebraic group  $\bar{D}^{r,j}$  is equal to  $\bar{\mathbb{G}}_a^r$ . For  $\sigma \in I_K$ , the geometric monodromy action  $\sigma_{\text{geom}}$  on  $\bar{D}^{r,j}$  coincides with the multiplication  $[\theta_j(\sigma)]$ .*

*Proof.* Put  $\mathbb{A} = \mathcal{O}_K[[T_1, \dots, T_r]]$  and  $j = k/l$  with  $(k, l) = 1$ . Let  $L$  be a finite Galois extension of  $K$  containing  $\pi^{1/l}$  and  $e' = e(L/K)$  be its ramification index over  $K$ . Then  $e'k/l \in \mathbb{Z}$  and the stable normalized integral model of  $D^{r,j}$  over  $\mathcal{O}_L$  is  $\mathcal{O}_L\langle T_1/(\pi_L)^{e'k/l}, \dots, T_r/(\pi_L)^{e'k/l} \rangle = \mathcal{O}_L\langle W_1, \dots, W_r \rangle$ , where  $W_i = T_i/(\pi^{1/l})^k$ . Set  $\mu_{\mathbb{A}}$  to be the coproduct of  $\mathbb{A}$ . We have

$$\mu_{\mathbb{A}}(T_i) = T_i \hat{\otimes} 1 + 1 \hat{\otimes} T_i + (\text{higher degree})$$

and then  $\mu_{\mathbb{A}}(W_i) = \mu_{\mathbb{A}}((\pi^{1/l})^k W_i)/(\pi^{1/l})^k$  is equal to

$$W_i \hat{\otimes}_{\pi} 1 + 1 \hat{\otimes}_{\pi} W_i + (\pi^{1/l})^k (\text{higher degree})$$

in this  $\mathcal{O}_L$ -algebra. This shows the first assertion.

On the other hand, we know the geometric monodromy action on  $\bar{D}^{r,j}$  is tame ([2, Lemma 7.7]). Write  $l = p^m l_0$  with  $p \nmid l_0$ . Then for  $\sigma \in I_K$ , we have  $\sigma((\pi^{1/l})^k)/(\pi^{1/l})^k \equiv \theta_{l_0}(\sigma)^{kp^{-m}} \zeta_{p^m}^N \bmod \mathfrak{m}_{\bar{K}}$  with some  $N$  and this is equal to  $\theta_j(\sigma)$ . Thus the action on the affine algebra  $\bar{F}[W_1, \dots, W_r]$  of  $\bar{D}^{r,j}$  is given by  $\sigma_{\text{geom}}^*(W_i) = \theta_j(\sigma)W_i$ . This coincides with  $[\theta_j(\sigma)]$ .  $\square$

Next we consider the geometric monodromy action on  $\bar{X}_{\mathcal{G}}^j$ . Let  $\bar{X}_{\mathcal{G}}^{j,0}$  denote the unit component of the algebraic group  $\bar{X}_{\mathcal{G}}^j$  and  $\bar{\mathcal{H}}^j$  be the geometric closed fiber of  $\mathcal{H}_{\mathcal{O}_L}^j$ . We begin with the following lemma.

**Lemma 4.2.** *If  $\psi \in \text{End}(\bar{X}_{\mathcal{G}}^{j,0})$  induces the zero map on  $\bar{D}^{r,j}$ , then  $\psi = 0$ .*

*Proof.* Put  $\bar{\mathcal{H}}_0^j = \bar{\mathcal{H}}^j \cap \bar{X}_{\mathcal{G}}^{j,0}$ . This is the kernel of the faithfully flat map  $\bar{X}_{\mathcal{G}}^{j,0} \rightarrow \bar{D}^{r,j}$  and by assumption we have the following commutative diagram whose rows are exact.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bar{\mathcal{H}}_0^j & \longrightarrow & \bar{X}_{\mathcal{G}}^{j,0} & \longrightarrow & \bar{D}^{r,j} \longrightarrow 0 \\ & & \downarrow & & \psi \downarrow & & 0 \downarrow \\ 0 & \longrightarrow & \bar{\mathcal{H}}^j & \longrightarrow & \bar{X}_{\mathcal{G}}^j & \longrightarrow & \bar{D}^{r,j} \longrightarrow 0 \end{array}$$

Thus  $\psi$  factors through  $\bar{\mathcal{H}}_0^j$ . Put  $\bar{C} = \text{Im}(\psi)$ . Then this is a closed subgroup scheme of  $\bar{\mathcal{H}}_0^j$  and the map  $\bar{X}_{\mathcal{G}}^{j,0} \rightarrow \bar{C}$  is faithfully flat. Since  $\bar{X}_{\mathcal{G}}^{j,0}$

is regular and connected, we see that  $\bar{C}$  is also regular and connected by [13, Theorem 23.7]. Hence  $\bar{C} = 0$  and we have  $\psi = 0$ .  $\square$

