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# Wavelet bases in the weighted Besov and Triebel-Lizorkin spaces with $A_p^{\text{loc}}$ -weights

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## Abstract

The aim of this paper is to obtain the wavelet expansion in the Besov spaces and the Triebel-Lizorkin spaces coming with  $A_p^{\text{loc}}$ -weights. After characterizing these spaces in terms of wavelet, we shall obtain unconditional bases and greedy bases.

**Keywords** Besov space, Triebel-Lizorkin space, wavelet, unconditional basis, greedy basis.

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## 1 Introduction

In this paper we find a basis on  $A_{p,q}^{s,w}$ , the weighted Besov and Triebel-Lizorkin

space, where  $w$  is an  $A_p$ -local weight defined by Rychkov [21]. As is well-known, the class  $A_p$  was defined [19] in connection with the Hardy-Littlewood maximal operator given by

$$Mf(x) := \sup_{x \in Q} m_Q(|f|) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy, f \in L_1^{\text{loc}}$$

where  $Q$  runs over all cubes whose edges are parallel to the coordinate axis and  $m_Q(g)$  denotes the average of  $g$  over a cube  $Q$ . Below we mean a compact cube whose edges are parallel to the coordinate axis simply by “cube”. We also mean by “weight” a non-negative measurable function which is locally integrable on  $\mathbb{R}^n$  and does not vanish  $dx$ -almost everywhere in  $\mathbb{R}^n$ . A weight  $w$  is said to be an  $A_1$ -weight, if there exists a constant  $c > 0$  such that  $Mw(x) \leq cw(x)$  for  $dx$ -almost everywhere  $x \in \mathbb{R}^n$ . Let  $p > 1$ . A weight  $w$  is said to be an  $A_p$ -weight if

$$\sup_Q m_Q(w) \cdot m_Q(w^{-\frac{1}{p-1}})^{p-1} < \infty,$$

where  $Q$  runs over all cubes. We refer to [8] for more information of this class of weights.

The present paper deals with its local version. To formulate the local version of  $A_p$ -class, first we introduce the following maximal operator due to Rychkov [21].

$$M_{\text{loc}}f(x) := \sup_{\substack{x \in Q \\ \ell(Q) \leq 1}} m_Q(|f|).$$

Here  $Q$  runs over all cubes whose edges are parallel to the coordinate axis and which have sidelength less than 1. Motivated by the definition due to Muckenhoupt, Rychkov defined the class of the weights as follows:

$$A_p^{\text{loc}} := \{w : w \text{ is a weight with } A_p^{\text{loc}}(w) < \infty\}, 1 \leq p < \infty,$$

where

$$A_1^{\text{loc}}(w) := \text{esssup}_{x \in \mathbb{R}^n} \frac{M_{\text{loc}}w(x)}{w(x)}$$

$$A_p^{\text{loc}}(w) := \sup_{\ell(Q) \leq 1} m_Q(w) \cdot m_Q(w^{-\frac{1}{p-1}})^{p-1}, p > 1.$$

We also define  $A_\infty^{\text{loc}} := \bigcup_{1 \leq p < \infty} A_p^{\text{loc}}$  as a set. This class of weights enjoys properties analogous to  $A_p$  such as the openness property and the (local) reverse Hölder inequality.

In [21] Rychkov defined the weighted Besov-spaces  $B_{p,q}^{s,w}$  and the weighted Triebel-Lizorkin spaces  $F_{p,q}^{s,w}$ . We remark that for some smaller class of weights these function spaces are investigated in [22].

The first and second authors investigated the atomic decomposition of this space [11]. For the unweighted case we refer to the pioneering works [5, 25].

The aim of the present paper is to characterize these weighted function spaces in terms of wavelets. We always place ourselves in the setting of function spaces coming with an  $A_\infty^{\text{loc}}$ -weight  $w$ . This type of approach can be found in the textbook [26] when  $w \equiv 1$ . In [26] we can find more about wavelet characterization for the unweighted case. In particular [5, 6, 20, 23, 25] are pioneering works of this attempt with  $w \equiv 1$ . If  $w \in A_p$ , then Deng, Xu and Yan obtained some characterization of 1-dimensional dotted function spaces  $\dot{F}_{p,q}^s(\mathbb{R}, w)$  with  $|s|$  restricted to very small values [4]. A minor modification shows that our characterization is applicable to the dotted function spaces  $\dot{A}_{p,q}^s(\mathbb{R}^n, w)$  with  $w \in A_p$  even if we do not assume that  $|s|$  is small.

Finally we describe the organization of this paper. In Section 2 we recall the definition of the function spaces  $B_{p,q}^{s,w}$  and  $F_{p,q}^{s,w}$  with  $w \in A_\infty^{\text{loc}}$ . As well as we define the function spaces, we make a brief review of the maximal operator, the notion of bases in Banach spaces. In Section 3 we give an auxiliary result which is interesting of its own right, where we identify the dual space of the function spaces. This result will serve to define  $\langle f, \tau \rangle$ , where  $f$  is a function in the weighted Besov spaces or the weighted Triebel-Lizorkin spaces and  $\tau$  is a wavelet or a scaling function. Section 4 devotes to the wavelet characterizations. Finally in Section 5 we obtain a greedy basis, which enjoys a very nice property. Greedy bases, which was introduced in [14], begin to be studied recently. For example Garrigós, Hernández and Martell investigated greedy bases in Orlicz spaces in [7]. In [9, 10, 12] we investigated ones in Lebesgue, weighted Sobolev spaces and Herz spaces. The present paper will deal with those in weighted Besov and Triebel-Lizorkin spaces.

## 2 Preliminaries

### Function spaces $A_{p,q}^{s,w}$ with $w \in A_p^{\text{loc}}$

Let us describe precisely the function spaces  $A_{p,q}^{s,w}$ . Throughout this paper, unless additionally stated, we assume that  $w$  is an  $A_\infty^{\text{loc}}$ -weight.

We keep to the following notations.

1. We set  $\mathbb{N}_0 = \{0, 1, \dots\}$  and  $\mathbb{N} = \{1, 2, \dots\}$ .
2.  $B(R) := \{x \in \mathbb{R}^n : |x| < R\}$ .
3. Let  $f$  be a measurable function. Then define

$$\|f : L_p^w\| := \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{\frac{1}{p}}$$

for  $1 < p < \infty$ .

4. Let  $1 < p < \infty$  and  $1 \leq q \leq \infty$ . Given a sequence of measurable functions

$\{f_j\}_{j \in \mathbb{N}_0}$ , we define

$$\begin{aligned} \|\{f_j\}_{j \in \mathbb{N}_0} : l_q(L_p^w)\| &:= \left( \sum_{j \in \mathbb{N}_0} \|f_j : L_p^w\|^q \right)^{\frac{1}{q}} \\ \|\{f_j\}_{j \in \mathbb{N}_0} : L_p^w(l_q)\| &:= \left\| \left( \sum_{j \in \mathbb{N}_0} |f_j|^q \right)^{\frac{1}{q}} : L_p^w \right\|. \end{aligned}$$

5.  $\mathcal{S}_e$  denotes the set of all smooth functions  $\phi$  satisfying

$$q_N(\phi) := \sup_{x \in \mathbb{R}^n} \sum_{|\alpha| \leq N} e^{N|x|} |\partial^\alpha \phi(x)| < \infty$$

for all  $N \in \mathbb{N}_0$ . Topologize  $\mathcal{S}_e$  with  $\{q_N\}_{N \in \mathbb{N}_0}$ .  $\mathcal{S}'_e$  denotes the topological dual of  $\mathcal{S}_e$ .

**Definition 2.1.** Let  $L \in \mathbb{N}_0 \cup \{-1\}$  and  $s \in \mathbb{R}$ . Take a sequence  $\{\phi_j\}_{j \in \mathbb{N}_0}$  of  $C_c^\infty(\mathbb{R}^n)$ -functions satisfying the following conditions.

1.  $L \geq [s]$ .
2.  $\text{supp}(\phi_0) \subset B(1)$ .
3.  $\phi_1(x) = \phi_0(x) - 2^n \phi_0(2x)$ .
4.  $\phi_j(x) = 2^{(j-1)n} \phi_1(2^{j-1}x)$  for  $j \in \mathbb{N}$ .
5.  $\int_{\mathbb{R}^n} x^\beta \phi_1(x) dx = 0$  for all  $\beta \in \mathbb{N}_0^n$  with  $|\beta| \leq L$ . If  $L = -1$ , then this condition means no condition.
6.  $\int_{\mathbb{R}^n} \phi_0(x) dx \neq 0$ .

