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TAME CHARACTERS AND RAMIFICATION OF
FINITE FLAT GROUP SCHEMES

SHIN HATTORI

1. Introduction

Let $K$ be a complete discrete valuation field of mixed characteristic $(0, p)$ with residue field $F$ which may be imperfect, $G_K$ be its absolute Galois group and $I_K$ be its inertia subgroup. Let $\mathcal{G}$ be a finite flat group scheme over $O_K$. When $\mathcal{G}$ is monogenic, that is to say, when the affine algebra of $\mathcal{G}$ is generated over $O_K$ by one element, it is well-known that the tame characters appearing in the $I_K$-module $\mathcal{G}(\bar{K})$ are determined by the slopes of the Newton polygon of a defining equation of $\mathcal{G}$ ([15, Proposition 10]).

On the other hand, for an elliptic modular form $f$ of level $N$ prime to $p$, we also have a description of the tame characters of the associated mod $p$ Galois representation $\bar{\rho}_f$ ([10, Theorem 2.5, Theorem 2.6], [9, Section 4.3]). This is based on Raynaud’s theory of prolongations of finite flat group schemes or the integral $p$-adic Hodge theory. However, for an analogous study of the associated mod $p$ Galois representation of a Hilbert modular form over a totally real number field, we encounter a local field of higher absolute ramification index. In this case, these two theories no longer work well and we need some other techniques.

In this paper, to propose a new approach toward this problem, we generalize the aforementioned result of [15] to the higher dimensional case using the ramification theory of Abbes and Saito ([2], [3]) and determine the tame characters appearing in $\mathcal{G}(\bar{K})$ in terms of the ramification of $\mathcal{G}$ without any restriction on the absolute ramification index of $K$. Namely, we show the following theorem.

Theorem 1.1. Let $\mathcal{G}$ be a finite flat group scheme over $O_K$. Write $\{\mathcal{G}^j\}_{j \in \mathbb{Q}_{>0}}$ for the ramification filtration of $\mathcal{G}$ in the sense of [2] and [3]. Then the graded piece $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K})$ is killed by $p$ and the $I_K$-module $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K}) \otimes_{\mathbb{F}_p} \mathbb{F}_p$ is the direct sum of $\mathbb{F}_p$-conjugates of the fundamental character $\theta_j$ of level $j$.

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In view of the following corollary, we can regard this theorem as a counterpart for finite flat group schemes of the Hasse-Arf theorem in the classical ramification theory.

**Corollary 1.2.** Let $L$ be an abelian extension of $K$. Suppose that its integer ring $\mathcal{O}_L$ is a $G$-torsor over $\mathcal{O}_K$. Then the denominator of every jump of the upper numbering ramification filtration $\{\text{Gal}(L/K)^j\}_{j \in \mathbb{Q}_{>0}}$ ([2]) is a power of $p$.

To prove the main theorem, we firstly show that the tubular neighborhood of $G$ can be chosen to have a rigid analytic group structure. Passing to the closed fiber, we realize the graded piece of the ramification filtration of $G$ as the kernel of an etale isogeny of the additive groups $\mathbb{G}_a^r$ over $\mathbb{F}$. Then we determine the tame characters by comparing the $I_K$-action on the graded piece with the $\mathbb{G}_m$-action on $\mathbb{G}_a^r$ given by the multiplication. As an appendix, we also give an explicit formula for the conductor of a Raynaud $\mathbb{F}$-vector space scheme over $\mathcal{O}_K$ ([14]).

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2. REVIEW OF THE RAMIFICATION THEORY OF ABBES AND SAITO

Let $K$ be a complete discrete valuation field with residue field $F$ which may be imperfect. Set $\pi = \pi_K$ to be an uniformizer of $K$. The separable closure of $K$ is denoted by $\overline{K}$ and the absolute Galois group of $K$ by $G_K$. Let $m_K$ and $\overline{F}$ be the maximal ideal and the residue field of $\mathcal{O}_K$ respectively. In [2] and [3], Abbes and Saito defined the ramification theory of a finite flat $\mathcal{O}_K$-algebra of relative complete intersection. In this section, we gather the necessary definitions and briefly recall their theory.

Let $A$ be a finite flat $\mathcal{O}_K$-algebra and $\mathbb{A}$ be a complete Noetherian semi-local ring (with its topology defined by $\text{rad}(\mathbb{A})$) which is of formally smooth over $\mathcal{O}_K$ and whose quotient ring $\mathbb{A}/\text{rad}(\mathbb{A})$ is of finite type over $F$. A surjection of $\mathcal{O}_K$-algebras $\mathbb{A} \to A$ is called an embedding if $\mathbb{A}/\text{rad}(\mathbb{A}) \to A/\text{rad}(A)$ is an isomorphism. For an embedding $(\mathbb{A} \to A)$ and $j \in \mathbb{Q}_{>0}$, the $j$-th tubular neighborhood of $(\mathbb{A} \to A)$ is the $K$-affinoid variety $X^j(\mathbb{A} \to A)$ constructed as follows. Write $j = k/l$ with $k, l$ non-negative integers. Put $I = \text{Ker}(\mathbb{A} \to A)$ and $A^k_l = A[I^l/\pi^k]$, where $\wedge$ means the $\pi$-adic completion. Then $A^k_l$ is a quotient ring of the Tate algebra $\mathcal{O}_K \langle T_1, \ldots, T_r \rangle$ for some $r$. Its generic fiber $A^k_l = A^1_l \otimes_{\mathcal{O}_K} K$ is independent of the choice of a representation...
j = k/l ([3, Lemma 1.4]) and set $X^j(\mathbb{A} \to A) = \text{Sp}(\mathcal{A}_K^j)$. This affinoid variety is geometrically regular ([3, Lemma 1.6]).

We put $F(A) = \text{Hom}_{\mathcal{O}_K\text{-alg}}(A, \mathcal{O}_K)$ and $F^j(A) = \varprojlim \pi_0(X^j(\mathbb{A} \to A))_K$. Here $\pi_0(X)_K$ denotes the set of geometric connected components of a $K$-affinoid variety $X$ and the projective limit is taken in the category of embeddings of $A$. Note that the projective family $\pi_0(X^j(\mathbb{A} \to A))_K$ is constant ([3, Section 1.2]). These define contravariant functors $F$ and $F^j$ from the category of finite flat $\mathcal{O}_K$-algebras to the category of finite $G_K$-sets. Moreover, there are morphisms of functors $F \to F^j$ and $F^j \to F^{j'}$ for $j' \geq j > 0$.

By the finiteness theorem of Grauert and Remmert ([8, Theorem 1.3]), there exists a finite separable extension $L$ of $K$ such that the geometric closed fiber of the unit disc $\mathcal{X}^j(\mathbb{A} \to A)_{O_L}$ for the supremum norm in $X^j(\mathbb{A} \to A)_L = X^j(\mathbb{A} \to A) \times_K L$ is reduced. Then for any finite separable extension $L'$ of $L$, the $\pi_{L'}$-adic formal scheme $\mathcal{X}^j(\mathbb{A} \to A)_{O_{L}} \otimes_{O_L} O_{L'}$ coincides with the unit disc for the supremum norm in $X^j(\mathbb{A} \to A)_{L'}$ and thus is normal. The $\pi_{L'}$-adic formal scheme $\mathcal{X}^j(\mathbb{A} \to A)_{O_L}$ is referred as the stable normalized integral model of $X^j(\mathbb{A} \to A)$ over $O_L$ and its geometric closed fiber is denoted by $\tilde{X}^j(\mathbb{A} \to A)$. If $L/K$ is Galois, the Galois group $\text{Gal}(L/K)$ acts on it by the functoriality of the unit disc for the supremum norm. We have the $G_K$-equivariant isomorphism $\pi_0(\tilde{X}^j(\mathbb{A} \to A))_F \to \pi_0(X^j(\mathbb{A} \to A))_K$, where the former is the set of geometric connected components of $\tilde{X}^j(\mathbb{A} \to A)$ ([3, Corollary 1.11]).