**Corollary 4.3.** *Let  $\mathcal{G}$  be a connected finite flat group scheme over  $\mathcal{O}_K$ . Take a formal resolution  $(\mathcal{G} \rightarrow \Gamma)$  of dimension  $r$ . Then the algebraic group  $\bar{X}_{\mathcal{G}}^{j,0}$  is isomorphic to  $\bar{\mathbb{G}}_a^r$ .*

*Proof.* By the previous lemma and Lemma 4.1, we see that  $\bar{X}_{\mathcal{G}}^{j,0}$  is killed by  $p$ . Hence the assertion follows from [12, Lemma 1.7.1].  $\square$

**Corollary 4.4.** *The geometric monodromy action of  $I_K$  on  $\bar{X}_{\mathcal{G}}^{j,0}$  is tame.*

*Proof.* For an element  $\sigma$  of the wild inertia subgroup  $P_K$ , the geometric monodromy action  $\sigma_{\text{geom}}$  on  $\bar{D}^{r,j}$  is trivial. Applying the lemma to  $\sigma_{\text{geom}} - \text{id} \in \text{End}(\bar{X}_{\mathcal{G}}^{j,0})$  shows the assertion.  $\square$

**Corollary 4.5.** *Let  $J$  be a finite cyclic quotient of  $I_K$  through which the tame character  $\theta_j$  factors and  $\tau$  be a generator of  $J$ . Let  $F(t)$  denote the minimal polynomial of  $\theta_j(\tau) \in \bar{\mathbb{F}}_p$  over  $\mathbb{F}_p$ . Then the geometric monodromy action of  $I_K$  on  $\bar{X}_{\mathcal{G}}^{j,0}$  also factors through  $J$  and the equation  $F(\tau_{\text{geom}}) = 0$  holds in  $\text{End}(\bar{X}_{\mathcal{G}}^{j,0})$ .*

*Proof.* The first assertion follows from Lemma 4.2 as the previous corollary. The second assertion also follows from this lemma using Lemma 4.1.  $\square$

Let  $c = c(\mathcal{G})$  be the conductor of  $\mathcal{G}$ . The lemma below enables us to realize  $\mathcal{G}^c(\bar{K})$  as a subgroup of  $\bar{X}_{\mathcal{G}}^{c,0}$ .

**Lemma 4.6.** *The specialization map  $\text{sp}_c : X_{\mathcal{G},\bar{K}}^c \rightarrow \bar{X}_{\mathcal{G}}^c$  induces an  $I_K$ -equivariant isomorphism  $\mathcal{G}(\bar{K}) \rightarrow \bar{\mathcal{H}}^c(\bar{F})$  and  $\mathcal{G}^c(\bar{K}) \rightarrow \bar{\mathcal{H}}_0^c(\bar{F})$ . Here we consider on the left-hand side the natural action as the  $\bar{K}$ -valued points of  $\mathcal{G}$  (resp.  $\mathcal{G}^c$ ) and on the right-hand side the restriction of the geometric monodromy action on  $\bar{X}_{\mathcal{G}}^c$ .*

*Proof.* By definition, the generic fiber of  $\mathcal{H}_{\mathcal{O}_L}^c$  is equal to  $\mathcal{G}_L$ . From the exact sequence (1) and Lemma 3.6, we know that  $\mathcal{H}_{\mathcal{O}_L}^c$  is etale over  $\mathcal{O}_L$  and there is the following exact sequence of algebraic groups over  $\bar{F}$ .

$$(2) \quad 0 \rightarrow \bar{\mathcal{H}}^c \rightarrow \bar{X}_{\mathcal{G}}^c \rightarrow \bar{D}^{r,j} \rightarrow 0$$

Thus we have a natural isomorphism  $\mathcal{H}_{\mathcal{O}_L}^c(\bar{K}) \rightarrow \bar{\mathcal{H}}^c(\bar{F})$  and the composite  $\mathcal{G}(\bar{K}) = \mathcal{H}_{\mathcal{O}_L}^c(\bar{K}) \rightarrow \bar{\mathcal{H}}^c(\bar{F}) \rightarrow \bar{X}_{\mathcal{G}}^c(\bar{F})$  coincides with the map

$\text{sp}_c$ . From [2, Corollary 4.4], we see that this map sends  $\mathcal{G}^c(\bar{K})$  isomorphically onto  $\mathcal{H}_0^c(\bar{F})$ .

