



Title	Shift Operators on the $\mathbb{C}^2$ -valued Hardy Space
Author(s)	Douglas, Ronald G.; Nakazi, Takahiko; Seto, Michio
Citation	Hokkaido University Preprint Series in Mathematics, 850, 1-16
Issue Date	2007
DOI	10.14943/84000
Doc URL	<a href="http://hdl.handle.net/2115/69659">http://hdl.handle.net/2115/69659</a>
Type	bulletin (article)
File Information	pre850.pdf



[Instructions for use](#)

# Shift Operators on the $\mathbb{C}^2$ -valued Hardy Space

RONALD G. DOUGLAS, TAKAHIKO NAKAZI  
and MICHIO SETO \*

## Abstract

In this paper, we study closed invariant subspaces under the action of a unilateral shift and a truncated shift in the Hardy space that takes values in a two dimensional Hilbert space. We deal with characteristic functions, unitary equivalence and  $C^*$ -algebras on these spaces.

2000 *Mathematics subject classification*: primary 47B38, secondary 47B35.

*Keywords and phrases*: vector valued Hardy spaces and shift operators.

## 0 Introduction

In the classical Hardy space theory on the unit disk, A. Beurling proved that every submodule, which is a closed invariant subspace under multiplication operators by bounded analytic functions, is characterized by an inner function in [5]. This result is very important for the structure theory of a single operator on a Hilbert space (cf. [11]), so that many researchers have studied submodules in other Hilbert function spaces. However, the general structure of submodules is extremely complicated in multivariables. Recent development and an extensive list of references concerning this topic can be found in [7].

In this paper, we study closed invariant subspaces under the action of a unilateral shift and a truncated shift in the Hardy space that takes values in a two dimensional Hilbert space. We call such an invariant subspace a submodule for short. Although this topic is very restricted, the authors consider that this research will help us to understand the general structure of submodules in the Hardy space over the bidisk. Because, the Hardy space over the bidisk can be regarded as the limit of the tensor product Hilbert space of the classical Hardy space  $H^2$  and the  $n$  dimensional Hilbert space  $\mathbb{C}^n$  as  $n$  tends to infinity in the weak topology. More precisely, the  $\mathbb{C}^2$ -valued Hardy space  $H^2 \otimes \mathbb{C}^2$

---

\*The authors were partially supported by the Grant-in-Aid for Scientific Research, Japan Society for Promotion of Science.

can be identified with the quotient module  $(H^2 \otimes H^2)/w^2(H^2 \otimes H^2)$ , and  $H^2 \otimes H^2$  is actually equal to the Hardy space over the bidisk with variables  $z$  and  $w$ . We also want to emphasize that some difficulties in general cases appear in this two dimensional setting, and our results can be generalized easily to the case where  $n$  is any positive integer.

Section 1 is the preliminaries, in which we introduce notions used in this paper and translate well known results on the Sz.-Nagy-Foiaş characteristic function into our setting. In Section 2, we obtain a Beurling type theorem under some condition. In Section 3, we focus on some properties of row operators whose entries are Toeplitz operators. In Section 4, we discuss unitary equivalence on submodules, and obtain a result similar to Agrawal-Clark-Douglas theorem given in [1]. In Section 5, we deal with an example in detail. Submodules of finite codimension are described in Section 6. In Section 7, isomorphisms of the  $C^*$ -algebras generated by the operators defined by module multiplication on submodules are considered.

## 1 Preliminaries

Let  $\mathbb{C}$  denote the complex plane, and let  $\mathbb{D}$  denote the open unit disk in  $\mathbb{C}$ , that is,  $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ . If  $\mathcal{H}$  is a Hilbert space, then  $H^2(\mathcal{H})$  denotes the  $\mathcal{H}$ -valued Hardy space over  $\mathbb{D}$ . An  $\mathcal{H}$ -valued function  $f$  belongs to  $H^2(\mathcal{H})$  if and only if  $f$  is analytic in  $\mathbb{D}$  and

$$\|f\|^2 := \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|^2 d\theta < +\infty.$$

In other words,  $H^2(\mathcal{H})$  is the tensor product Hilbert space of the classical Hardy space  $H^2 = H^2(\mathbb{C})$  and  $\mathcal{H}$ . Let  $T_z$  be the Toeplitz operator of the coordinate function  $z$ , and let  $S$  be the truncated shift operator on  $\mathbb{C}^2$  which maps  $(1, 0)$  to  $(0, 1)$  and  $(0, 1)$  to  $(0, 0)$ . Then  $H^2(\mathbb{C}^2)$  has a module structure over the commutative algebra generated by  $T_z$  and  $S$  by the usual action of operators.

**Definition 1.1** *A closed subspace  $\mathcal{M}$  of  $H^2(\mathbb{C}^2)$  is called a submodule if  $\mathcal{M}$  is invariant under the action of operators  $T_z$  and  $S$ .*

Let  $V_z$  (resp.  $R$ ) denote the restriction of  $T_z$  (resp.  $S$ ) on a submodule  $\mathcal{M}$ , that is,  $V_z = T_z|_{\mathcal{M}}$  (resp.  $R = S|_{\mathcal{M}}$ ). Then  $V_z$  commutes with  $R$  on  $\mathcal{M}$ .

If  $\mathcal{M}$  is a submodule of  $H^2(\mathbb{C}^2)$ , then by the Kolmogorov-Wold decomposition, we have

$$\mathcal{M} = \sum_{n \geq 0} \oplus T_z^n(\mathcal{M} \ominus T_z \mathcal{M}).$$

Hence  $\mathcal{M}$  can be identified with  $H^2(\mathcal{D})$ , where we set  $\mathcal{D} = \mathcal{M} \ominus T_z \mathcal{M}$ . It is easy to check that  $\dim \mathcal{D} \leq 2$ . We define an operator  $\Omega(\zeta)$  from  $\mathcal{D}$  to  $\mathbb{C}^2$  for  $\zeta$  a.e. in  $\mathbb{T} = \partial\mathbb{D}$  as

follows:

$$\Omega(\zeta)d := d(\zeta),$$

where  $d$  is in  $\mathcal{D}$ . Then we can define an operator valued function  $\Omega$  which maps the function  $z^j \otimes d$  in  $H^2(\mathcal{D})$  to a function in  $H^2(\mathbb{C}^2)$  as follows:

$$\Omega(\zeta) ((z^j \otimes d)(\zeta)) = \Omega(\zeta)\zeta^j d = \zeta^j d(\zeta),$$

a.e. on  $\mathbb{T}$ , where  $d$  is in  $\mathcal{D}$  and  $j \geq 0$ . Then  $\Omega$  has the following properties:

- (i)  $\Omega$  can be extended analytically to  $\mathbb{D}$ ,
- (ii)  $\Omega^*(\zeta)\Omega(\zeta) = I_{\mathcal{D}}$  a.e. on  $\mathbb{T}$ ,
- (iii)  $\mathcal{M} = \Omega H^2(\mathcal{D})$ .

