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Citation	Hokkaido University Preprint Series in Mathematics, 857, 1-43
Issue Date	2007
DOI	10.14943/84007
Doc URL	<a href="http://hdl.handle.net/2115/69666">http://hdl.handle.net/2115/69666</a>
Type	bulletin (article)
File Information	pre857.pdf



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# INSTABILITY OF BOUND STATES OF A NONLINEAR SCHRÖDINGER EQUATION WITH A DIRAC POTENTIAL

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ABSTRACT. We study analytically and numerically the stability of the standing waves for a nonlinear Schrödinger equation with a point defect and a power type nonlinearity. A main difficulty is to compute the number of negative eigenvalues of the linearized operator around the standing waves, and it is overcome by a perturbation method and continuation arguments. Among others, in the case of a repulsive defect, we show that the standing wave solution is stable in  $H_{\text{rad}}^1(\mathbb{R})$  and unstable in  $H^1(\mathbb{R})$  under subcritical nonlinearity. Further we investigate the nature of instability: under critical or supercritical nonlinear interaction, we prove the instability by blow-up in the repulsive case by showing a virial theorem and using a minimization method involving two constraints. In the subcritical radial case, unstable bound states cannot collapse, but rather narrow down until they reach the stable regime (a *finite-width instability*). In the non-radial repulsive case, all bound states are unstable, and the instability is manifested by a lateral drift away from the defect, sometimes in combination with a finite-width instability or a blowup instability.

## 1. INTRODUCTION

We consider a nonlinear Schrödinger equation with a delta function potential

$$(1) \quad \begin{cases} i\partial_t u(t, x) = -\partial_x^2 u - \gamma u \delta(x) - |u|^{p-1} u, \\ u(0, x) = u_0, \end{cases}$$

where  $\gamma \in \mathbb{R}$ ,  $1 < p < +\infty$  and  $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ . Here,  $\delta$  is the Dirac distribution at the origin. Namely,  $\langle \delta, v \rangle = v(0)$  for  $v \in H^1(\mathbb{R})$ .

When  $\gamma = 0$ , this type of equations arises in various physical situations in the description of nonlinear waves (see [37] and the references therein); especially in nonlinear optics, it describes the propagation of a laser beam in a homogeneous medium. When  $\gamma \neq 0$ , equation (1) models the nonlinear propagation of light through optical waveguides with a localized defect (see [5, 19, 22, 31] and

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*Date:* June 3, 2007.

the references therein for more detailed considerations on the physical background). The authors in [5, 19, 22, 23, 24, 34, 35] observed the phenomenon of soliton scattering by the effect of the defect, namely, interactions between the defect and the soliton (the standing wave solution of the case  $\gamma = 0$ ). For example, varying amplitude and velocity of the soliton, they studied how the defect is separating the soliton into two parts : one part is transmitted past the defect, the other one is captured at the defect. Holmer, Marzuola and Zworski [22, 23] gave numerical simulations and theoretical arguments on this subject. In this paper, we study the stability of the standing wave solution of (1) created by the Dirac delta.

A standing wave for (1) is a solution of the form  $u(t, x) = e^{i\omega t}\varphi(x)$  where  $\varphi$  is required to satisfy

$$(2) \quad \begin{cases} -\partial_x^2\varphi + \omega\varphi - \gamma\delta(x)\varphi - |\varphi|^{p-1}\varphi = 0, \\ \varphi \in H^1(\mathbb{R}) \setminus \{0\}. \end{cases}$$

Before stating our results, we introduce some notations and recall some previous results.

The space  $L^r(\mathbb{R}, \mathbb{C})$  will be denoted by  $L^r(\mathbb{R})$  and its norm by  $\|\cdot\|_r$ . When  $r = 2$ , the space  $L^2(\mathbb{R})$  will be endowed with the scalar product

$$(u, v)_2 = \operatorname{Re} \int_{\mathbb{R}} u\bar{v}dx \text{ for } u, v \in L^2(\mathbb{R}).$$