For  $x \in X_{\mathcal{G}}^c(\bar{K})$  and  $\sigma \in I_K$ , let  $\sigma(x)$  denote the natural action of  $\sigma$  on  $\bar{K}$ -valued points. Then we have  $\sigma_{\text{geom}}(x) \circ \sigma = \sigma(x)$ . Taking its specialization shows the  $I_K$ -equivariance. □

The following theorem can be regarded as a generalization for a finite flat group scheme over  $\mathcal{O}_K$  of the structure theorem of the graded piece of the classical upper numbering ramification filtration.

**Theorem 4.7.** *Let  $\mathcal{G}$  be a finite flat group scheme over  $\mathcal{O}_K$  and  $j \in \mathbb{Q}_{>0}$ . Then the  $G_K$ -module  $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K})$  is tame and killed by  $p$ .*

*Proof.* Since  $\mathcal{G}^j = (\mathcal{G}^0)^j$ , where  $\mathcal{G}^0$  denotes the unit component of  $\mathcal{G}$ , we may assume  $\mathcal{G}$  is connected. Suppose that  $j$  is a jump of the ramification filtration on  $\mathcal{G}$  and consider the quotient  $\mathcal{G}/\mathcal{G}^{j+}$ . The subgroup  $\mathcal{G}^j(\bar{K}) \subseteq \mathcal{G}(\bar{K})$  has a non-trivial image in  $(\mathcal{G}/\mathcal{G}^{j+})(\bar{K})$ . By the Herbrand theorem ([1, Lemme 2.10]), the natural map  $\mathcal{G}^t(\bar{K}) \rightarrow (\mathcal{G}/\mathcal{G}^{j+})^t(\bar{K})$  is surjective for any  $t > 0$ . We have  $(\mathcal{G}/\mathcal{G}^{j+})^t = 0$  for  $t > j$  and  $(\mathcal{G}/\mathcal{G}^{j+})^j \neq 0$ . Thus the ramification filtration on  $\mathcal{G}/\mathcal{G}^{j+}$  jumps at  $j$  and  $(\mathcal{G}/\mathcal{G}^{j+})^j(\bar{K}) = \mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K})$ . Replacing  $\mathcal{G}$  with  $\mathcal{G}/\mathcal{G}^{j+}$ , we may assume  $j = c = c(\mathcal{G})$ .

Take a formal resolution  $(\mathcal{G} \rightarrow \Gamma)$  of dimension  $r$  and consider its associated affinoid homomorphism  $X_{\mathcal{G}}^c \rightarrow D^{r,c}$ . Then the theorem follows from Corollary 4.3, Corollary 4.4, and Lemma 4.6. □

From this theorem, we see that the inertia subgroup  $I_K$  acts on  $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$  by the direct sum of tame characters. The theorem below determines these characters up to  $p$ -power exponent.

**Theorem 4.8.** *Let  $\mathcal{G}$  be a finite flat group scheme over  $\mathcal{O}_K$  and  $j \in \mathbb{Q}_{>0}$ . Then  $I_K$  acts on  $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$  by the direct sum of  $\mathbb{F}_p$ -conjugates of the fundamental character  $\theta_j$ .*

*Proof.* We may assume that  $\mathcal{G}$  is connected. The same argument as in the proof of Theorem 4.7 using the Herbrand theorem reduces the claim to the case where  $j = c = c(\mathcal{G})$ . Take a formal resolution  $(\mathcal{G} \rightarrow \Gamma)$  of dimension  $r$ . Let  $J$  and  $\tau$  be as in Corollary 4.5. Then Corollary 4.5 and Lemma 4.6 show that every eigenvalue of the action of  $\tau_{\text{geom}}$  on the finite dimensional  $\mathbb{F}_p$ -vector space  $\mathcal{G}^c(\bar{K})$  is a conjugate of  $\theta_j(\tau)$  over  $\mathbb{F}_p$ . Since the order of  $J$  is prime to  $p$ , we conclude that  $I_K$  acts on  $\mathcal{G}^c(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$  by the direct sum of  $\mathbb{F}_p$ -conjugates of  $\theta_c$ . □

**Corollary 4.9.** *Let  $\mathcal{G}$  be a finite flat group scheme over  $\mathcal{O}_K$ . Then the order of the image of the homomorphism  $I_K \rightarrow \text{Aut}(\mathcal{G}(\bar{K}))$  is a power of  $p$  if and only if every jump  $j$  of the ramification filtration  $\{\mathcal{G}^j\}_{j \in \mathbb{Q}_{>0}}$  is an element of  $\mathbb{Z}[1/p]$ .*

*Proof.* From Theorem 4.8, we see that the jumps are in  $\mathbb{Z}[1/p]$  if and only if the graded pieces  $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K})$  are unramified. By Theorem 4.7, this is equivalent to the condition that  $\sharp\text{Im}(I_K \rightarrow \text{Aut}(\mathcal{G}(\bar{K})))$  is a  $p$ -power. □

When  $\mathcal{G}(\bar{K})$  is unramified and killed by  $p$ , we have the following reinforcement of this result, which is an easy corollary of the Herbrand theorem. The author does not know if every jump is an integer whenever  $\mathcal{G}(\bar{K})$  is unramified. If  $\mathcal{G}$  is monogenic, then we see that this holds true from [11, Theorem 4].