Let  $U$  be an isometry from  $\mathcal{D}$  to  $\mathbb{C}^2$ . Then there exists an operator valued function  $\Theta$  defined by the following commutative diagram:

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\Omega(\zeta)} & \mathbb{C}^2 \\ U \downarrow & & \parallel \\ \mathbb{C}^2 & \xrightarrow{\Theta(\zeta)} & \mathbb{C}^2. \end{array}$$

Moreover  $\Theta$  is a partial isometry valued function having the following properties:

- (i)  $\Theta$  can be extended analytically to  $\mathbb{D}$ ,
- (ii)  $\Theta^*(\zeta)\Theta(\zeta)$  is the orthogonal projection onto  $U\mathcal{D}$  a.e. on  $\mathbb{T}$ ,
- (iii)  $\mathcal{M} = \Theta H^2(\mathbb{C}^2)$ .

This  $\Theta$  is called the characteristic function of  $\mathcal{M}$  in the sense of Sz.-Nagy and Foiaş [11], which is determined uniquely up to unitary matrices. Moreover  $\Theta$  has a matrix representation depending on  $U$ . The entries of  $\Theta$  are Toeplitz operators of bounded analytic functions on  $\mathbb{D}$ , that is,

$$\Theta = \begin{pmatrix} T_{\theta_{11}} & T_{\theta_{12}} \\ T_{\theta_{21}} & T_{\theta_{22}} \end{pmatrix},$$

where every  $\theta_{ij}$  is a bounded analytic function on  $\mathbb{D}$ . Hence it follows that  $\text{tr } \Theta$  and  $\det \Theta$  are bounded analytic functions. It is easy to check the following:

- (i) if  $\dim \mathcal{D} = 2$ , then  $\det \Theta$  is an inner function,
- (ii) For any  $d$  in  $\mathcal{D}$ , entries of  $d$  are bounded analytic functions.

## 2 A Beurling type theorem

In this section, we consider the condition that  $V_z^*$  commutes with  $R$ . This condition is similar to that in Mandrekar [9] and Nakazi [10].

**Theorem 2.1**  $V_z^*$  commutes with  $R$  if and only if  $\Theta$  has one of the following matrix representations:

$$\Theta = \begin{pmatrix} T_q & 0 \\ 0 & T_q \end{pmatrix} \quad \text{or} \quad \Theta = \begin{pmatrix} 0 & 0 \\ T_q & 0 \end{pmatrix},$$

where  $q$  is an inner function on  $\mathbb{D}$ .

**Proof** It is easy to show the ‘‘if’’ part. We show the ‘‘only if’’ part. We suppose that  $V_z^*$  commutes with  $R$ , which implies that  $\mathcal{D}$  is invariant under  $R$ . Let  $d$  be a normalized vector in  $\mathcal{D} \ominus R\mathcal{D}$ . Then  $d$  is orthogonal to  $Rd$ .

First, we show that  $\dim \mathcal{D} \ominus R\mathcal{D} = 1$ . Let  $f$  be an element in  $\mathcal{D} \ominus R\mathcal{D}$  orthogonal to  $d$ , and let  $S^{(1)}$ ,  $S^{(0)}$  and  $S^{(-1)}$  denote  $S$ ,  $I_{\mathcal{M}}$  and  $S^*$ , respectively. Then

$$\int_{\mathbb{T}} \langle f(\zeta), S^{(n)}d(\zeta) \rangle \zeta^k |d\zeta| = 0,$$

for any integer  $k$  and  $n = -1, 0, 1$ . Since  $d$  is a non-zero vector, we have  $f = 0$ .

(Case 1.  $Rd \neq 0$ )

Let  $d_1 = d$  and  $d_2 = Rd/\|Rd\|$ . We define a unitary operator  $U$  from  $\mathcal{D}$  onto  $\mathbb{C}^2$  as follows:

$$U : \begin{aligned} d_1 &\mapsto e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ d_2 &\mapsto e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Next we show that  $\Theta$  commutes with  $S$ .

$$\begin{aligned} S\Theta(\zeta) : \begin{pmatrix} f(\zeta) \\ g(\zeta) \end{pmatrix} &\mapsto S\Omega(\zeta)(f(\zeta)d_1 \oplus g(\zeta)d_2) = f(\zeta)d_2(\zeta), \\ \Theta(\zeta)S : \begin{pmatrix} f(\zeta) \\ g(\zeta) \end{pmatrix} &\mapsto \Omega(\zeta)(f(\zeta)d_2) = f(\zeta)d_2(\zeta). \end{aligned}$$

Therefore  $\Theta$  commutes with  $S$  on  $H^2(\mathbb{C}^2)$ . Hence  $\Theta$  can be represented by the following Toeplitz form:

$$\Theta = \begin{pmatrix} T_{\theta_1} & 0 \\ T_{\theta_2} & T_{\theta_1} \end{pmatrix}.$$

Since  $\Theta(\zeta)$  is an isometry a.e. on  $\mathbb{T}$ , its column vectors are orthogonal. Hence

$$\theta_2(\zeta)\overline{\theta_1(\zeta)} = 0.$$

Since every  $\theta_i$  is bounded analytic, we have  $\theta_1 = 0$  or  $\theta_2 = 0$ . Hence we have  $\theta_2 = 0$  by the assumption  $Rd \neq 0$ .

(Case 2.  $Rd = 0$ ) We define an isometry from  $\mathcal{D}$  to  $\mathbb{C}^2$  as follows:

$$U : d \mapsto e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then it is easy to check that  $\Theta S = S\Theta = 0$ . Therefore  $\Theta$  has a Toeplitz form. Moreover, the assumption  $Rd = 0$  implies that  $R\mathcal{M} = S\mathcal{M} = \{0\}$ . Then, by the Beurling theorem, there exists an inner function  $q$  such that

$$\mathcal{M} = \begin{pmatrix} 0 \\ qH^2(\mathbb{C}) \end{pmatrix}.$$

Hence we have

$$\Theta = \begin{pmatrix} 0 & 0 \\ T_q & 0 \end{pmatrix}.$$

This completes the proof.