This condition for  $\phi_0$  is referred to as  $M_s$ -condition in [21].

Using the sequence  $\{\phi_j\}_{j \in \mathbb{N}_0}$  above, Rychkov defined the function space  $A_{p,q}^{s,w}$  as follows:

**Definition 2.2.** Suppose that the parameters  $p, q, s$  satisfy

$$1 \leq p < \infty, 1 \leq q \leq \infty, s \in \mathbb{R}.$$

Assume that  $L$  appearing in Definition 2.1 is greater than  $\max(-1, [s])$ . Then define

$$\begin{aligned} \|f : B_{p,q}^{s,w}\| &:= \|\{2^{js} \phi_j * f\}_{j \in \mathbb{N}_0} : l_q(L_p^w)\| \\ \|f : F_{p,q}^{s,w}\| &:= \|\{2^{js} \phi_j * f\}_{j \in \mathbb{N}_0} : L_p^w(l_q)\| \end{aligned}$$

for  $f \in \mathcal{S}'_e$ . To unify the notation below  $A_{p,q}^{s,w}$  is used to denote either  $B_{p,q}^{s,w}$  or  $F_{p,q}^{s,w}$ .

Rychkov actually defined  $A_{p,q}^{s,w}$  with  $0 < p < \infty$ ,  $0 < q \leq \infty$ ,  $s \in \mathbb{R}$  and  $w \in A_\infty^{\text{loc}}$ . However, in the present paper we take up  $A_{p,q}^{s,w}$  with the parameter indicated in the definition above.

Rychkov proved the following theorem for  $w \in A_\infty^{\text{loc}}$ .

**Theorem 2.3.** [21] *Suppose that the parameters  $p, q, s$  satisfy*

$$1 \leq p < \infty, 1 \leq q \leq \infty, s \in \mathbb{R}$$

and  $w \in A_\infty^{\text{loc}}$ .

1. *The sequence of  $\{\phi_j\}_{j \in \mathbb{N}_0}$  in Definition 2.1 does exist.*
2. *Different admissible choice of  $\{\phi_j\}_{j \in \mathbb{N}_0}$  in Definition 2.1 will yield equivalent norms, provided  $\phi_0$  satisfies the  $M_s$ -condition.*

As for the weighted function spaces, we remark  $h_p^w$  was investigated by Bui [1], which was equivalent to  $F_{p,2}^{0,w}$  [21].

We also remark that the following atomic decomposition for these function spaces is obtained. To formulate the result, we need some notations. Let

$$Q_{\nu,m} = \prod_{j=1}^n \left[ \frac{m_j}{2^\nu}, \frac{m_j + 1}{2^\nu} \right].$$

Denote by  $\chi_{\nu,m}^{(p)}$  the  $p$ -normalized indicator of this cube:  $\chi_{\nu,m}^{(p)}(x) := 2^{\frac{n\nu}{p}} \chi_{Q_{\nu,m}}(x)$ . Given a doubly indexed sequence  $\lambda = \{\lambda_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ , define

$$\begin{aligned} \|\lambda : b_{p,q}^w\| &:= \left\| \left\{ \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{\nu,m}^{(p)} \right\}_{\nu \in \mathbb{N}_0} : l_q(L_p^w) \right\| \\ \|\lambda : f_{p,q}^w\| &:= \left\| \left\{ \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{\nu,m}^{(p)} \right\}_{\nu \in \mathbb{N}_0} : L_p^w(l_q) \right\|. \end{aligned}$$

$a_{p,q}^w$  denotes either  $b_{p,q}^w$  or  $f_{p,q}^w$ .

**Proposition 2.4.** [11] *Suppose that the parameters  $p, q, s$  and an integer  $L$  satisfy*

$$1 \leq p < \infty, 1 \leq q \leq \infty, s \in \mathbb{R}, L \geq -1.$$

*Assume in addition that  $w \in A_\infty^{\text{loc}}$ . Then there exist constants  $c_0, c_1, c_2 > 0$  and  $c_\alpha, \alpha \in \mathbb{N}_0^n$  with the following property.*

1. *Let  $f \in A_{p,q}^{s,w}$ . Then  $f$  admits the following decomposition.*

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m},$$

where the coefficient  $\lambda = \{\lambda_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$  and  $\{a_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$  satisfy conditions (1)–(4) given below.

$$\|\lambda : a_{p,q}^w\| \leq c_0 \|f : A_{p,q}^{s,w}\| \quad (1)$$

$$\text{supp}(a_{\nu m}) \subset c_1 Q_{\nu,m} \quad (2)$$

$$a_{\nu m} \in C^\infty, \|\partial^\alpha a_{\nu m} : L_\infty\| \leq c_\alpha 2^{-\nu(s-\frac{n}{p})+\nu|\alpha|} \text{ for all } \alpha \in \mathbb{N}_0^n \quad (3)$$

$$\int_{\mathbb{R}^n} x^\beta a_{\nu m}(x) dx = 0 \text{ for all } \nu \in \mathbb{N}, m \in \mathbb{Z}^n, \beta \in \mathbb{N}_0^n \text{ with } |\beta| \leq L. \quad (4)$$

If  $L = -1$ , then (4) means no condition.

2. Assume that the coefficient  $\lambda = \{\lambda_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$  and  $\{a_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$  satisfy conditions (2), (4),

$$\|\lambda : a_{p,q}^{s,w}\| < \infty \quad (5)$$

and

$$a_{\nu m} \in C^{(1+[s])_+}, \|\partial^\alpha a_{\nu m} : L_\infty\| \leq c_\alpha 2^{-\nu(s-\frac{n}{p})+\nu|\alpha|}. \quad (6)$$

for all  $\alpha \in \mathbb{N}_0$  with  $|\alpha| \leq (1+[s])_+$ . Then

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}$$

converges in  $\mathcal{S}'_e$  and satisfies the norm estimate

$$\|f : A_{p,q}^{s,w}\| \leq c_2 \|\lambda : a_{p,q}^w\|.$$

**Local maximal operators** Now we collect some auxiliary results concerning the local maximal operator, which is also due to Rychkov [21].

**Definition 2.5.** Let  $r > 0$  and  $\eta > 0$ . Then define

$$M_{\text{loc}}^{(\eta)} f(x) := \sup_{\substack{x \in Q \\ \ell(Q) \leq r}} m_Q(|f|^\eta)^{\frac{1}{\eta}}.$$

Define also  $M_{\text{loc}} f(x) := M_{\text{loc}}^{(1)} f(x)$ .

It can happen that the precise values of  $r$  appearing implicitly in  $M_{\text{loc}}^{(\eta)} f(x)$  are different in every occurrence.

In this paper we use the following boundedness of this maximal operator.

**Proposition 2.6.** Suppose that  $1 < p < \infty$  and  $1 < q \leq \infty$ . Assume that  $w \in A_p^{\text{loc}}$ . Then there exists a constant  $c > 0$  such that

$$\|\{M_{\text{loc}} f_j\}_{j \in \mathbb{N}_0} : L_p(l_q)\| \leq c \|\{f_j\}_{j \in \mathbb{N}_0} : L_p(l_q)\|$$

for all sequence of measurable functions  $\{f_j\}_{j \in \mathbb{N}_0}$ .

## Unconditional bases

Finally to formulate our results, let us recall the definition of unconditional bases. Let  $X$  be a Banach space and  $A$  a countable index set in this section.

We recall the definition of several kinds of bases. The first one is unconditional basis. It is known that there are several equivalent definitions of unconditional basis in Banach spaces [13, 17].

**Definition 2.7.** Let  $\{x_m\}_{m \in A}$  be a sequence of elements in  $X$ . The series  $\sum_{m \in A} x_m$  is said to converge unconditionally in  $X$ , if  $\sum_{i=1}^{\infty} x_{\sigma(i)}$  converges for every bijection from  $\mathbb{N}$  to  $A$ .

Next we introduce several kinds of bases, some of which are defined by Konyagin and Temlyakov [14]. Now we follow closely to [14].

**Definition 2.8.**  $\{x_k\}_{k=1}^{\infty} \subset X$  is said to be a Schauder basis if there exists a unique sequence  $\{c_k(x)\}_{k=1}^{\infty} \subset \mathbb{C}$  such that  $x = \sum_{k=1}^{\infty} c_k(x)x_k$  in  $X$  for all  $x \in X$ . Furthermore, if the convergence above is always unconditional, then the basis is said to be unconditional.