**Proposition 4.10.** *Let  $\mathcal{G}$  be a finite flat group scheme over  $\mathcal{O}_K$  which is killed by  $p$ . Suppose that the  $G_K$ -module  $\mathcal{G}(\bar{K})$  is unramified. Then every jump  $j$  of the ramification filtration  $\{\mathcal{G}^j\}_{j \in \mathbb{Q}_{>0}}$  is an element of  $p\mathbb{Z}$ .*

*Proof.* We may assume  $K = K^{\text{nr}}$  and  $G_K$  acts trivially on  $\mathcal{G}(\bar{K})$ . There is a quotient  $W$  of  $\mathcal{G}(\bar{K})/\mathcal{G}^{j+}(\bar{K})$  where  $\mathcal{G}^j(\bar{K})$  has a non-trivial image and of rank one over  $\mathbb{F}_p$ . Taking the schematic closure,  $W$  extends to a finite flat group scheme  $\mathcal{W}$  over  $\mathcal{O}_K$  which is a quotient of  $\mathcal{G}/\mathcal{G}^{j+}$ . By the Herbrand theorem, we see that the ramification filtration of  $\mathcal{W}$  jumps at  $j$ . On the other hand,  $\mathcal{W}$  is a Raynaud  $\mathbb{F}_p$ -vector space scheme with unramified generic fiber. Thus the proposition follows from [11, Theorem 4]. □

For the rest of this section, we state some corollaries in the case where  $\mathcal{G}$  is an  $\mathbb{F}$ -vector space scheme of rank one or two for a finite extension  $\mathbb{F}$  over  $\mathbb{F}_p$ . In the case of rank one, Theorem 4.8 directly shows the claim below, while this can be shown also as a corollary of a determination of the conductor of a Raynaud  $\mathbb{F}$ -vector space scheme (Theorem 5.5).

**Corollary 4.11.** *Let  $\mathcal{G}$  be an  $\mathbb{F}$ -vector space scheme of rank one over  $\mathcal{O}_K$  and  $c = c(\mathcal{G})$ . Then the  $I_K$ -action on the  $\mathbb{F}$ -vector space  $\mathcal{G}(\bar{K})$  of rank one is given by the character  $\theta_c^{p^n}$  for some  $n$ .*

In the case of rank two, we have the following.

**Corollary 4.12.** *Let  $\mathcal{G}$  be a finite flat  $\mathbb{F}$ -vector space scheme of rank two over  $\mathcal{O}_K$  and  $c = c(\mathcal{G})$ . Then the  $I_K$ -module  $\mathcal{G}(\bar{K}) \otimes_{\mathbb{F}} \bar{\mathbb{F}}_p$  contains*

the character  $\theta_c^{p^n}$  for some  $n$ . If the  $G_K$ -module  $\mathcal{G}(\bar{K})$  is reducible, this holds true for  $\mathcal{G}(\bar{K})$  itself.

*Proof.* The first assertion follows easily from Theorem 4.8 and the surjection  $\mathcal{G}^c(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p \rightarrow \mathcal{G}^c(\bar{K}) \otimes_{\mathbb{F}} \bar{\mathbb{F}}_p$ . Suppose the  $I_K$ -module  $\mathcal{G}(\bar{K})$  is reducible. When  $\mathcal{G}^c$  is of rank one, the assertion is clear from Theorem 4.8. If  $\mathcal{G}^c = \mathcal{G}$ , then  $\mathcal{G}^c$  is reducible and the assertion follows also from Theorem 4.8. □

The corollary below indicates that the conductor  $c(\mathcal{G})$  carries information about not only the tame characters but also their extension structures in the  $I_K$ -module  $\mathcal{G}(\bar{K})$ .

**Corollary 4.13.** *Consider an exact sequence of finite flat  $\mathbb{F}$ -vector space schemes over  $\mathcal{O}_K$*

$$0 \rightarrow \mathcal{G}_1 \rightarrow \mathcal{G} \rightarrow \mathcal{G}_2 \rightarrow 0$$

where  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are connected of rank one. If  $c(\mathcal{G}) = c(\mathcal{G}_2)$ , then the  $I_K$ -module  $\mathcal{G}(\bar{K})$  splits.