The space  $H^1(\mathbb{R}, \mathbb{C})$  will be denoted by  $H^1(\mathbb{R})$ , its norm by  $\|\cdot\|_{H^1(\mathbb{R})}$  and the duality product between  $H^{-1}(\mathbb{R})$  and  $H^1(\mathbb{R})$  by  $\langle \cdot, \cdot \rangle$ . We write  $H_{\text{rad}}^1(\mathbb{R})$  for the space of radial (even) functions of  $H^1(\mathbb{R})$  :

$$H_{\text{rad}}^1(\mathbb{R}) = \{v \in H^1(\mathbb{R}); v(x) = v(-x), \quad x \in \mathbb{R}\}.$$

When  $\gamma = 0$ , the set of solutions of (2) has been known for a long time. In particular, modulo translation and phase, there exists a unique positive solution, which is explicitly known. This solution is even and is a ground state (see, for example, [3, 6, 26] for such results). When  $\gamma \neq 0$ , an explicit solution of (2) was presented in [13, 19] and the following was proved in [12, 13].

**Proposition 1.** *Let  $\omega > \gamma^2/4$ . Then there exists a unique positive solution  $\varphi_{\omega, \gamma}$  of (2). This solution is the unique positive minimizer of*

$$d(\omega) = \begin{cases} \inf \{S_{\omega, \gamma}(v); v \in H^1(\mathbb{R}) \setminus \{0\}, I_{\omega, \gamma}(v) = 0\} & \text{if } \gamma \geq 0, \\ \inf \{S_{\omega, \gamma}(v); v \in H_{\text{rad}}^1(\mathbb{R}) \setminus \{0\}, I_{\omega, \gamma}(v) = 0\} & \text{if } \gamma < 0, \end{cases}$$

where  $S_{\omega,\gamma}$  and  $I_{\omega,\gamma}$  are defined for  $v \in H^1(\mathbb{R})$  by

$$\begin{aligned} S_{\omega,\gamma}(v) &= \frac{1}{2}\|\partial_x v\|_2^2 + \frac{\omega}{2}\|v\|_2^2 - \frac{\gamma}{2}|v(0)|^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1}, \\ I_{\omega,\gamma}(v) &= \|\partial_x v\|_2^2 + \omega\|v\|_2^2 - \gamma|v(0)|^2 - \|v\|_{p+1}^{p+1}. \end{aligned}$$

Furthermore, we have an explicit formula for  $\varphi_{\omega,\gamma}$

$$(3) \quad \varphi_{\omega,\gamma}(x) = \left[ \frac{(p+1)\omega}{2} \operatorname{sech}^2 \left( \frac{(p-1)\sqrt{\omega}}{2}|x| + \tanh^{-1} \left( \frac{\gamma}{2\sqrt{\omega}} \right) \right) \right]^{\frac{1}{p-1}}.$$

The dependence of  $\varphi_{\omega,\gamma}$  on  $\omega$  and  $\gamma$  can be seen in Figure 1. The parameter  $\omega$  affects the width and height of  $\varphi_{\omega,\gamma}$ : the larger  $\omega$  is, the narrower and higher  $\varphi_{\omega,\gamma}$  becomes, and vice versa. The sign of  $\gamma$  determines the profile of  $\varphi_{\omega,\gamma}$  near  $x = 0$ : It has a "v" shape when  $\gamma < 0$ , and a "\^" shape when  $\gamma > 0$ .

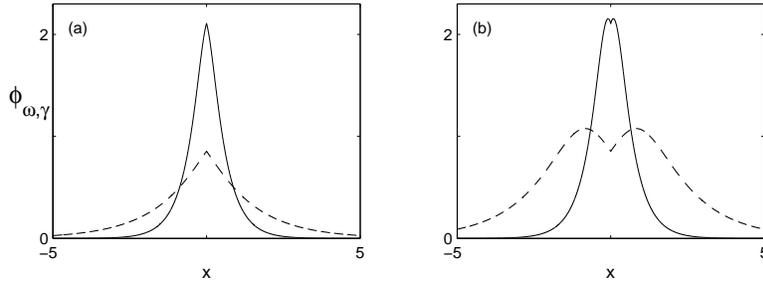


FIGURE 1.  $\varphi_{\omega,\gamma}$  as a function of  $x$  for  $\omega = 4$  (solid line) and  $\omega = 0.5$  (dashed line). (a)  $\gamma = 1$ ; (b)  $\gamma = -1$ . Here,  $p = 4$ .