Now we recall the definition of greedy basis and democratic basis.

**Definition 2.9.** Let  $\{x_k\}_{k=1}^{\infty}$  be a Schauder basis in  $X$  normalized as  $\|x_k\|_X = 1$  for all  $k \in \mathbb{N}$ .

1.  $\{x_k\}_{k=1}^{\infty}$  is called greedy for  $X$  if there exists a constant  $C > 0$  such that for every  $x \in X$  there exists a permutation  $\rho$  of  $\mathbb{N}$  which satisfies

$$|c_{\rho(1)}(x)| \geq |c_{\rho(2)}(x)| \geq \dots \geq |c_{\rho(N)}(x)| \geq \dots$$

and

$$\left\| x - \sum_{k=1}^N c_{\rho(k)}(x)x_{\rho(k)} \right\|_X \leq C \inf_{y \in \Sigma_N} \|x - y\|_X,$$

for every  $N \in \mathbb{N}$ , where  $\Sigma_N := \left\{ \sum_{\nu \in \Lambda} \alpha_{\nu} x_{\nu} : \alpha_{\nu} \in \mathbb{C}, \#\Lambda \leq N, \Lambda \subset \mathbb{N} \right\}$ .

2.  $\{x_k\}_{k=1}^{\infty}$  is called democratic for  $X$  if there exists a constant  $C > 0$  independent of  $P$  and  $Q$  such that

$$\left\| \sum_{k \in P} x_k \right\|_X \leq C \left\| \sum_{k \in Q} x_k \right\|_X$$

for any finite subsets  $P, Q \subset \mathbb{N}$  with the same cardinality  $\#P = \#Q$ .



We remark that “unconditional ” and “democratic ” are independent notions. Indeed, in [14, Section 3] Konyagin and Temlyakov gave some examples of bases, which are not democratic but unconditional, or which are not unconditional but democratic. However, if we assume that the basis is unconditional. Then the above two notions agree. That is, we have the following.

**Proposition 2.10.** [14, Section 1] *Let  $\{x_k\}_{k=1}^\infty$  be a Schauder basis in  $X$  such that  $\|x_k\|_X = 1$  for all  $k \in \mathbb{N}$ . Then  $\{x_k\}_{k=1}^\infty$  is a greedy basis if and only if it is unconditional and democratic.*

### 3 Duality

In this section we prove an auxiliary result which is interesting of its own right.

**Lemma 3.1.** *Suppose that  $\theta > 0$ ,  $1 \leq p < \infty$ ,  $1 \leq q \leq \infty$  and  $w \in A_\infty^{\text{loc}}$ . Then we have*

$$\left\| \left\{ \sum_{l=0}^{\infty} 2^{-\theta|l-j|} |f_l| \right\}_{j \in \mathbb{N}_0} : L_p^w(l_q) \right\| \leq c \|\{f_j\}_{j \in \mathbb{N}} : L_p^w(l_q)\|$$

$$\left\| \left\{ \sum_{l=0}^{\infty} 2^{-\theta|l-j|} |f_l| \right\}_{j \in \mathbb{N}_0} : l_q(L_p^w) \right\| \leq c \|\{f_j\}_{j \in \mathbb{N}} : l_q(L_p^w)\|$$

for all sequences of measurable functions  $\{f_j\}_{j \in \mathbb{N}_0}$ .

*Proof.* This lemma is proved easily by Hölder’s inequality and we omit the proof. ■

**Theorem 3.2 (Duality  $A_{p,q}^{s,w}$ - $A_{p',q'}^{-s,w^{-\frac{1}{p-1}}}$ ).** *Let  $w$  be an  $A_p^{\text{loc}}$ -weight. Suppose that the parameters  $p, q, s$  satisfy*

$$1 < p < \infty, 1 \leq q < \infty, s \in \mathbb{R}.$$

Set  $v := w^{-\frac{1}{p-1}}$ . Then we have the following.

1. For all  $g \in A_{p',q'}^{-s,v}$ , the mapping

$$f \in \mathcal{S}_e \mapsto \langle g, f \rangle \in \mathbb{C}$$

extends to a continuous linear functional on  $A_{p,q}^{s,w}$ . Speaking precisely, we have

$$|\langle g, f \rangle| \leq c \|g : A_{p',q'}^{-s,v}\| \cdot \|f : A_{p,q}^{s,w}\|$$

for all  $f \in \mathcal{S}_e$ .

2. Conversely any continuous functional  $\Phi$  can be realized by some  $g \in A_{p',q'}^{-s,v}$  satisfying

$$\|g : A_{p',q'}^{-s,v}\| \leq c \|\Phi : A_{p,q}^{s,w}\|,$$

where  $c$  depends only on the parameters  $p, q, s$  and the  $A_p^{\text{loc}}$ -constant of  $w$ .

*Proof.* We concentrate on the  $F$ -scale, the counterpart for the  $B$ -scale being the same. Let  $f \in \mathcal{S}_e$ . By the local reproducing formula [21, Theorem 1.6], we have

$$f = \sum_{j=0}^{\infty} \psi_j * \phi_j * f,$$

where the convergence takes place in  $\mathcal{S}_e$ . Therefore, we obtain

$$\langle g, f \rangle = \sum_{j=0}^{\infty} \langle g, \psi_j * \phi_j * f \rangle = \sum_{j=0}^{\infty} \int_{\mathbb{R}^n} \check{\psi}_j * g(x) \cdot \phi_j * f(x) dx,$$

where  $\check{\psi}_j(x) := \psi_j(-x)$ . A repeated application of the Hölder inequality gives us

$$\begin{aligned} & \left\| \sum_{j=0}^{\infty} \int_{\mathbb{R}^n} \check{\psi}_j * g(x) \cdot \phi_j * f(x) dx \right\| \\ & \leq \|\{2^{-js} \check{\psi}_j * g\}_{j=0}^{\infty} : L_{p'}^v(l_{q'})\| \cdot \|\{2^{js} \phi_j * f\}_{j=0}^{\infty} : L_p^w(l_q)\| \\ & \leq c \|g : F_{p',q'}^{-s,v}\| \cdot \|f : F_{p,q}^{s,w}\|, \end{aligned}$$

proving 1. Here we have used

$$\|\{2^{-js} \check{\psi}_j * g\}_{j=0}^{\infty} : L_{p'}^v(l_{q'})\| \leq c \|g : F_{p',q'}^{-s,v}\|$$

to obtain the second inequality.

Let us prove 2. To do this, we pick a continuous functional  $\Phi$  on  $F_{p,q}^{s,w}$  arbitrarily. Let  $Y$  be the closure of the image of the mapping

$$f \in F_{p,q}^{s,w} \mapsto \{2^{js} \phi_j * f\}_{j=0}^{\infty} \in L_p^w(l_q).$$

Then we define a functional  $\Psi$  on  $Y$  uniquely so that it satisfies

$$\Psi(\{2^{js} \phi_j * f\}_{j=0}^{\infty}) = \Phi(f)$$

for all  $f \in F_{p,q}^{s,w}$ . Observe that

$$|\Psi(\{2^{js} \phi_j * f\}_{j=0}^{\infty})| = |\Phi(f)| \leq c \|\{2^{js} \phi_j * f\}_{j=0}^{\infty} : L_p^w(l_q)\|.$$

Therefore  $\Psi$  extends to a continuous linear functional on  $Y$ . By the Hahn-Banach extension theorem,  $\Psi$  extends even to a continuous linear functional on  $L_p^w(l_q)$ . For the

sake of simplicity let us denote it by  $\Psi$  again. By the duality  $L_p^w(l_q)$ - $L_{p'}^v(l_{q'})$ , which is well-known, there exists  $\{g_j\}_{j=0}^\infty \in L_{p'}^v(l_{q'})$  such that

$$\|\{g_j\}_{j=0}^\infty : L_{p'}^v(l_{q'})\| = \|\Psi : L_p^w(l_q)^*\| \quad \text{and} \quad \Psi(\{f_j\}_{j=0}^\infty) = \sum_{j=0}^\infty \int_{\mathbb{R}^n} f_j(x) g_j(x) dx$$

for all  $\{f_j\}_{j=0}^\infty \in L_p^w(l_q)$ . In particular, putting the observations above together, we obtain

$$\Phi(f) = \Psi(\{2^{js} \phi_j * f\}_{j=0}^\infty) = \sum_{j=0}^\infty \int_{\mathbb{R}^n} 2^{js} \phi_j * f(x) g_j(x) dx.$$

In view of the observation above, we are to define

$$g := \sum_{j=0}^\infty 2^{js} \check{\phi}_j * g_j. \quad (7)$$

In order that  $g$  makes sense at least in  $\mathcal{S}'_e$ , we shall claim that the sum converges at least in  $\mathcal{S}'_e$  and then we shall establish

$$\|g : F_{p',q'}^{-s,v}\| \leq c \|\Phi : F_{p,q}^{s,w*}\|. \quad (8)$$

Once we prove that (7) converges in  $\mathcal{S}'_e$  and that inequality (8) holds, then the theorem will have been proved.