*Proof.* Put  $c = c(\mathcal{G})$ . Take a formal resolution  $(\mathcal{G} \rightarrow \Gamma)$  of dimension  $r$  and put  $\Gamma_2 = \Gamma/\mathcal{G}_1$ . Then we get a finite flat map of formal resolutions

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \mathcal{G}_2 & \longrightarrow & \Gamma_2. \end{array}$$

Therefore we have a finite flat homomorphism of rigid  $K$ -analytic groups  $X^j(\mathcal{G} \rightarrow \Gamma) \rightarrow X^j(\mathcal{G}_2 \rightarrow \Gamma_2)$  by Corollary 3.2. As in the proof of Lemma 3.3, we see that this map is finite etale.

Suppose  $\mathcal{G}^c(\bar{K})$  is of rank one. If  $\mathcal{G}^c(\bar{K}) \neq \mathcal{G}_1(\bar{K})$  as an  $\mathbb{F}$ -subspace of  $\mathcal{G}(\bar{K})$ , the  $I_K$ -module  $\mathcal{G}(\bar{K})$  splits and the proposition follows. Suppose  $\mathcal{G}^c(\bar{K}) = \mathcal{G}_1(\bar{K})$ . The affinoid variety  $X^c(\mathcal{G} \rightarrow \Gamma)$  decomposes to  $\sharp\mathbb{F}$  components over some finite extension  $K'$  of  $K$ . Each component is a Zariski open and closed subset of  $X^c(\mathcal{G} \rightarrow \Gamma)_{K'}$ . As the map  $f : X^c(\mathcal{G} \rightarrow \Gamma)_{K'} \rightarrow X^c(\mathcal{G}_2 \rightarrow \Gamma_2)_{K'}$  is finite etale and  $X^c(\mathcal{G}_2 \rightarrow \Gamma_2)_{K'}$  is connected, every component  $X^{c,i}(\mathcal{G} \rightarrow \Gamma)_{K'}$  maps surjectively to  $X^c(\mathcal{G}_2 \rightarrow \Gamma_2)_{K'}$ . Take some  $g_i \in \mathcal{G}(\bar{K}) \cap X^{c,i}(\mathcal{G} \rightarrow \Gamma)_{K'}$ . Using the group structure, we see that  $\mathcal{G}(\bar{K}) \cap X^{c,i}(\mathcal{G} \rightarrow \Gamma)_{K'} = g_i + \mathcal{G}^c(\bar{K}) = g_i + \mathcal{G}_1(\bar{K})$  and  $f(\mathcal{G}(\bar{K}) \cap X^{c,i}(\mathcal{G} \rightarrow \Gamma)_{K'}) = f(g_i)$ . However, we have  $f^{-1}(\mathcal{G}_2(\bar{K})) = \mathcal{G}(\bar{K})$  and thus  $f(\mathcal{G}(\bar{K}) \cap X^{c,i}(\mathcal{G} \rightarrow \Gamma)_{K'}) = \mathcal{G}_2(\bar{K})$ . This is a contradiction. Therefore we may assume  $\mathcal{G}^c(\bar{K}) = \mathcal{G}(\bar{K})$ . In this case, the proposition follows from Theorem 4.7. □

## 5. EXAMPLE: RANK ONE CALCULATION

In this section, we calculate the conductor of a Raynaud  $\mathbb{F}$ -vector space scheme over  $\mathcal{O}_K$ . The point is that, as we can see from the bound of the conductor ([11, Theorem 7]), it is enough to consider the  $j$ -th tubular neighborhood only for  $j \leq pe/(p-1) + \varepsilon$  with sufficiently small  $\varepsilon > 0$ . For such  $j$ , we can compute the tubular neighborhood easily by Lemma 5.4 below.

Let  $K$  be a complete discrete valuation field of mixed characteristic  $(0, p)$ . We write  $\pi = \pi_K$  for its uniformizer and  $e$  for its absolute ramification index. We normalize a valuation  $v_K$  of  $K$  as  $v_K(\pi) = 1$  and extend it to the algebraic closure  $\bar{K}$  of  $K$ . For  $a \in \bar{K}$  and  $j \in \mathbb{R}$ , let  $D(a, j)$  denote the closed disc  $\{z \in \mathcal{O}_{\bar{K}} \mid v_K(z-a) \geq j\}$ . This is the underlying subset of a  $K(a)$ -affinoid subdomain of the unit disc over  $K(a)$ .