**Remark 2.** (i) As it was stated in [12, Remark 8 and Lemma 26], the set of solutions of (2)

$$\{v \in H^1(\mathbb{R}) \setminus \{0\} \text{ such that } -\partial_x^2 v + \omega v - \gamma v \delta - |v|^{p-1}v = 0\}$$

is explicitly given by  $\{e^{i\theta}\varphi_{\omega,\gamma} \mid \theta \in \mathbb{R}\}$ .

(ii) There is no nontrivial solution in  $H^1(\mathbb{R})$  for  $\omega \leq \gamma^2/4$ .

The local well-posedness of the Cauchy problem for (1) is ensured by [6, Theorem 4.6.1]. Indeed, the operator  $-\partial_x^2 - \gamma\delta$  is a self-adjoint operator on  $L^2(\mathbb{R})$  (see [1, Chapter I.3.1] and Section 2 for details). Precisely, we have

**Proposition 3.** *For any  $u_0 \in H^1(\mathbb{R})$ , there exist  $T_{u_0} > 0$  and a unique solution  $u \in \mathcal{C}([0, T_{u_0}), H^1(\mathbb{R})) \cap \mathcal{C}^1([0, T_{u_0}), H^{-1}(\mathbb{R}))$  of (1) such that*

$\lim_{t \uparrow T_{u_0}} \|\partial_x u\|_2 = +\infty$  if  $T_{u_0} < +\infty$ . Furthermore, the conservation of energy and charge hold, that is, for any  $t \in [0, T_{u_0})$  we have

$$(4) \quad E(u(t)) = E(u_0),$$

$$(5) \quad \|u(t)\|_2^2 = \|u_0\|_2^2,$$

where the energy  $E$  is defined by

$$E(v) = \frac{1}{2} \|\partial_x v\|_2^2 - \frac{\gamma}{2} |v(0)|^2 - \frac{1}{p+1} \|v\|_{p+1}^{p+1}, \quad \text{for } v \in H^1(\mathbb{R}).$$

(see also a verification of this proposition in [13, Proposition 1]).

**Remark 4.** From the uniqueness result of Proposition 3 it follows that if an initial data  $u_0$  belongs to  $H_{\text{rad}}^1(\mathbb{R})$  then  $u(t)$  also belongs to  $H_{\text{rad}}^1(\mathbb{R})$  for all  $t \in [0, T_{u_0})$ .

We consider the stability in the following sense.

**Definition 5.** Let  $\varphi$  be a solution of (2). We say that the standing wave  $u(x, t) = e^{i\omega t} \varphi(x)$  is (orbitally) stable in  $H^1(\mathbb{R})$  (resp.  $H_{\text{rad}}^1(\mathbb{R})$ ) if for any  $\varepsilon > 0$  there exists  $\eta > 0$  with the following property : if  $u_0 \in H^1(\mathbb{R})$  (resp.  $H_{\text{rad}}^1(\mathbb{R})$ ) satisfies  $\|u_0 - \varphi\|_{H^1(\mathbb{R})} < \eta$ , then the solution  $u(t)$  of (1) with  $u(0) = u_0$  exists for any  $t \geq 0$  and

$$\sup_{t \in [0, +\infty)} \inf_{\theta \in \mathbb{R}} \|u(t) - e^{i\theta} \varphi\|_{H^1(\mathbb{R})} < \varepsilon.$$

Otherwise, the standing wave  $u(x, t) = e^{i\omega t} \varphi(x)$  is said to be (orbitally) unstable in  $H^1(\mathbb{R})$  (resp.  $H_{\text{rad}}^1(\mathbb{R})$ ).

**Remark 6.** With this definition and Remark 4, it is clear that stability in  $H^1(\mathbb{R})$  implies stability in  $H_{\text{rad}}^1(\mathbb{R})$  and conversely that instability in  $H_{\text{rad}}^1(\mathbb{R})$  implies instability in  $H^1(\mathbb{R})$ .

When  $\gamma = 0$ , the orbital stability for (1) has been extensively studied (see [2, 6, 7, 37, 38] and the references therein). In particular, from [7] we know that  $e^{i\omega t} \varphi_{\omega, 0}(x)$  is stable in  $H^1(\mathbb{R})$  for any  $\omega > 0$  if  $1 < p < 5$ . On the other hand, it was shown that  $e^{i\omega t} \varphi_{\omega, 0}(x)$  is unstable in  $H^1(\mathbb{R})$  for any  $\omega > 0$  if  $p \geq 5$  (see [2] for  $p > 5$  and [38] for  $p = 5$ ).

