Invariant Subspaces In The Bidisc And Wandering Subspaces

By

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* This research was partially supported by Grant-in-Aid for Scientific Research, Japan Society for the Promotion of Science

2000 Mathematics Subject Classification : Primary 47 A 15, 46 J 15 ; Secondary 47 A 20

Abstract. Let M be a forward shift invariant subspace and N a backward shift invariant subspace in the Hardy space H^2 on the bidisc. We assume that $H^2 = N \oplus M$. Using the wandering subspace of M and N, we study the relations between M and N. Moreover we study M and N using several natural operators which are defined by shift operators on H^2.

§1. Introduction

Let T^2 be the torus that is the Cartesian product of two unit circles T in $\mathbb{C}$. Let $p = 2$ or $p = \infty$. The usual Lebesgue spaces, with respect to the Haar measure m on T^2, are denoted by $L^p = L^p(T^2)$, and $H^p = H^p(T^2)$ is the space of all f in L^p whose Fourier coefficients

$$\hat{f}(j, \ell) = \int_{T^2} f(z, w)\bar{z}^j \bar{w}^\ell dm(z, w)$$

are 0 as soon as at least one component of (j, ℓ) is negative. Then H^p is called the Hardy space. As $T^2 = (z, T) \times (w, T), H^p(z, T)$ and $H^p(w, T)$ denote the one variable Hardy spaces.

Let P_{H^2} be the orthogonal projection from L^2 onto H^2. For ϕ in L^∞, the Toeplitz operator T_ϕ is defined by

$$T_\phi f = P_{H^2}(\phi f) \quad (f \in H^2).$$

A closed subspace M of H^2 is said to be forward shift invariant if $T_z M \subset M$ and $T_w M \subset M$, and a closed subspace N of H^2 is said to be backward shift invariant if $T_z^* N \subset N$ and $T_w^* N \subset N$. Let P_M and P_N be the orthogonal projections from H^2 onto M and N, respectively. In this paper, we assume that $M \oplus N = H^2$, that is, $P_M + P_N = I$ where I is the identity operator on H^2. Let

$$A = P_M T_z P_N \text{ and } B = P_N T_w^* P_M.$$

For ϕ in H^∞

$$V_\phi f = P_M(\phi f) \quad (f \in M)$$

and

$$S_\phi f = P_N(\phi f) \quad (f \in N).$$

Suppose that

$$\mathcal{V} = V_z V_w^* - V_w^* V_z \text{ and } \mathcal{S} = S_z S_w^* - S_w^* S_z.$$

It is known [4] that $AB \mid M = \mathcal{V}$ and $BA \mid N = \mathcal{S}$. K.Guo and R.Yang [3] showed that AB is Hilbert-Schmidt under some mild condition. In this paper, we study M or N when A, B, AB or BA is of finite rank. K.Izuchi and T.Nakazi [4] described an invariant subspace M or N with $A = 0$ or $B = 0$. V.Mandrekar [6], P.Ghatage and V.Mandrekar [2], and T.Nakazi ([7], [8]) described an invariant subspace M with $AB = 0$. K.Izuchi and T.Nakazi [4] and K.Izuchi, T.Nakazi and M.Seto [5] described an invariant subspace N with $BA = 0$.

For a forward shift invariant subspace M, put

$$M_1 = \ker V_z^*, M_2 = \ker V_w^* \text{ and } M_0 = M_1 \cap M_2.$$

then these are called wandering subspaces for M. For a backward shift invariant subspace N, with $M = H^2 \ominus N$, put

$$N_1 = [T_z^* M_1]_2, N_2 = [T_w^* M_2]_2 \text{ and } N_0 = N_1 \cap N_2,$$

then these should be called wandering subspaces for N.

In §2 we decompose and study M and N using wandering subspaces M_1, M_2, N_1 and N_2. In §3 we study M and N when A or B is of finite rank. For an operator K, $r(K)$ denotes the rank of K. In §4 we show that $r(AB) = \dim N_1 \cap N_2$ in general and $r(BA) = \dim M_1 \cap M_2$ under some mild conditions.

