Fermionic renormalization group method based on the smooth Feshbach map

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Abstract

For a fermion system, an operator theoretic renormalization group method based on the smooth Feshbach map is constructed. By using the fermionic renormalization group method, the closed operator of the form: $H_g(\theta) = H_{\rm S} \otimes \mathbf{1} + e^{\theta\nu}\mathbf{1} \otimes H_{\rm f} + W_g(\theta)$ is analyzed, where $H_{\rm S}$ is a self-adjoint operator on a separable Hilbert space and bounded from below, $H_{\rm f}$ denotes the fermionic quantization of the one fermion kinetic energy $c|\mathbf{k}|^\nu, \mathbf{k} \in \mathbb{R}^d$ $(c, \nu > 0)$, $W_g(\theta)$ is a small perturbation with respect to $H_{\rm S} \otimes \mathbf{1} + e^{\theta\nu}\mathbf{1} \otimes H_{\rm f}$ and $\theta \in \mathbb{C}$ is a complex scaling parameter. The constant $g \in \mathbb{R}$ denotes a coupling constant such that $W_g(\theta) \to 0 (g \to 0)$ in some sense. It is assumed that $H_{\rm S}$ has a discrete simple eigenvalue $E \in \sigma_{\rm d}(H_{\rm S})$, and proved that $H_g(\theta)$ has an eigenvalue $E_g(\theta)$ close to E for a small coupling constant g. Moreover, the eigenvalue $E_g(\theta)$ and the corresponding eigenvector $\Psi(\theta)$ is constructed by the process of the operator theoretic renormalization group method.

Keywords: smooth Feshbach map, renormalization group, fermionic renormalization group

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Contents

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1 Introduction

In this paper, for a fermion system, we construct an operator theoretic renormalization group method proposed in [3].

We consider a system which a fermion field coupled to a quantum system S. The Hilbert space of the total system is given by

$$\mathcal{H} = \mathcal{H}_{\mathrm{S}} \otimes \mathcal{F}, \tag{1.1}$$

where $\mathcal{H}_{\mathrm{S}}$ denotes the Hilbert space for the quantum system S which is a separable Hilbert space, and $\mathcal{F}$ denotes the fermion Fock space:

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \bigwedge^{n} L^2(\mathbb{M}),$$

where $\wedge^n L^2(\mathbb{M})$ denotes the n-fold antisymmetric tensor product of $L^2(\mathbb{M})$ with $\wedge^0 L^2(\mathbb{M}) = \mathbb{C}$, $\mathbb{M} := \mathbb{R}^d \times \mathbb{L}$ is the momentum-spin arguments of a single fermion with $\mathbb{L} := \{-s, -s+1, \dots, s-1, s\}$ and s denotes a non-negative half-integer. The Hamiltonian of the system S is denoted by H_{S} which is a given self-adjoint operator on $\mathcal{H}_{\mathrm{S}}$ and bounded from below. Let $b^*(k), b(k),\ k \in \mathbb{M}$ be the kernels of the fermion creation and annihilation operators, which obey the canonical anticommutation relations:

$$\begin{aligned} &\{b(k), b^*(\tilde{k})\} = \delta_{l,\tilde{l}}\delta(\mathbf{k} - \tilde{\mathbf{k}}), \quad \{b(k), b(\tilde{k})\} = \{b^*(k), b^*(\tilde{k})\} = 0, \\ &k = (\mathbf{k}, l),\ k = (\tilde{\mathbf{k}}, \tilde{l}) \in \mathbb{M}. \end{aligned} \tag{1.2}$$

Let $\Omega = (1, 0, 0, \dots) \in \mathcal{F}$ be the vacuum vector. The vacuum vector is specified by the condition

$$b(k)\Omega = 0, \quad k \in \mathbb{M}. \tag{1.3}$$

The free Hamiltonian of the fermion field H_{f} is defined by

$$H_{\mathrm{f}} = \int_{\mathbb{R}^d} \sum_{l \in \mathbb{L}} \omega(\mathbf{k}, l) b^*(\mathbf{k}, l) b(\mathbf{k}, l) d\mathbf{k},$$

with the single free fermion energy $\omega(k) = c|\mathbf{k}|^\nu,\ k = (\mathbf{k}, l) \in \mathbb{M}$.

The operator for the coupled system is defined by

$$H_g(\theta) = H_{\mathrm{S}} \otimes \mathbf{1} + e^{\theta\nu} \mathbf{1} \otimes H_{\mathrm{f}} + W_g(\theta). \tag{1.4}$$

Here, the operator $W_g(\theta)$ is the interaction Hamiltonian between the system S and the fermion field, and $\theta \in \mathbb{C}$ is a complex scaling parameter. We suppose that the interaction $W_g(\theta)$ has the form

$$W_g(\theta) = \sum_{M+N=1}^{\infty} g^{M+N} W_{M,N}(\theta), \tag{1.5}$$

$$W_{M,N}(\theta) = \int_{\mathbb{M}^{M+N}} dK^{(M,N)} G^{(\theta)}_{M,N}(K^{(M,N)}) \otimes b^*(k_1)\cdots b^*(k_M) b(\tilde{k}_1)\cdots b(\tilde{k}_N), \tag{1.6}$$

where $g \in \mathbb{R}$ is the coupling constant and

$$K^{(M,N)} = (k_1, \cdots, k_M, \tilde{k}_1, \cdots, \tilde{k}_N) \in \mathbb{M}^{M+N},$$

$$\int_{\mathbb{M}^{M+N}} dK^{(M,N)} := \int_{\mathbb{R}^{d(M+N)}} \sum_{\substack{(l_1,\ldots,l_M)\in\mathbb{L}^M, \\ (\tilde{l}_1,\ldots,\tilde{l}_N)\in\mathbb{L}^N}} d\mathbf{k}_1 \cdots d\mathbf{k}_M d\tilde{\mathbf{k}}_1 \cdots d\tilde{\mathbf{k}}_N, \tag{1.7}$$

and $G^{(\theta)}_{M,N}$ are functions with values in operators on $\mathcal{H}_{\mathrm{S}}$. The precise conditions for $G^{(\theta)}_{M,N}$ are written in the next section.

Suppose that H_{S} has a non-degenerate discrete eigenvalue $E \in \sigma_{\mathrm{d}}(H_{\mathrm{S}})$. Since the vacuum vector Ω is an eigenvector of H_{f} with eigenvalue 0, $H_0(\theta)$ has a eigenvalue E. We are interested in the fate of the eigenvalue E under influence of the perturbation $W_g(\theta)$.

The fermionic renormalization group which we proposed in this paper is constructed for the operator (1.4), and under suitable conditions, it is proved that $H_g(\theta)$ has an eigenvalue $E_g(\theta)$ closed to E for small $g \in \mathbb{R}$. The eigenvalue $E_g(\theta)$ and the corresponding eigenvector $\Psi_g(\theta)$ is constructed by the same process as in [3].

The (bosonic) operator theoretic renormalization group was invented by V. Bach, J. Fröhlich, and I. M. Sigal [1, 2]. In [2], the operator of the similar form (1.4)-(1.6) is considered, but boson is treated instead of fermionand $M + N \leq 2$ is assumed. They proved the existence of an eigenvalue of the (complex scaled) Hamiltonian, and constructed the eignvalue and the corresponding eigenvector. Moreover, they gave the range of the continuous spectrum which extended from the eigenvalue.

In the paper [3], V. Bach, T. Chen, J. Fröhlich, and I. M. Sigal introduced the smooth Feshbach map and largely improved the proof of the convergence of the renormalization group.

Our paper is based on the smooth Feshbach map and the improved renormalization group method [3]. Our construction for the fermionic operator theoretic renormalization group is similar as in [3] without the Wick ordering and its related estimate.

The feature of this paper is that we can treat a large class of interactions. In particular, the interaction Hamiltonian $W_g(\theta)$ includes arbitrary order of the creation and annihilation operators.

In the following Section 2, to explain the problem in detail, we give the precise definitions of $H_g(\theta)$. In order to explain and use the smooth Feshbach map, we review it in Section 3.

Section 5 is devoted to the construction of the renormalization group.

The main originality of this paper is to procure the Wick ordering formula for fermion. The Wick ordering formula for fermion and related formula is written in the Appendix A.

2 Hypotheses and Main Results

Through this paper, we denote the inner product and the norm of a Hilbert space $\mathcal{X}$ by $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ and $\| \cdot \|$ respectively, where we use the convention that the inner product is antilinear (respectively linear) in the first (respectively second) variable. If there is no danger of confusion, then we omit the subscript $\mathcal{X}$ in $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ and $\| \cdot \|$. For a linear operator T on a Hilbert space, we denote its domain, spectrum and resolvent by $\mathrm{dom}(T)$, $\sigma(T)$ and $\mathrm{Res}(T)$, respectively. If T is densely defined, then the adjoint of T is denoted by T^*.

One can identify a vector $\Psi \in \mathcal{F}$ with a sequence $(\Psi^{(n)})_{n=0}^{\infty}$ of n-fermion state $\Psi^{(n)} \in \wedge^n L^2(\mathbb{M}) \subset L^2(\mathbb{M}^n)$. We observe that, for all $\psi \in \wedge^n L^2(\mathbb{M})$ and $\pi \in \mathcal{S}_n$,

$$\psi(k_{\pi(1)}, \cdots, k_{\pi(n)}) = \mathrm{sgn}(\pi)\psi(k_1, \cdots, k_n), \quad \text{a.e.} \tag{2.1}$$

where $\mathcal{S}_n$ is the group of permutations of n elements and $\mathrm{sgn}(\pi)$ the sign of the permutation π. The inner product of $\mathcal{F}$ is defined by

$$\langle \Psi, \Phi \rangle = \sum_{n=0}^{\infty} \langle \Psi^{(n)}, \Phi^{(n)} \rangle_{\wedge^n L^2(\mathbb{M})} \tag{2.2}$$

for $\Psi, \Phi \in \mathcal{F}$, where

$$\langle \Psi^{(n)}, \Phi^{(n)} \rangle_{\wedge^n L^2(\mathbb{M})} = \int_{\mathbb{M}^n} \prod_{j=1}^{n} dk_j \Psi^{(n)}(k_1, \cdots, k_n)^* \Phi^{(n)}(k_1, \cdots, k_n). \tag{2.3}$$

We define the free Hamiltonian of the fermion field H_{f} by

$$\mathrm{dom}(H_{\mathrm{f}}) := \left\{ \Psi \in \mathcal{F} \middle| \sum_{n=0}^{\infty} \|(H_{\mathrm{f}}\Psi)^{(n)}\|^2 < \infty \right\}, \tag{2.4}$$

$$(H_{\mathrm{f}}\Psi)^{(n)}(k_1, \cdots, k_n) = \left(\sum_{j=1}^{n} \omega(k_j) \right) \Psi^{(n)}(k_1, \cdots, k_n), \quad n \in \mathbb{N} \tag{2.5}$$

$$(H_{\mathrm{f}}\Psi)^{(0)} = 0, \tag{2.6}$$

where

$$\omega(k) := c|\mathbf{k}|^{\nu}, \quad k = (\mathbf{k}, l) \in \mathbb{M},$$

with a positive constant $c, \nu > 0$. For a nonrelativistic fermion, the choice of the constants c, ν are $c = 1/2m$ and $\nu = 2$, where m denotes the mass of the fermion. In this paper, for any $\Psi \in \mathcal{F}$, $b(k)\Psi$ is regarded as a $\times_{n=0}^{\infty} \wedge^n L^2(\mathbb{M})$-valued function:

$$b(k) : \mathbb{M} \ni k \longmapsto b(k)\Psi \in \mathop{\times}_{n=0}^{\infty} \wedge^n L^2(\mathbb{M}), \quad \text{a.e.}, \tag{2.7}$$

$$(b(k)\Psi)^{(n)}(k_1, \cdots, k_n) = \sqrt{n+1}\Psi^{(n+1)}(k, k_1, \cdots, k_n), \tag{2.8}$$

where the symbol "×" denotes the Cartesian product. We set

$$\mathrm{dom}(b(k)) := \{\Psi \in \mathcal{F} | b(k')\Psi \in \mathcal{F} \text{ a.e.} k' \in \mathbb{M}\}.$$

Note that $\mathrm{dom}(b(k))$ is independent of $k \in \mathbb{M}$. We observe that, for all $\Psi \in \mathcal{F}$ and $\Phi \in \mathrm{dom}(H_{\mathrm{f}})$,

$$\begin{aligned}
\langle \Psi, H_{\mathrm{f}}\Phi\rangle =& \sum_{n=0}^{\infty} \int_{\mathbb{M}^{(n+1)}} \prod_{j=1}^{n+1} dk_j \Psi^{(n+1)}(k_1, \cdots, k_{n+1})^* \\
& \times \left(\sum_{j=1}^{n+1} \omega(k_j)\right) \Psi^{(n+1)}(k_1, \cdots, k_{n+1}) \\
=& \sum_{n=0}^{\infty} \int_{\mathbb{M}\times\mathbb{M}^n} dk \prod_{j=1}^{n} dk_j (b(k)\Psi)^{(n)}(k_1, \cdots, k_n)^* \\
& \times \omega(k)(b(k)\Psi)^{(n)}(k_1, \cdots, k_n)
\end{aligned} \tag{2.9}$$

where we have used the antisymmetry (2.1). Hence we have

$$\langle \Psi, H_{\mathrm{f}}\Phi\rangle = \int_{\mathbb{M}} dk\omega(k)\langle b(k)\Psi, b(k)\Phi\rangle \tag{2.10}$$

and, in this sense, write symbolically

$$H_{\mathrm{f}} = \int_{\mathbb{M}} dk\omega(k)b^*(k)b(k). \tag{2.11}$$

In the same way as (2.11), the number operator, N_{f}, is defined by

$$N_{\mathrm{f}} = \int_{\mathbb{M}} dkb^*(k)b(k). \tag{2.12}$$

We remark that

$$\mathrm{dom}(H_{\mathrm{f}}^{1/2}),\ \ \mathrm{dom}(N_{\mathrm{f}}^{1/2}) \subset \mathrm{dom}(b(k)), \tag{2.13}$$

since, for all $\Psi \in \mathrm{dom}(H_{\mathrm{f}}^{1/2})$ and $\Phi \in \mathrm{dom}(N_{\mathrm{f}}^{1/2})$,

$$\begin{aligned}
\|H_{\mathrm{f}}^{1/2}\Psi\|^2 &= \int_{\mathbb{M}} dk\omega(k)\|b(k)\Psi\|^2 < \infty, \\
\|N_{\mathrm{f}}^{1/2}\Phi\|^2 &= \int_{\mathbb{M}} dk\|b(k)\Phi\|^2 < \infty.
\end{aligned}$$

The (smeared) annihilation operator $b(f)$ ($f \in L^2(\mathbb{M})$) defined by

$$b(f) = \int_{\mathbb{M}} f(k)^* b(k) dk, \tag{2.14}$$

and the adjoint $b^*(f)$, called the (smeared) creation operator, obey the canonical anti-commutation relations (CAR):

$$\{b(f), b(g)\} = \langle f, g\rangle, \quad \{b(f), b(g)\} = \{b^*(f), b^*(g)\} = 0 \tag{2.15}$$

for all $f, g \in L^2(\mathbb{M})$, where $\{X, Y\} = XY + YX$.

The Hamiltonian of the total system is defined by

$$H_g := H_{\mathrm{S}} \otimes \mathbf{1} + \mathbf{1} \otimes H_{\mathrm{f}} + W_g,$$

where the symmetric operator W_g is of the form:

$$W_g = \sum_{M+N=1}^{\infty} g^{M+N} W_{M,N}, \tag{2.16}$$

$$W_{M,N} = \int_{\mathbb{M}^{M+N}} dK^{(M,N)} G_{M,N}(K^{(M,N)}) \otimes b^*(k_1) \cdots b^*(k_M) b(\tilde{k}_1) \cdots b(\tilde{k}_N), \tag{2.17}$$

and

$$K^{(M,N)} = (k_1, \cdots, k_M, \tilde{k}_1, \cdots, \tilde{k}_N) \in \mathbb{M}^{M+N},$$

$$\int_{\mathbb{M}^{M+N}} dK^{(M,N)} := \int_{\mathbb{R}^{d(M+N)}} \sum_{\substack{(l_1,\dots,l_M)\in\mathbb{L}^M,\\ (\tilde{l}_1,\dots,\tilde{l}_N)\in\mathbb{L}^N}} d\mathbf{k}_1 \cdots d\mathbf{k}_M d\tilde{\mathbf{k}}_1 \cdots d\tilde{\mathbf{k}}_N. \tag{2.18}$$

Here, for almost every $K^{(M,N)} \in \mathbb{M}^{M+N}$, $G_{M,N}(K^{(M,N)})$ is a densely defined closable operator on $\mathcal{H}_{\mathrm{S}}$. $H_0 := H_{\mathrm{S}} \otimes \mathbf{1} + \mathbf{1} \otimes H_{\mathrm{f}}$ is regarded to the unperturbed Hamiltonian, and W_g is regarded to the perturbation Hamiltonian.

In what follows we formulate hypotheses of main theorem and introduce some objects.

Hypothesis 1. (spectrum) *Assume that H_{S} has a non-degenerate isolate eigenvalue $E \in \sigma_{\mathrm{d}}(H_{\mathrm{S}})$ such that*

$$\mathrm{dist}(E, \sigma(H_{\mathrm{S}}))\backslash\{E\}) \geq 1. \tag{2.19}$$

In general, if the operator H_{S} has a discrete eigenvalue E, it holds that $c_1 := \mathrm{dist}(E, \sigma(H_{\mathrm{S}})\backslash\{E\}) > 0$ and $\mathrm{dist}(c_1^{-1}E, \sigma(c_1^{-1}H_{\mathrm{S}}))\backslash\{c_1^{-1}E\} \geq 1$. We can assume (2.19) without loss of generality.