**Remark 2.1** In this remark, we suppose that  $\dim \mathcal{D} = 1$ . Then we have the complete description of  $\mathcal{M}$ . Indeed, Let  $d = (d_1, d_2)$  be a normalized vector in  $\mathcal{D}$ . Then, for an isometry  $U$  which maps  $d$  to  $(1, 0)$ , we have

$$\Theta = \begin{pmatrix} T_{d_1} & 0 \\ T_{d_2} & 0 \end{pmatrix}.$$

Therefore

$$\mathcal{M} = \{(d_1 f, d_2 f) : f \in H^2\}.$$

However  $\mathcal{M}$  is invariant under  $S$ . It follows that  $d_1 = 0$ . Hence we have

$$\Theta = \begin{pmatrix} 0 & 0 \\ T_{d_2} & 0 \end{pmatrix}.$$

That is, if  $\dim \mathcal{D} = 1$  then  $\mathcal{M} = \{0\} \oplus qH^2$  for some inner function  $q$ . Hence we have the following:

**Theorem 2.2** *Let  $\mathcal{M}$  be a submodule. Then  $\dim \mathcal{D} = 1$  if and only if there exists an inner function  $q$  such that  $\mathcal{M} = \{0\} \oplus qH^2$ .*

**Corollary 2.1** *If  $\dim \mathcal{D} = 1$ , then  $[V_z^*, R] = 0$ .*

### 3 Submodules and row operators

We consider general cases where  $\dim \mathcal{D} = 2$ . Then the characteristic function  $\Theta$  of a submodule  $\mathcal{M}$  takes values in unitary matrices on  $\mathbb{C}^2$ . The matrix representation of  $\Theta$  is uniquely determined up to unitary matrices on  $\mathbb{C}^2$ . In this section, we focus on the following observation:

Since  $\mathcal{M}$  is invariant under  $S$ , for any  $f_1$  and  $f_2$  in  $H^2$ , there exist functions  $g_1$  and  $g_2$  in  $H^2$  such that

$$S\Theta \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \Theta \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} 0 \\ \theta_{11}f_1 + \theta_{12}f_2 \end{pmatrix} = \begin{pmatrix} \theta_{11}g_1 + \theta_{12}g_2 \\ \theta_{21}g_1 + \theta_{22}g_2 \end{pmatrix}, \quad (3.1)$$

where each  $\theta_{ij}$  is an entry of  $\Theta$ . Let  $T_j$  denote the operator from  $H^2 \oplus H^2$  to  $H^2$  defined as follows:

$$T_j : \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \mapsto \theta_{j1}f_1 + \theta_{j2}f_2.$$

Then (3.1) is equivalent to the following:

$$\text{Im } T_1 \subseteq T_2(\ker T_1). \quad (3.2)$$

Note that  $\ker T_j$  includes the following non trivial vector space for each  $j$ :

$$\{(-\theta_{j2}h, \theta_{j1}h) : h \in H^2\}.$$

It is easy to show the following:

- (i)  $\ker T_1$  is invariant under  $T_z$  in  $H^2(\mathbb{C}^2)$ ,
- (ii)  $\text{Im } T_1$  and  $T_2(\ker T_1)$  are invariant under  $T_z$  in  $H^2$ ,
- (iii) the closure of  $T_2(\ker T_1)$  includes  $(\det \Theta)H^2$ .

Let  $\mathcal{M}_j = \mathcal{M}(\theta_{j1}, \theta_{j2})$  denote the submodule generated by  $\theta_{j1}$  and  $\theta_{j2}$  in  $H^2$ , that is,  $\mathcal{M}_j$  is the closure of the subspace  $\{\theta_{j1}f + \theta_{j2}g : f, g \in H^2\}$ . Then there exists an inner function  $q_j$  such that  $\mathcal{M}_j = q_j H^2$  by Beurling's theorem. Since  $q_1 H^2 \subseteq q_2 H^2$  by (3.2), it follows that  $q_2$  is a divisor of  $q_1$ . Hence  $\Theta$  can be decomposed as follows:

$$\Theta = \begin{pmatrix} T_{q_1} & 0 \\ 0 & T_{q_2} \end{pmatrix} \begin{pmatrix} T_{\eta_{11}} & T_{\eta_{12}} \\ T_{\eta_{21}} & T_{\eta_{22}} \end{pmatrix},$$

where  $q_1$  is divided by  $q_2$  and  $\mathcal{M}(\eta_{j1}, \eta_{j2}) = H^2$  for every  $j$ . We note that inner functions  $q_1$  and  $q_2$  are uniquely determined by  $\mathcal{M}$  up to constants in  $\mathbb{T}$ .

**Proposition 3.1** *Let  $\mathcal{M}$  be a submodule of  $H^2(\mathbb{C}^2)$  with  $\dim \mathcal{D} = 2$ , and let  $\Theta = (T_{\theta_{ij}})$  be the characteristic function of  $\mathcal{M}$ . Then row operators  $T_1 = (T_{\theta_{11}} \ T_{\theta_{12}})$  and  $T_2 = (T_{\theta_{21}} \ T_{\theta_{22}})$  satisfy the following two conditions:*

- (i) *the pair of row operators  $(T_1, T_2)$  is a joint isometry, that is,  $T_1^* T_1 + T_2^* T_2 = I$ ,*
- (ii)  *$\text{Im } T_1 \subseteq T_2(\ker T_1)$ .*

*Conversely if two row operators  $T_1 = (T_{\theta_{11}} \ T_{\theta_{12}})$  and  $T_2 = (T_{\theta_{21}} \ T_{\theta_{22}})$  whose entries are Toeplitz operators of bounded analytic functions satisfy the above conditions (i) and (ii), then the range of the operator matrix  $(T_{\theta_{ij}})$  is a submodule in  $H^2(\mathbb{C}^2)$  with  $\dim \mathcal{D} = 2$ .*

## 4 Unitary equivalence

In this section, we classify submodules in the sense of unitary equivalence. If  $\mathcal{H}$  is a Hilbert space, then  $L^2(\mathcal{H})$  denotes the Hilbert space consisting of  $\mathcal{H}$ -valued square integrable functions with respect to the normalized Lebesgue measure  $d\theta/2\pi$  on  $\mathbb{T}$ . Then it is easy to check that a co-isometric extension of  $V_z$  on a submodule  $\mathcal{M}$  of  $H^2(\mathbb{C}^2)$  is the multiplication operator by  $z$  on  $L^2(\mathbb{C}^2)$ .