For integers  $0 \leq s_1, \dots, s_r \leq e$ , let  $\mathcal{G}(s_1, \dots, s_r)$  denote the Raynaud  $\mathbb{F}_{p^r}$ -vector space scheme over  $\mathcal{O}_K$  defined by the  $r$  equations  $T_1^p = \pi^{s_1} T_2, T_2^p = \pi^{s_2} T_3, \dots, T_r^p = \pi^{s_r} T_1$  ([14]). We set  $j_k = (ps_k + p^2 s_{k-1} + \dots + p^k s_1 + p^{k+1} s_r + p^{k+2} s_{r-1} + \dots + p^r s_{k+1}) / (p^r - 1)$ . Before the calculation of  $c(\mathcal{G}(s_1, \dots, s_r))$ , we gather some elementary lemmas.

**Lemma 5.1.** *Let  $a \in \mathcal{O}_K$  and  $s = v_K(a)$ . Then the affinoid variety  $X^j(\bar{K}) = \{x \in \mathcal{O}_{\bar{K}} \mid v_K(x^p - a) \geq j\}$  is equal to*

$$\begin{cases} D(a^{1/p}, j/p) & \text{if } j \leq s + pe/(p-1), \\ \prod_{i=0}^{p-1} D(a^{1/p} \zeta_p^i, j - e - (p-1)s/p) & \text{if } j > s + pe/(p-1). \end{cases}$$

*Proof.* We have  $v_K(x^p - a) = \sum_i v_K(x - a^{1/p} \zeta_p^i)$ . If  $v_K(x - a^{1/p} \zeta_p^i) \geq v_K(x - a^{1/p} \zeta_p^{i'})$  for any  $i' \neq i$ , then  $v_K(x - a^{1/p} \zeta_p^{i'}) \leq v_K(a^{1/p} \zeta_p^{i'} (1 - \zeta_p^{i-i'})) = s/p + e/(p-1)$ . Thus we have  $v_K(x - a^{1/p} \zeta_p^i) \geq \sup(j/p, j - (p-1)s/p - e)$  and

$$X^j(\bar{K}) \subseteq \bigcup_i D(a^{1/p} \zeta_p^i, \sup(j/p, j - (p-1)s/p - e)).$$

Suppose that  $j/p \geq j - (p-1)s/p - e$ . Then we have  $v_K(a^{1/p}(1 - \zeta_p^i)) = s/p + e/(p-1) \geq j/p$ ,  $D(a^{1/p}, j/p) = D(a^{1/p} \zeta_p^i, j/p)$  for any  $i$  and thus

$$X^j(\bar{K}) = D(a^{1/p}, j/p).$$

When  $j/p < j - (p-1)s/p - e$ , we have  $v_K(a^{1/p}(1 - \zeta_p^i)) = s/p + e/(p-1) < j - (p-1)s/p - e$ . This means that if  $w \in D(a^{1/p} \zeta_p^i, j - (p-1)s/p - e)$  for some  $i$ , then  $v_K(w - a^{1/p} \zeta_p^{i'}) < j - (p-1)s/p - e$  for any other  $i'$ .

Thus the discs  $D(a^{1/p}\zeta_p^i, j - (p-1)s/p - e)$  are disjoint and

$$X^j(\bar{K}) = \coprod_i D(a^{1/p}\zeta_p^i, j - (p-1)s/p - e).$$

These are equalities of the underlying sets of affinoid subdomains of the unit disc over  $K(a^{1/p}, \zeta_p)$ . By the universality of an affinoid subdomain, this extends to an isomorphism of affinoid varieties.  $\square$

We can prove the following lemma just in the same way.

**Lemma 5.2.** *The affinoid variety  $\{x \in \mathcal{O}_{\bar{K}} \mid v_K(x^{p^r} - ax) \geq j\}$  is equal to*

$$\begin{cases} D(0, j/p^r) & \text{if } j \leq p^r v(a)/(p^r - 1), \\ \prod_{i=0}^{p^r-1} D(\sigma_i, j - v(a)) & \text{if } j > p^r v(a)/(p^r - 1), \end{cases}$$

where  $\sigma_i$ 's are the roots of  $X^{p^r} = aX$ .