Suppose that $A$ is of relative complete intersection over $\mathcal{O}_K$ and $A \otimes_{\mathcal{O}_K} K$ is etale over $K$. Then the natural map $F(A) \to F^j(A)$ is surjective. The family $\{F(A) \to F^j(A)\}_{j \in \mathbb{Q}_{>0}}$ is separated, exhaustive and its jumps are rational ([2, Proposition 6.4]). The conductor of $A$ is defined to be $c(A) = \inf\{j \in \mathbb{Q}_{>0} \mid F(A) \to F^j(A) \text{ is an isomorphism}\}$. If $B$ is the affine algebra of a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$ which is generically etale, then $B$ is of relative complete intersection (for example, [5, Proposition 2.2.2]) and the theory above can all be applied to $B$. By the functoriality, $F^j(B)$ is endowed with a $G_K$-module structure (1, Lemme 2.1) and the natural map $\mathcal{G}(\tilde{K}) = F(B) \to F^j(B)$ is a $G_K$-homomorphism. Let $\mathcal{G}^j$ denote the schematic closure ([14]) in $\mathcal{G}$ of the kernel of this homomorphism. It is called the $j$-th ramification filtration of $\mathcal{G}$. We refer $c(B)$ as the conductor of $\mathcal{G}$, which is denoted also by $c(\mathcal{G})$. We put $\mathcal{G}^{j+}(\tilde{K}) = \bigcup_{j' > j} \mathcal{G}^{j'}(\tilde{K})$ and define $\mathcal{G}^{j+}$ to be the schematic closure of $\mathcal{G}^{j+}(\tilde{K})$ in $\mathcal{G}$. 
3. Group structure on the tubular neighborhood of a finite flat group scheme

Let $K$ denote a complete discrete valuation field of mixed characteristic $(0, p)$ whose residue field $F$ may be imperfect and $v_K$ the valuation of $K$ extended to $K$ which is normalized as $v_K(\pi) = 1$. Let $\mathcal{G} = \text{Spec}(B)$ be a connected finite flat group scheme over $\mathcal{O}_K$. We define a formal resolution of $\mathcal{G}$ to be a closed immersion $\mathcal{G} \to \Gamma$ of (profinite) formal group schemes over $\mathcal{O}_K$, where $\Gamma = \text{Spf}(B)$ is connected and smooth. Such an immersion can be constructed as follows. By a theorem of Raynaud ([4, Théorème 3.1.1]), we can find an abelian scheme $A$ over $\mathcal{O}_K$ and a closed immersion of group schemes $\mathcal{G} \to A$. Taking the formal completion of $A$ along the zero section, we get a formal resolution of $\mathcal{G}$. We define the relative dimension of $\mathcal{G}$ over $\mathcal{O}_K$ as the dimension of a formal resolution ($\mathcal{G} \to \Gamma$). We define a morphism of formal resolutions to be a pair of group homomorphisms $(f, \mathfrak{f})$ which makes the following diagram commutative.

\[
\begin{array}{ccc}
\mathcal{G} & \longrightarrow & \Gamma \\
\downarrow f & & \downarrow \mathfrak{f} \\
\mathcal{G}' & \longrightarrow & \Gamma'
\end{array}
\]

Note that a formal resolution of $\mathcal{G}$ is also an embedding of $B$ in the sense of Section 2. We say $(f, \mathfrak{f})$ is finite flat if this is finite flat as a map of embeddings ([3]). Consider the $j$-th tubular neighborhood $X^j(B \to B)$ of the embedding $(B \to B)$, which we also write as $X^j(\mathcal{G} \to \Gamma)$. The following lemma, whose first part is implicit in [1], enables us to introduce a group structure on this affinoid variety.

**Lemma 3.1.** Let $(\mathbb{A} \to A)$ and $(\mathbb{B} \to B)$ be embeddings of finite flat $\mathcal{O}_K$-algebras. Put $C = \mathbb{A} \otimes \mathcal{O}_K \mathbb{B}$ and $C = A \otimes \mathcal{O}_K B$. Then the surjection $C \to C$ is also an embedding and we have a canonical isomorphism $X^j(C \to C) \to X^j(A \to A) \times_K X^j(B \to B)$. Moreover, this isomorphism extends to a canonical isomorphism between their stable normalized integral models $\mathcal{X}^j(C \to C)_{\mathcal{O}_K} \to \mathcal{X}^j(A \to A)_{\mathcal{O}_K} \times_{\mathcal{O}_K} \mathcal{X}^j(B \to B)_{\mathcal{O}_K}$.

**Proof.** By the functoriality of a tubular neighborhood, we have an affinoid map $\Phi: X^j(C \to C) \to X^j(A \to A) \times_K X^j(B \to B)$. To see that $\Phi$ is an isomorphism, we may replace $K$ with a finite separable extension and suppose that $A$ and $B$ are local, $j$ is an integer and $A/\text{rad}(A) = B/\text{rad}(B) = F$. Then we may identify $\mathbb{A}$ with $\mathcal{O}_K[[T_1, \ldots , T_r]]$ and $\mathbb{B}$ with $\mathcal{O}_K[[T_1', \ldots , T_{r'}]]$ for some $r$ and $r'$. Let $J = (f_1, \ldots , f_s)$ (resp. $J = (g_1, \ldots , g_{s'})$) be the kernel of the surjection $\mathbb{A} =
Corollary 3.2. Let $(\mathcal{G} \rightarrow \Gamma)$ be a formal resolution of $\mathcal{G}$. Then the group structure of $\Gamma$ induces a rigid $K$-analytic group structure on the tubular neighborhood $X^j(\mathcal{G} \rightarrow \Gamma)$. This group structure also extends to $X^j(\mathcal{G} \rightarrow \Gamma)_{\mathcal{O}_K}$ (resp. $X^j(\mathcal{G} \rightarrow \Gamma)$) and endows it with a $\pi$-adic formal group scheme structure over $\mathcal{O}_K$ (resp. an algebraic group structure over $\bar{F}$).

Moreover, for a morphism of formal resolutions $(\mathcal{G} \rightarrow \Gamma) \rightarrow (\mathcal{G}' \rightarrow \Gamma')$, the induced affinoid map $X^j(\mathcal{G} \rightarrow \Gamma) \rightarrow X^j(\mathcal{G}' \rightarrow \Gamma')$ is a homomorphism of rigid $K$-analytic groups. This homomorphism also extends to their stable normalized integral models as a homomorphism $X^j(\mathcal{G} \rightarrow \Gamma)_{\mathcal{O}_K} \rightarrow X^j(\mathcal{G}' \rightarrow \Gamma')_{\mathcal{O}_K}$ of $\pi$-adic formal group schemes and to their geometric closed fibers as a homomorphism $X^j(\mathcal{G} \rightarrow \Gamma) \rightarrow X^j(\mathcal{G}' \rightarrow \Gamma')$ of algebraic groups over $\bar{F}$.