**Definition 4.1** *Two submodules  $\mathcal{M}$  and  $\tilde{\mathcal{M}}$  are said to be unitarily equivalent as modules if there exists a unitary operator  $W$  from  $\mathcal{M}$  onto  $\tilde{\mathcal{M}}$  such that  $\tilde{V}_z W = W V_z$  and  $\tilde{R} W = W R$ , where  $\tilde{V}_z$  and  $\tilde{R}$  denote restrictions of  $T_z$  and  $S$  on  $\tilde{\mathcal{M}}$ , respectively. Then the above  $W$  is called a unitary module map from  $\mathcal{M}$  onto  $\tilde{\mathcal{M}}$ .*

**Lemma 4.1** *If two submodules  $\mathcal{M}$  and  $\tilde{\mathcal{M}}$  are unitarily equivalent as modules, then  $\dim \mathcal{D} = \dim \tilde{\mathcal{D}}$ , where we set  $\tilde{\mathcal{D}} = \tilde{\mathcal{M}} \ominus z\tilde{\mathcal{M}}$ .*

**Proof** Let  $W$  be a unitary module map from  $\mathcal{M}$  onto  $\tilde{\mathcal{M}}$ . Then it is easy to show that  $W(\mathcal{M} \ominus z\mathcal{M}) = \tilde{\mathcal{M}} \ominus z\tilde{\mathcal{M}}$ .

By Theorem 2.2 and Lemma 4.1, it suffices to consider unitary equivalence only on submodules with  $\dim \mathcal{D} = 2$ .

**Theorem 4.1** *If two submodules  $\mathcal{M}$  and  $\tilde{\mathcal{M}}$  with  $\dim \mathcal{D} = 2$  are unitarily equivalent as modules by a unitary module map  $W$ , then there exists a unimodular Borel function  $u$  on  $\mathbb{T}$  such that*

$$W = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix},$$

*where we identified a bounded Borel function  $u$  with its multiplication operator.*



**Proof** First, we note that  $W$  intertwines  $V_z$  and  $\tilde{V}_z$ , that is,  $WV_z = \tilde{V}_zW$ . By the commutant lifting theorem ([11]), there exists an operator  $X$  which commutes with the multiplication by  $z$  on  $L^2(\mathbb{C}^4) = L^2(\mathbb{C}^2) \oplus L^2(\mathbb{C}^2)$  and agrees with  $W$  on  $\mathcal{M}$ , that is,

$$\begin{aligned} X : \mathcal{M} \oplus \{0\} &\rightarrow \{0\} \oplus \tilde{\mathcal{M}} \\ f \oplus 0 &\mapsto 0 \oplus Wf \end{aligned}$$

for any  $f$  in  $\mathcal{M}$ . Then each entry of  $X$  is a multiplication operator by a bounded Borel function. We set

$$X = \begin{pmatrix} * & * \\ U & * \end{pmatrix} \text{ and } U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}.$$

Next, we show that  $u_{11} = u_{22}$  and  $u_{12} = 0$ . Since

$$\begin{aligned} WS &: \begin{pmatrix} f \\ g \end{pmatrix} \mapsto \begin{pmatrix} u_{12}f \\ u_{22}f \end{pmatrix} \\ SW &: \begin{pmatrix} f \\ g \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ u_{11}f + u_{12}g \end{pmatrix} \end{aligned}$$

and  $W$  commutes with  $S$  on  $\mathcal{M}$ , we have  $u_{12}f = 0$  and  $u_{22}f = u_{11}f + u_{12}g$  for any  $(f, g)$  in  $\mathcal{M}$ . Since entries of elements of  $\mathcal{M}$  are in  $H^2$ , we have that  $u_{12} = 0$  and  $u_{11} = u_{22}$ .

Lastly, we show  $u_{21} = 0$ . We note that there exists an inner function  $q_1$  such that  $\mathcal{M}(\theta_{11}, \theta_{12}) = q_1H^2$ , and  $S\mathcal{M}$  is dense in  $\{(0, q_1f) : f \in H^2\}$ . Since  $W$  is a unitary module map from  $\mathcal{M}$  onto  $\tilde{\mathcal{M}}$ , we have

$$\|f\| = \|q_1f\| = \left\| \begin{pmatrix} u_{11} & 0 \\ u_{21} & u_{11} \end{pmatrix} \begin{pmatrix} 0 \\ q_1f \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ u_{11}q_1f \end{pmatrix} \right\| = \|u_{11}q_1f\|,$$

for any vector  $(0, q_1f)$  in  $\mathcal{M}$ . Hence  $u_{11}q_1$  is an isometry as a multiplication operator on  $H^2$ , that is,  $u_{11}q_1$  is inner. Hence  $u_{11}$  is a unimodular function. Further, we have

$$\begin{aligned} \|q_1f\|^2 + \|q_2g\|^2 &= \left\| \begin{pmatrix} u_{11} & 0 \\ u_{21} & u_{11} \end{pmatrix} \begin{pmatrix} q_1f \\ q_2g \end{pmatrix} \right\|^2 \\ &= \left\| \begin{pmatrix} u_{11}q_1f \\ u_{21}q_1f + u_{11}q_2g \end{pmatrix} \right\|^2 = \|u_{11}q_1f\|^2 + \|u_{21}q_1f + u_{11}q_2g\|^2, \end{aligned}$$

for any  $(q_1f, q_2g)$  in  $\mathcal{M}$ . Hence we have  $\|q_2g\| = \|u_{21}q_1f + u_{11}q_2g\|$ . Therefore we have  $2\|q_2g\| \geq \|u_{21}q_1f\|$ . If  $u_{21}$  were not the zero function, then  $\ker T_2$  would be a subspace of  $\ker T_1$ . However  $\ker T_2$  is not the null space and  $\Theta = (\theta_{ij})$  is isometric by the assumption  $\dim \mathcal{D} = 2$ . This is a contradiction. Hence  $u_{21} = 0$ . This concludes the proof.