Since $\sigma(H_{\mathrm{f}}) = [0, \infty)$, the spectrum of the unperturbed Hamiltonian is $\sigma(H_0) = [E_0, \infty)$ with $E_0 := \inf \sigma(H_{\mathrm{S}})$. The vector Ω is an eigenvector of H_0 with eigenvalue 0. Hence, H_0 has an embedded eigenvalue E. In this paper, we study the fate of E under the perturbation $W_g(\theta)$. To analyze the perturbed Hamiltonian H_g, for $\theta \in \mathbb{R}$, we introduce the family of operators $H_g(\theta)$ of the form

$$H_g(\theta) \equiv (\mathbf{1} \otimes \Gamma_\theta) H_g (\mathbf{1} \otimes \Gamma_\theta^*) = H_0(\theta) + W_g(\theta), \tag{2.20}$$

where Γ_ρ is the dilation operator, i.e.,

$$\Gamma_\rho b(\mathbf{k}, l) \Gamma_\rho^* = \rho^{-d/2} b(\rho^{-1}\mathbf{k}, l), \tag{2.21}$$

and

$$H_0(\theta) \equiv H_{\mathrm{S}} \otimes \mathbf{1} + e^{\theta\nu} \mathbf{1} \otimes H_{\mathrm{f}} \tag{2.22}$$

$$W_g(\theta) \equiv (\mathbf{1} \otimes \Gamma_{e^\theta}) W_g (\mathbf{1} \otimes \Gamma_{e^\theta}^*) = \sum_{M+N=1}^{\infty} g^{M+N} W_{M,N}(\theta), \tag{2.23}$$

$$
\begin{aligned}
W_{M,N}(\theta) &\equiv \Gamma_{e^\theta} W_{M,N} \Gamma_{e^\theta}^* \\
&= \int_{\mathbb{M}^{M+N}} dK^{(M,N)} G_{M,N}^{(\theta)}(K^{(M,N)}) \otimes b^*(k_1)\cdots b^*(k_M) b(\tilde{k}_1)\cdots b(\tilde{k}_N), \quad (2.24)\\
G_{M,N}^{(\theta)}(K^{(M,N)}) &:= e^{d(M+N)\theta/2} G_{M,N}(e^\theta K^{(M,N)}), \quad (2.25)\\
e^\theta K^{(M,N)} &:= (e^\theta \mathbf{k}_1, l_1; \ldots; e^\theta \mathbf{k}_M, l_M; e^\theta \tilde{\mathbf{k}}_1 \tilde{l}_1; \ldots; e^\theta \tilde{\mathbf{k}}_N, \tilde{l}_N). \quad (2.26)
\end{aligned}
$$

Hypothesis 2. *Assume that, for every θ in some complex neighborhood of 0,*

(i) $G_{M,N}(e^\theta K^{(M,N)})$ *is defined on* $\mathrm{dom}(G_{M,N})$ *that contains* $\mathrm{dom}(H_0(\theta))$,

(ii) For all $M+N \geq 1$, $W_{M,N}(\theta)$ *is relatively bounded with respect to* $H_0(\theta)$ *and*

$$
\sum_{M+N=1}^{\infty} g^{M+M} \|W_{M,N}(\theta)\Psi\| \leq a_g(\theta)\|H_0(\theta)\Psi\| + b_g(\theta)\|\Psi\|, \quad (2.27)
$$

for all $\Psi \in \mathrm{dom}(H_0(\theta))$, *with some constants* $a_g(\theta), b_g(\theta) \geq 0$,

(iii) $H_g(\theta)$ *is an analytic family of type A [6] near* $\theta = 0$.

(iv) $a_g(\theta) < 1$.

(v) There exists a constant $\gamma > 1/2$ *such that*

$$
\int_{\mathbb{M}^{M+N}} \frac{dK^{(M,N)}}{\left[\prod_{j=1}^M \omega(k_j) \prod_{j=1}^N \omega(k_j)\right]^{1+2\gamma}} \|G_{M,N}^{(\theta)}(K^{(M,N)})(H_f+1)^{-1}\|_{\mathrm{op}}^2 < \infty,
$$

holds for all $M+N \geq 1$.

By the hypothesis above, one can show that, $H_g(\theta)$ is closed operator with the domain $\mathrm{dom}(H_g(\theta)) = \mathrm{dom}(H_0)$. In particular, H_g is a self-adjoint operator on $\mathrm{dom}(H_0)$.

By Hypothesis 2, we can consider the case $\theta = -i\vartheta/\nu$ $(0 < \vartheta < \pi/2)$. In what follows, we set $\theta = -i\vartheta/\nu$ and fix the parameter $\vartheta \in (0, \pi/2)$ so that Hypothesis 2 holds. Then, the spectrum $\sigma(H_0(-i\vartheta/\nu))$ contains separate rays of continuous spectrum and the eigenvalue E of $H_0(-i\vartheta/\nu)$ are located at tip of a branch of a continuous spectrum. Indeed, we observe

$$
\begin{aligned}
\sigma(H_0(-i\vartheta/\nu)) &= \{\lambda_1 + e^{-i\vartheta}\lambda_2 | \lambda_1 \in \sigma(H_{\mathrm{S}}), \lambda_2 \in \sigma(H_{\mathrm{f}})\} \\
&\supset \left\{E + e^{-i\vartheta}\lambda \mid \lambda \in [0, \infty)\right\}.
\end{aligned}
$$

In order to study the fate of E under the perturbation of W_g, we introduce a spectral parameter $z \in \mathbb{C}$, and define a family of operators $H[z]$ by

$$
H[z] = H_g(-i\vartheta/\nu) - E - z, \quad (2.28)
$$

where $0 < \vartheta < \pi/2$.

By using the fermionic renormalization group method established in this paper, we will construct a constant $e_g(\theta)$ and a vector $\Psi_g(\theta) \in \mathrm{dom}(H_g(-i\vartheta/\nu)) \setminus \{0\}$ such that

$$
H[e_g(\theta)]\Psi_g(\theta) = 0,
$$

which implies that $E_g(\theta) := E + e_g(\theta)$ is an eigenvalue of $H_g(-i\vartheta/\nu)$ and $\Psi_g(\theta)$ is corresponding eigenvector.

The following theorem is our main result:

Theorem 2.1. *There exists a constant $g_0 > 0$ such that, for all g with $|g| \le g_0$, the limits*

$$e_g(\theta) := \lim_{\beta\to\infty} e_{(0,\beta)} : = \lim_{\beta\to\infty} J_{(0)}^{-1} \circ J_{(1)}^{-1} \circ \cdots \circ J_{(\beta)}^{-1}[0] \in \mathbb{C} \tag{2.29}$$

and

$$\Psi_{(0,\infty)} : = \lim_{\beta\to\infty} Q_{(0)}\Gamma_\rho^* Q_{(1)}\Gamma_\rho^* \cdots Q_{(\beta-1)}\Omega \in \mathrm{Ran}\mathbf{1}_{[H_\mathrm{f}<1]} \tag{2.30}$$

exist. Moreover, $\Psi_g(\theta) := Q_\chi \phi \otimes \Psi_{(0,\infty)} \neq 0$ and

$$H[e_g(\theta)]\Psi_g(\theta) = 0, \tag{2.31}$$

where the functions $J_{(\beta)}$ and the operators $Q_{(\beta)}$ are defined by (5.52) *and* (5.61)*, respectively, and the operator Q_χ is defined by*

$$Q_\chi := \chi - \bar{\chi} H_{\bar{\chi}}^{-1}[e_{(0,\infty)}]\bar{\chi} W_g(\theta)\chi$$

with $\chi, \bar{\chi}$ and $H_{\bar{\chi}}^{-1}$ given by (4.1)*,* (4.3)*, and* (4.18)*, respectively.*

3 Smooth Feshbach map

In this section we review the smooth Feshbach map [3]. The smooth Feshbach map is the main ingredient to construct the operator theoretic renormalization group. Let χ be a bounded self-adjoint operator on a separable Hilbert space $\mathcal{H}$ such that $0 \le \chi \le 1$. We set

$$\bar{\chi} := \sqrt{1-\chi^2}.$$

Suppose that χ and $\bar{\chi}$ are non-zero operators. Let T be a closed operator on $\mathcal{H}$. We assume that

$$\chi T \subset T\chi,$$

and hence $\bar{\chi}T \subset T\bar{\chi}$, which mean that χ and $\bar{\chi}$ leave $\mathrm{dom}(T)$ invariant and commute with T. Let H be a closed operator on $\mathcal{H}$ such that $\mathrm{dom}(H) = \mathrm{dom}(T)$ and we set

$$H_\chi := T + \chi W \chi, \qquad H_{\bar{\chi}} := T + \bar{\chi} W \bar{\chi},$$

where $W := H - T$. We observe that, by the assumptions, the operators W, H_χ and $H_{\bar{\chi}}$ are defined on $\mathrm{dom}(T)$ and H_χ (resp. $H_{\bar{\chi}}$) is reduced by $\overline{\mathrm{Ran}\,\chi}$ (resp. $\overline{\mathrm{Ran}\,\bar{\chi}}$). We denote the projection onto $\overline{\mathrm{Ran}\,\chi}$ (resp. $\overline{\mathrm{Ran}\,\bar{\chi}}$) by P (resp. $\bar{P}$) and have

$$H_\chi \subset PH_\chi P + P^\perp T P^\perp, \qquad H_{\bar{\chi}} \subset \bar{P} H_{\bar{\chi}} \bar{P} + \bar{P}^\perp T \bar{P}^\perp,$$

where $P^\perp := 1 - P$ (resp. $\bar{P}^\perp := 1 - \bar{P}$) is the projection on $\ker\chi$ (resp. $\ker\bar{\chi}$).

We now introduce the *Feshbach triple* $\langle \chi, T, H \rangle$ as follows:

Definition 3.1. *Let χ, T and H as above. Then, we call $\langle \chi, H, T\rangle$ a Feshbach triple if $H_{\bar\chi}$ is bounded invertible on $\overline{\mathrm{Ran}\,\bar\chi}$ and the following conditions hold: the operators $\chi W\bar\chi H_{\bar\chi}^{-1}\bar\chi$ and $\chi W\bar\chi H_{\bar\chi}^{-1}\bar\chi W\chi$ extend to bounded operators from $\mathcal{H}$ to $\overline{\mathrm{Ran}\,\chi}$ and $\bar\chi H_{\bar\chi}^{-1}\bar\chi W\chi$ to bounded operators from $\mathcal{H}$ to $\overline{\mathrm{Ran}\,\bar\chi}$, where $H_{\bar\chi}^{-1}$ denotes the inverse operator of $\bar P H_{\bar\chi}\bar P$.*

We remark that, if $H_{\bar\chi}$ is bounded invertible on $\overline{\mathrm{Ran}\,\bar\chi}$, then the operators $\chi W\bar\chi H_{\bar\chi}^{-1}\bar\chi$, $\bar\chi H_{\bar\chi}^{-1}\bar\chi W\chi$ and $\chi W\bar\chi H_{\bar\chi}^{-1}\bar\chi W\chi$ are defined on $\mathrm{dom}(T)$.

For a Feshbach triple $\langle \chi, H, T\rangle$, we denote the closures of the operators $\chi W\bar\chi H_{\bar\chi}^{-1}\bar\chi$, $\chi W\bar\chi H_{\bar\chi}^{-1}\bar\chi W\chi$ and $\bar\chi H_{\bar\chi}^{-1}\bar\chi W\chi$ by the same symbols.

The definition of the Feshbach triple as above implies

$$\chi W\bar\chi H_{\bar\chi}^{-1}\bar\chi,\ \chi W\bar\chi H_{\bar\chi}^{-1}\bar\chi W\chi \in \mathcal{B}(\mathcal{H};\overline{\mathrm{Ran}\,\chi}),\ \bar\chi H_{\bar\chi}^{-1}\bar\chi W\chi \in \mathcal{B}(\mathcal{H};\overline{\mathrm{Ran}\,\bar\chi}). \tag{3.1}$$

For a Feshbach triple $\langle \chi, H, T\rangle$, we define the operator

$$F_\chi(H,T) := H_\chi - \chi W\bar\chi H_{\bar\chi}^{-1}\bar\chi W\chi, \tag{3.2}$$

acting on $\mathcal{H}$. We observe, by the definition of the Feshbach triple, that $F_\chi(H,T)$ is defined on $\mathrm{dom}(T)$.

The map from Feshbach pairs to operators on $\mathcal{H}$

$$\langle \chi, H, T\rangle \longmapsto F_\chi(H,T) \tag{3.3}$$

is called the *smooth Feshbach map (SFM)*. We remark that $F_\chi(H,T)$ is reduced by $\overline{\mathrm{Ran}\,\chi}$ and

$$F_\chi(H,T) \subset PF_\chi(H,T)P + P^\perp T P^\perp.$$

The SFM is an isospectral map in the sense of the following theorem.

Theorem 3.2. (SFM [3]) *Let $\langle \chi, H, T\rangle$ be a Feshbach triple. Then the following (i)-(v) hold:*

(i) If T is bounded invertible on $\overline{\mathrm{Ran}\,\bar\chi}$ and H is bounded invertible on $\mathcal{H}$ then $F_\chi(H,T)$ is bounded invertible on $\mathcal{H}$. In this case,

$$F_\chi(H,T)^{-1} = \chi H^{-1}\chi + \bar\chi T^{-1}\bar\chi. \tag{3.4}$$

If $F_\chi(H,T)$ is bounded invertible on $\overline{\mathrm{Ran}\,\chi}$, then H is bounded invertible on $\mathcal{H}$. In this case,

$$H^{-1} = Q_\chi(H,T)F_\chi(H,T)^{-1}Q_\chi^\#(H,T) + \bar\chi H_{\bar\chi}^{-1}\bar\chi, \tag{3.5}$$

where we set

$$Q_\chi(H,T) := \chi - \bar\chi H_{\bar\chi}^{-1}\bar\chi W\chi \in \mathcal{B}(\overline{\mathrm{Ran}\,\chi},\mathcal{H}), \tag{3.6}$$

$$Q_\chi^\#(H,T) := \chi - \chi W\bar\chi H_{\bar\chi}^{-1}\bar\chi \in \mathcal{B}(\mathcal{H},\overline{\mathrm{Ran}\,\chi}). \tag{3.7}$$

(ii) If $\psi \in \ker H \setminus \{0\}$, then $\chi\psi \in \ker F_\chi(H,T)\setminus\{0\}$:

$$F_\chi(H,T)\chi\psi = 0. \tag{3.8}$$

(iii) If $\phi \in \ker F_\chi(H,T)\setminus\{0\}$, then $Q_\chi(H,T)\phi \in \ker H$:

$$HQ_\chi(H,T)\phi = 0. \tag{3.9}$$

Assume, in addition that, T is bounded invertible on $\overline{\mathrm{Ran}\,\bar\chi}$. Then, $\phi \in \overline{\mathrm{Ran}\,\chi}\setminus\{0\}$ and $Q_\chi(H,T)\phi \neq 0$.

(iv) If T is bounded invertible on $\overline{\operatorname{Ran}\bar{\chi}}$, then

$$\dim\ker H = \dim\ker F. \tag{3.10}$$

Moreover, if $\dim\ker H < \infty$ *or* $\dim\ker F_\chi(H,T) < \infty$*, then the maps*

$$\chi : \ker H \longrightarrow \ker F_\chi(H,T)$$

and

$$Q_\chi(H,T) : \ker F_\chi(H,T) \longrightarrow \ker H$$

are bijective.

(v) Assume that H and T are self-adjoint operator and set

$$M := H_{\bar{\chi}}^{-1}\bar{\chi}W\chi \in \mathcal{B}(\mathcal{H}, \overline{\operatorname{Ran}\bar{\chi}})$$

and

$$N := (1 + M^*M)^{-1/2} \in \mathcal{B}(\mathcal{H}).$$

If T is bounded invertible on $\overline{\operatorname{Ran}\bar{\chi}}$ and H_χ is self-adjoint, then, for all $\psi \in \mathcal{H}$,

$$\lim_{\epsilon\downarrow 0} \operatorname{Im}\langle \psi, (H - i\epsilon)^{-1}\psi\rangle = \lim_{\epsilon\downarrow 0} \operatorname{Im}\langle NQ^*\psi, (NF_\chi(H,T) - i\epsilon)^{-1}NQ^*\psi\rangle \tag{3.11}$$

and

$$\lim_{\epsilon\downarrow 0} \operatorname{Im}\langle \psi, (NF_\chi(H,T)N - i\varepsilon)^{-1}\psi\rangle = \lim_{\epsilon\downarrow 0} \operatorname{Im}\langle \chi N^{-1}\psi, (H - i\epsilon)^{-1}\chi N^{-1}\psi\rangle. \tag{3.12}$$

4 First Reduction Step

We hereafter assume Hypotheses 1-2. By using the smooth Feshbach map, we eliminate the degree of high energy fermion, and restrict the degree of the system S to the eigenstate φ. Let

$$\chi := P \otimes \sin\left[\frac{\pi}{2}\Xi\left(H_{\mathrm{f}}\right)\right], \tag{4.1}$$

where P is the orthogonal projection onto the eigenspace $\ker(H_{\mathrm{S}} - E)$ and the function $\Xi : \mathbb{R} \to [0,1]$ is smooth in $(0,1)$ and obeys

$$\Xi(r) = \begin{cases} 1 & \left(0 \le r < \frac{3}{4}\right), \\ 0 & (r < 0, \tau \le r), \end{cases} \tag{4.2}$$

where $3/4 < \tau < 1$. Then we have

$$\bar{\chi} := \sqrt{1 - \chi^2} = P \otimes \cos\left[\frac{\pi}{2}\Xi\left(H_{\mathrm{f}}\right)\right] + P^\perp \otimes \mathbf{1}. \tag{4.3}$$

Let

$$T[z] := H_0(-i\vartheta/\nu) - E - z \tag{4.4}$$

and

$$W := H[z] - T[z] = W_g(-i\vartheta/\nu). \tag{4.5}$$

It is evident that $T[z]$ is closed, commuting with χ. Furthermore, we have the following lemma.