**Lemma 5.3.** *For  $g_1(Y_1, \dots, Y_d), g_2(Y_1, \dots, Y_d) \in K[Y_1, \dots, Y_d]$  and  $j_1 \geq j_2$ , the affinoid variety  $\{(x, y_1, \dots, y_d) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}}^d \mid v_K(x - g_1(y_1, \dots, y_d)) \geq j_1, v_K(x - g_2(y_1, \dots, y_d)) \geq j_2\}$  is equal to  $\{(x, y_1, \dots, y_d) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}}^d \mid v_K(x - g_1(y_1, \dots, y_d)) \geq j_1, v_K(g_1(y_1, \dots, y_d) - g_2(y_1, \dots, y_d)) \geq j_2\}$ .*

*Proof.* For fixed  $(x, y)$ , these two conditions are equivalent. The universality of an affinoid subdomain proves the lemma.  $\square$

**Lemma 5.4.** *Let  $a \in \mathcal{O}_{\bar{K}}$  and  $s = v_K(a)$ . If  $j \leq pe/(p-1) + s$ , then the affinoid variety  $X^j(\bar{K}) = \{(x, y) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}} \mid v_K(x^p - ay^{p^n}) \geq j\}$  is equal to  $\{(x, y) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}} \mid v_K(x - a^{1/p}y^{p^{n-1}}) \geq j/p\}$ .*

*Proof.* Lemma 5.1 shows that the fiber of the second projection  $X^j(\bar{K}) \rightarrow \mathcal{O}_{\bar{K}}$  at  $y$  is equal to

$$\begin{cases} D(a^{1/p}y^{p^{n-1}}, j/p) & \text{if } j \leq s + p^{n-1}v_K(y) + pe/(p-1), \\ \prod_{i=0}^{p-1} D(a^{1/p}\zeta_p^i y^{p^{n-1}}, j - e - (p-1)(s + p^{n-1}v_K(y))/p) & \text{otherwise.} \end{cases}$$

Thus we have  $X^j(\bar{K}) = \{(x, y) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}} \mid v_K(x - a^{1/p}y^{p^{n-1}}) \geq j/p\}$  for  $j \leq pe/(p-1) + s$ . This is the underlying set of a  $K(a^{1/p})$ -affinoid variety. Again this equality extends to an isomorphism over  $K(a^{1/p})$ .  $\square$

Now we proceed to the proof of the main theorem of this section.

**Theorem 5.5.**  $c(\mathcal{G}(s_1, \dots, s_r)) = \sup_k j_k$ .



*Proof.* We may assume that  $j_r$  is the supremum of  $j_k$ 's. If  $j_r = 0$ , then  $\mathcal{G}(s_1, \dots, s_r)$  is etale and  $c(\mathcal{G}(s_1, \dots, s_r)) = 0$ . Thus we may assume  $j_r > 0$ . Consider the homomorphism of  $\mathcal{O}_K$ -algebras

$$\begin{aligned} A &= \mathcal{O}_K[T_1, \dots, T_r]/(T_1^p - \pi^{s_1}T_2, \dots, T_r^p - \pi^{s_r}T_1) \rightarrow \\ B &= \mathcal{O}_K[W, T_2, \dots, T_r]/(W^{p^r} - \pi^{s_1}T_2, T_2^p - \pi^{s_2}T_3, \dots, \\ &\quad T_{r-1}^p - \pi^{s_{r-1}}T_r, T_r^p - \pi^{s_r}W^{p^{r-1}}), \end{aligned}$$

defined by  $T_1 \mapsto W^{p^{r-1}}$ . This induces a surjection of  $K$ -affinoid varieties

$$X_B^j(\bar{K}) \ni (w, t_2, \dots, t_r) \mapsto (w^{p^{r-1}}, t_2, \dots, t_r) \in X_A^j(\bar{K}),$$

where

$$\begin{aligned} X_A^j(\bar{K}) &= \{(t_1, \dots, t_r) \in \mathcal{O}_{\bar{K}}^r \mid v_K(t_1^p - \pi^{s_1}t_2) \geq j, \dots, \\ &\quad v_K(t_{r-1}^p - \pi^{s_{r-1}}t_r) \geq j, v_K(t_r^p - \pi^{s_r}t_1) \geq j\} \end{aligned}$$

and

$$\begin{aligned} X_B^j(\bar{K}) &= \{(w, t_2, \dots, t_r) \in \mathcal{O}_{\bar{K}}^r \mid v_K(w^{p^r} - \pi^{s_1}t_2) \geq j, \\ &\quad v_K(t_2^p - \pi^{s_2}t_3) \geq j, \dots, v_K(t_r^p - \pi^{s_r}w^{p^{r-1}}) \geq j\}. \end{aligned}$$

These are affinoid subdomains of the  $r$ -dimensional unit polydisc over  $K$ . We calculate a jump of  $\{F^j(B)\}_{j \in \mathbb{Q}_{>0}}$  at first.