We shall give two criterions of unitary equivalence of submodules as corollaries of Theorem 4.1. To begin, we note the following basic fact as a lemma:

**Lemma 4.2** *Let  $\mathcal{M}$  and  $\tilde{\mathcal{M}}$  be submodules with  $\dim \mathcal{D} = 2$ . If they are unitarily equivalent as modules by a unitary module map  $W$ , then  $W\Theta$  is the characteristic function of  $\tilde{\mathcal{M}}$ .*

**Proof** Trivially,  $W\Theta$  is an isometry from  $H^2(\mathbb{C}^2)$  onto  $\tilde{\mathcal{M}}$ . We note that  $(W\Theta)(\zeta) = W(\zeta)\Theta(\zeta)$  for  $\zeta$  a.e. in  $\mathbb{T}$ . Further,  $W\Theta$  can be extended analytically to  $\mathbb{D}$ , and it is easy to show that  $((W\Theta)(\zeta))^* ((W\Theta)(\zeta)) = I_{\mathbb{C}^2}$ .

For a submodule  $\tilde{\mathcal{M}}$ , we use the notations  $\tilde{\Theta}$ ,  $\tilde{T}_1$ ,  $\tilde{q}_1$  and  $\tilde{q}_2$  defined by  $\tilde{\mathcal{M}}$ , as in the case of  $\mathcal{M}$ .

**Corollary 4.1** *Suppose that  $\mathcal{M}$  and  $\tilde{\mathcal{M}}$  are two submodules in  $H^2(\mathbb{C}^2)$ . If they are unitarily equivalent as modules, then  $\tilde{q}_1/q_1 = \tilde{q}_2/q_2$ .*

**Proof** By Theorem 4.1 and Lemma 4.2, we have the conclusion.

**Corollary 4.2** *If  $\mathcal{M}$  and  $\tilde{\mathcal{M}}$  are unitarily equivalent as modules, then there exists a unitary matrix  $V$  in  $M_2(\mathbb{C})$  such that  $V \ker T_1 = \ker \tilde{T}_1$ .*

**Proof** Suppose that  $\mathcal{M} = \Theta H^2(\mathbb{C}^2)$  and  $\tilde{\mathcal{M}} = \tilde{\Theta} H^2(\mathbb{C}^2)$  are unitarily equivalent as modules by a unitary module map  $W$ . Then, by Theorem 4.1 and Lemma 4.2, there exist a bounded Borel function  $u$  on  $\mathbb{T}$  and a constant unitary matrix  $V$  such that

$$W\Theta = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \Theta = \tilde{\Theta}V.$$

First, we suppose that  $(f, g)$  is in  $\ker \tilde{T}_1$ . Then

$$\tilde{\Theta} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ * \end{pmatrix}.$$

Since  $W\Theta V^* = \tilde{\Theta}$ , we have  $T_1 V^*(f, g) = 0$ . Hence we have  $\ker \tilde{T}_1 \subseteq V \ker T_1$ .

Conversely, if  $(f, g)$  is in  $V \ker T_1$ , then we have

$$\Theta V^* \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ * \end{pmatrix}.$$

Since  $W\Theta V^* = \tilde{\Theta}$ , we have  $\tilde{T}_1(f, g) = 0$ . Hence we have  $V \ker T_1 \subseteq \ker \tilde{T}_1$ . This concludes the proof.

## 5 Example: the submodule generated by $(z, -1)$

In this section, we deal with an example. Let  $\mathcal{M}$  be the submodule generated by  $(z, -1)$ . Then, by some calculations, we have  $\mathcal{D} = \vee\{(1/\sqrt{2})(z, -1), (1/\sqrt{2})(z^2, z)\}$ . Hence we have the following matrix representation of the characteristic function  $\Theta$  of  $\mathcal{M}$  with respect to the canonical orthonormal basis  $(1, 0)$  and  $(0, 1)$  of  $\mathbb{C}^2$ :

$$\Theta = \frac{1}{\sqrt{2}} \begin{pmatrix} T_z & T_z^2 \\ -1 & T_z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} T_z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & T_z \\ -1 & T_z \end{pmatrix}.$$

By Corollaries 4.1 and 4.2, we have that the following three submodules

$$\frac{1}{\sqrt{2}} \begin{pmatrix} T_z & T_z^2 \\ -1 & T_z \end{pmatrix} H^2(\mathbb{C}^2), \begin{pmatrix} T_z & 0 \\ 0 & 1 \end{pmatrix} H^2(\mathbb{C}^2) \text{ and } H^2(\mathbb{C}^2)$$

are mutually different in the sense of unitary equivalence, that is, they are not unitarily equivalent as modules.

However there exists a module map, which is not unitary, from  $H^2(\mathbb{C}^2)$  onto  $\mathcal{M}$ . Indeed, closed subspaces  $\{(0, zf) : f \in H^2\}$  and  $\{(zf, -f) : f \in H^2\}$  are contained in  $\mathcal{M}$ . Hence we have

$$\mathcal{M} = \{(zf, -f + zg) : f, g \in H^2\} = \Delta H^2(\mathbb{C}^2),$$

where  $\Delta$  is the following operator matrix:

$$\Delta = \begin{pmatrix} T_z & 0 \\ -1 & T_z \end{pmatrix}.$$

We note that  $\Delta$  is a module map from  $H^2(\mathbb{C}^2)$  onto  $\mathcal{M}$ , that is,  $\Delta$  commutes with  $T_z$  and  $S$  on  $H^2(\mathbb{C}^2)$ . Moreover  $\Delta$  has the left inverse as follows:

$$\begin{pmatrix} T_z^* & 0 \\ T_z^{*2} & T_z^* \end{pmatrix} \begin{pmatrix} T_z & 0 \\ -1 & T_z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

## 6 Submodules of finite codimension

In this section, we study submodules of finite codimension in  $H^2(\mathbb{C}^2)$ .