§2. Wandering subspace

Let M be a forward shift invariant subspace and N be a backward shift invariant subspace with $H^2 = M \oplus N$. Put

$$M_z^\infty = \bigcap_{n=1}^\infty \{f \in M \ ; \ \bar{z}^n f \in M\} \ \text{ and } \ M_w^\infty = \bigcap_{n=1}^\infty \{f \in M \ ; \ \bar{w}^n f \in M\}$$

, and

$$N_z^\infty = \bigcap_{n=1}^\infty \{f \in N \ ; \ z^n f \in N\} \ \text{ and } \ N_w^\infty = \bigcap_{n=1}^\infty \{f \in N \ ; \ w^n f \in N\}.$$

In the case of one variable, $M_z^\infty = N_z^\infty = \{0\}$. In the case of two variables, M_z^∞ is also always $\{0\}$ but N_z^∞ may not be $\{0\}$. In fact, if $N \supset q_1 H^2(z,T)$ then $N_z^\infty \supset q_1 H^2(z,T)$ where $q_1 = q_1(z)$ is one variable inner function.

Theorem 1. *Let N be a backward shift invariant subspace and $M = H^2 \ominus N$.*

(1) $M_z^\infty = M_w^\infty = \{0\}$ and $M = \sum_{n=0}^\infty \oplus T_z^n M_1 = \sum_{n=0}^\infty \oplus T_w^n M_2$.

(2) $N = \left[\bigcup_{n=0}^\infty T_z^{*n} N_1\right]_2 \oplus N_z^\infty = \left[\bigcup_{n=0}^\infty T_w^{*n} N_2\right]_2 \oplus N_w^\infty$.

Proof. (1) is well known. (2) If $f \in N_z^\infty$ then by deffinition $z^n f \in N$ for any $n \geq 1$ and hence f is orthogonal to $\left[\bigcup_{n=0}^\infty T_z^{*n} N_1\right]_2$. Conversely suppose that f is orthogonal to $\bigcup_{n=0}^\infty T_z^{*n} N_1$. Since $f \perp N_1$, zf is orthogonal to $M_1 + zM$ because $N_1 = T_z^* M_1$ and $f \in N$. Hence $zf \in N$. Since $f \perp T_z^* N_1$, $z^2 f$ is orthogonal to $M_1 + zM$ because $T_z^* N_1 = T_z^{*2} M_1$ and $zf \in N$. Hence $z^2 f \in N$. By repeating the same argument, we can show that $z^n f$ belongs to N for any $n \geq 1$. This implies (2).

Corollary 1. *Let N be a backward shift invariant subspace.*

(1) $N = N_z^\infty$ if and only if $N = H^2(z,T) \otimes (H^2(w,T) \ominus q_2 H^2(w,T))$ where $q_2 = q_2(w)$ is a one variable inner function.

(2) $N = \left[\bigcup_{n=0}^{\infty} T_z^{*n} N_1\right]_2$ *if and only if for each nonzero* f *in* N *there exists* $n \geq 1$ *such that* $z^n f \notin N$.

Proof (1) If $N = N_z^{\infty}$ then $N_1 = 0$ and so $T_z^* M_1 = 0$. Hence $M_1 \subset H^2(w,T)$ and so $M_1 = q_2 H^2(w,T)$ by a well known theorem of A.Beurling [1]. Therefore $M = q_2 H^2$ and so $N = H^2(z,T) \otimes (H^2(w,T) \ominus q_2 H^2(w,T))$. Conversely if $M = q_2 H^2$ then $M_1 = q_2 H^2(w,T)$ and so $N_1 = T_z^* M_1 = 0$. (2) is clear by (2) of Theorem 1.

By (1) of Theorem 1, both M_1 and M_2 are cyclic subspaces for T_z and T_w, that is,

$$\left[\bigcup_{(n,m)\geq(0,0)} T_z^n T_w^m M_j\right]_2 = M \quad \text{for} \quad j = 1, 2.$$

It may happen that $\left[\bigcup_{(n,m)\geq(0,0)} T_z^n T_w^m M_0\right]_2 = M$ where $M_0 = M_1 \cap M_2$. By (2) of Theorem 1,if $N_z^{\infty} = \{0\}$ or $N_w^{\infty} = \{0\}$ then N_1 or N_2 is a cyclic subspace for T_z^* and T_w^*, that is,

$$\left[\bigcup_{(n,m)\geq(0,0)} T_z^{*n} T_w^{*m} N_j\right]_2 = N \quad \text{for} \quad j = 1, 2.$$

In general, N_0 may not be a cyclic subspace because $N_0 = \langle 0 \rangle$ may happen. We can ask whether $T_z^* M_0$ or $T_w^* M_0$ is a cyclic subspace for T_z^* and T_w^* because $N_1 \supset T_z^* M_0$ and $N_2 \supset T_w^* M_0$. However this is not true. If $M = zH^2$ then $N = H^2(w,T)$ and $M_0 = \langle z \rangle$. Then $T_w^* M_0 = \langle 0 \rangle$ and $T_z^* M_0 = \langle 1 \rangle$.