Lemma 4.1. *$T[z]$ is bounded invertible on* $\operatorname{Ran}\bar{\chi}$ *for all z with*

$$|z| < \min\{3/4, \sin(\vartheta/\nu)\}.$$

Proof. Let us first note that the orthogonal projection $P_{\bar{\chi}}$ onto $\overline{\operatorname{Ran}\bar{\chi}}$ is of the following form

$$P_{\bar{\chi}} = P \otimes \mathbf{1}_{[H_{\mathrm{f}} > \frac{3}{4}]} + P^{\perp} \otimes \mathbf{1}, \tag{4.6}$$

and hence

$$P_{\bar{\chi}} T[z] P_{\bar{\chi}} = L_1 + L_2, \tag{4.7}$$

where the function $\mathbf{1}_A$ is the indicator of a set A and

$$L_1 = P \otimes \mathbf{1}_{[H_{\mathrm{f}} > \frac{3}{4}]} \left(e^{-i\vartheta} H_{\mathrm{f}} - z\right) \mathbf{1}_{[H_{\mathrm{f}} > \frac{3}{4}]}, \tag{4.8}$$

$$L_2 = P^{\perp} \left(H_{\mathrm{S}} - E\right) P^{\perp} \otimes \mathbf{1} + P^{\perp} \otimes \left(e^{-i\vartheta} H_{\mathrm{f}} - z\right). \tag{4.9}$$

We need only to prove L_1 and L_2 are bounded invertible, i.e., $z \in \operatorname{Res}(L_1) \cap \operatorname{Res}(L_2)$, since, by (4.7), (4.8) and (4.9), $P_{\bar{\chi}} T[z] P_{\bar{\chi}}$ is reduced by $\operatorname{Ran} P \otimes \mathbf{1}_{[H_{\mathrm{f}} > \frac{3}{4}]}$ and $\operatorname{Ran} P^{\perp} \otimes 1$. Indeed, we observe $z \in \operatorname{Res}(L_1)$ and $z \in \operatorname{Res}(L_2)$ provided $|z| < 3/4$ and $|z| < \sin(\vartheta/\nu)$, respectively. □

Let $T^{-1}[z]$ be the inverse of $P_{\bar{\chi}} T[z] P_{\bar{\chi}}$ for all z with $|z| < \rho_0$:

$$T^{-1}[z] := \left(P_{\bar{\chi}} T[z] P_{\bar{\chi}}\right)^{-1}, \tag{4.10}$$

where we set

$$\rho_0 := \min\left\{\frac{3}{4}, \sin(\vartheta/\nu)\right\}. \tag{4.11}$$

Then, we have, for all z with $|z| < \rho_0/2$,

$$\operatorname{Res}(P_{\bar{\chi}} T[z] P_{\bar{\chi}}) \supset D_{\rho_0/2}, \tag{4.12}$$

where

$$D_\epsilon := \{z \in \mathbb{C} \mid |z| \le \epsilon\} \tag{4.13}$$

for all $\epsilon > 0$. By Hypothesis 2, we have

$$\begin{aligned} &\left\|W \bar{\chi} T^{-1}[z] \bar{\chi} \Psi\right\| \\ &\le a_g(-i\vartheta/\nu) \left\|H_0(-i\vartheta/\nu)\ \bar{\chi} T^{-1}[z] \bar{\chi}\Psi\right\| + b_g(-i\vartheta/\nu) \left\|\bar{\chi} T^{-1}[z]\bar{\chi}\Psi\right\| \\ &\le \left\{a_g(-i\vartheta/\nu) + (a_g(-i\vartheta/\nu)|E+z| + b_g(-i\vartheta/\nu)) \left\|T^{-1}[z]\right\|\right\} \left\|\bar{\chi}\Psi\right\|, \end{aligned} \tag{4.14}$$

where $a_g(-i\vartheta/\nu)$ and $b_g(-i\vartheta/\nu)$ are defined by (2.27). We next require the following.

Hypothesis 3. (Feshbach triple) *The triple $\langle H[z], T[z], \chi\rangle$ is a Feshbach triple and*

$$2a_g(-i\vartheta/\nu) + \frac{2}{\rho_0}\left(|E| a_g(-i\vartheta/\nu) + b_g(-i\vartheta/\nu)\right) < 1. \tag{4.15}$$

Let

$$H_{\bar{\chi}}[z] := T[z] + \bar{\chi} W \bar{\chi}. \tag{4.16}$$

Then we have the following lemma.

Lemma 4.2. *Assume that Hypothesis 3. Then, for all* $z \in D_{\rho_0/2}$,

$$F_\chi(H[z], T[z]) = T[z] + \sum_{L=1}^{\infty} (-1)^{L-1} \chi W \left(\bar\chi T^{-1}[z] \bar\chi W\right)^{L-1} \chi. \tag{4.17}$$

Proof. We note that $H_{\bar\chi}[z]$ is bounded invertible on $\operatorname{Ran} \bar\chi$ and the Neumann series expansion of the inverse

$$H_{\bar\chi}^{-1}[z] = \sum_{L=0}^{\infty} (-1)^L T^{-1}[z] \left(\bar\chi W \bar\chi T^{-1}[z]\right)^L \tag{4.18}$$

is norm convergent since, by Hypothesis 3 and (4.14), we have

$$\left\| W \bar\chi T^{-1}[z] \right\|_{\mathcal{B}(\operatorname{Ran} \bar\chi; \mathcal{F})} < 1. \tag{4.19}$$

Then, by the definition of the Feshbach map (3.2) and (4.18), we obtain

$$\begin{aligned}
F_\chi(H[z], T[z]) &= T[z] + \chi W \chi - \chi W \bar\chi H_{\bar\chi}^{-1}[z] \bar\chi W \chi \\
&= T[z] + \chi W \chi + \sum_{L=0}^{\infty} (-1)^{L+1} \chi W \bar\chi T^{-1}[z] \left(\bar\chi W \bar\chi T^{-1}[z]\right)^L \bar\chi W \chi \\
&= T[z] + \chi W \chi + \sum_{L=0}^{\infty} (-1)^{L+1} \chi W \left(\bar\chi T^{-1}[z] \bar\chi W\right)^{L+1} \chi,
\end{aligned}$$

which is equivalent to (4.17). $\square$

Let P_χ be the orthogonal projection onto $\operatorname{Ran} \chi$:

$$P_\chi = P \otimes \mathbf{1}_{[H_\mathrm{f} < \tau]}, \tag{4.20}$$

where the constant $3/4 < \tau < 1$ is defined in (4.2). According to Theorem 3.2 (iii), we need only to consider the spectrum of $P_\chi F_\chi(H[z], T[z]) P_\chi$ since $T^{-1}[z]$ is bounded invertible on $\operatorname{Ran} \bar\chi$ with $z \in D_{\rho_0/2}$. We note that the operator $H_{(0)}[z]$ on $\operatorname{Ran} \mathbf{1}_{[H_\mathrm{f} < \tau]}$ can be defined by

$$P \otimes H_{(0)}[z] = P_\chi F_\chi(H[z], T[z]) P_\chi \tag{4.21}$$

since, by Hypothesis 1, the eigenvalue E is simple.

Let us next derive $H_{(0)}$ from (4.21) and arrange the annihilation and creation operators in order. We observe, from Lemma 4.2 and (4.1), that

$$\begin{aligned}
&P_\chi F_\chi(H[z], T[z]) P_\chi \\
&= P_\chi T[z] P_\chi + \sum_{L=1}^{\infty} (-1)^{L-1} P_\chi \chi W \left(\bar\chi T^{-1}[z] \bar\chi W\right)^{L-1} \chi P_\chi \\
&= P \otimes \mathbf{1}_{[H_\mathrm{f} < \tau]} \left(e^{-i\vartheta} H_\mathrm{f} - z\right) \mathbf{1}_{[H_\mathrm{f} < \tau]} \\
&\quad + \sum_{L=1}^{\infty} (-1)^{L-1} \sum_{M_l + N_l \geq 1; l = 1, \cdots, L} g^{\sum_{l=1}^{L} (M_l + N_l)} \\
&\quad \times P \otimes \mathbf{1}_{[H_\mathrm{f} < \tau]} K(-i\vartheta/\nu; \{M_l, N_l\}_{l=1}^{L}) P \otimes \mathbf{1}_{[H_\mathrm{f} < \tau]},
\end{aligned} \tag{4.22}$$

where

$$
\begin{aligned}
&K(-i\vartheta/\nu;\{M_l,N_l\}_{l=1}^{L})\\
&= P\otimes\sin\left[\frac{\pi}{2}\Xi\left(H_{\mathrm{f}}\right)\right]W_{M_1,N_1}(-i\vartheta/\nu)RW_{M_2,N_2}(-i\vartheta/\nu)R\cdots\\
&\quad\times RW_{M_{l-1},N_{l-1}}(-i\vartheta/\nu)RW_{M_l,N_l}(-i\vartheta/\nu)P\otimes\sin\left[\frac{\pi}{2}\Xi\left(H_{\mathrm{f}}\right)\right]
\end{aligned}
\tag{4.23}
$$

and

$$
R := \bar{\chi}T^{-1}[z]\bar{\chi}. \tag{4.24}
$$

Lemma 4.3. (Wick ordering) *Let φ be the eigenvector of P. Let* $\mathrm{sgn}(\cdots)$*, $\mathcal{K}_{M,\ell}$, $\mathcal{K}_{N,\ell}$, r_ℓ, $\Sigma(\tilde{k}_\ell^{(n_\ell)})$ be symbols defined in Theorem A.3. Then*

$$
\begin{aligned}
&K\left(-i\vartheta/\nu;\{M_\ell,N_\ell\}_{\ell=1}^{L}\right)\\
&=\sum_{\substack{\mathcal{I}_{M,\ell}\subseteq\mathcal{K}_{M,\ell}\\ \ell=1,\dots,L}}\ \sum_{\substack{\mathcal{I}_{N,\ell}\subseteq\mathcal{K}_{N,\ell}\\ \ell=1,\dots,L}}\mathrm{sgn}(\mathcal{K}\backslash\mathcal{I},\boldsymbol{:}\,\mathcal{I}\,\boldsymbol{:})\prod_{\ell=1}^{L}\mathrm{sgn}\begin{pmatrix} & \mathcal{K}_{M,\ell}\\ \mathcal{I}_{M,\ell} & \mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}\end{pmatrix}\\
&\quad\times\mathrm{sgn}\begin{pmatrix} & \mathcal{K}_{N,\ell}\\ \mathcal{I}_{N,\ell} & \mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}\end{pmatrix}\\
&\quad\times P\otimes\int_{\mathbb{M}^{m+n}}\prod_{\ell=1}^{L}\left\{dk_\ell^{(m_\ell)}d\tilde{k}_\ell^{(n_\ell)}\right\}\prod_{\ell=1}^{L}b^*(k_\ell^{(m_\ell)})\\
&\quad\times\left\{\hat{D}_L\big[H_f;\{\hat{W}^{m_\ell,n_\ell}_{M_\ell-m_\ell,N_\ell-n_\ell};k_\ell^{(m_\ell)};\tilde{k}_\ell^{(n_\ell)}\}_{\ell=1}^{L};R\big]\right\}^{\mathrm{asym}}_{m,n}\prod_{\ell=1}^{L}b(\tilde{k}_\ell^{(n_\ell)}),
\end{aligned}
$$

where

$$
\begin{aligned}
&\hat{D}_L[r;\{\hat{W}^{m_\ell,n_\ell}_{M_\ell-m_\ell,N_\ell-n_\ell};k_\ell^{(m_\ell)};\tilde{k}_\ell^{(n_\ell)}\}_{\ell=1}^{L};R]\\
&:=\sin\left[\frac{\pi}{2}\Xi(r+\tilde{r}_0)\right]\Bigg\langle\varphi\otimes\Omega,\Bigg\{\prod_{\ell=1}^{L-1}\hat{W}^{m_\ell,n_\ell}_{M_\ell-m_\ell,N_\ell-n_\ell}\big[k_\ell^{(m_\ell)};\tilde{k}_\ell^{(n_\ell)}\big]\\
&\qquad\times R\big[H_f+r+\tilde{r}_\ell+\Sigma(\tilde{k}_\ell^{(m_\ell)})\big]\Bigg\}\\
&\qquad\times\hat{W}^{m_L,n_L}_{M_L-m_L,N_L-n_L}\big[k_L^{(m_L)};\tilde{k}_L^{(n_L)}\big]\varphi\otimes\Omega\Bigg\rangle\sin\left[\frac{\pi}{2}\Xi(r+r_L)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{W}^{m,n}_{p,q}\big[k_\ell^{(m)};\tilde{k}_\ell^{(n)}\big] &:= \int_{\mathbb{M}^{m+n}}dx^{(m)}d\tilde{x}^{(n)}b^+(x^{(p)})G^{(\vartheta)}_{m+p,n+q}[K^{(m+p,n+q)}]b^-(\tilde{x}^{(q)}),\\
G^{(\vartheta)}_{M,N}[K^{(M,N)}] &:= e^{-i\frac{d(M+N)}{2\nu}}G_{M,N}\left(e^{-i\vartheta/2\nu}K^{(M,N)}\right),\\
R[r] &:= \bar{\chi}[r](H_{\mathrm{S}}+e^{-i\vartheta/\nu}r-E-z)^{-1}\bar{\chi}[r]\otimes\mathbf{1}.
\end{aligned}
$$

Proof. Similar to the proof of Theorem A.3. □

5 Renormalization group method

This section is devoted to the fermionic renormalization group method based on the smooth Feshbach map [3]. In Section 4, the operator $H_{(0)}[z]$ on $\mathcal{H}_{\rm red} := \mathbf{1}_{[H_{\rm f}<1]}\mathcal{F}$ is derived from the Feshbach map of the Hamiltonian $H[z]$ on $\mathcal{H}$. By Theorem 3.2 and the simplicity of the eigenvalue E, we observe that $H[z]$ has the eigenvalue 0 if $H_{(0)}[z]$ has the eigenvalue 0. By using the renormalization group method, one can prove that there exists a complex number $e_{(0,\infty)} \in \mathbb{C}$ such that $H_{(0)}[e_{(0,\infty)}]$ has the eigenvalue 0. Moreover, one can construct the corresponding eigenvector $\Psi_{(0,\infty)}$:

$$H_{(0)}[e_{(0,\infty)}]\Psi_{(0,\infty)} = 0.$$

Hence, we obtain the eigenvalue $E_g(\theta)$ of the Hamiltonian $H_g(\theta)$ by $E_g(\theta) = E + e_{(0,\infty)}$, and, thanks to Theorem 3.2, reconstruct the eigenvector $\Psi_g(\theta)$ by $Q_\chi \phi \otimes \Psi_{(0,\infty)}$, where Q_χ is defined in Theorem 2.1.

The operator $H_{(0)}[z]$ indicates that one define a class of Hamiltonians $H[z] = H[(w_{m,n}[z])_{m+n\geq 0}] \in \mathcal{B}(\mathcal{H}_{\rm red})$ on $\mathcal{H}_{\rm red}$ of the form

$$H[z] = T[z; H_{\rm f}] - E[z] + W[(w_{m,n}[z])_{m+n\geq 1}], \quad z \in D_{1/2},$$

where functions $w_{m,n}[z] : [0,1] \times \mathbb{R}^{d(m+n)} \mapsto \mathbb{C}$ are elements of Banach spaces $\mathcal{W}^{\#}_{m,n}$ $(m+n \geq 1)$, $w_{0,0}[z] : [0,1] \mapsto \mathbb{C}$ an element of a Banach space $\mathcal{W}^{\#}_{0,0}$ and denote $T[z;r] := w_{0,0}[z;r] - w_{0,0}[z;0]$, $E[z] := -w_{0,0}[z;0]$ and

$$D_{1/2} = \{z \in \mathbb{C} \mid |z| \leq 1/2\}.$$

The operator norm of $H[z]$ is controlled by the norm of the Banach space $\mathcal{W}_{\geq 0}$ consisting of analytic functions $\underline{w} : D_{1/2} \ni z \mapsto \underline{w}[z] = (w_{m,n}[z])_{m+n\geq 0}$. We construct the renormalization transformation $\mathcal{R}_\rho : \mathcal{W}_{\geq 0} \mapsto \mathcal{W}_{\geq 0}$ in subsection 5.3, via the renormalization map $\mathcal{R}^H_\rho : \mathcal{W}_{\geq 0} \to H[\mathcal{W}_{\geq 0}]$, which is given by a scaling transformation S_ρ and the Feshbach map of the Hamiltonian $H[z]$, and satisfy

$$\mathcal{R}^H_\rho(\underline{w}) = H[\mathcal{R}_\rho(\underline{w})[\cdot]].$$

We show that the renormalization transformation $\mathcal{R}_\rho$ has a contractivity in Theorem 5.7. Let $\underline{w}^{(0)}$ satisfy $H_{(0)}[z] = H[\underline{w}^{(0)}[z]]$. Iterating the renormalization transformation, we have a sequence $(\mathcal{R}^\alpha_\rho(\underline{w}^{(0)}))_{\alpha=0}^\infty$, by which, together with the contractivity of $\mathcal{R}_\rho$, one can construct the complex number $e_{(0,\infty)}$ and the eigenvector $\Psi_{(0,\infty)}$.