**Theorem 6.1** *Let  $\mathcal{M}$  be a submodule in  $H^2(\mathbb{C}^2)$ . Then the following three assertions are equivalent:*

- (i)  $\mathcal{M}$  is generated by some non-zero vectors whose entries are polynomials,
- (ii)  $\mathcal{M}$  is of finite codimension in  $H^2(\mathbb{C}^2)$ ,

(iii) *there exists a finite Blaschke product  $q$  such that  $\mathcal{M}$  contains  $qH^2(\mathbb{C}^2)$ .*

**Proof** First, we show (i)  $\Rightarrow$  (ii). It suffices to assume that  $\mathcal{M}$  is generated by a vector  $(p_1, p_2)$  where  $p_1$  and  $p_2$  are polynomials. Let  $\mathcal{N}$  denote the orthogonal complement of  $\mathcal{M}$  in  $H^2(\mathbb{C}^2)$ . If a vector  $(x, y)$  is in  $\mathcal{N}$ , then the following equations hold for any  $h$  in  $H^2$ :

$$\begin{aligned} \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} p_1 h \\ p_2 h \end{pmatrix} \right\rangle &= \langle x, p_1 h \rangle + \langle y, p_2 h \rangle = 0 \\ \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ p_1 h \end{pmatrix} \right\rangle &= \langle y, p_1 h \rangle = 0. \end{aligned}$$

Hence we have that  $T_{p_1}^* x + T_{p_2}^* y = 0$  and  $y$  is orthogonal to  $p_1 H^2$  in  $H^2$ . Let  $(p_1 H^2)^\perp$  denote the orthogonal complement of  $p_1 H^2$  in  $H^2$ . Then  $x$  belongs to the vector space

$$\mathcal{X} := (T_{p_1}^*)^{-1} (\{-T_{p_2}^* y : y \in (p_1 H^2)^\perp\}).$$

Hence we have that  $\mathcal{N}$  is contained in  $\mathcal{X} \times (p_1 H^2)^\perp$ . Since  $p_1$  is a polynomial and  $\mathcal{X}$  is orthogonal to  $p_1^2 H^2$ , we have that the dimension of  $\mathcal{X} \times (p_1 H^2)^\perp$  is finite.

Next, we show (ii)  $\Rightarrow$  (iii). For any bounded analytic function  $\varphi$ ,  $S_\varphi$  will denote the operator defined as follows:  $S_\varphi f = P_{\mathcal{N}} \varphi f$  for any  $f$  in  $\mathcal{N}$ . Since  $\mathcal{N}$  is of finite dimension, there exists a polynomial  $p$  such that  $S_p = p(S_z) = 0$ . Let  $q$  be the inner factor of  $p$ . Then  $q$  is a finite Blaschke product and  $S_q = 0$ . Hence  $q\mathcal{N}$  is a subspace of  $\mathcal{M}$ . Further,  $\mathcal{M}$  is invariant under  $T_q$ . Therefore  $qH^2(\mathbb{C}^2)$  is contained in  $\mathcal{M}$ .

Lastly, we show (iii)  $\Rightarrow$  (i). Let  $q$  be a finite Blaschke product. Since  $\mathcal{M}$  is invariant under multiplications by bounded analytic functions,  $qH^2(\mathbb{C}^2)$  is generated by some vectors whose entries are polynomials. Hence, it suffices to show that entries of every element in  $H^2(\mathbb{C}^2) \ominus qH^2(\mathbb{C}^2)$  are rational functions whose poles are not in  $\mathbb{D}$ . Let  $(x, y)$  be in  $H^2(\mathbb{C}^2) \ominus qH^2(\mathbb{C}^2)$ . Then the following equations hold for any  $f$  and  $g$  in  $H^2$ :

$$\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} qf \\ qg \end{pmatrix} \right\rangle = \langle x, qf \rangle + \langle y, qg \rangle = 0.$$

Hence  $x$  and  $y$  are orthogonal to  $qH^2$ , that is,  $x$  and  $y$  are rational functions whose poles are not in  $\mathbb{D}$ .

**Remark 6.1** Theorem 6.1 should be compared with the following facts in Hardy spaces:

- In the classical Hardy space, a submodule is generated by polynomials if and only if it is of finite codimension. This is known as Kronecker's theorem.
- In the Hardy space over the bidisk, there exist submodules which are generated by polynomials and of infinite codimension (cf. [2]).

## 7 $C^*$ -algebras on submodules

Let  $\mathfrak{T}$  denote the Toeplitz algebra. Then the  $C^*$ -algebra generated by  $T_z$  and  $S$  is the tensor product  $C^*$ -algebra of  $\mathfrak{T}$  and  $M_2(\mathbb{C})$ .  $\mathfrak{A}(\mathcal{M})$  will denote the  $C^*$ -algebra generated by  $V_z$  and  $R$  in the algebra of all bounded linear operators on  $\mathcal{M}$ .

**Definition 7.1** *For two submodules  $\mathcal{M}_1$  and  $\mathcal{M}_2$  in  $H^2(\mathbb{C}^2)$ , two  $C^*$ -algebras  $\mathfrak{A}(\mathcal{M}_1)$  and  $\mathfrak{A}(\mathcal{M}_2)$  are said to be unitarily equivalent if there exists a unitary operator  $U$  from  $\mathcal{M}_1$  onto  $\mathcal{M}_2$  such that  $U\mathfrak{A}(\mathcal{M}_1)U^* = \mathfrak{A}(\mathcal{M}_2)$ .*

Trivially, if two submodules are unitarily equivalent as modules, then their  $C^*$ -algebras are unitarily equivalent. Furthermore, it is not difficult to see that  $C^*$ -algebras on submodules given in Section 5 are unitarily equivalent to  $\mathfrak{A}(H^2(\mathbb{C}^2))$ , so it will be natural that one asks the following question:

**Question** For any submodule  $\mathcal{M}$  with  $\dim \mathcal{D} = 2$ , is  $\mathfrak{A}(\mathcal{M})$  unitarily equivalent to  $\mathfrak{A}(H^2(\mathbb{C}^2))$  ?

We will give a negative answer to this question, that is, we will see that there exist submodules whose  $C^*$ -algebras are not unitarily equivalent to  $\mathfrak{A}(H^2(\mathbb{C}^2))$ . In order to deal with this problem, we need several lemmas.

**Lemma 7.1** *If a submodule  $\mathcal{M}$  is an orthogonal direct sum of two submodules  $\mathcal{M}_1$  and  $\mathcal{M}_2$  in  $H^2(\mathbb{C}^2)$ , then  $\mathcal{M}_1 = \{0\}$  or  $\mathcal{M}_2 = \{0\}$ .*

**Proof** Let  $f_j = (f_{1j}, f_{2j})$  be an element in  $\mathcal{M}_j$  for  $j = 1, 2$ . Then, for any non-negative integers  $k$  and  $l$ , we have

$$\int_{\mathbb{T}} (f_{11}\overline{f_{12}} + f_{21}\overline{f_{22}})\zeta^{k-l} |d\zeta| = \langle T_z^k f_1, T_z^l f_2 \rangle = 0,$$

$$\int_{\mathbb{T}} f_{11}\overline{f_{22}}\zeta^{k-l} |d\zeta| = \langle T_z^k S f_1, T_z^l f_2 \rangle = 0$$

and

$$\int_{\mathbb{T}} f_{21}\overline{f_{12}}\zeta^{k-l} |d\zeta| = \langle T_z^k f_1, T_z^l S f_2 \rangle = 0.$$

It follows that  $f_{11}\overline{f_{12}} + f_{21}\overline{f_{22}} = 0$ ,  $f_{11}\overline{f_{22}} = 0$  and  $f_{21}\overline{f_{12}} = 0$  on  $\mathbb{T}$ . Since every  $f_{ij}$  belongs to  $H^2$ ,  $f_{ij}(\zeta)$  does not take zero a.e. on  $\mathbb{T}$ . Hence we have  $f_1 = 0$  or  $f_2 = 0$ . This concludes the proof.