Let $\mathcal{G} = \text{Spec}(B)$ be a connected finite flat group scheme over $\mathcal{O}_K$ and $(\mathcal{G} \rightarrow \Gamma = \text{Spf}(B))$ be a formal resolution of dimension $r$. Set $\text{Spf}(\mathcal{A}) = \Gamma/\mathcal{G}$ and regard the zero section $\text{Spec}(\mathcal{O}_K) \rightarrow \text{Spf}(\mathcal{A})$ as a formal resolution of the trivial group. Then we have a finite flat map
of formal resolutions

\[ G \longrightarrow \text{Spf}(\mathbb{B}) \]

\[ \text{Spec}(\mathcal{O}_K) \longrightarrow \text{Spf}(A) \]

By Corollary 3.2 and [3, Lemma 1.8], we get a finite flat map of rigid \( K \)-analytic groups \( f^j : X^j_G = X^j(G \to \Gamma) \to D^{r,j} = X^j(\text{Spec}(\mathcal{O}_K) \to \text{Spf}(A)) \), where \( D^{r,j} \) denotes the \( r \)-dimensional polydisc \( \{(z_1, \ldots, z_r) \in \mathcal{O}_K^r \mid v_K(z_i) \geq j \text{ for any } i \} \). We call this the affinoid homomorphism associated to a formal resolution \((G \to \Gamma)\). Write \( B^j_K \) and \( A^j_K \) for the \( K \)-affinoid algebras of \( X^j_G \) and \( D^{r,j} \) respectively. The stable normalized integral model over \( \mathcal{O}_L \) of \( X^j_G \) (resp. \( D^{r,j} \)) is denoted by \( X^j_{G,K} \) (resp. \( D^{r,j}_{K} \)) and its geometric closed fiber by \( \hat{X}^j_G \) (resp. \( D^{r,j}_{K} \)). Note that the algebraic group \( \hat{X}^j_G \) is reduced, hence smooth by [16, Theorem 11.6].

**Lemma 3.3.** The affinoid homomorphism \( f^j : X^j_G \to D^{r,j} \) is etale for any \( j > 0 \). Moreover, for \( j > C(G) \), there exists a finite extension \( K' / K \) such that \( X^j_{G,K'} \) is isomorphic to the disjoint sum of finitely many copies of \( D^{r,j}_{K'} \).

**Proof.** We have \( \Omega^1_B / A_K = B^j_K B / B \). It is enough to show that \( \hat{\Omega}_B / A_K \) is a torsion \( \mathcal{O}_K \)-module. Let \( J_A \) and \( J_B \) be the augmentation ideals of \( A \) and \( B \) respectively. Set \( I = \text{Ker}(B \to B) \). Then \( \hat{\Omega}_B / A_K = \text{Coker}(B \otimes \mathcal{O}_K \to \hat{\Omega}_B / \mathcal{O}_K) \) is equal to \( B \otimes \mathcal{O}_K \text{Coker}(\text{Cot}(A) \to \text{Cot}(B)) = B \otimes \mathcal{O}_K J_B / (I + J_B^2) = B \otimes \mathcal{O}_K \text{Cot}(B) \). This shows the first assertion. For the second assertion, take a finite extension \( K' \) of \( K \) where the geometric connected components of \( X^j_G \) are defined. By assumption, each of the connected components of \( X^j_{G,K'} \) is a finite etale cover of \( D^{r,j}_{K'} \) whose degree is one. Thus this is isomorphic to \( D^{r,j}_{K'} \).

Take a finite extension \( L \) of \( K \) where the stable normalized integral models of \( X^j_G \) and \( D^{r,j} \) are defined. The generic fiber \( \mathcal{G}_L = \mathcal{G} \times_{\mathcal{O}_K} L \) can be regarded as a rigid \( L \)-analytic subgroup of \( X^j_{G,L} \) defined by an ideal \( J_L \) of \( B^j_L \). Put \( \mathcal{J} = J_L \cap B^j_{\mathcal{O}_L} \) and \( \hat{\mathcal{B}}^j = B^j_{\mathcal{O}_L} / \mathcal{J} \). The latter is a subring of \( B_L = B \otimes_K L \). Since we have a commutative diagram with
surjective horizontal arrows and injective vertical arrows
\[
\begin{array}{ccc}
B_0^j \otimes_{\mathcal{O}_K} \mathcal{O}_L & \longrightarrow & B \otimes_{\mathcal{O}_K} \mathcal{O}_L \\
\downarrow & & \downarrow \\
\mathcal{B}_0^j & \longrightarrow & \mathcal{B}_L^j,
\end{array}
\]
we see that $\mathcal{B}_0^j$ is integral over $B \otimes_{\mathcal{O}_K} \mathcal{O}_L$. Thus the $\mathcal{O}_K$-algebra $\mathcal{B}_0^j$ is finite flat. Set $\mathcal{H}_0^j = \text{Spec}(\mathcal{B}_0^j)$. This can be regarded as a closed $\pi_L$-adic formal subscheme of $\mathcal{X}_G^{j,\mathcal{O}_L}$.

**Lemma 3.4.** The group structure of $G$ extends to $\mathcal{H}_0^j$. The group scheme $\mathcal{H}_0^j$ is a closed $\pi_L$-adic formal subgroup scheme of $\mathcal{X}_G^{j,\mathcal{O}_L}$.

**Proof.** Put $K = \text{Ker}(\mathcal{B}_0^j \otimes_{\mathcal{O}_L} \mathcal{B}_0^j \rightarrow \mathcal{B}_L^j \otimes_{\mathcal{O}_L} \mathcal{B}_L^j)$ and $K_L = K \otimes_{\mathcal{O}_L} L$. Let $\mu$ be the coproduct of $\mathcal{B}_0^j$. We must show $\mu(J) \subseteq K$. This follows from the commutative diagram below whose rows are exact and vertical arrows are injective.

\[
\begin{array}{cccccc}
0 & \longrightarrow & K & \longrightarrow & \mathcal{B}_0^j \otimes_{\mathcal{O}_L} \mathcal{B}_0^j & \longrightarrow & \mathcal{B}_L^j \otimes_{\mathcal{O}_L} \mathcal{B}_L^j & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_L & \longrightarrow & \mathcal{B}_L^j \otimes_{\mathcal{O}_L} \mathcal{B}_L^j & \longrightarrow & B_L \otimes_{\mathcal{O}_L} B_L & \longrightarrow & 0
\end{array}
\]

Passing to the generic fiber, we see that the second assertion holds. $\square$

**Lemma 3.5.** The associated homomorphism $\tilde{f}^j : \mathcal{X}_G^{j,\mathcal{O}_L} \rightarrow \mathcal{D}_0^{r,j}$ is finite flat. Moreover, there exists an exact sequence of $\pi_L$-adic formal group schemes

\[(1) \quad 0 \rightarrow \mathcal{H}_0^j \rightarrow \mathcal{X}_G^{j,\mathcal{O}_L} \rightarrow \mathcal{D}_0^{r,j} \rightarrow 0.\]

**Proof.** From Lemma 3.3, the associated affinoid map $f^j : X_G^j \rightarrow D^{r,j}$ is finite etale. Let $\mathcal{B}_L^j$ and $\mathcal{A}_L^j$ be their affinoid algebras as above. Since $D^{r,j}$ is integral, we see that $f^j$ is surjective and the ring homomorphism $\mathcal{A}_L^j \rightarrow \mathcal{B}_L^j$ is injective. Thus we have an injection $\tilde{A}_L^j \rightarrow \tilde{B}_L^j$, which is finite by [6, Corollary 6.4.1/6]. Hence $\tilde{f}^j : \tilde{X}_G^j \rightarrow \tilde{D}^{r,j}$ is a surjective homomorphism of algebraic groups over $\tilde{F}$. Since $\tilde{X}_G^j$ and $\tilde{D}^{r,j}$ are regular, we see that $\tilde{f}^j$ is faithfully flat by [13, Theorem 23.1]. Since $\tilde{A}_L^j$ and $\tilde{B}_L^j$ is $\pi_L$-torsion free, the map $\tilde{f}^j$ is flat by the local criterion of flatness. Put $\mathcal{H}' = \text{Ker}(f^j)$. This is a closed $\pi_L$-adic formal subgroup scheme of $\mathcal{X}_G^{j,\mathcal{O}_L}$ and can be regarded also as a finite flat group scheme over $\mathcal{O}_L$. Passing to the generic fiber, we see that $\mathcal{H}_0^j$ is a closed subgroup scheme of $\mathcal{H}'$. Comparing these ranks concludes the lemma.
From Lemma 3.3, we see that for \( j > c = c(G) \), the map \( \hat{f}^j \) identifies \( X^j_G,\mathcal{O}_L \) with the direct sum of finitely many copies of \( D_{r;j}^c,\mathcal{O}_L \). More precisely, we have the following.