**Lemma 5.6.** *If  $j_r < pe/(p-1)$ , then the first jump of  $\{F^j(B)\}_{j \in \mathbb{Q}_{>0}}$  occurs at  $j = j_r$  and  $\sharp F^{j_r}(B) = p^r$ .*

Note that the base change from  $K$  to a finite extension  $L$  multiplies  $s_i$ 's,  $j_i$ 's and  $e$  by the ramification index of  $L/K$ . Thus, to prove Lemma 5.6 and Theorem 5.5, we may assume that  $p^{r-1}$  divides  $s_i$ 's and  $e$ .

*Proof.* Consider the  $K$ -affinoid variety  $X_B^j$  for  $j \leq pe/(p-1)$ . Then the iterative use of Lemma 5.4 and Lemma 5.3 shows that the affinoid variety  $X_B^j(\bar{K})$  is equal to

$$\begin{aligned} \{v_K(w^{p^r} - \pi^{(s_r + ps_{r-1} + \dots + p^{r-1}s_1)/p^{r-1}}w) \geq pl_1(j), v_K(t_2 - g_2(w)) \geq u_2, \\ v_K(t_3 - g_3(t_2, w)) \geq u_3, \dots, v_K(t_r - g_r(t_{r-1}, w)) \geq u_r\}, \end{aligned}$$

where  $l_i(j)$ ,  $g_i(t_{i-1}, w)$ ,  $g_2(w)$  and  $u_i$  are defined as follows;

- $l_r(j) = j/p$ ,
- $l_{i-1}(j) = \inf(j, l_i(j) + s_{i-1})/p$ ,
- $g_i(t_{i-1}, w) = t_{i-1}^p/\pi^{s_{i-1}}$  and  $u_i = j - s_{i-1}$  if  $j \geq l_i(j) + s_{i-1}$ ,
- $g_i(t_{i-1}, w) = \pi^{s_r + ps_{r-1} + \dots + p^{r-i}s_i/p^{r-i+1}}w^{p^{i-2}}$  and  $u_i = l_i(j)$  if  $j < l_i(j) + s_{i-1}$ ,

- $g_2(w) = g_2(w^{p^{r-1}}, w)$ .

Note that  $l_i(j)$  is a strictly monotone increasing function of  $j$ . This affinoid variety is isomorphic to the product of the affinoid variety  $\{w \in \mathcal{O}_{\bar{K}} \mid v(w^{p^r} - \pi^{(s_r + ps_{r-1} + \dots + p^{r-1}s_1)/p^{r-1}} w) \geq pl_1(j)\}$  and discs. Therefore, from Lemma 5.2, we see that the first jump of  $\{F^j(B)\}_{j \in \mathbb{Q}_{>0}}$  occurs at  $j$  such that  $pl_1(j) = j_r$ , provided this  $j$  satisfies  $0 < j < pe/(p-1)$ . Moreover, then we have  $\sharp F^j(B) = p^r$ . Thus the following lemma and the strict monotonicity of  $l_1$  terminate the proof of Lemma 5.6.  $\square$

**Lemma 5.7.**  $l_1(j_r) = j_r/p$ .

*Proof.* Suppose that there is  $k$  such that  $l_k(j_r) = j_r/p$  and  $j_r \geq l_{k'}(j_r) + s_{k'}$  for any  $1 < k' \leq k$ . Then we have  $l_1(j_r) = \inf(j_r, (j_r + ps_{k-1} + p^2s_{k-2} + \dots + p^{k-1}s_1)/p^{k-1})/p$  and the assumption  $j_{k-1} \leq j_r$  implies  $l_1(j_r) = j_r/p$ .  $\square$

On the other hand, let  $s = (s_r + ps_{r-1} + \dots + p^{r-1}s_1)/p^{r-1}$  and  $\sigma_0, \dots, \sigma_{p^r-1}$  be the roots of the equation  $X^{p^r} - \pi^s X = 0$ . Then we see that the images by  $w \mapsto w^{p^{r-1}}$  of the discs  $D(\sigma_i, pl_1(j) - s)$  are disjoint for  $j > j_r$ . Hence the surjection  $\pi_0(X_B^j(\bar{K})) \rightarrow \pi_0(X_A^j(\bar{K}))$  is bijective for  $0 < j \leq pe/(p-1)$  and the first (and the last) jump of  $\{F^j(A)\}_{j \in \mathbb{Q}_{>0}}$  also occurs at  $j_r$ , provided  $j_r < pe/(p-1)$ .

When  $j_r = pe/(p-1)$ , we see that  $s_k = e > 0$  for any  $k$ . Thus we can use Lemma 5.4 for  $j < pe/(p-1) + \varepsilon$  with sufficiently small  $\varepsilon > 0$ . Then, by the same reasoning as above, we conclude that  $c(A) = pe/(p-1)$ .  $\square$

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