**Lemma 7.2** *Let  $\mathcal{M}$  be a submodule in  $H^2(\mathbb{C}^2)$ . Then  $\mathfrak{A}(\mathcal{M})$  contains the ideal of all compact operators on  $\mathcal{M}$ .*

**Proof** Since  $\mathfrak{A}(\mathcal{M})$  is irreducible by Lemma 7.1 and contains a non-zero finite rank operator  $[V_z^*, V_z]$ , we have the conclusion.

Let  $C(X)$  be the  $C^*$ -algebra of all continuous functions defined on a compact subset  $X$  of  $\mathbb{C}$ , let  $M_2(C(\mathbb{T}))$  denote the two-by-two matrix algebra whose entries are functions in  $C(\mathbb{T})$ , and  $\mathfrak{K}$  be the ideal of all compact operators. For an inner function  $q$ ,  $X_q$  denotes the set of all  $\zeta$  in  $\mathbb{T}$  such that  $q$  can not be extended analytically from  $\mathbb{D}$  to  $\zeta$ .

**Lemma 7.3** *Let  $q$  be an inner function. If  $\mathcal{M} = qH^2 \oplus H^2$ , then  $\mathfrak{A}(\mathcal{M})/\mathfrak{K}$  is isomorphic to  $M_2(C(\mathbb{T})) \oplus C(X_q)$ , where we set  $C(X_q) = \{0\}$  if  $X_q$  is the empty set.*

**Proof** Since  $\mathcal{M} = qH^2 \oplus H^2 = qH^2 \oplus qH^2 \oplus K(q)$  where we set  $K(q) = H^2 \ominus qH^2$ ,  $V_z$  and  $R$  have the following matrix representation:

$$V_z = \begin{pmatrix} V_z & 0 & 0 \\ 0 & V_z & A \\ 0 & 0 & S_z \end{pmatrix} \text{ and } R = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ on } qH^2 \oplus qH^2 \oplus K(q).$$

Then it is easy to show that the above operator  $A$  has rank one. Therefore we have

$$V_z \equiv \begin{pmatrix} V_z & 0 & 0 \\ 0 & V_z & 0 \\ 0 & 0 & S_z \end{pmatrix} \pmod{\mathfrak{K}}.$$

Further,  $C^*(S_z)/\mathfrak{K}$  is isomorphic to  $C(X_q)$  by Arveson's theorem in [4], where  $C^*(S_z)$  is the  $C^*$ -algebra generated by  $S_z$  and the identity operator on  $K(q)$ . Hence  $\mathfrak{A}(\mathcal{M})/\mathfrak{K}$  is isomorphic to  $M_2(C(\mathbb{T})) \oplus C(X_q)$ . This completes the proof.

**Lemma 7.4** *Set  $Q_j = M_2(C(\mathbb{T})) \oplus C(X_j)$  for  $j = 1, 2$ . Then  $Q_1$  and  $Q_2$  are isomorphic if and only if  $X_1$  and  $X_2$  are homeomorphic.*

**Proof** The “if” part is trivial. We show the “only if” part. Assume that there exists a  $*$ -isomorphism  $\varphi$  from  $Q_1$  onto  $Q_2$ . It suffices to show that the restriction of  $\varphi$  on  $\{0\} \oplus C(X_1)$  induces a  $*$ -isomorphism from  $C(X_1)$  onto  $C(X_2)$ .

First, we show that  $\varphi(M_2(C(\mathbb{T})) \oplus \{0\}) = M_2(C(\mathbb{T})) \oplus \{0\}$ . Let  $e_{ij}^k$  denote the matrix unit whose non-zero entry is  $z^k$  for  $1 \leq i, j \leq 2$  and  $k \in \mathbb{Z}$ . Then, for  $i \neq j$ ,  $\varphi(e_{ij}^k \oplus 0)$  is in  $M_2(C(\mathbb{T})) \oplus \{0\}$ , because  $e_{ij}^k \oplus 0$  is nilpotent. Since  $\{e_{ij}^k : 1 \leq i, j \leq 2, i \neq j, k \in \mathbb{Z}\}$  generates  $M_2(C(\mathbb{T})) \oplus \{0\}$  as  $C^*$ -algebras, we have  $\varphi(M_2(C(\mathbb{T})) \oplus \{0\}) \subseteq M_2(C(\mathbb{T})) \oplus \{0\}$ . Similarly we have that  $\varphi^{-1}(M_2(C(\mathbb{T})) \oplus \{0\}) \subseteq M_2(C(\mathbb{T})) \oplus \{0\}$ . Therefore we have that

$$\varphi(M_2(C(\mathbb{T})) \oplus \{0\}) = M_2(C(\mathbb{T})) \oplus \{0\}. \quad (7.1)$$

Next we show that  $\varphi(\{0\} \oplus C(X_1)) = \{0\} \oplus C(X_2)$ . For any  $f$  in  $C(X_1)$ , we set  $\varphi(0 \oplus f) = A \oplus g \in M_2(C(\mathbb{T})) \oplus C(X_2)$ . Then  $(e_{ij} \oplus 0)(A \oplus g)(e_{kl} \oplus 0) = (e_{ij} \oplus 0)\varphi(0 \oplus f)(e_{kl} \oplus 0) = 0$  by (7.1). Therefore we have that  $\varphi(\{0\} \oplus C(X_1)) \subseteq \{0\} \oplus C(X_2)$ . Similarly, we have that  $\varphi^{-1}(\{0\} \oplus C(X_2)) \subseteq \{0\} \oplus C(X_1)$ . This completes the proof.

Now we are in a position to prove the main theorem of this section.