**Lemma 3.6.** Let \( c = c(G) \) be the conductor of \( G \). Then the associated homomorphism \( \hat{f}^j : X^j_G,\mathcal{O}_L \to D_{r;j}^c,\mathcal{O}_L \) is finite etale if and only if \( j \geq c \).

**Proof.** Let \( sp_j : X^j_G \to \tilde{X}^j_G \) be the specialization map. By Lemma 3.3, we see that \( \hat{f}^c \) is finite etale at \( sp_c(x) \) for any \( x \in G(K) \) as in the proof of [2, Theorem 7.2], using a theorem of Bosch ([7, Lemma 2.1]). We conclude the etaleness of \( \hat{f}^c \) by the existence of the group structure. We have \( \Omega^1_{B^c,\mathcal{O}_L} = A^c,\mathcal{O}_L \otimes,\mathcal{O}_L = 0 \). Since \( B^c,\mathcal{O}_L \) is \( \pi_L \)-adic complete and Noetherian, we see that \( \Omega^1_{B^c,\mathcal{O}_L} = 0 \) and the homomorphism \( \hat{f}^c : X^j_G,\mathcal{O}_L \to D_{r;j}^c,\mathcal{O}_L \) is finite etale.

Let \( \hat{0} \) be the zero section of \( \tilde{D}^{r;j} \) and set \( X^{j+}_G = \cup_{j' > j} X^{j'}_G \). Then we have \( (\hat{f}^j)^{-1}(\hat{0}) = sp_j(X^{j+}_G) = sp_j(G(K)) \). If \( \hat{f}^j \) is etale, then \( \#(\hat{f}^j)^{-1}(\hat{0}) \) equals the degree of \( \hat{f}^j \), namely \( \#G(K) \). Thus \( X^{j+}_G \) splits and we have \( j \geq c \).

\( \square \)

4. **Ramification and the \( I_K \)-module structure of a finite flat group scheme**

Consider the right action of \( I_K \) on \( K \) defined by \( \sigma z = \sigma^{-1}(z) \) for \( \sigma \in I_K \). This action induces a \( K \)-semilinear left action of \( I_K \) on \( X^j_G,K = X^j_G \times_K K \), which also extends to an \( \mathcal{O}_K \)-semilinear action on its stable normalized integral model \( X^j_G,\mathcal{O}_K \). Thus we have an \( \bar{F} \)-linear left action of \( I_K \) on its closed fiber \( \tilde{X}^j_G \). We call this the geometric monodromy action of \( I_K \) and write the action of \( \sigma \in I_K \) as \( \sigma_{\text{geom}} \) (note that here we follow the terminology in [2], and not in [3] where the monodromy action is called arithmetic, since in our case there is no "geometric" action other than the monodromy action). Similarly, we have the geometric monodromy action of \( I_K \) on \( \tilde{D}^{r;j} \).

The latter action is described as follows. Let the additive group (resp. multiplicative group) over \( \bar{F} \) be denoted by \( \mathbb{G}_a \) (resp. \( \mathbb{G}_m \)). Consider the left action \( \mathbb{G}_m \times \mathbb{G}_a \to \mathbb{G}_a \) given by the multiplication. Write this action of \( \lambda \in \bar{F}^\times \) as \( [\lambda] \). This action is defined by \( T_i \mapsto \lambda T_i \), where \( \mathbb{G}_a = \text{Spec}(\bar{F}[T_1,\ldots, T_r]) \). For \( j \in \mathbb{Q}_{>0}, \) we define the fundamental character \( \theta_j : I_K \to \bar{F}^\times \) to be \( \theta_k^p \), where \( k' / l' \) is the prime-to-\( p \)-denominator part.
Thus diagram whose rows are exact.

\[ \text{X\textsuperscript{ramification index over }K} \]

Proof. Lemma 4.2. If \( H \) geometric closed fiber of \( \text{denote the unit component of the algebraic group} \)

\[ K \]

\[ \text{be a finite Galois extension of } K \text{ containing } \pi^{1/l} \text{ and } e' = e(L/K) \text{ be its} \]

\[ \text{ramification index over } K \text{. Then } e'k/l \in \mathbb{Z} \text{ and the stable normalized integral model of } \)

\[ D^{r,j} \text{ over } \mathcal{O}_L \text{ is } \mathcal{O}_L(T_1/(\pi_L)^{e'k/l}, \ldots, T_r/(\pi_L)^{e'k/l}) = \]

\[ \mathcal{O}_L(W_1, \ldots, W_r), \text{ where } W_i = T_i/(\pi^{1/l})^k. \text{ Set } \mu_k \text{ to be the coproduct of } \mathbb{A}. \text{ We have} \]

\[ \mu_k(T_i) = T_i \otimes 1 + 1 \otimes T_i \text{ (higher degree)} \]

\[ \text{and then } \mu_k(W_i) = \mu_k((\pi^{1/l})^kW_i)/(\pi^{1/l})^k \text{ is equal to} \]

\[ W_i \otimes 1 + 1 \otimes W_i + (\pi^{1/l})^k \text{ (higher degree)} \]

in this \( \mathcal{O}_L \)-algebra. This shows the first assertion.

On the other hand, we know the geometric monodromy action on \( D^{r,j} \) is tame ([2, Lemma 7.7]). Write \( l = p^m l_0 \) with \( p \nmid l_0 \). Then for \( \sigma \in I_K \), we have \( \sigma((\pi^{1/l})^k)/(\pi^{1/l})^k \equiv \theta_{l_0}(\sigma)^{kpr^{-m}} \zeta_{p^m} \mod \mathfrak{m}_K \) with some \( N \) and this is equal to \( \theta_j(\sigma) \). Thus the action on the affine algebra \( \overline{F}[W_1, \ldots, W_r] \) of \( D^{r,j} \) is given by \( \sigma_{\text{geom}}(W_i) = \theta_j(\sigma)W_i \). This coincides with \( [\theta_j(\sigma)] \). \( \square \)

Next we consider the geometric monodromy action on \( \overline{X}_G^j \). Let \( \overline{X}_G^{j,0} \) denote the unit component of the algebraic group \( \overline{X}_G^j \) and \( \overline{H}^j \) be the geometric closed fiber of \( \mathcal{H}_0^j \). We begin with the following lemma.

Lemma 4.2. If \( \psi \in \text{End}(\overline{X}_G^{j,0}) \) induces the zero map on \( D^{r,j} \), then \( \psi = 0 \).

Proof. Put \( \mathcal{H}_0^j = \overline{H}^j \cap \overline{X}_G^{j,0} \). This is the kernel of the faithfully flat map \( \overline{X}_G^{j,0} \to D^{r,j} \) and by assumption we have the following commutative diagram whose rows are exact.