In this section, we review the renormalization group analysis based on the smooth Feshbach map developed by [3] with a little modeification in order to apply the method to our model. We refer the reader to [3] for details.

5.1 A Banach space of sequences of functions

We first define the Banach space $\mathcal{W}^{\#}_{\geq 0} = \bigoplus_{m+n\geq 0} \mathcal{W}^{\#}_{m,n}$ as follows. Let $\mathcal{W}^{\#}_{0,0}$ be the function space $C^1([0,1])$ of continuously differentiable functions on $[0,1]$ with the norm

$$\|w_{0,0}\| := |w_{0,0}[0]| + \sup_{r\in[0,1]} |\partial_r (w_{0,0}[r] - w_{0,0}[0])|, \quad w_{0,0} \in \mathcal{W}^{\#}_{0,0}, \tag{5.1}$$

which is equivalent to the norm

$$\|w_{0,0}\|_{C^1([0,1])} := \sup_{r\in[0,1]} |w_{0,0}[r]| + \sup_{r\in[0,1]} |\partial_r w_{0,0}[r]|\,,$$

since

$$|w_{0,0}[0]| \leq \sup_{r\in[0,1]} |w_{0,0}[r]| \leq |w_{0,0}[0]| + 1 \times \sup_{r\in[0,1]} |\partial_r w_{0,0}[r]| = \|w_{0,0}\|,$$

where we denote the derivative $w_{0,0}$ by $\partial_r w_{0,0}$. In this sense, we write $w_{0,0} \in \mathcal{W}^{\#}_{0,0}$ as

$$w_{0,0}[r] = T[r] - E, \tag{5.2}$$

where $E := -w_{0,0}[0]$ and $T[r] := w_{0,0}[r] - w_{0,0}[0]$; hence $T \in C^1([0,1])$ and $T[0] = 0$.

For all $m, n \in \mathbb{Z}$ with $m + n \geq 1$ and $m, n \geq 0$, $\mathcal{W}^{\#}_{m,n}$ denotes the Banach space which consists of functions $w_{m,n} : [0,1] \times (B_1 \times \mathbb{L})^{m+n} \to \mathbb{C}$ obeying the following properties: (a) for a.e. $K^{(m,n)} \in (B_1 \times \mathbb{L})^{m+n}$, $w_{m,n}[\cdot; K^{(m,n)}] \in C^1([0,1])$ is continuously differentiable, where B_1 denotes the unit ball in $\mathbb{R}^d$; (b) for each $r \in [0,1]$, $w_{m,n}[r; K^{(m,n)}]$ is antisymmetric with respect to the variables $K^{(m,n)} = (k_1, \cdots, k_m, \tilde{k}_1, \cdots, \tilde{k}_n) \in (B_1 \times \mathbb{L})^{m+n}$ in the following sense; for all permutations $\pi \in S_m, \tilde{\pi} \in S_n$

$$w_{m,n}[r; K^{(m,n)}_{\pi,\tilde{\pi}}] = \mathrm{sgn}(\pi)\mathrm{sgn}(\tilde{\pi}) w_{m,n}[r; K^{(m,n)}], \tag{5.3}$$

where we denote the group of permutations of n elements by S_n and

$$K^{(m,n)}_{\pi,\tilde{\pi}} = (k_{\pi(1)}, \cdots, k_{\pi(m)}, \tilde{k}_{\tilde{\pi}(1)}, \cdots, \tilde{k}_{\tilde{\pi}(n)});$$

(c) for $\gamma > 0$ fixed, $w_{m,n}$ satisfies the following norm bound

$$\|w_{m,n}\|^{\#}_{\gamma} := \|w_{M,N}\|_{\gamma} + \|\partial_r w_{M,N}\|_{\gamma} < \infty, \tag{5.4}$$

where

$$\|w_{m,n}\|_{\gamma} := \left(\int_{(B_1\times\mathbb{L})^{m+n}} dK^{(m,n)} \frac{\sup_{r\in[0,1]} |w_{m,n}[r; K^{(m,n)}]|^2}{\left[\prod_{j=1}^m w(k_j) \prod_{j=1}^n w(\tilde{k}_j)\right]^{1+2\gamma}} \right)^{1/2}. \tag{5.5}$$

We define the Banach space

$$\mathcal{W}^{\#}_{\geq 0} = \bigoplus_{m+n\geq 0} W^{\#}_{m,n} \tag{5.6}$$

with the norm

$$\|w\|^{\#}_{\gamma,\xi} := \sum_{m+n\geq 1} \frac{\|w_{m,n}\|^{\#}_{\gamma}}{\xi^{m+n}}, \quad w = \{w_{m,n}\}_{m+n\geq 1} \in \mathcal{W}^{\#}, \tag{5.7}$$

where we fix $\gamma > 0$ and $0 < \xi < 1$.

5.2 Hamiltonians defined by an operator-valued function

Let $\mathcal{H}_{\mathrm{red}}$ be the closed subspace of $\mathcal{F}$ given by

$$\mathcal{H}_{\mathrm{red}} := \mathrm{Ran}\, \mathbf{1}_{[H_{\mathrm{f}}<1]} = \mathbf{1}_{[H_{\mathrm{f}}<1]}\mathcal{F}. \tag{5.8}$$

For all $w = (w_{m,n})_{m+n\geq 0} \in \mathcal{W}_{\geq 0}^{\#}$ we define a Hamiltonian $H \in \mathcal{B}(\mathcal{H}_{\mathrm{red}})$ by

$$H = T[H_{\mathrm{f}}] + W - E, \tag{5.9}$$

where $E \in \mathbb{C}$ and $T \in C^1([0,1])$ with $T[0] = 0$ are given by (5.2) and hence $T[H_{\mathrm{f}}] \in \mathcal{B}(\mathcal{H}_{\mathrm{red}})$ is defined by functional calculus. Here

$$W = \sum_{m+n\geq 1} W_{m,n} \tag{5.10}$$

and $W_{m,n} \in \mathcal{B}(\mathcal{H}_{\mathrm{red}})$ is given by

$$\begin{aligned} &W_{m,n} \equiv W_{m,n}[w_{m,n}] \\ &= \mathbf{1}_{[H_{\mathrm{f}}<1]} \int_{(B_1\times\mathbb{L})^{m+n}} dK^{(m,n)} b^*(k^{(m)}) w_{m,n}[H_{\mathrm{f}}; K^{(m,n)}] b(\tilde{k}^{(n)}) \mathbf{1}_{[H_{\mathrm{f}}<1]}, \end{aligned} \tag{5.11}$$

where we denote

$$b^*(k^{(m)}) = b^*(k_1)\cdots b^*(k_m), \qquad b(\tilde{k}^{(n)}) = b(\tilde{k}_1)\cdots b(\tilde{k}_m) \tag{5.12}$$

for a.e. $K^{(m,n)} = (k^{(m)}, \tilde{k}^{(n)}) = (k_1, \cdots, k_m, \tilde{k}_1, \cdots, \tilde{k}_n) \in (B_1 \times \mathbb{L})^{m+n}$.

Theorem 5.1. *For $\gamma > 0$, $m, n \geq 0$ with $m + n \geq 1$ and $w_{m,n} \in \mathcal{W}_{m,n}^{\#}$*

$$\|W[w_{m,n}]\|_{\mathcal{B}(\mathcal{H}_{\mathrm{red}})} \leq \frac{\|w_{m,n}\|_\gamma}{(m^m n^n)^\gamma}. \tag{5.13}$$

Proof. Let us note that

$$\|b(k^{(m)})\mathbf{1}_{[H_{\mathrm{f}}<1]}\psi\| \leq \frac{\|b(k^{(m)})\mathbf{1}_{[H_{\mathrm{f}}<1]}\psi\|}{m^{\gamma m}\left[\prod_{j=1}^m \omega(k_j)\right]^\gamma},$$

which implies that

$$\begin{aligned} &|\langle \psi, W_{m,n}\phi\rangle| \\ &\leq \int_{(B_1\times\mathbb{L})^{m+n}} dK^{(m,n)} \sup_{r\in[0,1]} |w_{m,n}[r; K^{(m,n)}]| \\ &\times \left\|b(k^{(m)})\mathbf{1}_{[H_{\mathrm{f}}<1]}\psi\right\| \left\|b(k^{(m)})\mathbf{1}_{[H_{\mathrm{f}}<1]}\phi\right\| \\ &\leq \frac{1}{m^{\gamma m} n^{\gamma n}} \int_{(B_1\times\mathbb{L})^{m+n}} dK^{(m,n)} \frac{\sup_{r\in[0,1]} |w_{m,n}[r; K^{(m,n)}]|}{\left[\prod_{j=1}^m \omega(k_j) \prod_{j=1}^n \omega(\tilde{k}_j)\right]^\gamma} \\ &\times \left\|b(k^{(m)})\mathbf{1}_{[H_{\mathrm{f}}<1]}\psi\right\| \left\|b(k^{(m)})\mathbf{1}_{[H_{\mathrm{f}}<1]}\phi\right\| \\ &\leq \frac{\|w_{m,n}\|_\gamma B_m(\psi)^{1/2} B_n(\phi)^{1/2}}{m^{\gamma m} n^{\gamma n}}, \end{aligned}$$

where

$$\begin{aligned}
B_m(\psi) &:= \int_{(B_1\times\mathbb{L})^m} dk^{(m)} \left[\prod_{j=1}^{m} \omega(k_j)\right] \|b(k^{(m)})\mathbf{1}_{[H_\mathrm{f}<1]}\psi\|^2 \\
&= \int_{(B_1\times\mathbb{L})^{m-1}} dk^{(m-1)} \left[\prod_{j=1}^{m-1} \omega(k_j)\right] \|H_\mathrm{f}^{1/2} b(k^{(m-1)})\mathbf{1}_{[H_\mathrm{f}<1]}\psi\|^2 \\
&\le \int_{(B_1\times\mathbb{L})^{m-1}} dk^{(m-1)} \left[\prod_{j=1}^{m-1} \omega(k_j)\right] \|b(k^{(m-1)})H_\mathrm{f}^{1/2} P_\Omega^\perp \mathbf{1}_{[H_\mathrm{f}<1]}\psi\|^2 \\
&= B_{m-1}(H_\mathrm{f}^{1/2} P_\Omega^\perp \psi) \le \cdots \le \|(H_\mathrm{f} P_\Omega^\perp)^{m/2}\mathbf{1}_{[H_\mathrm{f}<1]}\psi\|^2
\end{aligned}$$

and $P_\Omega^\perp$ denotes the orthogonal projection onto the orthogonal complement of the linear span of the Fock vacuum. □

Let

$$\mathcal{W}_{\ge 1}^{\#} = \bigoplus_{m+n\ge 1} \mathcal{W}_{m,n}^{\#}. \tag{5.14}$$

Theorem 5.2. *For all $\gamma > 0$ and $0 < \xi < 1$, the map $H : \mathcal{W}_{\ge 0}^{\#} \to \mathcal{B}(\mathcal{H}_\mathrm{red})$ is injective and*

$$\|H[w]\|_{\mathcal{B}(\mathcal{H}_\mathrm{red})} \le \|w\|_{\gamma,\xi}^{\#} \tag{5.15}$$

for all $w = (w_{m,n})_{m+n\ge 0} \in \mathcal{W}_{\ge 0}^{\#}$ and

$$\|H[w]\|_{\mathcal{B}(\mathcal{H}_\mathrm{red})} \le \xi \|w\|_{\gamma,\xi}^{\#} \tag{5.16}$$

for all $w = (w_{m,n})_{m+n\ge 1} \in \mathcal{W}_{\ge 1}^{\#}$.

5.3 Renormalization transformation

Let $\mathcal{W}_{\ge 0}$ be the Banach space of $\mathcal{W}_{\ge 0}^{\#}$ -valued analytic functions $\underline{w}$ on $D_{1/2}$ with the norm

$$\|\underline{w}\|_{\gamma,\xi} := \sup_{z\in D_{1/2}} \|\underline{w}[z]\|_{\gamma,\xi}^{\#} \tag{5.17}$$

and $H[\mathcal{W}_{\ge 0}]$ the space of analytic functions $H[\underline{w}[\cdot]]$ with $\underline{w} \in \mathcal{W}_{\ge 0}$:

$$H[\underline{w}[\cdot]] : D_{1/2} \ni z \longmapsto H[\underline{w}[z]] \in H[\mathcal{W}_{\ge 0}^{\#}]. \tag{5.18}$$

We construct the renormalization transformation $\mathcal{R}_\rho$ as follows.

Let

$$\mathcal{U}[\underline{w}] := \left\{ z \in D_{1/2} \middle| |E[z]| \le \frac{\rho}{2} \right\} \tag{5.19}$$

and

$$\begin{aligned}
\mathcal{D}(\epsilon,\delta) := \Bigg\{ \underline{w} \in \mathcal{W}_{\ge 0} \Bigg| & \sup_{z\in D_{1/2}} \|T[z;r] - r\|_{\mathcal{T}} \le \epsilon, \ \sup_{z\in D_{1/2}} |E[z] - z| \le \delta, \\
& \sup_{z\in D_{1/2}} \|(w_{m,n}[z])_{m+n\ge 1}\|_{\gamma,\xi}^{\#} \le \delta \Bigg\},
\end{aligned} \tag{5.20}$$

where $\|f\|_{\mathcal{T}} := \sup_{r\in[0,1]} |f'[r]|$ for $f \in C^1([0,1])$.

We set

$$\chi_\rho[r] := \sin\left[\frac{\pi}{2}\Xi(r/\rho)\right]. \tag{5.21}$$

Lemma 5.3. *Let $\underline{w} \in \mathcal{D}(\epsilon,\delta)$ satisfy*

$$\frac{4\delta\xi}{1-3\epsilon} < \rho \tag{5.22}$$

with $0<\rho,\xi<1$. Then $\langle \chi_\rho[H_{\mathrm{f}}], H[\underline{w}[z]], T[z;H_{\mathrm{f}}]-E[z]\rangle$ is a Feshbach triple for all $z \in \mathcal{U}[z]$. In particular, if $\underline{w} \in \mathcal{D}(\epsilon,\epsilon)$ and

$$\epsilon < \frac{\rho}{4\xi+3\rho} \tag{5.23}$$

then $\langle \chi_\rho[H_{\mathrm{f}}], H[\underline{w}[z], T[z;H_{\mathrm{f}}]-E[z]\rangle$ is a Feshbach triple for all $z \in \mathcal{U}[\underline{w}]$.

By Lemma 5.3 we can define the Feshbach map of the triple $\langle \chi_\rho[H_{\mathrm{f}}], H[\underline{w}[z]], T[z;H_{\mathrm{f}}]-E[z]\rangle$ and have, in the same way as the proof of Lemma 4.2,

$$\begin{aligned}
&F_{\chi_\rho[H_{\mathrm{f}}]}(H[\underline{w}[z]], T[z;H_{\mathrm{f}}]-E[z]) \\
&= T[z;H_{\mathrm{f}}]-E[z] \qquad (5.24)\\
&\quad+\sum_{L=1}^{\infty}(-1)^{L-1}\chi_\rho[H_{\mathrm{f}}]W[z]\left(\bar{\chi}_\rho[H_{\mathrm{f}}](T[z;H_{\mathrm{f}}]-E[z])^{-1}\bar{\chi}_\rho[H_{\mathrm{f}}]W[z]\right)^{L-1}\chi_\rho[H_{\mathrm{f}}],
\end{aligned}$$

where

$$\bar{\chi}_\rho[r] := \sqrt{1-\chi_\rho[r]^2} = \cos\left[\frac{\pi}{2}\Xi(r/\rho)\right]. \tag{5.25}$$

By using of the Wick ordering formula, we find that, for each $z \in \mathcal{U}[\underline{w}]$, there exists a $\underline{\tilde{w}}[z] \in \mathcal{W}^{\#}_{\geq 0}$ such that

$$F_{\chi_\rho[H_{\mathrm{f}}]}(H[\underline{w}[z]], T[z;H_{\mathrm{f}}]-E[z]) = H[\underline{\tilde{w}}[z]] \tag{5.26}$$

and hence define the map

$$\mathcal{U}[\underline{\tilde{w}}] \ni z \longmapsto H[\underline{\tilde{w}}[z]] \in H[\mathcal{W}^{\#}_{\geq 0}]. \tag{5.27}$$

We next introduce a scaling transformation S_ρ and define the map $\underline{\hat{w}}[\cdot] \in \mathcal{W}_{\geq 0}$ from (5.27). Let $S_\rho : \mathcal{B}(\mathcal{F}) \to \mathcal{B}(\mathcal{F})$ be the scaling transformation defined by

$$S_\rho(A) := \frac{1}{\rho}\Gamma_{\rho^{1/\nu}} A \Gamma^*_{\rho^{1/\nu}}, \qquad A \in \mathcal{B}(\mathcal{F}), \tag{5.28}$$

where Γ_η $(\eta>0)$ is defined by (2.21) and satisfies

$$\Gamma_\eta H_{\mathrm{f}} \Gamma^*_\eta = \eta^\nu H_{\mathrm{f}}. \tag{5.29}$$

We define the map $s_\rho : \mathcal{W}^{\#}_{\geq 0} \ni w \mapsto s_\rho(w) \in \mathcal{W}^{\#}_{\geq 0}$ by

$$S_\rho(H[w]) =: H[s_\rho(w)], \tag{5.30}$$

and denote, for $w = \{w_{m,n}\}_{m+m\geq 0}$,

$$s_\rho((w_{m,n})_{m+m\geq 0}) = (s_\rho(w_{m,n}))_{m+m\geq 0}.$$

By the definition, we observe that

$$s_\rho(w_{m,n})[r;K^{(m,n)}] = \rho^{d(m+n)/(2\nu)-1} w_{m,n}[\rho r;\rho^{1/\nu}K^{(m,n)}], \quad m+n\geq 1 \tag{5.31}$$

and

$$s_\rho(w_{0,0})[r] = \rho^{-1}w_{0,0}[\rho r] = \rho^{-1}T[\rho r] - \rho^{-1}E, \tag{5.32}$$

where we write

$$\rho^{1/\nu}K^{(m,n)} = (\rho^{1/\nu}k_1,\cdots,\rho^{1/\nu}k_m,\rho^{1/\nu}\tilde{k}_1,\cdots,\rho^{1/\nu}\tilde{k}_N)$$

and

$$\rho^{1/\nu}k_j = (\rho^{1/\nu}\mathbf{k}_j, l_j), \qquad \rho^{1/\nu}\tilde{k}_j = (\rho^{1/\nu}\tilde{\mathbf{k}}_j, \tilde{l}_j).$$

Let us define the renormalized spectral parameter by

$$E_\rho : \mathcal{U}[\underline{w}] \ni z \longmapsto \rho^{-1}E[z] \in D_{1/2} \tag{5.33}$$

for $\underline{w} \in \mathcal{W}_{\geq 0}$.