**Theorem 7.1** *Set  $\mathcal{M}_1 = q_{11}H^2 \oplus q_{12}H^2$  and  $\mathcal{M}_2 = q_{21}H^2 \oplus q_{22}H^2$  where  $q_{j1}$  and  $q_{j2}$  are inner functions such that  $q_{j1}$  is divisible by  $q_{j2}$  for each  $j = 1, 2$ . Then two  $C^*$ -algebras  $\mathfrak{A}(\mathcal{M}_1)$  and  $\mathfrak{A}(\mathcal{M}_2)$  are unitarily equivalent if and only if  $X_{q_{11}/q_{12}}$  and  $X_{q_{21}/q_{22}}$  are homeomorphic.*

**Proof** Without loss of generality, we may assume that  $q_{12}$  and  $q_{22}$  are constant, and we set  $q_1 = q_{11}$  and  $q_2 = q_{21}$ , for short. Then the “only if” part is immediate from Lemmas 7.3 and 7.4. We show the “if” part, that is, we assume that  $X_{q_1}$  and  $X_{q_2}$  are homeomorphic. First, we mention the fact that  $S_z$  which appeared in the proof of Lemma 7.3 is the sum of a normal operator and a compact operator, which was first proved by Ahern and Clark in [3]. Hence, without loss of generality, we may assume that  $X_{q_1} = X_{q_2}$  by the standard spectral theory. Further, using the BDF theory (cf. [6] and [8]), it follows that there exists a unitary operator  $U_0$  from  $K(q_1)$  onto  $K(q_2)$  such that  $U_0 S_z^{(1)} U_0^* \equiv S_z^{(2)} \pmod{\mathfrak{K}}$ , where  $S_z^{(j)}$  denotes  $S_z$  acting on  $K(q_j)$ . Setting

$$U = \begin{pmatrix} T_{q_2} T_{q_1}^* & 0 & 0 \\ 0 & T_{q_2} T_{q_1}^* & 0 \\ 0 & 0 & U_0 \end{pmatrix},$$

then  $U$  is a unitary operator from  $\mathcal{M}_1$  onto  $\mathcal{M}_2$ , and it is easy to check that the following:

$$U \begin{pmatrix} V_z^{(1)} & 0 & 0 \\ 0 & V_z^{(1)} & 0 \\ 0 & 0 & S_z^{(1)} \end{pmatrix} U^* \equiv \begin{pmatrix} V_z^{(2)} & 0 & 0 \\ 0 & V_z^{(2)} & 0 \\ 0 & 0 & S_z^{(2)} \end{pmatrix} \pmod{\mathfrak{K}}$$

and

$$U \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^* \equiv \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \pmod{\mathfrak{K}}.$$

Therefore we have that  $U\mathfrak{A}(\mathcal{M}_1)U^* = \mathfrak{A}(\mathcal{M}_2)$  by Lemma 7.2. This completes the proof.

**Corollary 7.1** *There exist submodules whose  $C^*$ -algebras are not unitarily equivalent to  $\mathfrak{A}(H^2(\mathbb{C}^2))$ .*

**Remark 7.1** With a little more effort, one can prove that the two zero sets of quotients of inner functions are actually equal if and only if the  $C^*$ -algebras are isomorphic using a map which extends the identification of module multiplication.

**Remark 7.2** The proof of Theorem 7.1 can be applied to general cases in  $n \geq 3$ . Then the corresponding statement is as follows: Set  $\mathcal{M}_1 = q_{11}H^2 \oplus \cdots \oplus q_{1n}H^2$  and  $\mathcal{M}_2 = q_{21}H^2 \oplus \cdots \oplus q_{2n}H^2$  where  $q_{ij}$ 's are inner functions such that  $q_{ij}$  is divisible by  $q_{i,j+1}$  for each  $i = 1, 2$  and  $j = 1, 2, \dots, n-1$ . Then two  $C^*$ -algebras  $\mathfrak{A}(\mathcal{M}_1)$  and  $\mathfrak{A}(\mathcal{M}_2)$  are unitarily equivalent if and only if  $X_{q_{1j}/q_{1,j+1}}$  and  $X_{q_{2j}/q_{2,j+1}}$  are homeomorphic for every  $j = 1, 2, \dots, n-1$ .

**Acknowledgment** The authors would like to express gratitude to the referee for his comments and suggestions.

## References

- [1] O. P. Agrawal, D. N. Clark and R. G. Douglas, *Invariant subspaces in the polydisk*, Pacific J. Math. **121**, (1986), pp. 1–11.
- [2] P. R. Ahern and D. N. Clark, *Invariant subspaces and analytic continuation in several variables*, J. Math. Mech. **19**, (1969/70), pp. 963–969.
- [3] P. R. Ahern and D. N. Clark, *On functions orthogonal to invariant subspaces*, Acta Math. **124**, (1970), pp. 191–204.
- [4] W. B. Arveson, *Subalgebras of  $C^*$ -algebras*, Acta Math. **123**, (1969), pp. 141–224.
- [5] A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math., **81**, (1949), pp. 239–255.
- [6] L. G. Brown, R. G. Douglas and P. A. Fillmore, *Extensions of  $C^*$ -algebras and  $K$ -homology*, Ann. Math. **105**, (1977), pp. 265–324.
- [7] X. Chen and K. Guo, *Analytic Hilbert Modules*, Research Notes in Mathematics **433**, CHAPMAN and HALL/CRC, 2003.
- [8] K. R. Davidson,  *$C^*$ -algebras by example*, Fields Institute Monographs, 6. American Mathematical Society, Providence, RI, 1996.
- [9] V. Mandrekar, *The validity of Beurling theorems in polydiscs*, Proc. Amer. Math. Soc. **103**, (1988), pp. 145–148.



- [10] T. Nakazi, *Certain invariant subspaces of  $H^2$  and  $L^2$  on a bidisc*, Canad. J. Math. **40**, (1988), pp. 1272–1280.
- [11] B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space*, North-Holland, Amsterdam, 1970.

Ronald G. Douglas  
Texas A&M University  
College Station  
Texas 77843-3368  
e-mail: rgd@tamu.edu

Takahiko Nakazi  
Department of Mathematics  
Faculty of Science  
Hokkaido University  
Sapporo 060-0810, Japan  
e-mail: nakazi@math.sci.hokudai.ac.jp

Michio Seto  
Department of Mathematics  
Kanagawa University  
Yokohama 221-8686, Japan  
e-mail: seto@kanagawa-u.ac.jp