In [19], Goodman, Holmes and Weinstein focused on the special case  $p = 3$ ,  $\gamma > 0$  and proved that the standing wave  $e^{i\omega t} \varphi_{\omega, \gamma}(x)$  is orbitally stable in  $H^1(\mathbb{R})$ . When  $\gamma > 0$ , the orbital stability and instability were completely studied in [13] : the standing wave  $e^{i\omega t} \varphi_{\omega, \gamma}(x)$  is stable in  $H^1(\mathbb{R})$  for any  $\omega > \gamma^2/4$  if  $1 < p \leq 5$ , and if  $p > 5$ , there exists a critical frequency  $\omega_1 > \gamma^2/4$

such that  $e^{i\omega t}\varphi_{\omega,\gamma}(x)$  is stable in  $H^1(\mathbb{R})$  for any  $\omega \in (\gamma^2/4, \omega_1)$  and unstable in  $H^1(\mathbb{R})$  for any  $\omega > \omega_1$ .

When  $\gamma < 0$ , Fukuizumi and Jeanjean showed the following result in [12].

**Proposition 7.** *Let  $\gamma < 0$  and  $\omega > \gamma^2/4$ .*

- (i) *If  $1 < p \leq 3$  the standing wave  $e^{i\omega t}\varphi_{\omega,\gamma}(x)$  is stable in  $H_{\text{rad}}^1(\mathbb{R})$ .*
- (ii) *If  $3 < p < 5$ , there exists  $\omega_2 > \gamma^2/4$  such that the standing wave  $e^{i\omega t}\varphi_{\omega,\gamma}(x)$  is stable in  $H_{\text{rad}}^1(\mathbb{R})$  when  $\omega > \omega_2$  and unstable in  $H^1(\mathbb{R})$  when  $\gamma^2/4 < \omega < \omega_2$ .*
- (iii) *If  $p \geq 5$ , then the standing wave  $e^{i\omega t}\varphi_{\omega,\gamma}(x)$  is unstable in  $H^1(\mathbb{R})$ .*

The critical frequency  $\omega_2$  is given by

$$\frac{J(\omega_2)(p-5)}{p-1} = \frac{\gamma}{2\sqrt{\omega_2}} \left(1 - \frac{\gamma^2}{4\omega_2}\right)^{-(p-3)/(p-1)},$$

$$J(\omega_2) = \int_{A(\omega_2,\gamma)}^{+\infty} \text{sech}^{4/(p-1)}(y) dy, \quad A(\omega_2, \gamma) = \tanh^{-1}\left(\frac{\gamma}{2\sqrt{\omega_2}}\right).$$

The results of stability of [12] recalled in Proposition 7 assert only on stability under radial perturbations. Furthermore, the nature of instability is not revealed. In this paper, we prove that there is instability in the whole space when stability holds under radial perturbation (see Theorem 1), and that, when  $p \geq 5$ , the instability established in [12] is strong instability (see Definition 9 and Theorem 2).

Our first main result is the following.

**Theorem 1.** *Let  $\gamma < 0$  and  $\omega > \gamma^2/4$ .*

- (i) *If  $1 < p \leq 3$  the standing wave  $e^{i\omega t}\varphi_{\omega,\gamma}(x)$  is unstable in  $H^1(\mathbb{R})$ .*
- (ii) *If  $3 < p < 5$ , the standing wave  $e^{i\omega t}\varphi_{\omega,\gamma}(x)$  is unstable in  $H^1(\mathbb{R})$  for any  $\omega > \omega_2$ , where  $\omega_2$  is defined in Proposition 7.*

As in [12, 13], our stability analysis relies on the abstract theory by Grillakis, Shatah and Strauss [20, 21] for a Hamiltonian system which is invariant under a one-parameter group of operators. In trying to follow this approach the main point is to check the following two conditions:

- (1) *The slope condition:* The sign of  $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2$ .
- (2) *The spectral condition:* The number of negative eigenvalues of the linearized operator

$$L_{1,\omega}^\gamma v = -\partial_x^2 v + \omega v - \gamma \delta v - p\varphi_{\omega,\gamma}^{p-1} v.$$

We refer the reader to Section 2 for the precise criterion and a detailed explanation on how  $L_{1,\omega}^\gamma$  appears in this stability analysis. Making use of the explicit form (3) for  $\varphi_{\omega,\gamma}$ , the sign of  $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2$  was explicitly computed in [12, 13].