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{H}_0^j & \longrightarrow & \overline{X}_G^{j,0} & \longrightarrow & D^{r,j} & \longrightarrow & 0 \\
\downarrow & & \psi \downarrow & & \downarrow & & 0 \downarrow & & 0 \\
0 & \longrightarrow & \mathcal{H}_0^j & \longrightarrow & \overline{X}_G^{j,0} & \longrightarrow & D^{r,j} & \longrightarrow & 0 \\
\end{array}
\]

Thus \( \psi \) factors through \( \mathcal{H}_0^j \). Put \( \mathcal{C} = \text{Im}(\psi) \). Then this is a closed subgroup scheme of \( \mathcal{H}_0^j \) and the map \( \overline{X}_G^{j,0} \to \mathcal{C} \) is faithfully flat. Since \( \overline{X}_G^{j,0} \)
is regular and connected, we see that \( \bar{C} \) is also regular and connected by [13, Theorem 23.7]. Hence \( \bar{C} = 0 \) and we have \( \psi = 0 \).

Corollary 4.3. Let \( \mathcal{G} \) be a connected finite flat group scheme over \( \mathcal{O}_K \). Take a formal resolution \( (\mathcal{G} \to \Gamma) \) of dimension \( r \). Then the algebraic group \( \bar{X}_\mathcal{G}^{j,0} \) is isomorphic to \( \bar{G}^r \).

Proof. By the previous lemma and Lemma 4.1, we see that \( \bar{X}_\mathcal{G}^{j,0} \) is killed by \( p \). Hence the assertion follows from [12, Lemma 1.7.1].

Corollary 4.4. The geometric monodromy action of \( I_K \) on \( \bar{X}_\mathcal{G}^{j,0} \) is tame.

Proof. For an element \( \sigma \) of the wild inertia subgroup \( P_K \), the geometric monodromy action \( \sigma_{\text{geom}} \) on \( \bar{D}^{r,j} \) is trivial. Applying the lemma to \( \sigma_{\text{geom}} \) shows the assertion.

Corollary 4.5. Let \( J \) be a finite cyclic quotient of \( I_K \) through which the tame character \( \theta_j \) factors and \( \tau \) be a generator of \( J \). Let \( F(t) \) denote the minimal polynomial of \( \theta_j(\tau) \in \bar{F}_p \) over \( F_p \). Then the geometric monodromy action of \( I_K \) on \( \bar{X}_\mathcal{G}^{j,0} \) also factors through \( J \) and the equation \( F(\tau_{\text{geom}}) = 0 \) holds in \( \text{End}(\bar{X}_\mathcal{G}^{j,0}) \).

Proof. The first assertion follows from Lemma 4.2 as the previous corollary. The second assertion also follows from this lemma using Lemma 4.1.

Let \( c = c(\mathcal{G}) \) be the conductor of \( \mathcal{G} \). The lemma below enables us to realize \( \mathcal{G}^c(\bar{K}) \) as a subgroup of \( \bar{X}_\mathcal{G}^c \).

Lemma 4.6. The specialization map \( sp_c : X_{\mathcal{G}, \bar{K}}^c \to \bar{X}_\mathcal{G}^c \) induces an \( I_K \)-equivariant isomorphism \( \mathcal{G}(\bar{K}) \to \bar{H}^c(\bar{F}) \) and \( \mathcal{G}^c(\bar{K}) \to \bar{H}_0^c(\bar{F}) \). Here we consider on the left-hand side the natural action as the \( K \)-valued points of \( \mathcal{G} \) (resp. \( \mathcal{G}^c \)) and on the right-hand side the restriction of the geometric monodromy action on \( \bar{X}_\mathcal{G}^c \).

Proof. By definition, the generic fiber of \( \bar{H}_0^c \) is equal to \( \mathcal{G}_L \). From the exact sequence (1) and Lemma 3.6, we know that \( \bar{H}_0^c \) is etale over \( \mathcal{O}_L \) and there is the following exact sequence of algebraic groups over \( \bar{F} \).

\[
0 \to \bar{H}^c \to \bar{X}_\mathcal{G}^c \to \bar{D}^{r,j} \to 0
\]

Thus we have a natural isomorphism \( \bar{H}_0^c(\bar{K}) \to \bar{H}^c(\bar{F}) \) and the composite \( \mathcal{G}(\bar{K}) = \mathcal{H}_0^c(\bar{K}) \to \bar{H}^c(\bar{F}) \to \bar{X}_\mathcal{G}^c(\bar{F}) \) coincides with the map.
sp}. From [2, Corollary 4.4], we see that this map sends \( G^c(\overline{K}) \) isomorphically onto \( H_c^0(F) \).

For \( x \in X^c_0(\overline{K}) \) and \( \sigma \in I_K \), let \( \sigma(x) \) denote the natural action of \( \sigma \) on \( \overline{K} \)-valued points. Then we have \( \sigma_{\text{geom}}(x) \circ \sigma = \sigma(x) \). Taking its specialization shows the \( I_K \)-equivariance.

The following theorem can be regarded as a generalization for a finite flat group scheme over \( \mathcal{O}_K \) of the structure theorem of the graded piece of the classical upper numbering ramification filtration.

**Theorem 4.7.** Let \( \mathcal{G} \) be a finite flat group scheme over \( \mathcal{O}_K \) and \( j \in \mathbb{Q}_{>0} \). Then the \( G_K \)-module \( \mathcal{G}^j(\overline{K})/\mathcal{G}^{j+}(\overline{K}) \) is tame and killed by \( p \).

**Proof.** Since \( \mathcal{G}^j = (\mathcal{G}^0)^j \), where \( \mathcal{G}^0 \) denotes the unit component of \( \mathcal{G} \), we may assume \( \mathcal{G} \) is connected. Suppose that \( j \) is a jump of the ramification filtration on \( \mathcal{G} \) and consider the quotient \( \mathcal{G} = \mathcal{G}^j(\overline{K})/\mathcal{G}^{j+}(\overline{K}) \). The subgroup \( \mathcal{G}^j(\overline{K}) \subseteq \mathcal{G}(\overline{K}) \) has a non-trivial image in \( (\mathcal{G}/\mathcal{G}^{j+})(\overline{K}) \). By the Herbrand theorem ([1, Lemme 2.10]), the natural map \( \mathcal{G}^j(\overline{K}) \to (\mathcal{G}/\mathcal{G}^{j+})^j(\overline{K}) \) is surjective for any \( t > 0 \). We have \( (\mathcal{G}/\mathcal{G}^{j+})^t = 0 \) for \( t > j \) and \( (\mathcal{G}/\mathcal{G}^{j+})^j \neq 0 \). Thus the ramification filtration on \( \mathcal{G}/\mathcal{G}^{j+} \) jumps at \( j \) and \( (\mathcal{G}/\mathcal{G}^{j+})^j(\overline{K}) = \mathcal{G}^j(\overline{K})/\mathcal{G}^{j+}(\overline{K}) \). Replacing \( \mathcal{G} \) with \( \mathcal{G}/\mathcal{G}^{j+} \), we may assume \( j = c = c(\mathcal{G}) \).

Take a formal resolution \( (\mathcal{G} \to \Gamma) \) of dimension \( r \) and consider its associated affinoid homomorphism \( X^c_0 \to D^{\infty} \). Then the theorem follows from Corollary 4.3, Corollary 4.4, and Lemma 4.6.

From this theorem, we see that the inertia subgroup \( I_K \) acts on \( \mathcal{G}^j(\overline{K})/\mathcal{G}^{j+}(\overline{K}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \) by the direct sum of tame characters. The theorem below determines these characters up to \( p \)-power exponent.