Lemma 5.4. *Fix $0<\rho<1$ and let $\delta>0$ satisfy*

$$\rho + 2\delta < 1. \tag{5.34}$$

If $\underline{w}\in\mathcal{D}(\epsilon,\delta)$, then E_ρ is a surjection and

$$|\rho\partial_z E_\rho[z]-1| \leq \frac{2\delta}{(1-\rho-2\delta)^2}. \tag{5.35}$$

Assume, in addition, that $|\rho\partial_z E_\rho[z]-1|<1$. Then E_ρ is an injection.

Assuming that, (5.34) and

$$\frac{2\delta}{(1-\rho-2\delta)^2} < 1 \tag{5.36}$$

hold with $0<\rho<1$ fixed, by Lemma 5.4, $E_\rho : \mathcal{U}[\underline{w}] \to D_{1/2}$ is an analytic bijection. Furthermore, assumet that $\epsilon,\delta>0$ satisfy (5.22). Then, by Lemma 5.3, $\langle \chi_\rho[H_\mathrm{f}], H[\underline{w}[z]], T[z;H_\mathrm{f}]-E[z]\rangle$ is a Feshbach triple for all $\underline{w}\in(\epsilon,\delta)$ and $z\in\mathcal{U}[\underline{w}]$.

Remark 5.1. *Choosing, for example, $\epsilon=\delta=1/16$ and $\rho=1/3$, $\epsilon,\delta,\rho>0$ satisfy* (5.23)*,* (5.34) *and* (5.36)*.*

Let $\mathcal{R}_\rho : \mathcal{D}(\epsilon,\delta)\to\mathcal{W}_{\geq 0}$ be the renormalization map given by

$$\mathcal{R}_\rho(\underline{w})[\zeta] = s_\rho(\underline{\tilde{w}}[E_\rho^{-1}[\zeta]]), \quad \zeta\in D_{1/2}, \tag{5.37}$$

where $\underline{\tilde{w}}$ is defined in (5.26) such that

$$\begin{aligned} \mathcal{R}^H_\rho(\underline{w})[\zeta] :&= S_\rho(F_{\chi_\rho[H_\mathrm{f}]}(H[\underline{w}[z]], T[z;H_\mathrm{f}]-E[z])) \\ &= S_\rho(H[\underline{\tilde{w}}[z]]) \\ &= H[\mathcal{R}_\rho(\underline{w})[\zeta]] \end{aligned} \tag{5.38}$$

with

$$\zeta = E_\rho[z]\in D_{1/2}, \quad z\in\mathcal{U}[\underline{w}]. \tag{5.39}$$

Denoting

$$W^{m,n}_{p,q}[z;r;k^{(m)};\tilde{k}^{(n)}] := \mathbf{1}_{[H_\mathrm{f}<1]}\int_{(B_1\times\mathbb{L})^{p+q}} dx^{(p)}d\tilde{x}^{(q)}b^*(x^{(p)}) \times w_{m+p,n+q}[z;r;k^{(m)},x^{(p)};\tilde{k}^{(n)},\tilde{x}^{(q)}]b(\tilde{x}^{(q)})\mathbf{1}_{[H_\mathrm{f}<1]}$$

and

$$W_{p,q}[z;r] := \mathbf{1}_{[H_\mathrm{f}<1]}\int_{(B_1\times\mathbb{L})^{p+q}} dx^{(p)}d\tilde{x}^{(q)}b^*(x^{(p)})w_{p,q}[z;r;x^{(p)};\tilde{x}^{(q)}]b(\tilde{x}^{(q)})\mathbf{1}_{[H_\mathrm{f}<1]},$$

we have, by Theorem A.3, the following theorem:

Theorem 5.5. *Fix $0<\rho<1$ and let $\zeta = E_\rho[z] \in D_{1/2}$ ($z\in\mathcal{U}[\underline{w}]$) and $\underline{w}\in\mathcal{D}(\epsilon,\delta)$, where $\epsilon,\delta>0$ satisfy* (5.22), (5.34) *and* (5.36). *Then, $\mathcal{R}_\rho(\underline{w}) =: \underline{\hat{w}} =)(\hat{w}^{\mathrm{asym}}_{m,n})_{m+n\geq 0}$ is given by*

$$\hat{w}_{m,n}[\zeta;r;K^{(m,n)}] = \rho^{\frac{d(m+n)}{2\nu}-1}\sum_{L=1}^{\infty}(-1)^{L-1}\sum_{\substack{m_1+\cdots+m_L=m,\\ n_1+\cdots+n_L=n}} \tag{5.40}$$

$$\sum_{\substack{p_l,q_l\geq 0;\\ m_l+p_l+m_l+q_l\geq 1,\\ l=1,\ldots,L}} \mathrm{sgn}(\{m_l\}_{l=1}^L;\{n_l\}_{l=1}^L)V_{\underline{m,p,n,q}}[z;r;K^{(m,n)}] \tag{5.41}$$

for $m+n\geq 1$, and

$$\hat{w}_{0,0}[\zeta;r] = \rho^{-1}w_{0,0}[z;\rho r] + \rho^{-1}\sum_{L=1}^{\infty}(-1)^{L-1}\sum_{\substack{p_l,q_l\geq 0;\\ p_l+q_l\geq 1,\\ l=1,\ldots,L}} V_{\underline{0,p,0,q}}[z;r], \tag{5.42}$$

where $\hat{w}^{\mathrm{asym}}_{m,n}$ denotes the antisymmetrization of $\hat{w}_{m,n}$;

$$\hat{w}^{\mathrm{asym}}_{m,n}[\zeta;r;K^{(m,n)}] = \frac{1}{m!n!}\sum_{\pi\in S_m}\sum_{\tilde{\pi}\in S_n}\mathrm{sgn}(\pi)\mathrm{sgn}(\tilde{\pi})w_{m,n}[\zeta;r;K^{(m,n)}_{\pi,\tilde{\pi}}],$$

$\mathrm{sgn}(\{m_l\}_{l=1}^L;\{n_l\}_{l=1}^L)$ *is defined by* (A.29), *the quadruplet $\underline{m,p,n,q}\in\mathbb{N}_0^{4L}$ and the function $V_{\underline{m,p,n,q}}$ are given by*

$$\underline{m,p,n,q} = (m_1,\cdots,m_L,p_1,\cdots,p_L,n_1,\cdots,n_L,q_1,\cdots,q_L)\in\mathbb{N}_0^{4L}$$

and

$$
\begin{aligned}
&V_{\underline{m,\,p,\,n,\,q}}[z;r;K^{(m,n)}] \\
&= \chi_\rho[r+\tilde{\lambda}_0]\Bigg\langle \Omega, \Bigg\{ \prod_{l=1}^{L-1} W_{p_l,q_l}^{m_l,n_l}[z;H_\mathrm{f}+\rho(r+\lambda_l);\rho^{1/\nu}k_l^{(m_l)};\tilde{k}_l^{(n_l)}] \\
&\quad \times \left(\frac{\bar{\chi}_\rho^2[H_\mathrm{f}+\rho(r+\tilde{\lambda}_l)]}{T[z;H_\mathrm{f}+\rho(r+\tilde{\lambda}_l)]-E[z]} \right) \Bigg\} \\
&\quad \times W_{p_L,q_L}^{m_L,n_L}[z;H_\mathrm{f}+\rho(r+\lambda_L);\rho^{1/\nu}k_L^{(m_L)};\rho^{1/\nu}\tilde{k}_L^{(n_L)}]\Omega \Bigg\rangle \chi_\rho[\rho(r+\tilde{\lambda}_L)], \\
&V_{\underline{0,\,p,\,0,\,q}}[z;r] \\
&= \chi_\rho[\rho r]\Bigg\langle \Omega, \Bigg\{ \prod_{l=1}^{L-1} W_{p_l,q_l}[z;H_\mathrm{f}+\rho r] \left(\frac{\bar{\chi}_\rho^2[H_\mathrm{f}+\rho r]}{T[z;H_\mathrm{f}+\rho r]-E[z]} \right) \Bigg\} \\
&\quad \times W_{p_L,q_L}[z;H_\mathrm{f}+\rho r]\Omega \Bigg\rangle \chi_\rho[\rho r]
\end{aligned}
$$

with

$$
\begin{aligned}
&K^{(m,n)} := (k_1^{(m_1)},\dots,k_L^{(m_L)},\tilde{k}_1^{(n_1)},\dots,\tilde{k}_L^{(n_L)}) \in (B_1\times\mathbb{L})^{m+n}, \\
&k_l^{(m_l)} := (k_{l,1},\dots,k_{l,m_l}) \in (B_1\times\mathbb{L})^{m_l}, \\
&\tilde{k}_l^{(n_l)} := (\tilde{k}_{l,1},\dots,\tilde{k}_{l,n_L}) \in (B_1\times\mathbb{L})^{n_l}, \quad l=1,\dots,L-1, \\
&\lambda_l := \sum_{l'=1}^{l-1}\sum_{j=1}^{n_{l'}} \omega(\tilde{k}_{l',j}) + \sum_{l'=l+1}^{L}\sum_{j=1}^{m_{l'}} \omega(k_{l',j}), \quad l=2,3,\dots,L-1, \\
&\lambda_0 := \sum_{l'=1}^{L}\sum_{j=1}^{m_{l'}} \omega(k_{l',j}), \quad \lambda_1 := \sum_{l'=2}^{L}\sum_{j=1}^{m_{l'}} \omega(k_{l',j}), \quad \lambda_L := \sum_{l'=1}^{L-1}\sum_{j=1}^{n_{l'}} \omega(\tilde{k}_{l',j}), \\
&\tilde{\lambda}_l := \lambda_l + \sum_{l=1}^{n_l} \omega(\tilde{k}_{l,j}), \quad l=1,\dots,L-1, \\
&\tilde{\lambda}_0 := \lambda_0, \quad \tilde{\lambda}_L := \sum_{l'=1}^{L}\sum_{j=1}^{n_{l'}} \omega(\tilde{k}_{l',j})
\end{aligned}
$$

Remark 5.2. *By* (A.29)*, we observe*

$$
\left|\mathrm{sgn}(\{m_l\}_{l=1}^L;\{n_l\}_{l=1}^L)\right| \le \prod_{j=1}^{L} \binom{m_l+p_l}{p_l}\binom{n_l+q_l}{q_l}. \tag{5.43}
$$

Lemma 5.6. *There exists some constant $C \ge 1$ independent of $L \ge 1$ and $\underline{m,p,n,q} \in \mathbb{N}_0^4$ such that*

$$
\begin{aligned}
&\rho^{\frac{d}{2\nu}(m+n)-1}\|V_{\underline{m,\,p,\,n,\,q}}[z]\|_\gamma^\# \\
&\quad \le 2(L+1)C^{L+1}\rho^{(m+n)(\gamma+1/2)-L}\prod_{l=1}^{L}\frac{\|w_{m_l+p_l,n_l+q_l}[z]\|_\gamma^\#}{(p_l^{p_l}q_l^{q_l})^\gamma}.
\end{aligned} \tag{5.44}
$$

5.4 Construction of the eigenvalue and the eigenstate

Theorem 5.7. (Codimension-1 contractivity) *Let $0 < \rho < 1$, $0 < \xi < 1/2$, $\epsilon > 0$ and $\delta > 0$ satisfy the conditions in Theorem 5.5. Assume that*

$$B_1 := \frac{C\xi\delta}{\rho} < 1, \quad B_2 := \frac{C\delta}{\rho(1-2\xi)^2} < 1, \tag{5.45}$$

and let

$$\Delta_1 := \frac{2C(3-2B_1)B_1^2}{(1-B_1)^2}, \quad \Delta_2 := \max\left\{\Delta_1, \frac{4C\rho^{\gamma+1/2}B_2(2-B_2)}{(1-B_2)^2}\right\}. \tag{5.46}$$

Then, for all $\underline{w} \in \mathcal{D}(\epsilon, \delta)$,

$$\mathcal{R}_\rho[\underline{w}] \in \mathcal{D}(\epsilon + \Delta_1, \Delta_2). \tag{5.47}$$

Remark 5.3. *Fixing $\mu > 0$ and setting $\gamma = \mu + 1/2$, by Theorem 5.7, if $0 < \rho < 1$, $\epsilon_0 > 0$ and $\xi > 0$ Then, for all $0 \le \epsilon, \delta \le \epsilon_0$,*

$$\mathcal{R}_\rho : \mathcal{D}(\epsilon, \delta) \to \mathcal{D}\left(\epsilon + \frac{\delta}{2}, \frac{\delta}{2}\right). \tag{5.48}$$

Let $0 < \rho < 1$, ϵ_0 and $\xi > 0$ be sufficiently small such that $\mathcal{D}(\epsilon_0, \epsilon_0)$ satisfy the conditions in Theorem 5.5 and, for all $0 \le \epsilon, \delta \le \epsilon_0$, (5.48) follows. Fix $\underline{w} \in \mathcal{D}(\epsilon_0/2, \epsilon_0/2)$, and set

$$\underline{w}^{(\alpha)} := \mathcal{R}_\rho^k(\underline{w}) \in \mathcal{D}((2^{-1}+2^{-2}+\cdots+2^{-\alpha-1})\epsilon_0, 2^{-\alpha-1}\epsilon_0) \subset \mathcal{D}\left(\epsilon_0, \frac{\epsilon_0}{2^\alpha}\right) \tag{5.49}$$

for all $j \in \mathbb{N}$ with $\underline{w}^{(0)} := \underline{w}$. Let $\underline{w}^{(j)} = (w_{m,n}^{(j)})_{m+n\ge 0}$ and set

$$E_{(\alpha)}[z] := -\underline{w}_{0,0}^{(\alpha)}[z; 0] \tag{5.50}$$

$$\mathcal{U}_{(\alpha)} := \mathcal{U}[\underline{w}^{(\alpha)}] = \left\{z \in D_{1/2} \big| |E_{(\alpha)}[z]| \le \frac{\rho}{2}\right\}. \tag{5.51}$$

and

$$J_{(\alpha)} : \mathcal{U}_{(\alpha)} \ni z \longmapsto \rho^{-1}E_{(\alpha)}[z] \in D_{1/2}. \tag{5.52}$$

By Lemma 5.4, $J_{(\alpha)}$ ($\alpha \in \mathbb{N}_0$) are analytic bijections. Let

$$e_{(\alpha,\beta)} := J_{(\alpha)}^{-1} \circ \cdots \circ J_{(\beta)}^{-1}[0] \tag{5.53}$$

for all $0 \le \alpha \le \beta$ with $e_{(\alpha,\alpha)} = J_{(\alpha)}^{-1}[0]$.

Lemma 5.8. *Let $0 < \rho < 1$ and ϵ_0 be as above and*

$$d_0 := \frac{2\epsilon_0}{(1-\rho-2\epsilon_0)^2} \quad (< 1). \tag{5.54}$$

Assume that

$$\frac{\rho}{1-d_0} < 1. \tag{5.55}$$

Then, there exist the limits

$$e_{(\alpha,\infty)} := \lim_{\beta\to\infty} e_{(\alpha,\beta)} \tag{5.56}$$

for all $\alpha \in \mathbb{N}_0$.

Let us assume that the limits $e_{(\alpha,\infty)}$ ($\alpha \in \mathbb{N}_0$) exist and

$$\begin{aligned} H_{(\alpha)} &:= T_{(\alpha)}[H_\mathrm{f}] - E_{(\alpha)} + W_{(\alpha)} \\ &:= H(\underline{w}^{(\alpha)})[e_{(\alpha,\infty)}], \end{aligned} \tag{5.57}$$

where

$$T_{(\alpha)}[r] := w_{0,0}^{(\alpha)}[e_{(\alpha,\infty)};r] - w_{0,0}^{(\alpha)}[e_{(\alpha,\infty)};0], \tag{5.58}$$

$$E_{(\alpha)} := -w_{0,0}^{(\alpha)}[e_{(\alpha,\infty)};0] = E_{(\alpha)}[e_{(\alpha,\infty)}], \tag{5.59}$$

$$W_{(\alpha)} := \sum_{m+n\geq 1} W_{m,n}[\underline{w}^{(\alpha)}[e_{(\alpha,\infty)}]]. \tag{5.60}$$

Moreover, we define the operators $Q_{(\alpha)}$ by

$$\begin{aligned} &Q_{(\alpha)} = Q_{\chi_\rho[H_\mathrm{f}]}\left(H_{(\alpha)}, T_{(\alpha)}[H_\mathrm{f}] - E_{(\alpha)}\right) \\ &:= \chi_\rho[H_\mathrm{f}] - \bar{\chi}_\rho[H_\mathrm{f}]\left(T_{(\alpha)}[H_\mathrm{f}] - E_{(\alpha)} + \bar{\chi}_\rho[H_\mathrm{f}]W_{(\alpha)}\bar{\chi}_\rho[H_\mathrm{f}]\right)^{-1}\bar{\chi}_\rho[H_\mathrm{f}]W_{(\alpha)}\chi_\rho[H_\mathrm{f}] \end{aligned} \tag{5.61}$$

and let

$$\Psi_{(\alpha,\beta)} := Q_{(\alpha)}\Gamma_\rho^* Q_{(\alpha+1)}\Gamma_\rho^* \cdots Q_{(\beta-1)}\Omega \tag{5.62}$$

for all $0 \leq \alpha \leq \beta$ with $\Psi_{(\alpha,\alpha)} := \Omega$.