Example 1. *Let* $N = H^2(z,T) + H^2(w,T)$. *Then the following (1) $\sim$ (3) are valid.*

(1) $N_1 = wH^2(w,T)$, $N_2 = zH^2(z,T)$ *and* $N_0 = \langle 0 \rangle$.

(2) $\left[\bigcup_{n\geq 0} T_z^{*n} N_1\right]_2 = wH^2(w,T)$, $\left[\bigcup_{n\geq 0} T_w^{*n} N_2\right]_2 = zH^2(z,T)$ *and* $\left[\bigcup_{(n,m)\geq 0} T_z^{*n} T_w^{*m} N_0\right]_2 = \langle 0 \rangle$.

(3) $N_z^{\infty} = H^2(z,T)$ *and* $N_w^{\infty} = H^2(w,T)$

Example 2. *Let* $N = \not{C}$ *and* $M = zH^2 + wH^2$. *Then the following (1) $\sim$ (3) are valid.*

(1) $N_1 = N_2 = N_0 = \not{C}$.

(2) $\left[\bigcup_{n\geq 0} T_z^{*n} N_1\right]_2 = \left[\bigcup_{n\geq 0} T_w^{*n} N_2\right]_2 = \left[\bigcup_{(n,m)\geq(0,0)} T_z^{*n} T_w^{*m} N_0\right]_2 = N$.

(3) $N_z^\infty = N_w^\infty = \langle 0 \rangle$.

Example 3. *Let* $N = (H^2(z,T) \ominus q_1 H^2(z,T)) \otimes (H^2(w,T) \ominus q_2 H^2(w,T))$ *and* $M = q_1 H^2 + q_2 H^2$ *where* $q_1 = q_1(z)$ *and* $q_2 = q_2(w)$ *are one variable inner functions.*

(1) $M_1 = q_1(H^2(w,T) \ominus q_2 H^2(w,T)) \oplus q_2 H^2(w,T)$ *and* $M_2 = q_2(H^2(z,T) \ominus q_1 H^2(z,T)) \oplus q_1 H^2(z,T)$.

(2) $N_1 = (T_z^* q_1)(H^2(w,T) \ominus q_2 H^2(w,T)),\ N_2 = (T_w^* q_2)(H^2(z,T) \ominus q_1 H^2(z,T))$ *and* $N_0 = \langle (T_z^* q_1)(T_w^* q_2) \rangle$.

(3) $\left[\bigcup_{n\geq 0} T_z^{*n} N_1\right]_2 = \left[\bigcup_{n\geq 0} T_w^{*n} N_2\right]_2 = \left[\bigcup_{(n,m)\geq(0,0)} T_z^{*n} T_w^{*m} N_0\right]_2 = N.$

Proof. (2) and (3) follow from (1). It is known [4] that $M = q_2 H^2 \oplus q_1(H^2 \ominus q_2 H^2) = (H^2(z,T) \otimes q_2 H^2(w,T)) \oplus \{q_1 H^2(z,T) \otimes (H^2(w,T) \ominus q_2 H^2(w,T))\}$. Hence (1) follows.

§3. $r(A) < \infty$ or $r(B) < \infty$

Recall that $A = P_M T_z P_N$ and $B = P_N T_w^* P_M$ (see Introduction). In this section, we are interested in when A or B is of finite rank. We know a characterization of $A = 0$ or $B = 0$ (see [3]). In fact $A = 0$ if and only if $N = H^2$ or $N = H^2 \ominus qH^2$ where $q = q(w)$ is a one variable inner function, and $B = 0$ if and only if $M = \{0\}$ or $M = qH^2$ where $q = q(z)$ is a one variable functon. In one variable Hardy space, A is of rank one for any N or B is of rank one for any M.