First, let us verify that the sum converges. To do this we take  $\varphi \in \mathcal{S}_e$ . It is easy to show that each summand belongs to  $\mathcal{S}'_e$ . Therefore, we may assume that  $g_0 = 0$ . In view of this assumption let us assume  $j \geq 1$  below.

$$\langle 2^{js} \check{\phi}_j * g_j, \varphi \rangle = 2^{js} \int_{\mathbb{R}^n} \varphi * \phi_j(x) g_j(x) dx.$$

Observe that

$$\varphi * \phi_j(x) = \int_{\mathbb{R}^n} \left( \varphi(x-y) - \sum_{|\beta| \leq L} \frac{(-y)^\beta}{\beta!} \partial^\beta \varphi(x) \right) \phi_j(y) dy$$

by virtue of the moment condition of  $\phi_j$ . Using the Taylor expansion, we see

$$\begin{aligned} & e^{N|x|} \sup_{y \in B(2^{-j+1})} \left| \varphi(x-y) - \sum_{|\beta| \leq L} \frac{(-y)^\beta}{\beta!} \partial^\beta \varphi(x) \right| \\ & \leq c 2^{-j(L+1)} e^{N|x|} \sup_{|\beta| \leq L+1} \sup_{y \in B(2^{-j+1})} |\partial^\beta \varphi(x-y)| \\ & \leq c 2^{-j(L+1)} q_N(\varphi), \end{aligned}$$

for all  $N \gg L$ . Hence, we obtain

$$|\langle 2^{js} \check{\phi}_j * g_j, \varphi \rangle| \leq c 2^{j(s-L-1)} q_N(\varphi) \cdot \|g_j : L_{p'}^v\|.$$

Adding this over  $j \in \mathbb{N}$ , we obtain

$$g = \sum_{j=0}^{\infty} 2^{js} \check{\phi}_j * g_j$$

converges at least in  $\mathcal{S}'_e$ .

Having checked that the sum converges, let us turn to the proof that the sum belongs to  $F_{p',q'}^{-s,v}$ . To do this, we observe that

$$|\phi_j * \check{\phi}_k(x)| \leq c 2^{jn-(L+1)(k-j)} \chi_{B(2^{-j+2})}(x),$$

whenever  $j \leq k$ . Indeed, a simple calculus shows

$$\text{supp}(\phi_j * \check{\phi}_k) \subset \text{supp}(\phi_j) + \text{supp}(\check{\phi}_k) \subset B(2^{-j+1}) + B(2^{-k+1}) \subset B(2^{-j+2}).$$

To see that the  $L^\infty$ -norm is less than  $c 2^{jn-(L+1)(k-j)}$ , by dilation we may assume  $j = 1$ . We remark that the case when  $j = 0$  can be incorporated readily. Then in this case, going through the same argument as before, we obtain the desired upper bound of  $\phi_j * \check{\phi}_k$ . That is, we use the Taylor expansion: Letting  $L = L_{\phi_1}$ , we obtain

$$\phi_j * \check{\phi}_k(x) = \int_{\mathbb{R}^n} \phi_1(x-y) \left( \check{\phi}_k(y) - \sum_{|\beta| \leq L} \frac{\partial^\beta \check{\phi}_k(0)}{\beta!} y^\beta \right) dy.$$

Since the mean value theorem gives us

$$\left| \check{\phi}_k(y) - \sum_{|\beta| \leq L} \frac{\partial^\beta \check{\phi}_k(0)}{\beta!} y^\beta \right| \leq c 2^{-(L+1)k},$$

the desired estimate  $|\phi_j * \check{\phi}_k(x)| \leq c 2^{jn-(L+1)(k-j)}$  holds for  $j, k \geq 0$ . Inserting the estimate, we finally obtain

$$|\phi_j * g(x)| \leq c \sum_{l=0}^{\infty} 2^{ls-(L+1)|j-l|} M_{\text{loc}} g_l(x).$$

Therefore, we obtain

$$\begin{aligned} \|\{2^{-js} \phi_j * g\}_{j=0}^{\infty} : L_{p'}^v(l_{q'})\| &\leq c \left\| \left\{ \sum_{l=0}^{\infty} 2^{(l-j)s-(L+1)|j-l|} M_{\text{loc}} g_l \right\}_{j=0}^{\infty} : L_{p'}^v(l_{q'}) \right\| \\ &\leq c \|\{M_{\text{loc}} g_j\}_{j=0}^{\infty} : L_{p'}^v(l_{q'})\| \\ &\leq c \|\Psi : (F_{p,q}^{s,w})^*\| \end{aligned}$$

by virtue of Proposition 2.6 and Lemma 3.1. Therefore, we conclude  $g \in F_{p',q'}^{-s,v}$  realizes  $\Psi$ . ■

Going through a similar argument, we can prove the following.

**Corollary 3.3.** *Suppose that the parameters  $p, q, s$  satisfy*

$$1 < p < \infty, 1 \leq q \leq \infty, s \in \mathbb{R}.$$

*Assume in addition that  $w \in A_p^{\text{loc}}$  and set  $v := w^{-\frac{1}{p-1}}$ . Then for  $f \in A_{p,q}^{s,w}$ , one has the following norm equivalence.*

$$c^{-1} \|f : A_{p,q}^{s,w}\| \leq \sup \left\{ |\langle f, g \rangle| : g \in \mathcal{S}_e, \|g : A_{p',q'}^{-s,v}\| \leq 1 \right\} \leq c \|f : A_{p,q}^{-s,v}\|.$$

## 4 Wavelet characterizations of $A_{p,q}^{s,w}$

*Notation .* Let  $j \in \mathbb{Z}$  and  $k \in \mathbb{Z}^n$ .

1. Given a function  $\psi$  defined on  $\mathbb{R}^n$ , we define

$$\psi_{j,k}(x) := 2^{jn/2} \psi(2^j x - k).$$

2. In order to describe wavelets conveniently, we define the index set  $E$  by

$$E := \{1, 2, \dots, 2^n - 1\}.$$

According to wavelet theory, there exists compactly supported  $C^r$ -functions  $\{\varphi, \psi^\epsilon : \epsilon \in E\}$  such that

$$\int_{\mathbb{R}^n} x^\alpha \psi^\epsilon(x) dx = 0 \text{ for all } \epsilon \in E \text{ and all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq r, \quad (9)$$

and the sequences

$$\{\varphi_{0,k}, \psi_{j,k}^\epsilon : \epsilon \in E, j \in \mathbb{N}_0, k \in \mathbb{Z}^n\} \quad (10)$$

form orthonormal bases in  $L^2(\mathbb{R}^n)$  [18, 27]. We often say that the function  $\varphi$  is a scaling function and each  $\psi^\epsilon$  is a wavelet in terms of a multiresolution analysis. In particular, we can construct them so that  $\varphi$  and each  $\psi^\epsilon$  are real-valued and compactly supported with  $\text{supp} \varphi = \text{supp} \psi^\epsilon = [0, 2N - 1]^n$  for any positive integer  $N \geq 2$ . In this case, the constant  $r$  is an increasing function of  $N$  [3, 16, 18].

In order to obtain characterizations of function spaces, we sometimes need not only wavelets but also scaling functions [15, 18]. Throughout this paper, we consider a set of functions  $\{\varphi, \psi^\epsilon : \epsilon \in E\}$  satisfying the following three conditions (A), (B) and (C):

- (A)  $\varphi$  and each  $\psi^\epsilon$  are compactly supported.
- (B)  $\varphi$  and each  $\psi^\epsilon$  belong to  $C^r(\mathbb{R}^n)$ , and each  $\psi^\epsilon$  satisfies condition (9) for some  $r \in \mathbb{N}_0$ .
- (C) The system (10) forms an orthonormal basis in  $L^2(\mathbb{R}^n)$ .