Theorem 5.9. *Fix $\gamma = \mu + 1/2$ with $\mu > 0$. Let $0 < \rho < 1$, $0 < \xi < 1/2$ and $\epsilon_0 > 0$ be sufficiently small such that $\mathcal{D}(\epsilon_0,\epsilon_0)$ satisfy (5.55) and the conditions in Theorem 5.5. Assume that, for all $0 \leq \epsilon,\delta \leq \epsilon_0$, (5.48) follows. If $\underline{w} \in \mathcal{D}(\epsilon_0/2,\epsilon_0/2)$, then the limits of $\Psi_{(\alpha,\beta)}$ as $\beta \to \infty$ for all $\alpha \in \mathbb{N}_0$ such that*

$$\Psi_{(\alpha,\infty)} := \lim_{\beta\to\infty} \Psi_{(\alpha,\beta)} \neq 0, \tag{5.63}$$

and

$$H(\underline{w}[e_{(0,\infty)}])\Psi_{(0,\infty)} = 0. \tag{5.64}$$

Remark 5.4. *The equation (5.64) means that the complex number $E[e_{(0,\infty)}]$ is an eigenvalue of $T[H_\mathrm{f}]+W$ and the vector $\Psi_{(0,\infty)}$ the eigenvector of $E[e_{(0,\infty)}]$.*

A Wick ordering

In this section, we give the Wick's theorem for fermion. Let $b^+(k)$, $b^-(k)$, $k \in \mathbb{M}$ be the kernels of the fermion creation and annihilation operators, respectively.

For $\mathcal{N} := \{1,\ldots,N\}$ and $(\sigma_1,\sigma_2,\ldots,\sigma_N) \in \{-1,+1\}^N$, we denote

$$\prod_{j\in\mathcal{N}} b^{\sigma_j}(k_j) := b^{\sigma_1}(k_1)b^{\sigma_2}(k_2)\cdots b^{\sigma_N}(k_N). \tag{A.1}$$

For any subset $\mathcal{I} \subseteq \mathcal{N}$, we denote

$$\prod_{j\in\mathcal{I}} b^{\sigma_j}(k_j) := \prod_{j\in\mathcal{N}} \chi(j\in\mathcal{I}) b^{\sigma_j}(k_j),$$

where $\chi(j\in\mathcal{I})$ is the characteristic function of $\mathcal{I}$. For $\mathcal{I}\subseteq\mathcal{N}$, we set $\mathcal{I}_\pm := \{j\in\mathcal{I}|\sigma_j=\pm1\}$. The Wick-ordered product of $\prod_{j\in\mathcal{I}} b^{\sigma_j}(k_j)$ is defined by

$$\mathbf{:}\prod_{j\in\mathcal{I}} b^{\sigma_j}(k_j)\mathbf{:} := \left(\prod_{j\in\mathcal{I}_+} b^+(k_j)\right)\left(\prod_{j\in\mathcal{I}_-} b^-(k_j)\right).$$

For $(\sigma_1,\dots,\sigma_N)\in\{-1,1\}^N$ and any subset $\mathcal{I}\in\mathcal{N}$, we define

$$\begin{aligned}
&\operatorname{sgn}(\mathcal{N}\setminus\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\\
&:= \begin{pmatrix} 1 & \cdots & N \\ \mathcal{N}\setminus\mathcal{I} & \mathcal{I}_+ & \mathcal{I}_- \end{pmatrix}\\
&:= \operatorname{sgn}\begin{pmatrix} 1 & 2 & \cdots & K & K+1 & \cdots & K+L & K+L+1 & \cdots & N \\ j_1 & j_2 & \cdots & j_K & j_{K+1} & \cdots & j_{K+L} & j_{K+L+1} & \cdots & j_N \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\{j_1,j_2,\dots,j_K\} &:= \mathcal{N}\setminus\mathcal{I}, && \text{with} \quad j_1<j_2<\dots<j_N,\\
\{j_{K+1},\dots,j_{K+L}\} &:= \mathcal{I}_+, && \text{with} \quad j_{K+1}<j_{K+2}\dots<j_{K+L},\\
\{j_{K+L+1},\dots,j_N\} &:= \mathcal{I}_-, && \text{with} \quad j_{K+L+1}<j_{K+L+2}<\dots<j_N.
\end{aligned}$$

The Wick-ordering of the Fermion product (A.1) is given by the following Theorem:

Theorem A.1. *For any $(\sigma_1,\dots,\sigma_N)\in\{+1,-1\}^N$, the formula*

$$\prod_{j\in\mathcal{N}} b^{\sigma_j}(k_j) = \sum_{\mathcal{I}\subseteq\mathcal{N}} \operatorname{sgn}(\mathcal{N}\setminus\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\left\langle\Omega, \prod_{j\in\mathcal{N}\setminus\mathcal{I}} b^{\sigma_j}(k_j)\Omega\right\rangle \mathbf{:}\prod_{j\in\mathcal{I}} b^{\sigma_j}(k_j)\mathbf{:} \tag{A.2}$$

holds.

Proof. We prove the theorem by induction with respect to $N\in\mathbb{N}$. For $N=1$, (A.2) is trivial. Assume that (A.2) is true for all products with up to N factors, for some $N\geq1$, and consider the product of $N+1$-factors. We set $\mathcal{N}+1 := \mathcal{N}\cup\{N+1\}$. For simplicity we write $b_j^{\sigma_j} := b^{\sigma_j}(k_j)$. In the case $\sigma_{N+1}=-1$, we have

$$\begin{aligned}
\prod_{j\in\mathcal{N}+1} b_j^{\sigma_j} &= \prod_{j\in\mathcal{N}} b_j^{\sigma_j} b_{N+1}^-\\
&= \sum_{\mathcal{I}\subseteq\mathcal{N}} \operatorname{sgn}(\mathcal{N}\setminus\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\left\langle\Omega, \prod_{j\in\mathcal{N}\setminus\mathcal{I}} b_j^{\sigma_j}\Omega\right\rangle \mathbf{:}\prod_{j\in\mathcal{I}} b_j^{\sigma_j}\mathbf{:}\, b_{N+1}^-\\
&= \sum_{\mathcal{I}\subseteq\mathcal{N}} \operatorname{sgn}(\mathcal{N}\setminus\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\left\langle\Omega, \prod_{j\in\mathcal{N}\setminus\mathcal{I}} b_j^{\sigma_j}\Omega\right\rangle \mathbf{:}\prod_{j\in\mathcal{I}} b_j^{\sigma_j} b_{N+1}^-\mathbf{:}\,.
\end{aligned}$$

On the other hand, for $\mathcal{I}' \subseteq \mathcal{N}+1$,

$$\operatorname{sgn}((\mathcal{N}+1)\backslash\mathcal{I}';\mathcal{I}'_+;\mathcal{I}'_-)\left\langle \Omega, \prod_{j\in(\mathcal{N}+1)\backslash\mathcal{I}'} b_j^{\sigma_j}\Omega\right\rangle :\prod_{j\in\mathcal{I}'} b_j^{\sigma_j} b_{N+1}^- : \tag{A.3}$$

vanishes if $N+1 \in (\mathcal{N}+1)\backslash\mathcal{I}'$. In the case $N+1\in\mathcal{I}'$, we have

$$(A.3) = \operatorname{sgn}(\mathcal{N}\backslash\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\left\langle \Omega, \prod_{j\in\mathcal{N}\backslash\mathcal{I}} b_j^{\sigma_j}\Omega\right\rangle :\prod_{j\in\mathcal{I}} b_j^{\sigma_j} b_{N+1}^- :,$$

with $\mathcal{I}=\mathcal{I}'\backslash\{N+1\}$, where we use the fact that $\operatorname{sgn}((\mathcal{N}+1)\backslash\mathcal{I}';\mathcal{I}'_+;\mathcal{I}'_-) = \operatorname{sgn}(\mathcal{N}\backslash\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)$. Hence, we obtain

$$\prod_{j\in\mathcal{N}+1} b_j^{\sigma_j} = \sum_{\mathcal{I}\subseteq\mathcal{N}+1}\operatorname{sgn}((\mathcal{N}+1)\backslash\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\left\langle \Omega, \prod_{j\in(\mathcal{N}+1)\backslash\mathcal{I}} b^{\sigma_j}(k_j)\Omega\right\rangle :\prod_{j\in\mathcal{I}} b^{\sigma_j}(k_j):.$$

Next we consider the case $\sigma_{N+1}=+1$. By the CAR, we have

$$\{b_i^{\sigma_i}, b_j^{\sigma_j}\} = \left\langle \Omega, b_i^{\sigma_i} b_j^{\sigma_j}\Omega\right\rangle.$$

By using this relation and the induction hypothesis, we have

$$\begin{aligned}
\prod_{j\in\mathcal{N}+1} b_j^{\sigma_j} &= \sum_{k=1}^{N}(-1)^{N-k}\left\langle \Omega, b_k^{\sigma_k} b_{N+1}^+\Omega\right\rangle \prod_{j\in\mathcal{N}\backslash\{k\}} b_j^{\sigma_j} + (-1)^N b_{N+1}^+\prod_{j\in\mathcal{N}} b_j^{\sigma_j}\\
&= \sum_{k=1}^{N}(-1)^{N-k}\left\langle \Omega, b_k^{\sigma_k} b_{N+1}^+\Omega\right\rangle \sum_{\mathcal{I}\subseteq\mathcal{N}\backslash\{k\}}\operatorname{sgn}((\mathcal{N}\backslash\{k\})\backslash\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\\
&\quad\times\left\langle \Omega, \prod_{j\in(\mathcal{N}\backslash\{k\})\backslash\mathcal{I}} b_j^{\sigma_j}\Omega\right\rangle :\prod_{j\in\mathcal{I}} b_j^{\sigma_j}:\\
&\quad+(-1)^N b_{N+1}^+\prod_{j\in\mathcal{N}} b_j^{\sigma_j}.
\end{aligned}$$

We note that

$$\sum_{k=1}^{N}\sum_{\mathcal{I}\subseteq\mathcal{N}\backslash\{k\}} F(k,\mathcal{I}) = \sum_{\mathcal{I}\subseteq\mathcal{N}}\sum_{k\in\mathcal{N}\backslash\mathcal{I}} F(k,\mathcal{I}), \tag{A.4}$$

for any function $F(k,\mathcal{I})$. By using (A.4), we observe

$$\begin{aligned}
\prod_{j\in\mathcal{N}+1} b_j^{\sigma_j} &= \sum_{\mathcal{I}\subseteq\mathcal{N}}\sum_{k\in\mathcal{N}\backslash\mathcal{I}}(-1)^{N-k}\left\langle \Omega, b_k^{\sigma_k} b_{N+1}^+\Omega\right\rangle \operatorname{sgn}((\mathcal{N}\backslash\{k\})\backslash\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\\
&\quad\times\left\langle \Omega, \prod_{j\in(\mathcal{N}\backslash\{k\})\backslash\mathcal{I}} b_j^{\sigma_j}\Omega\right\rangle :\prod_{j\in\mathcal{I}} b_j^{\sigma_j}: \qquad (A.5)\\
&\quad+(-1)^N b_{N+1}^+\prod_{j\in\mathcal{N}} b_j^{\sigma_j}. \qquad (A.6)
\end{aligned}$$

For $\mathcal{I}\subseteq\mathcal{N}\backslash\{k\}$, we set

$$\begin{aligned}
K-1 &:= |(\mathcal{N}\backslash\{k\})\backslash\mathcal{I}|,\\
\{\ell_1,\dots,\ell_{K-1}\} &:= (\mathcal{N}\backslash\{k\})\backslash\mathcal{I}, \quad \text{with } \ell_1<\cdots<\ell_{K-1}.
\end{aligned}$$

Let $\{j_{K+1},\dots,j_N\}$ be indexes such that

$$j_{K+1}<\cdots<j_N,\quad \text{and} \quad :\prod_{j\in\mathcal{I}} b_j^{\sigma_j}: \; = \prod_{s=K+1}^{N} b_{j_s}^{\sigma_{j_s}},$$

namely,

$$\left\langle \Omega, \prod_{j\in(\mathcal{N}\setminus\{k\})\setminus\mathcal{I}} b_j^{\sigma_j}\Omega \right\rangle :\prod_{j\in\mathcal{I}} b_j^{\sigma_j}: = \left\langle \Omega, \prod_{j=1}^{K-1} b_{\ell_j}^{\sigma_{\ell_j}}\Omega \right\rangle : \prod_{s=K+1}^{N} b_{j_s}^{\sigma_{j_s}} : . \tag{A.7}$$

The sign in Eq. (A.6) can be written as

$$\begin{aligned}&\operatorname{sgn}((\mathcal{N}\setminus\{k\})\setminus\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\\ &=\operatorname{sgn}\begin{pmatrix} 1 & \cdots & k-1 & k & k+1 & \cdots & K-1 & K & K+1 & \cdots & N \\ \ell_1 & \dots & \ell_{k-1} & k & \ell_k & \dots & \ell_{K-2} & \ell_{K-1} & j_{K+1} & \dots & j_N \end{pmatrix}\end{aligned}$$

For each fixed $k\in\mathcal{N}\setminus\mathcal{I}$, we set

$$n := \max\{s\in\{1,\dots,K-1\}|\ell_s<k\}$$

Then we have

$$\begin{aligned}&(-1)^{k-n}\operatorname{sgn}((\mathcal{N}\setminus\{k\})\setminus\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\\ &=\operatorname{sgn}\begin{pmatrix} 1\cdots n-1 & n & n+1 & \cdots & k & k+1\cdots & K & K+1\cdots & N \\ \ell_1\dots\ell_{n-1} & k & \ell_n & \dots & \ell_{k-1} & \ell_k & \dots\ell_{K-1} & j_{K+1} & \cdots j_N \end{pmatrix}.\end{aligned} \tag{A.8}$$

Note that

$$\ell_1<\cdots<\ell_{n-1}<k<\ell_n<\cdots<\ell_{K-1}.$$

By changing the names

$$\begin{aligned}&(\ell_1,\dots,\ell_{n-1},k,\ell_n,\dots,\ell_{k-1},\dots,\ell_{K-1})\\ \to&(j_1,\dots,j_{n-1},j_n,j_{n+1},\dots,j_k,\dots,j_{K-1}),\end{aligned} \tag{A.9}$$

we obtain that

$$\begin{aligned}\operatorname{sgn}((\mathcal{N}\setminus\{k\})\setminus\mathcal{I};\mathcal{I}_+;\mathcal{I}_-) =&(-1)^{k-n}\operatorname{sgn}\begin{pmatrix}1 & \cdots & N\\ j_1 & \cdots & j_N\end{pmatrix}\\ =&(-1)^{k-n}\operatorname{sgn}(\mathcal{N}\setminus\mathcal{I};\mathcal{I}_+;\mathcal{I}_-).\end{aligned} \tag{A.10}$$

By (A.7),(A.8), and (A.10), we have

$$
\begin{aligned}
(A.5) &= \sum_{\mathcal{I}\subseteq\mathcal{N}} \sum_{k\in\mathcal{N}\backslash\mathcal{I}} (-1)^{N-k}(-1)^{k-n}\mathrm{sgn}(\mathcal{N}\backslash\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\left\langle\Omega, b_k^{\sigma_k} b_{N+1}^{+}\Omega\right\rangle \\
&\quad\times\left\langle\Omega, \prod_{\substack{l=1\\ l\neq n}}^{K} b_{j_l}^{\sigma_{j_l}}\Omega\right\rangle : \prod_{l=K+1}^{N} b_{j_l}^{\sigma_{j_l}} : \\
&= \sum_{\mathcal{I}\subseteq\mathcal{N}} \mathrm{sgn}(\mathcal{N}\backslash\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\sum_{n=1}^{K}(-1)^{N-n}\left\langle\Omega, b_{j_n}^{\sigma_{j_n}} b_{N+1}^{+}\Omega\right\rangle\left\langle\Omega, \prod_{\substack{l=1\\ l\neq n}}^{K} b_{j_l}^{\sigma_{j_l}}\Omega\right\rangle \\
&\quad\times : \prod_{l=K+1}^{N} b_{j_l}^{\sigma_{j_l}} : \\
&= \sum_{\mathcal{I}\subseteq\mathcal{N}} \mathrm{sgn}(\mathcal{N}\backslash\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)(-1)^{N}\left\langle\Omega, \prod_{l=1}^{K} b_{j_l}^{\sigma_{j_l}} b_{N+1}^{+}\Omega\right\rangle : \prod_{l=K+1}^{N} b_{j_l}^{\sigma_{j_l}} : \\
&= \sum_{\mathcal{I}\subseteq\mathcal{N}} \mathrm{sgn}((\mathcal{N}+1)\backslash\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\left\langle\Omega, \prod_{j\in(\mathcal{N}+1)\backslash\mathcal{I}} b_j^{\sigma_j}\Omega\right\rangle : \prod_{j\in\mathcal{I}} b_j^{\sigma_j} :, \qquad (A.11)
\end{aligned}
$$

where we use the equation

$$
\begin{aligned}
&\sum_{n=1}^{K}(-1)^{N-n}\left\langle\Omega, b_{j_n}^{\sigma_{j_n}} b_{N+1}^{+}\Omega\right\rangle\left\langle\Omega, \prod_{\substack{l=1\\ l\neq n}}^{K} b_{j_l}^{\sigma_{j_l}}\Omega\right\rangle \\
&= \begin{cases}\left\langle\Omega, \prod_{l=1}^{K} b_{j_l}^{\sigma_{j_l}} b_{N+1}^{+}\Omega\right\rangle, & K \text{ is odd,}\\ 0 & K \text{ is even.}\end{cases}
\end{aligned}
$$