Lemma 1. *Let* M *be a forward shift invariant subspace of* H^2 *and* $N = H^2 \ominus M$.

(1) $[\mathrm{ran} A]_2 \subseteq M_1$ *and* $\ker A = \{f \in N\ ;\ T_z f \in N\} \oplus M$.

(2) $[\mathrm{ran} A^*]_2 = N_1$ *and* $\ker A^* = \{f \in M\ ;\ T_z^* f \in M\} \oplus N$.

(3) $M_1 = [\mathrm{ran} A]_2 \oplus \{\ker A^* \ominus (T_z M \oplus N)\}$.

(4) $M = [\mathrm{ran} A]_2 \oplus (\ker A^* \ominus N)$ *and* $N = [\mathrm{ran} A^*]_2 \oplus (\ker A \ominus M)$.

Proof. (1) By definitions, $[\mathrm{ran} A]_2 = [PT_z N]_2 \subseteq M_1$ because $T_z N$ is orthogonal to $T_z M$ and $\ker A = \{f \in N\ ;\ T_z f \in N\} \oplus M$. (2) Since $T_z^* M = T_z^* M_1 \oplus M$, $[\mathrm{ran} A^*]_2 = [T_z^* M_1]_2 = N_1$. By definition, $\ker A^* = \{f \in M\ ;\ T_z^* f \in M\} \oplus N$. (3) is clear by (1) and that $H^2 = [\mathrm{ran} A]_2 \oplus \ker A^*$. (4) is clear by (1),(2) and that $H^2 = [\mathrm{ran} A^*]_2 \oplus \ker A$.

Lemma 2. *Let* M *be a forward shift invariant subspace of* H^2 *and* $N = H^2 \ominus M$.

(1) $[\mathrm{ran} A]_2 = M_1 \ominus (M_1 \cap \ker T_z^*)$.

(2) $\ker A^* = (M_1 \cap \ker T_z^*) \oplus T_z M \oplus N$.

Proof.(1) Since $T_z N \perp \ker T_z^*$, $T_z N \perp M_1 \cap \ker T_z^*$ and so $P_M T_z N \perp M_1 \cap \ker T_z^*$. Hence by (1) of Lemma 1 $[\mathrm{ran} A]_2 \subseteq M_1 \ominus (M_1 \cap \ker T_z^*)$. If $f \in M_1$ and $f \perp \mathrm{ran} A$, then

$f \perp T_z N$ and so $T_z^* f \perp N$. Hence $T_z^* f \in N \cap M$ because $T_z^* M_1 \perp M$. Hence $T_z^* f = 0$. (2) is a result of (1) by (2) of Lemma 1.

Lemma 3. *Let M be a forward shift invariant subspace of H^2. Then if $[\mathrm{ran}A]_2 \neq M_1$ then $M_1 = [\mathrm{ran}A]_2 \oplus q_2 H^2(w,T)$.*

Proof. By Lemma 2, $M_1 \ominus [\mathrm{ran}A]_2 = M_1 \cap \ker T_z^*$ and $M_1 \cap \ker T_z^* \subset H^2(w,T)$ because $\ker T_z^* = H^2(w,T)$. Hence $w(M_1 \cap \ker T_z^*) \perp zM$ and so $w(M_1 \cap \ker T_z^*) \subseteq M_1 \cap \ker T_z^*$. By a theorem of Beurling [1] $M_1 \ominus [\mathrm{ran}A]_2 = q_2 H^2(w,T)$ for some one variable inner function $q_2 = q_2(w)$.

Theorem 2. *Let M be a nonzero forward shift invariant subspace.*

(1) If $r(A) < \infty$ then $M_1 = \mathrm{ran}A \oplus q_2 H^2(w,T)$ and $M = q_2 H^2 \oplus \left\{ \sum_{j=0}^{\infty} \oplus (\mathrm{ran}A) z^j \right\}$ where $q_2 = q_2(w)$ is a one variable inner function.

(2) If $r(B) < \infty$ then $M_2 = \mathrm{ran}B^* \oplus q_1 H^2(z,T)$ and $M = q_1 H^2 \oplus \left\{ \sum_{j=0}^{\infty} \oplus (\mathrm{ran}B^*) w^j \right\}$ where $q_1 = q_1(z)$ is a one variable inner function.