**Lemma 4.1.** Assume the functions  $\psi^\epsilon, \varphi \in C^{(1+[s])_+}$  are compactly supported functions such that

$$\int_{\mathbb{R}^n} x^\beta \psi^\epsilon(x) dx = 0$$

for all  $\epsilon \in E$  and  $\beta \in \mathbb{N}_0^n$  with  $|\beta| \leq [s]$ . Define

$$m_{j,A,B}(y) := (1 + 2^j |y|)^A 2^{|y|B},$$

where  $A, B$  are sufficiently large. Then we have

$$\left\| \sup_{y \in \mathbb{R}^n} \frac{|\langle f, \varphi(\cdot - x + y) \rangle|}{m_{0,A,B}(y)} : L_p^w(\mathbb{R}_x^n) \right\| \leq c \|f : A_{p,q}^{s,w}\|$$

and

$$\sup_{\epsilon \in E} \left\| \left\{ 2^{j(s+\frac{n}{2})} \sup_{y \in \mathbb{R}^n} \frac{|\langle f, \psi_{j,0}^\epsilon(\cdot - x + y) \rangle|}{m_{j,A,B}(y)} \right\}_{j \in \mathbb{N}_0} : l_q(L_p^w(\mathbb{R}_x^n)) \right\| \leq c \|f : B_{p,q}^{s,w}\| \quad (11)$$

$$\sup_{\epsilon \in E} \left\| \left\{ 2^{j(s+\frac{n}{2})} \sup_{y \in \mathbb{R}^n} \frac{|\langle f, \psi_{j,0}^\epsilon(\cdot - x + y) \rangle|}{m_{j,A,B}(y)} \right\}_{j \in \mathbb{N}_0} : L_p^w(\mathbb{R}_x^n, l_q) \right\| \leq c \|f : F_{p,q}^{s,w}\|. \quad (12)$$

for all  $f \in \mathcal{S}_e$ .

*Proof.* The proof is obtained by using the fact that  $\psi_{j,k}^\epsilon$  has a vanishing moment of order up to  $L(\geq [s])$ . The proof is similar to the one in [21, Theorem 2.5] and we omit it. ■

**Definition 4.2.** Suppose that the parameters  $p, q, s$  and the weight  $w$  satisfy

$$1 < p < \infty, 1 \leq q \leq \infty, s \in \mathbb{R}, w \in A_p^{\text{loc}}.$$

Assume in addition that  $\{\varphi, \psi^\epsilon : \epsilon \in E\}$  satisfy conditions (A), (B) and (C) with  $r = \max((1 + [s])_+, (1 + [-s])_+)$ . Define

$$\begin{aligned} & \mathcal{B}_{p,q}^{s,w}(f) \\ := & \left\| \sum_{k \in \mathbb{Z}^n} \langle f, \varphi_{0,k} \rangle \chi_{Q_{0,k}} : L_p^w \right\| + \sum_{\epsilon \in E} \left\| \left\{ 2^{j(s+\frac{n}{2})} \langle f, \psi_{j,k}^\epsilon \rangle \chi_{Q_{j,k}} \right\}_{j \in \mathbb{N}_0, k \in \mathbb{Z}^n} : b_{p,q}^w \right\| \\ & \mathcal{F}_{p,q}^{s,w}(f) \\ := & \left\| \sum_{k \in \mathbb{Z}^n} \langle f, \varphi_{0,k} \rangle \chi_{Q_{0,k}} : L_p^w \right\| + \sum_{\epsilon \in E} \left\| \left\{ 2^{j(s+\frac{n}{2})} \langle f, \psi_{j,k}^\epsilon \rangle \chi_{Q_{j,k}} \right\}_{j \in \mathbb{N}_0, k \in \mathbb{Z}^n} : f_{p,q}^w \right\|. \end{aligned}$$

for  $f \in B_{p,q}^{s,w}$  and  $f \in F_{p,q}^{s,w}$  respectively. To simplify our formulation, we denote by  $\mathcal{A}_{p,q}^{s,w}(f)$  either  $\mathcal{B}_{p,q}^{s,w}(f)$  or  $\mathcal{F}_{p,q}^{s,w}(f)$ .

A helpful remark may be in order.

**Remark 4.3.** Since  $\varphi$  and each  $\psi^\epsilon$  are bounded and compactly supported, we see that each  $\langle f, \varphi_{0,k} \rangle$  and  $\langle f, \psi_{j,k}^\epsilon \rangle$  make sense. Indeed, if  $\phi$  is a  $C^{([-s]+1)_+}$ -function with compact support, then it is not so hard to see that  $\phi \in A_{p,q}^{-s,w}$  by virtue of Proposition 2.4. Therefore, we are in the position of using Theorem 3.2 to see that each  $\langle f, \varphi_{0,k} \rangle$  and  $\langle f, \psi_{j,k}^\epsilon \rangle$  make sense.

**Theorem 4.4.** *Suppose that the parameters  $p, q, s$  and the weight  $w$  satisfy*

$$1 < p < \infty, 1 \leq q \leq \infty, s \in \mathbb{R}, w \in A_p^{\text{loc}}.$$

*Assume in addition that the functions  $\{\varphi, \psi^\epsilon : \epsilon \in E\}$  satisfy conditions (A), (B) and (C) with  $r = \max((1 + [s])_+, (1 + [-s])_+)$ . Then there exists a constant  $c > 0$  such that*

$$c^{-1} \|f : A_{p,q}^{s,w}\| \leq \mathcal{A}_{p,q}^{s,w}(f) \leq c \|f : A_{p,q}^{s,w}\|$$

for all  $f \in A_{p,q}^{s,w}$ .

*Proof.* We assume that the functions  $\varphi, \psi^\epsilon, \epsilon \in E$  are all real-valued. When they are complex-valued, a minor modification of the proof below works. We begin with the proof of the right-inequality. We observe that  $\delta > 0$  can be taken so that  $r = \max((1 + [s + \delta])_+, (1 + [-s + \delta])_+)$ . By Proposition 2.4 there exists a sequence of functions  $\{f_m\}_{m=1}^\infty \subset \mathcal{S}_e$  such that

$$\lim_{m \rightarrow \infty} f_m = f$$

in the topology of  $\mathcal{S}'$  and that

$$\|f_m : A_{p,q}^{s,w}\| \leq c \|f : A_{p,q}^{s,w}\|. \quad (13)$$

By Proposition 2.4 we may as well assume that the convergence in (13) takes place in  $A_{p,q}^{s-\delta}$ . Therefore, by Theorem 3.2 we see that

$$\begin{aligned} \lim_{m \rightarrow \infty} \sum_{k \in \mathbb{Z}^n} \langle f_m, \varphi_{0,k} \rangle \chi_{Q_{0,k}} &= \sum_{k \in \mathbb{Z}^n} \langle f, \varphi_{0,k} \rangle \chi_{Q_{0,k}} \\ \lim_{m \rightarrow \infty} \sum_{k \in \mathbb{Z}^n} 2^{j(s+\frac{n}{2})} \langle f_m, \psi_{j,k}^\epsilon \rangle \chi_{Q_{j,k}} &= \sum_{k \in \mathbb{Z}^n} 2^{j(s+\frac{n}{2})} \langle f, \psi_{j,k}^\epsilon \rangle \chi_{Q_{j,k}} \end{aligned}$$

in the sense of the pointwise convergence. By using Fatou's lemma we obtain

$$\mathcal{A}_{p,q}^{s,w}(f) \leq \liminf_{m \rightarrow \infty} \mathcal{A}_{p,q}^{s,w}(f_m). \quad (14)$$

Therefore, (13) and (14) justify that we may assume  $f \in \mathcal{S}_e$  even when  $q = \infty$ . With this in mind, let us prove the right-inequality. Now that

$$\sum_{k \in \mathbb{Z}^n} 2^{j(s+\frac{n}{2})} |\langle f, \psi_{j,k}^\epsilon \rangle| \chi_{Q_{j,k}}(x) \leq 2^{j(s+\frac{n}{2})} \sup_{y \in \mathbb{R}^n} \frac{|\langle f, \psi_{j,0}^\epsilon(\cdot - x + y) \rangle|}{m_{j,A,B}(y)}$$

is true, the right inequality is immediate by virtue of Lemma 4.1.