Similarly, we have

$$
(A.6) = \sum_{\mathcal{I}\subseteq\mathcal{N}} \mathrm{sgn}((\mathcal{N}+1)\backslash\mathcal{I}';\mathcal{I}'_+;\mathcal{I}'_-)\left\langle\Omega, \prod_{j\in(\mathcal{N}+1)\backslash\mathcal{I}} b_j^{\sigma_j}\Omega\right\rangle : \prod_{j\in\mathcal{I}'} b_j^{\sigma_j} :, \qquad (A.12)
$$

where $\mathcal{I}' := \mathcal{I}\cup\{N+1\}$. By (A.11), (A.12), we obtain the desired result:

$$
\prod_{j\in\mathcal{N}+1} b_j^{\sigma_j} = \sum_{\mathcal{I}\subseteq\mathcal{N}+1} \mathrm{sgn}(\mathcal{N}\backslash\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\left\langle\Omega, \prod_{j\in(\mathcal{N}+1)\backslash\mathcal{I}} b_j^{\sigma_j}\Omega\right\rangle : \prod_{j\in\mathcal{I}} b_j^{\sigma_j} :
$$

$\square$

Lemma A.2. *Let $f_j[r] : \mathbb{M} \to \mathbb{R}_+$, $j = 1, \ldots, N$ be Borel measurable functions.*

Then

$$\prod_{j=1}^{N}\{b^{\sigma_j}(k_j)f_j[H_f]\}$$
$$= \sum_{\mathcal{I}\subset\mathcal{N}} \mathrm{sgn}(\mathcal{N}\backslash\mathcal{I}, \boldsymbol{:}\,\mathcal{I}\,\boldsymbol{:}) \prod_{j\in I_+} b^+(k_j)$$
$$\times \left\langle \Omega, \prod_{j=1}^{N} \left\{ [b^{\sigma_j}(k_j)]^{\chi[j\notin\mathcal{I}]} f_j \left[H_f + r + \sum_{\substack{i=1\\ i\in\mathcal{I}_-}}^{j} \omega(k_i) + \sum_{\substack{i=j+1\\ i\in\mathcal{I}_+}}^{N} \omega(k_i) \right] \right\} \Omega \right\rangle \Bigg|_{r=H_f}$$
$$\times \prod_{j\in\mathcal{I}_-} b^-(k_j),$$

where $[b^{\sigma_j(k_j)}]^{\chi[j\notin\mathcal{I}]} = b^{\sigma_j}(k_j)$ *for* $j\notin\mathcal{I}$ *and* $[b^{\sigma_j}(k_j)]^{\chi[j\notin\mathcal{I}]} = 1$ *for* $j\in\mathcal{I}$.

Proof. Similar to the proof of [2, Lemma A.3]. □

Let

$$w_{m,n} : (\mathbb{R}_+) \times \mathbb{M}^m \times \mathbb{M}^n \to \mathbb{C}, \quad m,n\in\mathbb{N}_0, \tag{A.13}$$

be measurable functions. In the following, we use the notations

$$k^{(m)} := (k_1,\ldots,k_m) \in \mathbb{M}^m, \quad \tilde{k}^{(n)} := (\tilde{k}_1,\ldots,\tilde{k}_n) \in \mathbb{M}^n.$$

We assume that each function $w_{m,n}[r;k^{(m)};\tilde{k}^{(n)}]$ is antisymmetric with respect to $k^{(m)}\in\mathbb{M}^m$, $\tilde{k}^{(n)}\in\mathbb{M}^n$, respectively, i.e.,

$$\begin{aligned} w_{m,n}[r;k^{(m)};\tilde{k}^{(n)}] =& \{w_{m,n}[r;k^{(m)};\tilde{k}^{(n)}]\}^{\mathrm{asym}}_{m,n} \\ :=& \frac{1}{m!n!}\sum_{\pi\in S_m}\sum_{\tilde{\pi}\in S_n} \mathrm{sgn}(\pi)\mathrm{sgn}(\tilde{\pi}) w_{m,n}[r;k^{(m)}_{\pi};\tilde{k}^{(n)}_{\tilde{\pi}}], \end{aligned}$$

where

$$k^{(m)}_{\pi} := (k_{\pi(1)},\ldots,k_{\pi(m)}), \quad \tilde{k}^{(n)}_{\pi} := (\tilde{k}_{\pi(1)},\ldots,\tilde{k}_{\pi(n)}).$$

For $L\in\mathbb{N}_0$, we consider the operator

$$f_0[H_f]W_{M_1,N_1}f_1[H_f]W_{M_2,N_2}\cdots f_{L-1}[H_f]W_{M_L,N_L}f_L[H_f]. \tag{A.14}$$

We set

$$\begin{aligned} &K := M+N, \\ &M := \sum_{\ell=1}^{L} M_\ell, \quad N := \sum_{\ell=1}^{L} N_\ell. \end{aligned} \tag{A.15}$$

Corresponding to (A.15), we set

$$\begin{aligned} k^{(M)} :=& (k^{(M_\ell)}_\ell)_{\ell=1}^{L} \in \mathbb{M}^{M_1}\times\cdots\times\mathbb{M}^{M_L} \\ =& (k_{1,1},\ldots,k_{1,M_1};k_{2,1},\ldots,k_{2,M_2};\cdots;k_{L,1},\ldots,k_{L,M_L}), \\ \tilde{k}^{(N)} :=& (\tilde{k}^{(N_\ell)}_\ell)_{\ell=1}^{L} \in \mathbb{M}^{N_1}\times\cdots\times\mathbb{M}^{N_L} \\ =& (\tilde{k}_{1,1},\ldots,\tilde{k}_{1,N_1};\tilde{k}_{2,1},\ldots,\tilde{k}_{2,N_2};\cdots;\tilde{k}_{L,1},\ldots,\tilde{k}_{L,N_L}) \end{aligned}$$

We define

$$
\begin{aligned}
\mathcal{K} :=& \{1, \dots, K\}, \\
\mathcal{K}_{M,\ell} :=& \left\{ \sum_{j=1}^{\ell-1}(M_j + N_j) + 1, \dots, \sum_{j=1}^{\ell-1}(M_j + N_j) + M_\ell \right\}, \\
\mathcal{K}_{N,\ell} :=& \left\{ \sum_{j=1}^{\ell-1}(M_j + N_j) + M_j + 1, \dots, \sum_{j=1}^{\ell}(N_j + M_j) \right\}, \quad \ell = 1, \dots, L.
\end{aligned}
$$

Clearly,

$$
\begin{aligned}
\mathcal{K} &= \bigcup_{\ell=1}^{L} \bigcup_{\mu=M,N} \mathcal{K}_{\mu,\ell} \\
&= \{\mathcal{K}_{M,1}, \mathcal{K}_{N,1}, \mathcal{K}_{M,2}, \mathcal{K}_{N,2}, \cdots, \mathcal{K}_{M,L}, \mathcal{K}_{N,L}\}.
\end{aligned}
$$

For $m, n, p, q \in \mathbb{N}_0$ with $m + n + p + q \geq 1$, we define

$$
\begin{aligned}
& W_{p,q}^{m,n}[r; k^{(m)}; \tilde{k}^{(n)}] \\
:= & \int_{\mathbb{M}^{p+q}} dx^{(p)} d\tilde{x}^{(q)} b^+(x^{(p)}) w_{m+p,n+q}[r; k^{(m)}, x^{(p)}; \tilde{k}^{(n)}, \tilde{x}^{(q)}] b^-(\tilde{x}^{(q)}).
\end{aligned}
$$

The Wick ordering formula for the operator (A.14) is given by the following result:

Theorem A.3. *Let $L \in \mathbb{N}$ be a number. Suppose that $M_\ell \in \mathbb{N}_0$, $N_\ell \in \mathbb{N}_0$ are numbers such that $M_\ell + N_\ell \geq 1$. Let $\{w_{M_\ell,N_\ell}\}_{\ell=1}^{L}$ be functions defined in (A.13). Then,*

$$
\begin{aligned}
& f_0[H_f] W_{M_1,N_1} f_1[H_f] W_{M_2,N_2} \cdots f_{L-1}[H_f] W_{M_L,N_L} f_L[H_f] \\
= & \sum_{\substack{\mathcal{I}_{M,\ell} \subseteq \mathcal{K}_{M,\ell} \\ \ell=1,\dots,L}} \sum_{\substack{\mathcal{I}_{N,\ell} \subseteq \mathcal{K}_{N,\ell} \\ \ell=1,\dots,L}} \mathrm{sgn}(\mathcal{K} \backslash \mathcal{I}, \boldsymbol{:} \mathcal{I} \boldsymbol{:}) \prod_{\ell=1}^{L} \mathrm{sgn} \begin{pmatrix} & \mathcal{K}_{M,\ell} \\ \mathcal{I}_{M,\ell} & \mathcal{K}_{M,\ell} \backslash \mathcal{I}_{M,\ell} \end{pmatrix} \\
& \times \mathrm{sgn} \begin{pmatrix} & \mathcal{K}_{N,\ell} \\ \mathcal{I}_{N,\ell} & \mathcal{K}_{N,\ell} \backslash \mathcal{I}_{N,\ell} \end{pmatrix} \int_{\mathbb{M}^{m+n}} \prod_{\ell=1}^{L} \left\{ dk_\ell^{(m_\ell)} d\tilde{k}_\ell^{(n_\ell)} \right\} \prod_{\ell=1}^{L} b^+(k_\ell^{(m_\ell)}) \\
& \times \left\{ D_L \left[H_f; \{ W_{M_\ell - m_\ell, N_\ell - n_\ell}^{m_\ell, n_\ell}; k_\ell^{(m_\ell)}; \tilde{k}_\ell^{(n_\ell)} \}_{\ell=1}^{L}; \{f_\ell\}_{\ell=0}^{L} \right] \right\}_{m,n}^{\mathrm{asym}} \prod_{\ell=1}^{L} b^-(\tilde{k}_\ell^{(n_\ell)}),
\end{aligned}
\tag{A.16}
$$

where

$$
\begin{aligned}
&D_L[r;\{W^{m_\ell,n_\ell}_{M_\ell-m_\ell,N_\ell-n_\ell};k_\ell^{(m_\ell)};\tilde{k}_\ell^{(n_\ell)}\}_{\ell=1}^L;\{f_\ell\}_{\ell=1}^L] \\
&:= f_0[r+\tilde{r}_0]\Bigg\langle\Omega,\Bigg\{\prod_{\ell=1}^{L-1}W^{m_\ell,n_\ell}_{M_\ell-m_\ell,N_\ell-n_\ell}\big[H_f+r+r_\ell;k_\ell^{(m_\ell)};\tilde{k}_\ell^{(n_\ell)}\big] \\
&\quad\times f_\ell[H_f+r+\tilde{r}_\ell]\Bigg\} \\
&\quad\times W^{m_L,n_L}_{M_L-m_L,N_L-n_L}\big[r+r_L;k_L^{(m_L)};\tilde{k}_L^{(n_L)}\big]\Omega\Bigg\rangle f_L[r+\tilde{r}_L],
\end{aligned}
$$

and

$$\mathrm{sgn}(\mathcal{K}\backslash\mathcal{I},\boldsymbol{:}\,\mathcal{I}\,\boldsymbol{:}) := \mathrm{sgn}\begin{pmatrix} & \mathcal{K} & \\ \mathcal{K}\backslash\mathcal{I} & \bigcup_{\ell=1}^L\mathcal{I}_{M,\ell} & \bigcup_{\ell=1}^L\mathcal{I}_{N,\ell}\end{pmatrix} \tag{A.17}$$

$$r_\ell := \sum_{l=1}^{\ell-1}\Sigma[\tilde{k}_l^{(n_l)}] + \sum_{l=\ell+1}^{L}\Sigma[k_l^{(m_l)}], \quad \ell=2,3,\dots,L-1, \tag{A.18}$$

$$r_0 := \sum_{l=1}^{L}\Sigma[k_l^{(m_l)}], \quad r_1 := \sum_{l=2}^{L}\Sigma[k_l^{(m_l)}], \quad r_L := \sum_{l=1}^{L-1}\Sigma[\tilde{k}_l^{(n_l)}], \tag{A.19}$$

$$\tilde{r}_\ell := \sum_{l=1}^{\ell}\Sigma[\tilde{k}_l^{(n_l)}] + \sum_{l=\ell+1}^{L}\Sigma[k_l^{(m_l)}], \quad \ell=1,\dots,L-1. \tag{A.20}$$

$$\tilde{r}_0 := \sum_{l=1}^{L}\Sigma[k_l^{(m_l)}], \quad \tilde{r}_L := \sum_{l=1}^{L}\Sigma[\tilde{k}_l^{(n_l)}], \tag{A.21}$$

$$m_\ell := |\mathcal{I}_{M,\ell}|, \quad n_\ell := |\mathcal{I}_{N,\ell}|, \tag{A.22}$$

$$m := \sum_{\ell=1}^L m_\ell, \quad n := \sum_{\ell=1}^L n_\ell. \tag{A.23}$$

$$\tag{A.24}$$

Here, $\Sigma[\kappa^{(n)}] := \sum_{j=1}^n\omega(\kappa_j)$, $(\kappa = k_l, \tilde{k}_l)$.

Proof. By the definition of W_{M_ℓ,N_ℓ}, we have

$$
\begin{aligned}
&(\text{L.H.S. of (A.16)}) \\
&= \int_{\mathbb{M}^K}\prod_{\ell=1}^L\left\{\prod_{j=1}^{M_\ell}dk_{\ell,j}\prod_{j=1}^{N_\ell}d\tilde{k}_{\ell,j}\right\}f_0[H_f] \\
&\quad\times b^+(k_1^{(M_1)})w_{M_1,N_1}[H_f;k_1^{(M_1)};\tilde{k}_1^{(N_1)}]b^-(\tilde{k}_1^{(N_1)})f_1[H_f] \\
&\quad\times b^+(k_2^{(M_2)})w_{M_2,N_2}[H_f;k_2^{(M_2)};\tilde{k}_2^{(N_2)}]b^-(\tilde{k}_2^{(N_2)})f_2[H_f] \\
&\quad\times\cdots \\
&\quad\times b^+(k_{L-1}^{(M_{L-1})})w_{M_{L-1},N_{L-1}}[H_f;k_{L-1}^{(M_{L-1})};\tilde{k}_{L-1}^{(N_{L-1})}]b^-(\tilde{k}_{L-1}^{(N_{L-1})})f_{L-1}[H_f] \\
&\quad\times b^+(k_L^{(M_L)})w_{M_L,N_L}[H_f;k_L^{(M_L)};\tilde{k}_L^{(N_L)}]b(\tilde{k}_L^{(N_L)})f_L[H_f].
\end{aligned}
$$

By using Lemma (A.2), we have

$$
\begin{aligned}
&(\text{L.H.S. of (A.16)}) \\
&= \int_{\mathbb{M}^K} \prod_{\ell=1}^{L} \left\{ \prod_{j=1}^{M_\ell} dk_{\ell,j} \prod_{j=1}^{N_\ell} d\tilde{k}_{\ell,j} \right\} \sum_{\substack{\mathcal{I}_{M_\ell} \subseteq \mathcal{K}_{M,\ell} \\ \ell=1,\dots,L}} \sum_{\substack{\mathcal{I}_{N_\ell} \subseteq \mathcal{K}_{N,\ell} \\ \ell=1,\dots,L}} \mathrm{sgn}(\mathcal{K}\backslash\mathcal{I}, \boldsymbol{:} \mathcal{I} \boldsymbol{:}) \\
&\times \left[\prod_{\ell=1}^{L} \prod_{j \in \mathcal{I}_{M,\ell}} b^+(k_{\ell,j}) \right] \times f_0[r + \Lambda_0] \\
&\times \Big\langle \Omega, \Big\{ \prod_{\ell=1}^{L-1} \Big(\prod_{j \in \mathcal{K}_{M,\ell} \backslash \mathcal{I}_{M,\ell}} b^+(k_{\ell,j}) \Big) w_{M_\ell,N_\ell} \Big[H_f + r + \Lambda_\ell ; k_\ell^{(M_\ell)} ; \tilde{k}_\ell^{(N_\ell)} \Big] \\
&\times \Big(\prod_{j \in \mathcal{K}_{M,\ell} \backslash \mathcal{I}_{M,\ell}} b^-(\tilde{k}_{\ell,j}) \Big) f_\ell \Big[H_f + r + \Lambda_\ell + \sum_{j \in \mathcal{I}_{N,\ell}} \omega(\tilde{k}_{\ell,j}) \Big] \Big\} \\
&\times \Big(\prod_{j \in \mathcal{K}_{M,L} \backslash \mathcal{I}_{M,L}} b^+(k_{L,j}) \Big) w_{M_L,N_L} \Big[H_f + r + \Lambda_L ; k_L^{(M_L)} ; \tilde{k}_L^{(N_L)} \Big] \\
&\times \Big(\prod_{j \in \mathcal{K}_{M,L} \backslash \mathcal{I}_{M,L}} b^-(\tilde{k}_{L,j}) \Big) \Omega \Big\rangle \Big|_{r=H_f} f_L \left[r + \Lambda_L + \sum_{j \in \mathcal{I}_{N,L}} \omega(\tilde{k}_{L,j}) \right] \\
&\times \left[\prod_{\ell=1}^{L} \prod_{j \in \mathcal{I}_{N,\ell}} b^-(k_{\ell,j}) \right]
\end{aligned} \tag{A.25}
$$

where

$$
\Lambda_\ell := \sum_{l=1}^{\ell-1} \sum_{j \in \mathcal{I}_{N,l}} \omega(\tilde{k}_{l,j}) + \sum_{l=\ell+1}^{L} \sum_{j \in \mathcal{I}_{M,l}} \omega(k_{l,j}), \quad \ell = 2, 3, \dots, L-1,
$$

$$
\Lambda_0 := \sum_{l=1}^{L} \sum_{j \in \mathcal{I}_{M,l}} \omega(k_{l,j}), \quad \Lambda_1 := \sum_{l=2}^{L} \sum_{j \in \mathcal{I}_{M,l}} \omega(k_{l,j}), \quad \Lambda_L := \sum_{l=1}^{L-1} \sum_{j \in \mathcal{I}_{M,l}} \omega(\tilde{k}_{l,j}).
$$