(3) If $r(A) < \infty$ and $r(B) < \infty$ then there exist two inner functions $q_1 = q_1(z)$ and $q_2 = q_2(w)$ such that $q_1 H^2 + q_2 H^2$ is a closed forward shift invariant subspace, $M \supseteq q_1 H^2 + q_2 H^2$ and $\dim\{M_1 + M_2\}/\{q_1 H^2(z,T) + q_2 H^2(w,T)\} \leq r(A) + r(B)$.

Proof. Since $\dim M_1 = \infty$ by [7, Theorem 3], if $r(A) < \infty$ then $[\mathrm{ran}A]_2 \neq M_1$ and so by Lemm 3 $M_1 = [\mathrm{ran}A]_2 \oplus q_2 H^2(w,T)$ for some one variable inner function $q_2 = q_2(w)$. This implies (1). If $r(B) < \infty$ then $r(B^*) < \infty$. Since $B^* = P_M T_w P_N$, (1) implies (2). If $r(A) < \infty$ and $r(B) < \infty$, (1) and (2) imply (3)because it is known [4] that $q_1 H^2 + q_2 H^2$ is closed.

Corollary 2. *(1) If $A = 0$ then $M = \{0\}$ or $M = q_2 H^2$ for some one variable inner function $q_2 = q_2(w)$.*

(2) If $B = 0$ then $M = \{0\}$ or $M = q_1 H^2$ for some one variable inner function $q_1 = q_1(z)$.

Corollary 3. *(1) If $0 \leq n \leq \infty$ and $0 \leq m \leq \infty$, then there exist invariant subspaces M and N such that $r(A) = n$ and $r(B) = m$.*

(2) If $r(B) = 0$ then $r(A) = 0$ or $r(A) = \infty$. If $r(A) = 0$ then $r(B) = 0$ or $r(B) = \infty$.

Proof. (1) Let $1 \leq n < \infty$ and $1 \leq m < \infty$. Suppose that $M = z^m H^2 + w^n H^2$, then $M_1 = w^n H^2(w,T) + \langle 1, w, \cdots, w^n \rangle z^m$ and $M_2 = z^m H^2(z,T) + \langle 1, z, \cdots, z^m \rangle w^n$. By (1) and (2) of Theorem 2, $r(A) = n$ and $r(B) = m$.

(2) If $r(B) = 0$, then by (2) of Corollary 2 $M = \{0\}$ or $M = q_1 H^2$ where $q_1 = q_1(z)$ is a one variable inner function. If $M = \{0\}$ then $r(A) = 0$ by definition. If $M = q_1 H^2$

then $M_1 = q_1 H^2(w,T)$ and so if $r(A) < \infty$ then by (1) of Theorem 2 $M_1 \supset q_2 H^2(w,T)$ for some one variable inner function $q_2 = q_2(w)$. This implies that q_1 is constant. Hence $M = H^2$ and so $A = 0$.

Corollary 4. *If $M = q_1 H^2 + q_2 H^2$ where $q_1 = q_1(z)$ and $q_2 = q_2(w)$ are one variable inner functions, then $[\mathrm{ran} A]_2 = q_1(H^2(w,T) \ominus q_2 H^2(w,T))$ and $[\mathrm{ran} B^*]_2 = q_2(H^2(z,T) \ominus q_1 H^2(z,T))$. If $r(A) < \infty$ and $r(B) < \infty$ then $r(A) = \deg q_2$ and $r(B) = \deg q_1$.*

Corollary 5. *Let M be a forward shift invariant subspace. If M is of finite co-dimension n then $r(A) \leq n$, $r(B) \leq n$ and $M \supseteq q_1 H^2 + q_2 H^2$ where $q_1 = q_1(z)$ and $q_2 = q_2(w)$ are one variable finite Blaschke products.*

Proof. By the definitions of A and B, it is clear that $r(A) \leq n$ and $r(B) \leq n$. The second statement follows from (3) of Theorem 2.