If we assume that  $f \in A_{p,q}^{s,w}$ , then we can expand  $f$  into a wavelet series :

$$f = \sum_{k \in \mathbb{Z}^n} \langle f, \varphi_{0k} \rangle \varphi_{0,k} + \sum_{\epsilon \in E} \sum_{\nu \in \mathbb{N}} \sum_{k \in \mathbb{Z}^n} \langle f, \psi_{\nu k}^\epsilon \rangle \psi_{\nu k}^\epsilon.$$

Indeed, the right-hand side converges by virtue of the right inequality, which we have just established. To verify that the right-hand side agrees with  $f$ , we have only to check by evaluating at  $g \in \mathcal{S}_e$ . It is not so hard to see

$$\langle f, g \rangle = \sum_{k \in \mathbb{Z}^n} \langle f, \varphi_{0k} \rangle \langle \varphi_{0,k}, g \rangle + \sum_{\epsilon \in E} \sum_{\nu \in \mathbb{N}} \sum_{k \in \mathbb{Z}^n} \langle f, \psi_{\nu k}^\epsilon \rangle \langle \psi_{\nu k}^\epsilon, g \rangle.$$

Therefore, we are in a position of using Proposition 2.4 to see that

$$\|f : A_{p,q}^{s,w}\| \leq c \mathcal{A}_{p,q}^{s,w}(f).$$

This is the desired result. ■

## 5 Wavelet bases in $A_{p,q}^{s,w}$

### 5.1 Unconditional bases in $A_{p,q}^{s,w}$

**Theorem 5.1.** *Let  $1 < p < \infty$ ,  $1 \leq q \leq \infty$ ,  $s \in \mathbb{R}$ ,  $w \in A_p^{\text{loc}}$ , and the functions  $\{\varphi, \psi^\epsilon : \epsilon \in E\}$  satisfy conditions (A), (B) and (C) with  $r = \max((1+[s])_+, (1+[-s])_+)$ . Then the sequence*

$$\{\varphi_{0,k}, \psi_{j,k}^\epsilon : \epsilon \in E, j \in \mathbb{N}_0, k \in \mathbb{Z}^n\}$$

*forms an unconditional basis in  $A_{p,q}^{s,w}$ .*

*Proof.* There are some equivalent definitions of unconditional basis [13, 17, 27]. Now we shall check one of them. It suffices to check the following :

1. There exists a constant  $C > 0$  independent of  $f$ ,  $A$  and  $B$  such that

$$\|T_{A,B}f : A_{p,q}^{s,w}\| \leq C \|f : A_{p,q}^{s,w}\| \quad (15)$$

for all  $f \in A_{p,q}^{s,w}$  and all finite subsets  $A \subset \mathbb{Z}^n$  and  $B \subset E \times \mathbb{N}_0 \times \mathbb{Z}^n$ , where

$$T_{A,B}f := \sum_{k \in A} \langle f, \varphi_{0,k} \rangle \varphi_{0,k} + \sum_{(\epsilon,j,k) \in B} \langle f, \psi_{j,k}^\epsilon \rangle \psi_{j,k}^\epsilon.$$

2. The set  $\text{span}\{\varphi_{0,k}\}_{k \in \mathbb{Z}^n} \cup \text{span}\{\psi_{j,k}^\epsilon : \epsilon \in E, j \in \mathbb{N}_0, k \in \mathbb{Z}^n\}$  is dense in  $A_{p,q}^{s,w}$ .

We show (15) first. By the orthonormality and Theorem 4.4, we obtain

$$c^{-1} \|T_{A,B}f : A_{p,q}^{s,w}\| \leq \mathcal{A}_{p,q}^{s,w}(T_{A,B}f) \leq \mathcal{A}_{p,q}^{s,w}(f) \leq c \|f : A_{p,q}^{s,w}\|,$$



where  $c \geq 1$  is the constant appearing in Theorem 4.4.

Next we check the condition 2. It suffices to prove that for all  $f \in A_{p,q}^{s,w}$ ,

$$\lim_{A \rightarrow \mathbb{Z}^n, B \rightarrow E \times \mathbb{N}_0 \times \mathbb{Z}^n} \mathcal{A}_{p,q}^{s,w}(f - T_{A,B}f) = 0,$$

since  $\|f - T_{A,B}f : A_{p,q}^{s,w}\| \leq c \mathcal{A}_{p,q}^{s,w}(f - T_{A,B}f)$  by Theorem 4.4. Now define

$$\begin{aligned} \mathcal{A}_1(f) &:= \left\| \sum_{k \in \mathbb{Z}^n} \langle f, \varphi_{0,k} \rangle \chi_{Q_{0,k}} : L_p^w \right\| \\ \mathcal{A}_2(f) &:= \sum_{\epsilon \in E} \left\| \left\{ 2^{j(s+\frac{n}{2})} \langle f, \psi_{j,k}^\epsilon \rangle \chi_{Q_{j,k}} \right\}_{j \in \mathbb{N}_0, k \in \mathbb{Z}^n} : a_{p,q}^w \right\|, \end{aligned}$$

where  $a_{p,q}^w := f_{p,q}^w$  when  $A_{p,q}^{s,w} = F_{p,q}^{s,w}$ , and  $a_{p,q}^w := b_{p,q}^w$  when  $A_{p,q}^{s,w} = B_{p,q}^{s,w}$ . Then we have  $\mathcal{A}_{p,q}^{s,w}(f - T_{A,B}f) = \mathcal{A}_1(f - T_{A,B}f) + \mathcal{A}_2(f - T_{A,B}f)$ . We omit the detail because it is obvious that the orthonormality of the system  $\{\varphi_{0,k}\}_{k \in \mathbb{Z}^n} \cup \{\psi_{j,k}^\epsilon : \epsilon \in E, j \in \mathbb{N}_0, k \in \mathbb{Z}^n\}$  with regard to the  $L^2$ -inner product, the boundedness of  $\mathcal{A}_\nu(f - T_{A,B}f)$  for each  $\nu = 1, 2$  and Lebesgue's dominated convergence theorem give us the desired result. ■

## 5.2 Greedy bases in $F_{p,q}^{s,w}$

**Theorem 5.2.** *Assume the same condition as Theorem 5.1. Define*

$$\widetilde{\varphi}_{0,k} := \frac{\varphi_{0,k}}{\|\varphi_{0,k} : F_{p,q}^{s,w}\|} \quad \text{and} \quad \widetilde{\psi}_{j,k}^\epsilon := \frac{\psi_{j,k}^\epsilon}{\|\psi_{j,k}^\epsilon : F_{p,q}^{s,w}\|}.$$

Then the sequence

$$\left\{ \widetilde{\varphi}_{0,k}, \widetilde{\psi}_{j,k}^\epsilon : \epsilon \in E, j \in \mathbb{N}_0, k \in \mathbb{Z}^n \right\}$$

forms a greedy basis in  $F_{p,q}^{s,w}$ .

**Lemma 5.3.** *Let  $1 < p < \infty$  and  $w \in A_p^{\text{loc}}$ . Then there exists a constant  $1 < d < \infty$  such that for all dyadic cubes  $Q, Q'$  satisfying  $Q' \subsetneq Q$  and  $|Q'|, |Q| \leq 1$ ,*

$$d w(Q') \leq w(Q).$$

*Proof.* The proof of Lemma 5.3 is a local version of [8, p.141] and [24, Proof of Corollary 1.1]. Going through an argument similar to the ones in these literature, we can prove Lemma 5.3. ■

We need another characterization of  $F_{p,q}^{s,w}$  in order to obtain greedy bases in terms of wavelets. Let us write

$$\begin{aligned} \widetilde{\mathcal{F}}_1(f) &:= \left\| \left\{ \langle f, \varphi_{0,k} \rangle \|\varphi_{0,k} : F_{p,q}^{s,w}\| \right\}_{k \in \mathbb{Z}^n} : l^p(\mathbb{Z}^n) \right\| \\ \widetilde{\mathcal{F}}_2(f) &:= \sum_{\epsilon \in E} \left\| \left\{ w(Q_{j,k})^{-1/p} \|\psi_{j,k}^\epsilon : F_{p,q}^{s,w}\| \langle f, \psi_{j,k}^\epsilon \rangle \chi_{Q_{j,k}} \right\}_{j \in \mathbb{N}_0, k \in \mathbb{Z}^n} : f_{p,q}^w \right\|. \end{aligned}$$