**Theorem 4.8.** Let \( \mathcal{G} \) be a finite flat group scheme over \( \mathcal{O}_K \) and \( j \in \mathbb{Q}_{>0} \). Then \( I_K \) acts on \( \mathcal{G}^j(\overline{K})/\mathcal{G}^{j+}(\overline{K}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \) by the direct sum of \( \mathbb{F}_p \)-conjugates of the fundamental character \( \theta_j \).

**Proof.** We may assume that \( \mathcal{G} \) is connected. The same argument as in the proof of Theorem 4.7 using the Herbrand theorem reduces the claim to the case where \( j = c = c(\mathcal{G}) \). Take a formal resolution \( (\mathcal{G} \to \Gamma) \) of dimension \( r \). Let \( J \) and \( \tau \) be as in Corollary 4.5. Then Corollary 4.5 and Lemma 4.6 show that every eigenvalue of the action of \( \tau_{\text{geom}} \) on the finite dimensional \( \mathbb{F}_p \)-vector space \( \mathcal{G}^c(\overline{K}) \) is a conjugate of \( \theta_j(\tau) \) over \( \mathbb{F}_p \). Since the order of \( J \) is prime to \( p \), we conclude that \( I_K \) acts on \( \mathcal{G}^c(\overline{K}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \) by the direct sum of \( \mathbb{F}_p \)-conjugates of \( \theta_c \).
**Corollary 4.9.** Let $G$ be a finite flat group scheme over $\mathcal{O}_K$. Then the order of the image of the homomorphism $I_K \to \text{Aut}(G(\bar{K}))$ is a power of $p$ if and only if every jump $j$ of the ramification filtration $\{G^j\}_{j \in \mathbb{Q}_{>0}}$ is an element of $\mathbb{Z}[1/p]$.

**Proof.** From Theorem 4.8, we see that the jumps are in $\mathbb{Z}[1/p]$ if and only if the graded pieces $G^j(\bar{K})/G^{j+1}(\bar{K})$ are unramified. By Theorem 4.7, this is equivalent to the condition that $\sharp \text{Im}(I_K \to \text{Aut}(G(\bar{K})))$ is a $p$-power.

When $G(\bar{K})$ is unramified and killed by $p$, we have the following re-inforcement of this result, which is an easy corollary of the Herbrand theorem. The author does not know if every jump is an integer whenever $G(\bar{K})$ is unramified. If $G$ is monogenic, then we see that this holds true from [11, Theorem 4].

**Proposition 4.10.** Let $G$ be a finite flat group scheme over $\mathcal{O}_K$ which is killed by $p$. Suppose that the $G_K$-module $G(\bar{K})$ is unramified. Then every jump $j$ of the ramification filtration $\{G^j\}_{j \in \mathbb{Q}_{>0}}$ is an element of $p\mathbb{Z}$.

**Proof.** We may assume $K = K^{nr}$ and $G_K$ acts trivially on $G(\bar{K})$. There is a quotient $W$ of $G(\bar{K})/G^{j+1}(\bar{K})$ where $G^j(\bar{K})$ has a non-trivial image and of rank one over $\mathbb{F}_p$. Taking the schematic closure, $W$ extends to a finite flat group scheme $\mathcal{W}$ over $\mathcal{O}_K$ which is a quotient of $G/G^{j+1}$. By the Herbrand theorem, we see that the ramification filtration of $\mathcal{W}$ jumps at $j$. On the other hand, $W$ is a Raynaud $\mathbb{F}_p$-vector space scheme with unramified generic fiber. Thus the proposition follows from [11, Theorem 4].

For the rest of this section, we state some corollaries in the case where $G$ is a $\mathbb{F}$-vector space scheme of rank one or two for a finite extension $\mathbb{F}$ over $\mathbb{F}_p$. In the case of rank one, Theorem 4.8 directly shows the claim below, while this can be shown also as a corollary of a determination of the conductor of a Raynaud $\mathbb{F}$-vector space scheme (Theorem 5.5).

**Corollary 4.11.** Let $G$ be a $\mathbb{F}$-vector space scheme of rank one over $\mathcal{O}_K$ and $c = c(G)$. Then the $I_K$-action on the $\mathbb{F}$-vector space $G(\bar{K})$ of rank one is given by the character $\theta^m$ for some $n$.

In the case of rank two, we have the following.

**Corollary 4.12.** Let $G$ be a finite flat $\mathbb{F}$-vector space scheme of rank two over $\mathcal{O}_K$ and $c = c(G)$. Then the $I_K$-module $G(\bar{K}) \otimes_{\mathbb{F}} \mathbb{F}_p$ contains
the character $\theta_p^n$ for some $n$. If the $G_K$-module $G(\bar{K})$ is reducible, this holds true for $G(\bar{K})$ itself.

Proof. The first assertion follows easily from Theorem 4.8 and the surjection $G_c(\bar{K}) \otimes_{\mathbb{F}_p} \mathbb{F}_p \rightarrow G_c(\bar{K}) \otimes_{\mathbb{F}_p} \mathbb{F}_p$. Suppose the $I_K$-module $G(\bar{K})$ is reducible. When $G_c$ is of rank one, the assertion is clear from Theorem 4.8. If $G_c = G$, then $G_c$ is reducible and the assertion follows also from Theorem 4.8.

The corollary below indicates that the conductor $c(G)$ carries information about not only the tame characters but also their extension structures in the $I_K$-module $G(\bar{K})$.

**Corollary 4.13.** Consider an exact sequence of finite flat $\mathbb{F}$-vector space schemes over $\mathcal{O}_K$

$$0 \rightarrow G_1 \rightarrow G \rightarrow G_2 \rightarrow 0$$

where $G_1$ and $G_2$ are connected of rank one. If $c(G) = c(G_2)$, then the $I_K$-module $G(\bar{K})$ splits.

Proof. Put $c = c(G)$. Take a formal resolution $(G \rightarrow \Gamma)$ of dimension $r$ and put $\Gamma_2 = \Gamma / G_1$. Then we get a finite flat map of formal resolutions

$$G \longrightarrow \Gamma$$

$$\downarrow \quad \downarrow$$

$$G_2 \longrightarrow \Gamma_2.$$  

Therefore we have a finite flat homomorphism of rigid $K$-analytic groups $X^j(G \rightarrow \Gamma) \rightarrow X^j(G_2 \rightarrow \Gamma_2)$ by Corollary 3.2. As in the proof of Lemma 3.3, we see that this map is finite etale.

Suppose $G_c(\bar{K})$ is of rank one. If $G_c(\bar{K}) \neq G_1(\bar{K})$ as an $\mathbb{F}$-subspace of $G(\bar{K})$, the $I_K$-module $G(\bar{K})$ splits and the proposition follows. Suppose $G_c(\bar{K}) = G_1(\bar{K})$. The affinoid variety $X^c_i(G \rightarrow \Gamma)$ decomposes to $\sharp \mathbb{F}$ components over some finite extension $K'$ of $K$. Each component is a Zariski open and closed subset of $X^c_i(G \rightarrow \Gamma)_{K'}$. As the map $f : X^c_i(G \rightarrow \Gamma)_{K'} \rightarrow X^c_i(G_2 \rightarrow \Gamma_2)_{K'}$ is finite etale and $X^c_i(G_2 \rightarrow \Gamma_2)_{K'}$ is connected, every component $X^c_i(G \rightarrow \Gamma)_{K'}$ maps surjectively to $X^c_i(G_2 \rightarrow \Gamma_2)_{K'}$. Take some $g_i \in G(\bar{K}) \cap X^c_i(G \rightarrow \Gamma)_{K'}$. Using the group structure, we see that $f(G(\bar{K}) \cap X^c_i(G \rightarrow \Gamma)_{K'}) = g_i + G_c(\bar{K}) = g_i + G_1(\bar{K})$ and $f(G(\bar{K}) \cap X^c_i(G \rightarrow \Gamma)_{K'}) = f(g_i)$. However, we have $f^{-1}(G_2(\bar{K})) = G(\bar{K})$ and thus $f(G(\bar{K}) \cap X^c_i(G \rightarrow \Gamma)_{K'}) = G_2(\bar{K})$. This is a contradiction. Therefore we may assume $G_c(\bar{K}) = G(\bar{K})$. In this case, the proposition follows from Theorem 4.7.