Proposition 1. *Let M be a forward shift invariant subspace. Then $M \supseteq q_1 H^2 + q_2 H^2$ for some one variable inner functions $q_1 = q_1(z)$ and $q_2 = q_2(w)$ if and only if $[\mathrm{ran} A]_2 \neq M_1$ and $[\mathrm{ran} B^*]_2 \neq M_2$.*

Proof. The 'if' part is clear by Lemma 3. If $M \supseteq q_1 H^2$ then $q_1 H^2(z,T)$ is orthognal to wM and so $q_1 H^2(z,T) \subseteq M_2$. Hence Lemma 2 implies that $[\mathrm{ran} B^*]_2 \neq M_2$. Similarly we can prove that if $M \supseteq q_2 H^2$ then $[\mathrm{ran} A]_2 \neq M_1$.

Proposition 2. *$N_1 = [\mathrm{ran} A^*]_2$ and $N_2 = [\mathrm{ran} B]_2$. Hence $\dim N_1 = r(A)$ and $\dim N_2 = r(B)$.*

Proof. It is a result of (2) of Lemma 1.

§4. $r(AB) < \infty$ or $r(BA) < \infty$

Let M be a forward shift invariant subspace and $N = H^2 \ominus M$. Recall the definitions of $\mathcal{V}$ and $\mathcal{S}$ in Introduction. It is known [4] that $AB \mid M = \mathcal{V}$ and $BA \mid N = \mathcal{S}$. Then $AB = 0$ if and only if $\mathcal{V} = 0$, and $BA = 0$ if and only if $\mathcal{S} = 0$. We know the characterization of an invariant subspace such that $AB = 0$ or $BA = 0$. In fact, it is known (cf. [6],[7],[8]) that $AB = 0$ if and only if $M = qH^2$ for some inner function q. Recently it was proved (cf. [4],[5]) that $BA = 0$ if and only if $N = (H^2(z,T) \ominus q_1 H^2(z,T)) \otimes (H^2(w,T) \ominus q_2 H^2(w,T))$, $N = (H^2(z,T) \ominus q_1 H^2(z,T)) \otimes H^2(w,T)$ or $N = H^2(z,T) \otimes (H^2(w,T) \ominus q_2 H^2(w,T))$, where $q_1 = q_1(z)$ and $q_2 = q_2(w)$ are one variable inner functions. In this section, we study invariant subspaces such that $r(AB) < \infty$ or $r(BA) < \infty$.

Lemma 4. *Let M be a forward shift invariant subspace and $N = H^2 \ominus M$.*

(1) $r(BA) = \dim([P_M T_z N]_2 \cap [P_M T_w N]_2)$.

(2) $r(AB) = \dim([P_N T_z^* M]_2 \cap [P_N T_w^* M]_2)$.

Proof. (1) Since $[BAH^2]_2 = [B[\mathrm{ran} A]_2]_2,\ r(BA) = \dim((\ker B)^\perp \cap [\mathrm{ran} A]_2)$. This implies (1) because $(\ker B)^\perp = [\mathrm{ran} B^*]_2 = [P_M T_w N]_2$ and $[\mathrm{ran} A]_2 = [P_M T_z N]_2$. Similarly, (2) can be proved.

Theorem 3. *Let M be a forward shift invariant subspace of H^2 and $N = H^2 \ominus M$.*

(1) If $M_1 \cap \ker T_z^* = \{0\}$ and $M_2 \cap \ker T_w^* = \{0\}$ then $r(BA) = \dim M_1 \cap M_2$.

(2) $r(AB) = \dim N_1 \cap N_2$.

Proof. (1) By (1) and (2) of Lemma 1, $[\mathrm{ran} A]_2 = [P_M T_z N]_2 \subseteq M_1$ and $[\mathrm{ran} B^*]_2 = [P_M T_w N]_2 \subseteq M_2$. By Lemma 2, if $M_1 \cap \ker T_z^* = \{0\}$ then $[P_M T_z N]_2 = M_1$ and if $M_2 \cap \ker T_w^* = \{0\}$ then $[P_M T_w N]_2 = M_2$. Hence $r(BA) = \dim M_1 \cap M_2$ by Lemma 4.

(2) Since $[P_N T_z^* M]_2 = [P_N T_z^* M_1]_2 = N_1$ and $[P_N T_w^* M]_2 = [P_N T_w^* M_2] = N_2$, by Lemma 4 $r(AB) = \dim N_1 \cap N_2$.

In (1) of Theorem 3, we need the condition : $M_1 \cap \ker T_z^* = M_2 \cap \ker T_w^* = \{0\}$. In fact, $M_1 \cap M_2$ is always not trivial but BA may be zero.

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