**Lemma 5.4.** *Under the same condition as Theorem 5.1 there exists a constant  $c > 0$  such that for all  $f \in F_{p,q}^{s,w}$ ,*

$$c^{-1} \widetilde{\mathcal{F}}_1(f) \leq \left\| \sum_{k \in \mathbb{Z}^n} \langle f, \varphi_{0,k} \rangle \chi_{0,k} : L_p^w \right\| \leq c \widetilde{\mathcal{F}}_1(f) \quad (16)$$

and

$$c^{-1} \widetilde{\mathcal{F}}_2(f) \leq \sum_{\epsilon \in E} \left\| \left\{ 2^{j(s+\frac{n}{2})} \langle f, \psi_{j,k}^\epsilon \rangle \chi_{Q_{j,k}} \right\}_{j \in \mathbb{N}_0, k \in \mathbb{Z}^n} : F_{p,q}^w \right\| \leq c \widetilde{\mathcal{F}}_2(f). \quad (17)$$

*Proof of Lemma 5.4.* By Theorem 4.4, there exists a constant  $c \geq 1$  such that for each  $\epsilon \in E$ ,  $j \in \mathbb{N}_0$  and  $k \in \mathbb{Z}^n$ ,

$$c^{-1} \left\| \psi_{j,k}^\epsilon : F_{p,q}^{s,w} \right\| \leq \mathcal{F}_{p,q}^{s,w}(\psi_{j,k}^\epsilon) \leq c \left\| \psi_{j,k}^\epsilon : F_{p,q}^{s,w} \right\|.$$

On the other hand, we have  $\mathcal{F}_{p,q}^{s,w}(\psi_{j,k}^\epsilon) = 2^{j(s+n/2)} w(Q_{j,k})^{1/p}$ . Thus we can easily obtain the inequality (17).

Next we show the estimate (16). Using Theorem 4.4 again, we get

$$c^{-1} \left\| \varphi_{0,k} : F_{p,q}^{s,w} \right\| \leq \mathcal{F}_{p,q}^{s,w}(\varphi_{0,k}) \leq c \left\| \varphi_{0,k} : F_{p,q}^{s,w} \right\|.$$

Meanwhile we have  $\mathcal{F}_{p,q}^{s,w}(\varphi_{0,k}) = \left\| \chi_{Q_{0,k}} : L_p^w \right\|$  and

$$\left\| \sum_{k \in \mathbb{Z}^n} \langle f, \varphi_{0,k} \rangle \chi_{0,k} : L_p^w \right\| = \left\| \left\{ \langle f, \varphi_{0,k} \rangle \left\| \chi_{Q_{0,k}} : L_p^w \right\| \right\}_{k \in \mathbb{Z}^n} : l^p(\mathbb{Z}^n) \right\|.$$

These facts imply (16). ■

*Proof of Theorem 5.2.* The proof we give here is based on the proof of [2, Lemma 4.1]. In view of Theorem 5.1 and Proposition 2.10, it is enough to prove that the sequence

$$\left\{ \widetilde{\varphi}_{0,k}, \widetilde{\psi}_{j,k}^\epsilon : \epsilon \in E, j \in \mathbb{N}_0, k \in \mathbb{Z}^n \right\}$$

is democratic. We see that  $\{\varphi_{0,k}, \psi_{j,k}^\epsilon : \epsilon \in E, j \in \mathbb{N}_0, k \in \mathbb{Z}^n\}$  forms an unconditional basis for  $F_{p,q}^{s,w}$  by Theorem 5.1. Thus for all  $f \in F_{p,q}^{s,w}$  we can write

$$f = \sum_{k \in \mathbb{Z}^n} a_k(f) \widetilde{\varphi}_{0,k} + \sum_{\epsilon \in E} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} b_{j,k}^\epsilon(f) \widetilde{\psi}_{j,k}^\epsilon,$$

where  $a_k(f) := \langle f, \varphi_{0,k} \rangle \left\| \varphi_{0,k} : F_{p,q}^{s,w} \right\|$  and  $b_{j,k}^\epsilon(f) := \langle f, \psi_{j,k}^\epsilon \rangle \left\| \psi_{j,k}^\epsilon : F_{p,q}^{s,w} \right\|$ . Combining Theorem 4.4 and Lemma 5.4, we see that the norm

$$\left( \sum_{k \in \mathbb{Z}^n} |a_k(f)|^p \right)^{1/p} + \sum_{\epsilon \in E} \left\| \left( \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \left| w(Q_{j,k})^{-1/p} b_{j,k}^\epsilon(f) \chi_{Q_{j,k}} \right|^q \right)^{1/q} : L_p^w \right\|, \quad (18)$$

is equivalent to  $\|f : F_{p,q}^{s,w}\|$ . Let us denote  $\widetilde{\varphi}_Q := \widetilde{\varphi}_{j,k}$  and  $\widetilde{\psi}_Q^\epsilon := \widetilde{\psi}_{j,k}^\epsilon$  for a dyadic cube  $Q = Q_{j,k}$ . Now we take finite subsets  $A_\nu \subset \{Q_{0,k} : k \in \mathbb{Z}^n\}$  and  $\Lambda_\nu \subset E \times \{Q_{j,k} : j \in \mathbb{N}_0, k \in \mathbb{Z}^n\}$  satisfying  $\#A_1 + \#\Lambda_1 = \#A_2 + \#\Lambda_2$  arbitrarily, and write  $g_\nu := \sum_{I \in A_\nu} \widetilde{\varphi}_I + \sum_{(\epsilon, J) \in \Lambda_\nu} \widetilde{\psi}_J^\epsilon$  for  $\nu = 1, 2$ . We also denote

$$B_\nu := \{Q_{j,k} : (\epsilon, Q_{j,k}) \in \Lambda_\nu \text{ for some } \epsilon \in E\}.$$

Then we see that  $\#B_\nu \leq \#\Lambda_\nu \leq (2^n - 1)\#B_\nu$  for  $\nu = 1, 2$ . Using (18), we obtain

$$\begin{aligned} & c^{-1} \|g_1 : F_{p,q}^{s,w}\| \\ & \leq (\#A_1)^{1/p} + \left\| \left( \sum_{(\epsilon, J) \in \Lambda_1} |w(J)^{-1/p} \chi_J|^q \right)^{1/q} : L_p^w \right\| \\ & \leq (\#A_1)^{1/p} + (2^n - 1)^{1/q} \left\{ \int_{\bigcup_{J' \in B_1} J'} \left( \sum_{J \in B_1} w(J)^{-q/p} \chi_J(x) \right)^{p/q} w(x) dx \right\}^{1/p}. \end{aligned} \quad (19)$$

For each  $x \in \bigcup_{J \in B_1} J$ ,  $J_1(x)$  denotes the minimal dyadic cube in  $B_1$  with regard to the inclusion relation that contains  $x$ . Then we get

$$\sum_{J \in B_1} w(J)^{-q/p} \chi_J(x) \leq \sum_{r=0}^{\infty} w(J_r)^{-q/p}, \quad (20)$$

where  $J_0 := J_1(x)$ ,  $J_r$  is a dyadic cube satisfying  $J_{r-1} \subset J_r$  and  $2^n |J_{r-1}| = |J_r|$  for every  $r \in \mathbb{N}$ . By Lemma 5.3, there exists a constant  $1 < d < \infty$  such that for all  $r \in \mathbb{N}$ ,

$$w(J_r) \geq dw(J_{r-1}) \geq \dots \geq d^r w(J_0) = d^r w(J_1(x)).$$

Thus we have

$$\sum_{r=0}^{\infty} w(J_r)^{-q/p} \leq \sum_{r=0}^{\infty} (d^r w(J_1(x)))^{-q/p} = (1 - d^{-q/p})^{-1} w(J_1(x))^{-q/p}. \quad (21)$$

By virtue of (20) and (21), we obtain

$$\begin{aligned} & \int_{\bigcup_{J' \in B_1} J'} \left( \sum_{J \in B_1} w(J)^{-q/p} \chi_J(x) \right)^{p/q} w(x) dx \\ & \leq \int_{\bigcup_{J' \in B_1} J'} \left( c_1 w(J_1(x))^{-q/p} \right)^{p/q} w(x) dx \\ & = c^{p/q} \int_{\bigcup_{J' \in B_1} J'} w(J_1(x))^{-1} w(x) dx. \end{aligned} \quad (22)$$