\qed
5. Example: Rank One Calculation

In this section, we calculate the conductor of a Raynaud $\mathbb{F}$-vector space scheme over $\mathcal{O}_K$. The point is that, as we can see from the bound of the conductor ([11, Theorem 7]), it is enough to consider the $j$-th tubular neighborhood only for $j \leq pe/(p-1) + \varepsilon$ with sufficiently small $\varepsilon > 0$. For such $j$, we can compute the tubular neighborhood easily by Lemma 5.4 below.

Let $K$ be a complete discrete valuation field of mixed characteristic $(0, p)$. We write $\pi = \pi_K$ for its uniformizer and $e$ for its absolute ramification index. We normalize a valuation $v_K$ of $K$ as $v_K(\pi) = 1$ and extend it to the algebraic closure $\bar{K}$ of $K$. For $a \in \bar{K}$ and $j \in \mathbb{R}$, let $D(a, j)$ denote the closed disc \{ $z \in \mathcal{O}_K \mid v_K(z - a) \geq j$ \}. This is the underlying subset of a $K(a)$-affinoid subdomain of the unit disc over $K(a)$.

For integers $0 \leq s_1, \ldots, s_r \leq e$, let $\mathcal{G}(s_1, \ldots, s_r)$ denote the Raynaud $\mathbb{F}_p$-vector space scheme over $\mathcal{O}_K$ defined by the $r$ equations $T_1^p = \pi v_1 T_1, T_2^p = \pi v_2 T_2, \ldots, T_r^p = \pi v_r T_r$ ([14]). We set $j_k = (ps_k + p^2s_{k-1} + \ldots + p^ks_1 + p^{k+1}s_r + p^{k+2}s_{r-1} + \ldots + p^k s_{k+1})/(p^r - 1)$. Before the calculation of $c(\mathcal{G}(s_1, \ldots, s_r))$, we gather some elementary lemmas.

**Lemma 5.1.** Let $a \in \mathcal{O}_K$ and $s = v_K(a)$. Then the affinoid variety $X^j(K) = \{ x \in \mathcal{O}_K \mid v_K(x^p - a) \geq j \}$ is equal to

$$
\left\{ \begin{array}{ll}
D(a_1^{1/p}, j/p) & \text{if } j \leq s + pe/(p-1), \\
\prod_{i=0}^{p-1} D(a_i^{1/p}, s_i j - e - (p-1)s/p) & \text{if } j > s + pe/(p-1).
\end{array} \right.
$$

**Proof.** We have $v_K(x^p - a) = \sum_i v_K(x - a_1^{1/p} \zeta_i^{s_i})$. If $v_K(x - a_1^{1/p} \zeta_i^{s_i}) \geq v_K(x - a_1^{1/p} \zeta_i^{s_i})$ for any $i' \neq i$, then $v_K(x - a_1^{1/p} \zeta_i^{s_i}) = v_K(a_1^{1/p} \zeta_i^{s_i}(1 - \zeta_i^{s_i}r')) = s/p + e/(p-1)$. Thus we have $v_K(x - a_1^{1/p} \zeta_i^{s_i}) \geq \sup(j/p, j - (p-1)s/p - e)$ and

$$
X^j(K) \subseteq \bigcup_i D(a_i^{1/p}, j/p, j - (p-1)s/p - e).
$$

Suppose that $j/p \geq j - (p-1)s/p - e$. Then we have $v_K(a_1^{1/p}(1 - \zeta_i^{s_i})) = s/p + e/(p-1) \geq j/p$, $D(a_1^{1/p}, j/p) = D(a_1^{1/p}, j/p)$ for any $i$ and thus

$$
X^j(K) = D(a_1^{1/p}, j/p).
$$

When $j/p < j - (p-1)s/p - e$, we have $v_K(a_1^{1/p}(1 - \zeta_i^{s_i})) = s/p + e/(p-1) < j - (p-1)s/p - e$. This means that if $w \in D(a_1^{1/p}, j - (p-1)s/p - e)$ for some $i$, then $v_K(w - a_1^{1/p} \zeta_i^{s_i}) < j - (p-1)s/p - e$ for any other $i'$. 
Thus the discs $D(a^{1/p^i}c_{sp}^j, j - (p - 1)s/p - e)$ are disjoint and

$$X^j(\mathbb{K}) = \prod_i D(a^{1/p^i}c_{sp}^j, j - (p - 1)s/p - e).$$

These are equalities of the underlying sets of affinoid subdomains of the unit disc over $K(a^{1/p}, \zeta_p)$. By the universality of an affinoid subdomain, this extends to an isomorphism of affinoid varieties.

We can prove the following lemma just in the same way.

**Lemma 5.2.** The affinoid variety $\{x \in \mathcal{O}_K \mid v_K(x^{p^r} - ax) \geq j\}$ is equal to

$$\left\{ \begin{array}{ll}
D(0, j/p^r) & \text{if } j \leq p^r v(a)/(p^r - 1), \\
\prod_{i=0}^{r-1} D(\sigma_i, j - v(a)) & \text{if } j > p^r v(a)/(p^r - 1),
\end{array} \right.$$ 

where $\sigma_i$'s are the roots of $X^{p^r} = aX$.

**Lemma 5.3.** For $g_1(Y_1, \ldots, Y_d), g_2(Y_1, \ldots, Y_d) \in K[Y_1, \ldots, Y_d]$ and $j_1 \geq j_2$, the affinoid variety $\{(x, y_1, \ldots, y_d) \in \mathcal{O}_K \times \mathcal{O}_K^d \mid v_K(x - g_1(y_1, \ldots, y_d)) \geq j_1, v_K(x - g_2(y_1, \ldots, y_d)) \geq j_2\}$ is equal to $\{(x, y_1, \ldots, y_d) \in \mathcal{O}_K \times \mathcal{O}_K^d \mid v_K(x - g_1(y_1, \ldots, y_d)) \geq j_1, v_K((g_1(y_1, \ldots, y_d) - g_2(y_1, \ldots, y_d)) \geq j_2\}$.

**Proof.** For fixed $(x, y)$, these two conditions are equivalent. The universality of an affinoid subdomain proves the lemma.

**Lemma 5.4.** Let $a \in \mathcal{O}_K$ and $s = v_K(a)$. If $j \leq pe/(p - 1) + s$, then the affinoid variety $X^j(\mathbb{K}) = \{(x, y) \in \mathcal{O}_K \times \mathcal{O}_K \mid v_K(x^{p^r} - ay^{p^s}) \geq j\}$ is equal to $\{(x, y) \in \mathcal{O}_K \times \mathcal{O}_K \mid v_K(x - a^{1/p}y^{p^s}) \geq j/p\}$.