Next, we move the integral in the variables $\mathcal{K}_{M,\ell} \backslash \mathcal{I}_{M,\ell}$, $\mathcal{K}_{N,\ell} \backslash \mathcal{I}_{N,\ell}$ to the inside of the inner product $\langle \Omega, \cdots \Omega \rangle$:

$$
\begin{aligned}
&(\text{L.H.S. of (A.16)}) \\
&= \sum_{\substack{\mathcal{I}_{M_\ell} \subseteq \mathcal{K}_{M,\ell} \\ \ell=1,\dots,L}} \sum_{\substack{\mathcal{I}_{N_\ell} \subseteq \mathcal{K}_{N,\ell} \\ \ell=1,\dots,L}} \mathrm{sgn}(\mathcal{K}\backslash\mathcal{I}, \boldsymbol{:} \mathcal{I} \boldsymbol{:}) \int_{\mathbb{M}^{m+n}} \prod_{\ell=1}^{L} \left\{ \prod_{j \in \mathcal{I}_{M,\ell}} dk_{\ell,j} \prod_{j \in \mathcal{I}_{N_\ell}} d\tilde{k}_{\ell,j} \right\} \\
&\times \left[\prod_{\ell=1}^{L} \prod_{j \in \mathcal{I}_{M,\ell}} b^+(k_{\ell,j}) \right] G\Big[r; \Big\{ \{k_{\ell,j}\}_{j \in \mathcal{I}_{M,\ell}}, \{\tilde{k}_{\ell,j}\}_{j \in \mathcal{I}_{N,\ell}} \Big\}_{\ell=1}^{L} \Big] \Big|_{r=H_f} \\
&\times \left[\prod_{\ell=1}^{L} \prod_{j \in \mathcal{I}_{N,\ell}} b^-(\tilde{k}_{\ell,j}) \right],
\end{aligned} \tag{A.26}
$$

where

$$
\begin{aligned}
&G\Big[r;\Big\{\{k_{\ell,j}\}_{j\in\mathcal{I}_{M,\ell}},\{\tilde{k}_{\ell,j}\}_{j\in\mathcal{I}_{N,\ell}}\Big\}_{\ell=1}^{L}\Big]\\
&=f_0[r+\Lambda_0]\Big\langle\Omega,\Big\{\prod_{\ell=1}^{L-1}\int\Big[\prod_{j\in\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}}dk_{\ell,j}\prod_{j\in\mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}}d\tilde{k}_{\ell,j}\Big]\\
&\quad\times\Big(\prod_{j\in\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}}b^{+}(k_{\ell,j})\Big)\times w_{M_\ell,N_\ell}\Big[H_f+r+\Lambda_\ell;k_\ell^{(M_\ell)};\tilde{k}_\ell^{(N_\ell)}\Big]\\
&\quad\times\Big(\prod_{j\in\mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}}b^{-}(\tilde{k}_{\ell,j})\Big)f_\ell\Big[H_f+r+\Lambda_\ell+\sum_{j\in\mathcal{I}_{N,\ell}}\omega(\tilde{k}_{\ell,j})\Big]\Big\}\\
&\quad\times\int\Big[\prod_{j\in\mathcal{K}_{M,L}\backslash\mathcal{I}_{M,L}}dk_{L,j}\prod_{j\in\mathcal{K}_{N,L}\backslash\mathcal{I}_{N,L}}d\tilde{k}_{L,j}\Big]\Big(\prod_{j\in\mathcal{K}_{M,L}\backslash\mathcal{I}_{M,L}}b^{+}(k_{L,j})\Big)\\
&\quad\times w_{M_L,N_L}\Big[H_f+r+\Lambda_L;k_L^{(M_L)};\tilde{k}_L^{(N_L)}\Big]\Big(\prod_{j\in\mathcal{K}_{M,L}\backslash\mathcal{I}_{M,L}}b^{-}(\tilde{k}_{L,j})\Big)\Omega\Big\rangle\\
&\quad\times f_L\left[r+\Lambda_L+\sum_{j\in\mathcal{I}_{N,L}}\omega(\tilde{k}_{L,j})\right]
\end{aligned}
$$

Here we used the fact that Λ_ℓ, $\ell=1,\dots,L$ and $\sum_{j\in\mathcal{I}_{N,\ell}}\omega(\tilde{k}_{\ell,j})$ are independent of $k_{\ell,j}(j\in\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell})$, $\tilde{k}_{\ell,j}(j\in\mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell})$. We rename the variables in (A.25) as follows

$$
\begin{aligned}
k_{\ell,j}&\to x_{\ell,j},\quad j\in\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell},\\
\tilde{k}_{\ell,j}&\to \tilde{x}_{\ell,j},\quad j\in\mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&w_{M_\ell,N_\ell}[r;k_\ell^{(M_\ell)};\tilde{k}_\ell^{(N_\ell)}]\Big|_{\substack{k_{\ell,j}=x_{\ell,j},\,j\in\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}\\ \tilde{k}_{\ell,j}=\tilde{x}_{\ell,j},\,j\in\mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}}}\\
&=\operatorname{sgn}\begin{pmatrix}&\mathcal{K}_{M,\ell}\\ \mathcal{I}_{M,\ell}&\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}\end{pmatrix}\operatorname{sgn}\begin{pmatrix}&\mathcal{K}_{N,\ell}\\ \mathcal{I}_{N,\ell}&\mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}\end{pmatrix}\\
&\quad\times w_{M_\ell,N_\ell}\Big[r;\{k_{\ell,j}\}_{j\in\mathcal{I}_{M,\ell}},\{x_{\ell,j}\}_{j\in\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}}\Big|\{\tilde{k}_{\ell,j}\}_{j\in\mathcal{I}_{N,\ell}},\{\tilde{x}_{\ell,j}\}_{j\in\mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}}\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
&\int\Big[\prod_{j\in\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}}dk_{\ell,j}\prod_{j\in\mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}}d\tilde{k}_{\ell,j}\Big]\Big(\prod_{j\in\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}}b^{+}(k_{\ell,j})\Big)\\
&\times w_{M_\ell,N_\ell}\Big[H_f+r+\Lambda_\ell;k_\ell^{(M_\ell)};\tilde{k}_\ell^{(N_\ell)}\Big]\Big(\prod_{j\in\mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}}b^{-}(\tilde{k}_{\ell,j})\Big)\\
&=\operatorname{sgn}\begin{pmatrix}&\mathcal{K}_{M,\ell}\\ \mathcal{I}_{M,\ell}&\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}\end{pmatrix}\operatorname{sgn}\begin{pmatrix}&\mathcal{K}_{N,\ell}\\ \mathcal{I}_{N,\ell}&\mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}\end{pmatrix}\\
&\quad\times W^{m_\ell,n_\ell}_{M_\ell-m_\ell,N_\ell-n_\ell}\Big[H_f+r+\Lambda_\ell;\{k_{\ell,j}\}_{j\in\mathcal{I}_{M,\ell}};\{\tilde{k}_{\ell,j}\}_{j\in\mathcal{I}_{N,\ell}}\Big],
\end{aligned}
$$

where

$$m_\ell := |\mathcal{I}_{M,\ell}|, \quad |n_\ell| := |\mathcal{I}_{N,\ell}|, \quad \ell = 1, \dots, L.$$

Hence we have

$$\begin{aligned}
&G\Big[r; \Big\{\{k_{\ell,j}\}_{j\in\mathcal{I}_{M,\ell}}, \{\tilde{k}_{\ell,j}\}_{j\in\mathcal{I}_{N,\ell}}\Big\}_{\ell=1}^{L}\Big] \\
&= \left[\prod_{\ell=1}^{L} \mathrm{sgn}\begin{pmatrix} & \mathcal{K}_{M,\ell} \\ \mathcal{I}_{M,\ell} & \mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}\end{pmatrix} \mathrm{sgn}\begin{pmatrix} & \mathcal{K}_{N,\ell} \\ \mathcal{I}_{N,\ell} & \mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}\end{pmatrix}\right] f_0[r+\Lambda_0] \\
&\times \Bigg\langle \Omega, \prod_{\ell=1}^{L-1}\Bigg[W_{M_\ell-m_\ell,N_\ell-n_\ell}^{m_\ell,n_\ell}\Big[H_f + r + \Lambda_\ell; \{k_{\ell,j}\}_{j\in\mathcal{I}_{M,\ell}}; \{\tilde{k}_{\ell,j}\}_{j\in\mathcal{I}_{N,\ell}}\Big] \\
&\times f_\ell\Big[H_f + r + \Lambda_\ell + \sum_{j\in\mathcal{I}_{N,\ell}} \omega(\tilde{k}_{\ell,j})\Big]\Bigg] \\
&\times W_{M_L-m_L,N_L-n_L}^{m_L,n_L}\Big[r+\Lambda_L; \{k_{L,j}\}_{j\in\mathcal{I}_{M,L}}; \{\tilde{k}_{L,j}\}_{j\in\mathcal{I}_{N,L}}\Big]\Omega\Bigg\rangle \\
&\times f_L\left[r + \Lambda_L + \sum_{j\in\mathcal{I}_{N,L}} \omega(\tilde{k}_{L,j})\right]. \qquad \text{(A.27)}
\end{aligned}$$

By changing the names of the variables $\{k_{\ell,j}\}_{j\in\mathcal{I}_{M,\ell}}$, $\{\tilde{k}_{\ell,j}\}_{j\in\mathcal{I}_{N,\ell}}$ in (A.26) with (A.27):

$$\{k_{\ell,j}\}_{j\in\mathcal{I}_{M,\ell}} \to k_\ell^{(m_\ell)}, \quad \{\tilde{k}_{\ell,j}\}_{j\in\mathcal{I}_{N,\ell}} \to \tilde{k}_\ell^{(n_\ell)},$$

we have

$$\begin{aligned}
&(\text{L.H.S. of (A.16)}) \\
&= \sum_{\substack{\mathcal{I}_{M,\ell}\subseteq\mathcal{K}_{M,\ell} \\ \ell=1,\dots,L}} \sum_{\substack{\mathcal{I}_{N,\ell}\subseteq\mathcal{K}_{N,\ell} \\ \ell=1,\dots,L}} \mathrm{sgn}(\mathcal{K}\backslash\mathcal{I}, \boldsymbol{:}\, \mathcal{I}\,\boldsymbol{:}) \left[\prod_{\ell=1}^{L} \mathrm{sgn}\begin{pmatrix} & \mathcal{K}_{M,\ell} \\ \mathcal{I}_{M,\ell} & \mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}\end{pmatrix}\right. \\
&\left.\times\, \mathrm{sgn}\begin{pmatrix} & \mathcal{K}_{N,\ell} \\ \mathcal{I}_{N,\ell} & \mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}\end{pmatrix}\right] \int_{\mathbb{M}^{m+n}} \prod_{\ell=1}^{L}\Big\{dk_\ell^{(m_\ell)} d\tilde{k}_\ell^{(n_\ell)}\Big\} \prod_{\ell=1}^{L} b^+(k_\ell^{(m_\ell)}) \\
&\times D_L\Big[H_f; \{W_{M_\ell-m_\ell,N_\ell-n_\ell}^{m_\ell,n_\ell}; k_\ell^{(m_\ell)}; \tilde{k}_\ell^{(n_\ell)}\}_{\ell=1}^{L}; \{f_\ell\}_{\ell=1}^{L-1}\Big] \prod_{\ell=1}^{L} b^-(\tilde{k}^{(n)}).
\end{aligned}$$

Finally, by using this fact and the anticommutativity of b^-, b^+, we obtain the formula (A.16). $\square$

We set

$$W := \sum_{N+M\geq 1} W_{M,N}.$$

Theorem A.4. *Let W be a operator defined above. We write as*

$$f_0 W f_1 W \cdots W f_L = H[\underline{\tilde{w}}], \qquad \text{(A.28)}$$

where $\underline{\tilde{w}} = (\tilde{w}_{m,n})_{m+n\geq 0}$*. Then*

$$\tilde{w}_{m,n}(r; K^{(m,n)}) = \sum_{\substack{m_1+\cdots+m_L=m \\ n_1+\cdots+n_L=n}} \sum_{\substack{p_\ell,q_\ell\geq 0 \\ m_\ell+p_\ell+n_\ell+q_\ell\geq 1 \\ \ell=1,\ldots,L}} \mathrm{sgn}(\{m_\ell\}_{\ell=1}^L; \{n_\ell\}_{\ell=1}^L)$$
$$\int_{\mathbb{M}^{m+n}} \prod_{\ell=1}^L \left\{dk_\ell^{(m_\ell)} d\tilde{k}_\ell^{(n_\ell)}\right\} \prod_{\ell=1}^L b^+(k_\ell^{(m_\ell)})$$
$$\times \left\{D_L\left[H_f; \{W_{p_\ell,q_\ell}^{m_\ell,n_\ell}; k_\ell^{(m_\ell)}; \tilde{k}_\ell^{(n_\ell)}\}_{\ell=1}^L; \{f_\ell\}_{\ell=0}^L\right]\right\}_{m,n}^{\mathrm{asym}}$$
$$\times \prod_{\ell=1}^L b^-(\tilde{k}_\ell^{(n_\ell)}),$$

where $D_L[\cdots]$ *is the function defined in Theorem A.3,*

$$\mathrm{sgn}(\{m_\ell\}_{\ell=1}^L; \{n_\ell\}_{\ell=1}^L)$$
$$:= \sum_{\substack{\mathcal{I}_{M,\ell}\subseteq\mathcal{K}_{M,\ell} \\ m_\ell=|\mathcal{I}_{M,\ell}| \\ \ell=1,\ldots,L}} \sum_{\substack{\mathcal{I}_{N,\ell}\subseteq\mathcal{K}_{N,\ell} \\ n_\ell=|\mathcal{I}_{N,\ell}| \\ \ell=1,\ldots,L}} \mathrm{sgn}(\mathcal{K}\backslash\mathcal{I}, \boldsymbol{:} \mathcal{I} \boldsymbol{:})$$
$$\times \prod_{\ell=1}^L \mathrm{sgn}\begin{pmatrix} & \mathcal{K}_{M,\ell} \\ \mathcal{I}_{M,\ell} & \mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}\end{pmatrix} \mathrm{sgn}\begin{pmatrix} & \mathcal{K}_{N,\ell} \\ \mathcal{I}_{N,\ell} & \mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}\end{pmatrix}, \tag{A.29}$$

and $\mathrm{sgn}(\mathcal{K}\backslash\mathcal{I}, \boldsymbol{:} \mathcal{I} \boldsymbol{:})$ *is a constant defined in Theorem A.3.*

Proof. Note that

$$(\text{L. H. S. of } (A.28)) = \sum_{M_1+N_1\geq 1} \cdots \sum_{N_L+M_L\geq 1} (A.16). \tag{A.30}$$

It is easy to see that, for all $\ell = 1, \ldots, L$,

$$\sum_{M_\ell+N_\ell\geq 1} \sum_{\mathcal{I}_{M,\ell}\subseteq\mathcal{K}_{M,\ell}} \sum_{\mathcal{I}_{N,\ell}\subseteq\mathcal{K}_{N,\ell}}$$
$$= \sum_{M_\ell+N_\ell\geq 1} \sum_{m_\ell=0}^{M_\ell} \sum_{n_\ell=0}^{N_\ell} \sum_{\substack{\mathcal{I}_{M,\ell}\subseteq\mathcal{K}_{M,\ell} \\ |\mathcal{I}_{M,\ell}|=m_\ell}} \sum_{\substack{\mathcal{I}_{N,\ell}\subseteq\mathcal{K}_{N,\ell} \\ |\mathcal{I}_{N,\ell}|=n_\ell}}. \tag{A.31}$$

Furthermore, for any function $X[\cdots]$, we have

$$\sum_{M_\ell+N_\ell\geq 1} \sum_{m=0}^{M_\ell} \sum_{n=0}^{N_\ell} X(M_\ell, N_\ell, m_\ell, n_\ell) = \sum_{\substack{(M_\ell,N_\ell,m_\ell,n_\ell)\in\mathbb{N}_0^4 \\ M_\ell\geq m_\ell\geq 0;\, N_\ell\geq n_\ell\geq 0 \\ M_\ell+N_\ell\geq 1}} X(M_\ell, N_\ell, m_\ell, n_\ell)$$
$$= \sum_{\substack{(p_\ell,q_\ell,m_\ell,n_\ell)\in\mathbb{N}_0^4 \\ p_\ell+q_\ell+m_\ell+n_\ell\geq 1}} X(m_\ell+p_\ell, n_\ell+q_\ell, m_\ell, n_\ell). \tag{A.32}$$

By connecting (A.30)-(A.32) with Theorem A.3, one can obtain the desired result. $\square$

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