Now we set  $\tilde{J} := \left\{ x \in \bigcup_{J' \in B_1} J' : J_1(x) = J \right\}$  for each  $J \in B_1$ . Then, since  $\tilde{J} \subset J$  and

$\bigcup_{J' \in B_1} J' = \bigcup_{J \in B_1} \tilde{J}$ , it follows that

$$\begin{aligned} \int \bigcup_{J' \in B_1} J' w(J_1(x))^{-1} w(x) dx &= \int \bigcup_{J \in B_1} \tilde{J} w(J_1(x))^{-1} w(x) dx \\ &\leq \sum_{J \in B_1} \int_{\tilde{J}} w(J_1(x))^{-1} w(x) dx \\ &= \sum_{J \in B_1} \int_J w(J)^{-1} w(x) dx \\ &= \#B_1. \end{aligned} \quad (23)$$

By virtue of (19)–(23), we have

$$\begin{aligned} c \|g_1 : F_{p,q}^{s,w}\| &\leq (\#A_1)^{1/p} + (2^n - 1)^{1/q} c (\#B_1)^{1/p} \\ &\leq (\#A_1)^{1/p} + (2^n - 1)^{1/q} c (\#\Lambda_1)^{1/p}. \end{aligned}$$

Hence there exists a constant  $c_2 > 0$  independent of  $g_1$ ,  $A_1$  and  $\Lambda_1$  such that

$$\|g_1 : F_{p,q}^{s,w}\| \leq c (\#A_1 + \#\Lambda_1)^{1/p}. \quad (24)$$

On the other hand, applying (18) to  $f = g_2$ , we have

$$\begin{aligned} c \|g_2 : F_{p,q}^{s,w}\| &\geq (\#A_2)^{1/p} + \left\| \left( \sum_{(\epsilon, J) \in \Lambda_2} |w(J)^{-1/p} \chi_J|^q \right)^{1/q} : L_p^w \right\| \\ &\geq (\#A_2)^{1/p} + \left\{ \int \bigcup_{J' \in B_2} J' \left( \sum_{J \in B_2} w(J)^{-q/p} \chi_J(y) \right)^{p/q} w(y) dy \right\}^{1/p}. \end{aligned} \quad (25)$$

For each  $y \in \bigcup_{J \in B_2} J$ ,  $J_2(y)$  denotes the minimal dyadic cube in  $B_2$  with regard to the inclusion relation that contains  $y$ . Then we have

$$\left( \sum_{J \in B_2} w(J)^{-q/p} \chi_J(y) \right)^{p/q} \geq w(J_2(y))^{-1}. \quad (26)$$

Now going through the same argument as (20)–(21), replacing “ $B_1$ ,  $-q/p$  and  $J_1(x)$ ” by “ $B_2$ ,  $-1$  and  $J_2(y)$ ” respectively, we get

$$\sum_{J \in B_2} w(J)^{-1} \chi_J(y) \leq c w(J_2(y))^{-1}, \quad (27)$$

where  $c > 0$  is a constant independent of  $g_2$ ,  $A_2$  and  $\Lambda_2$ . Using (25)–(27), we obtain

$$\begin{aligned}
c\|g_2 : F_{p,q}^{s,w}\| &\geq (\sharp A_2)^{1/p} + \left( \int \bigcup_{J' \in B_2} J' c^{-1} \sum_{J \in B_2} w(J)^{-1} \chi_J(y) w(y) dy \right)^{1/p} \\
&= (\sharp A_2)^{1/p} + \left( c^{-1} \sum_{J \in B_2} w(J)^{-1} \int_J w(y) dy \right)^{1/p} \\
&= (\sharp A_2)^{1/p} + c^{-1/p} (\sharp B_2)^{1/p} \\
&\geq (\sharp A_2)^{1/p} + c^{-1/p} (\sharp \Lambda_2)^{1/p}.
\end{aligned}$$

Namely we can take a constant  $c > 0$  independent of  $g_2$ ,  $A_2$  and  $\Lambda_2$  so that

$$c\|g_2 : F_{p,q}^{s,w}\| \geq (\sharp A_2 + \sharp \Lambda_2)^{1/p}. \quad (28)$$

Since  $\sharp A_1 + \sharp \Lambda_1 = \sharp A_2 + \sharp \Lambda_2$ , (24) and (28) yield

$$\|g_1 : F_{p,q}^{s,w}\| \leq c\|g_2 : F_{p,q}^{s,w}\|.$$

Consequently we have proved that the sequence  $\{\widetilde{\varphi}_{0,k}, \widetilde{\psi}_{j,k}^\epsilon : \epsilon \in E, j \in \mathbb{N}_0, k \in \mathbb{Z}^n\}$  is democratic. ■

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## References

- [1] H. Q. Bui, *Weighted Hardy spaces*, Math. Nach. **103** (1981), 45–62.
- [2] A. Cohen, R. A. DeVore and R. Hochmuth, *Restricted nonlinear approximation*, Constr. Approx. **16** (2000), 85–113.
- [3] I. Daubechies, *Orthonormal bases of compactly supported wavelets*, Comm. Pure Appl. Math. **41** (1988), 909–996.
- [4] D. Deng, M. Xu and L. Yan, *Wavelet characterization of weighted Triebel-Lizorkin spaces*, Approx. Theory Appl. **18** (2002), no. 4, 76–92.
- [5] M. Frazier and B. Jawerth, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal. **93** (1990), 34–170.

- [6] M. Frazier, B. Jawerth and G. Weiss, *Littlewood-Paley theory and the study of function spaces*, CBMS-AMS Regional Conf. Ser. **79**, 1991.
- [7] G. Garrigós, E. Hernández and J. M. Martell, *Wavelets, Orlicz spaces, and greedy bases*, preprint.
- [8] J. García-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland, Amsterdam, 1985.
- [9] M. Izuki, *The Haar wavelets and the Haar scaling function in weighted  $L^p$  spaces with  $A_p^{\text{dy,m}}$  weights*, to appear in Hokkaido Math. J. **36**(2007).
- [10] M. Izuki, *The characterizations of weighted Sobolev spaces by wavelets and scaling functions*, preprint.
- [11] M. Izuki and Y. Sawano, *Atomic decomposition for the weighted Besov / Triebel-Lizorkin spaces with local class of weights*, preprint.
- [12] M. Izuki and K. Tachizawa, *Wavelet characterizations of weighted Herz spaces*, preprint.
- [13] K. S. Kashin and A. A. Saakyan, *Orthogonal Series*, Amer. Math. Soc., Providence, R. I., 1989.
- [14] S. V. Konyagin and V. N. Temlyakov, *A remark on greedy approximation in Banach spaces*, East. J. Approx. **5** (1999), 365–379.
- [15] P. G. Lemarié-Rieusset, *Ondelettes et poids de Muckenhoupt*, Studia Math. **108** (1994), 127–147.
- [16] P. G. Lemarié-Rieusset and G. Malgouyres, *Support des fonctions de base dans une analyse multi-résolution*, C. R. Acad. Sci. Paris **313** (1991), 377–380.
- [17] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer-Verlag, Berlin, 1977.
- [18] Y. Meyer, *Wavelets and Operators*, Cambridge University Press, Cambridge, 1992.
- [19] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.
- [20] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytzkij operators and nonlinear partial differential equations*, Berlin, de Gruyter, 1996.
- [21] V. S. Rychkov, *Littlewood-Paley theory and function spaces with  $A_p^{\text{loc}}$  weights*, Math. Nachr. **224** (2001), 145–180.
- [22] T. Schott, *Functions spaces with exponential weights I*, Math. Nachr, **189** (1998), 221–242.
- [23] W. Sickel, *Spline representations of functions in Besov-Triebel-Lizorkin spaces on  $\mathbb{R}^n$* , Forum Math. **2** (1990), 451–475.

- [24] K. Tachizawa, *On weighted dyadic Carleson's inequalities*, J. Inequal. Appl. **6** (2001), 415–433.
- [25] H. Triebel, *Fractal and Spectra*, Monographs in Math., Vol. 91, Birkhäuser, Basel/Boston, 1997.
- [26] H. Triebel, *Theory of function spaces*, Monographs in Math., Vol. 78, Birkhäuser, Basel/Boston, 1983.
- [27] P. Wojtaszczyk, *A Mathematical Introduction to Wavelets*, Cambridge University Press, Cambridge, 1997.