**Proof.** Lemma 5.1 shows that the fiber of the second projection $X^j(\mathbb{K}) \to \mathcal{O}_K$ at $y$ is equal to

$$\left\{ \begin{array}{ll}
D(a^{1/p}y^{p^s-1}, j/p) & \text{if } j \leq s + p^{s-1}v_K(y) + pe/(p - 1), \\
\prod_{i=0}^{r-1} D(a^{1/p}c_{sp}^{j-1}y^{p^s-1}, j - e - (p - 1)(s + p^{s-1}v_K(y))/p) & \text{otherwise}.
\end{array} \right.$$ 

Thus we have $X^j(\mathbb{K}) = \{(x, y) \in \mathcal{O}_K \times \mathcal{O}_K \mid v_K(x - a^{1/p}y^{p^s}) \geq j/p\}$ for $j \leq pe/(p - 1) + s$. This is the underlying set of a $K(a^{1/p})$-affinoid variety. Again this equality extends to an isomorphism over $K(a^{1/p})$.

Now we proceed to the proof of the main theorem of this section.

**Theorem 5.5.** $c(G(s_1, \ldots, s_r)) = \sup_kj_k$. 
Proof. We may assume that \( j_r \) is the supremum of \( j_k \)'s. If \( j_r = 0 \), then \( \mathcal{G}(s_1, \ldots, s_r) \) is etale and \( c(\mathcal{G}(s_1, \ldots, s_r)) = 0 \). Thus we may assume \( j_r > 0 \). Consider the homomorphism of \( \mathcal{O}_K \)-algebras

\[
A = \mathcal{O}_K[T_1, \ldots, T_r]/(T_1^p - \pi s_1 T_2, \ldots, T_r^p - \pi s_r T_1) \rightarrow B = \mathcal{O}_K[W, T_2, \ldots, T_r]/(W^{p^r} - \pi s_1 T_2^p - \pi s_2 T_3, \ldots,
\]

\[
T_{r-1}^p - \pi s_{r-1} T_r, T_r^p - \pi s_r W^{p^{r-1}}),
\]

defined by \( T_1 \mapsto W^{p^{r-1}} \). This induces a surjection of \( K \)-affinoid varieties

\[
X_B^j(\overline{K}) \ni (w, t_2, \ldots, t_r) \mapsto (w^{p^{r-1}}, t_2, \ldots, t_r) \in X_A^j(\overline{K}),
\]

where

\[
X_A^j(\overline{K}) = \{(t_1, \ldots, t_r) \in \mathcal{O}_K^r | v_K(t_1^p - \pi s_1 t_2) \geq j, \ldots, v_K(t_{r-1}^p - \pi s_{r-1} t_r) \geq j, v_K(t_r^p - \pi s_r t_1) \geq j \}
\]

and

\[
X_B^j(\overline{K}) = \{(w, t_2, \ldots, t_r) \in \mathcal{O}_K^r | v_K(w^{p^r} - \pi s_1 t_2) \geq j, v_K(t_2^p - \pi s_2 t_3) \geq j, \ldots, v_K(t_r^p - \pi s_r W^{p^{r-1}}) \geq j \}.
\]

These are affinoid subdomains of the \( r \)-dimensional unit polydisc over \( K \). We calculate a jump of \( \{F^j(B)\}_{j \in \mathbb{Q} > 0} \) at first.

**Lemma 5.6.** If \( j_r < pe/(p-1) \), then the first jump of \( \{F^j(B)\}_{j \in \mathbb{Q} > 0} \) occurs at \( j = j_r \) and \( 2F^{j_r}(B) = p^{r-1} \).

Note that the base change from \( K \) to a finite extension \( L \) multiplies \( s_i \)'s, \( j_i \)'s and \( e \) by the ramification index of \( L/K \). Thus, to prove Lemma 5.6 and Theorem 5.5, we may assume that \( p^{r-1} \) divides \( s_i \)'s and \( e \).

**Proof.** Consider the \( K \)-affinoid variety \( X_B^j(\overline{K}) \) for \( j \leq pe/(p-1) \). Then the iterative use of Lemma 5.4 and Lemma 5.3 shows that the affinoid variety \( X_B^j(\overline{K}) \) is equal to

\[
\{v_K(w^{p^r} - \pi^{s_r+p_{r-1}+\ldots+p_{r-1}-1} s_1)^{p^{r-1}} w) \geq pl_1(j), v_K(t_2 - g_2(w)) \geq u_2, v_K(t_3 - g_3(t_2, w)) \geq u_3, \ldots, v_K(t_r - g_r(t_{r-1}, w)) \geq u_r \},
\]

where \( l_i(j), g_i(t_{i-1}, w), g_2(w) \) and \( u_i \) are defined as follows:

- \( l_r(j) = j/p \),
- \( l_{i-1}(j) = \inf(j, l_i(j) + s_{i-1})/p \),
- \( g_i(t_{i-1}, w) = l_{i-1}^p/p^{s_{i-1}} \) and \( u_i = j - s_{i-1} \) if \( j \geq l_i(j) + s_{i-1} \),
- \( g_i(t_{i-1}, w) = \pi^{s_r+p_{r-1}+\ldots+p_{r-1}-1} s_i^{p^{r-1}} w^{p-2} \) and \( u_i = l_i(j) \) if \( j < l_i(j) + s_{i-1} \),
\[ g_2(w) = g_2(w^{p^{r-1}}, w). \]

Note that \( l_1(j) \) is a strictly monotone increasing function of \( j \). This affinoid variety is isomorphic to the product of the affinoid variety \( \{ w \in O_K \mid v(w^{p^r} - \pi^{s_r} + \ldots + s_1)\} \) and discs. Therefore, from Lemma 5.2, we see that the first jump of \( \{ F^j(B) \}_{j \in \mathbb{Q} > 0} \) occurs at \( j \) such that \( pl_1(j) = j_r \), provided this \( j \) satisfies \( 0 < j < pe/(p-1) \). Moreover, then we have \( F^j(B) = p^r \). Thus the following lemma and the strict monotonicity of \( l_1 \) terminate the proof of Lemma 5.6.

**Lemma 5.7.** \( l_1(j_r) = j_r/p \).

**Proof.** Suppose that there is \( k \) such that \( l_k(j_r) = j_r/p \) and \( j_r \geq l_{k'}(j_r) + s_{k'} \) for any \( 1 \leq k' \leq k \). Then we have \( l_1(j_r) = \inf(j_r, (j_r + ps_{k-1} + p^2s_{k-2} + \ldots + p^{k-1}s_1)/p^{k-1})/p \) and the assumption \( j_{k-1} \leq j_r \) implies \( l_1(j_r) = j_r/p \).

On the other hand, let \( s = (s_r + ps_{r-1} + \ldots + p^{r-1}s_1)/p^{r-1} \) and \( \sigma_0, \ldots, \sigma_{p^r-1} \) be the roots of the equation \( X^{p^r} - \pi^sX = 0 \). Then we see that the images by \( w \mapsto w^{p^{r-1}} \) of the discs \( D(\sigma_i, pl_1(j) - s) \) are disjoint for \( j > j_r \). Hence the surjection \( \pi_0(X_B^r(\bar{K})) \to \pi_0(X_B^r(\bar{K})) \) is bijective for \( 0 < j \leq pe/(p-1) \) and the first (and the last) jump of \( \{ F^j(A) \}_{j \in \mathbb{Q} > 0} \) also occurs at \( j_r \), provided \( j_r < pe/(p-1) \).

When \( j_r = pe/(p-1) \), we see that \( s_k = e > 0 \) for any \( k \). Thus we can use Lemma 5.4 for \( j < pe/(p-1) + \varepsilon \) with sufficiently small \( \varepsilon > 0 \). Then, by the same reasoning as above, we conclude that \( e(A) = pe/(p-1) \).

**References**


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