In [12], a spectral analysis is performed to count the number of negative eigenvalues, and it is proved that the number of negative eigenvalues of  $L_{1,\omega}^\gamma$  in  $H_{\text{rad}}^1(\mathbb{R})$  is one. This spectral analysis of  $L_{1,\omega}^\gamma$  is relying on the variational characterization of  $\varphi_{\omega,\gamma}$ . However, since  $\varphi_{\omega,\gamma}$  is a minimizer only in the space of radial (even) functions  $H_{\text{rad}}^1(\mathbb{R})$ , the result on the spectrum holds only in  $H_{\text{rad}}^1(\mathbb{R})$ , namely for even eigenfunctions. Therefore the number of negatives eigenvalues is known only for  $L_{1,\omega}^\gamma$  considered in  $H_{\text{rad}}^1(\mathbb{R})$ . With this approach, it is not possible to see whether other negative eigenvalues appear when the problem is considered on the whole space  $H^1(\mathbb{R})$ .

To overcome this difficulty, we develop a perturbation method. In the case  $\gamma = 0$ , the spectrum of  $L_{1,\omega}^0$  is well known by the work of Weinstein [39] (see Lemma 21) : there is only one negative eigenvalue, and 0 is a simple isolated eigenvalue (to see that, one proves that the kernel of  $L_{1,\omega}^0$  is spanned by  $\partial_x \varphi_{\omega,0}$ , that  $\partial_x \varphi_{\omega,0}$  has only one zero, and apply the Sturm Oscillation Theorem). When  $\gamma$  is small,  $L_{1,\omega}^\gamma$  can be considered as a holomorphic perturbation of  $L_{1,\omega}^0$ . Using the theory of holomorphic perturbations for linear operators, we prove that the spectrum of  $L_{1,\omega}^\gamma$  depends holomorphically on the spectrum of  $L_{1,\omega}^0$  (see Lemma 22). Then the use of Taylor expansion for the second eigenvalue of  $L_{1,\omega}^\gamma$  allows us to get the sign of the second eigenvalue when  $\gamma$  is small (see Lemma 23). A continuity argument combined with the fact that if  $\gamma \neq 0$  the nullspace of  $L_{1,\omega}^\gamma$  is zero extends the result to all  $\gamma \in \mathbb{R}$  (see the proof of Lemma 18). See subsection 2.3 for details. We will see that there are two negative eigenvalues of  $L_{1,\omega}^\gamma$  in  $H^1(\mathbb{R})$  if  $\gamma < 0$ .

- Remark 8.** (i) Our method can be applied as well in  $H^1(\mathbb{R})$  or in  $H_{\text{rad}}^1(\mathbb{R})$ , and for  $\gamma$  negative or positive (see subsections 2.4 and 2.5). Thus we can give another proof of the result of [13] in the case  $\gamma > 0$  and of Proposition 7.
- (ii) The study of the spectrum of linearized operators is often a central point when one wants to use the abstract theory of [20, 21]. See [9, 14, 15, 16, 25] among many others for related results.

The results of instability given in Theorem 1 and Proposition 7 say only that a certain solution which starts close to  $\varphi_{\omega,\gamma}$  will exit from a tubular neighborhood of the orbit of the standing wave in finite time. However, as this might be

of importance for the applications, we want to understand further the nature of instability. For that, we recall the concept of strong instability.

**Definition 9.** *A standing wave  $e^{i\omega t}\varphi(x)$  of (1) is said to be strongly unstable in  $H^1(\mathbb{R})$  if for any  $\varepsilon > 0$  there exist  $u_\varepsilon \in H^1(\mathbb{R})$  with  $\|u_\varepsilon - \varphi\|_{H^1(\mathbb{R})} < \varepsilon$  and  $T_{u_\varepsilon} < +\infty$  such that  $\lim_{t \uparrow T_{u_\varepsilon}} \|\partial_x u(t)\|_2 = +\infty$ , where  $u(t)$  is the solution of (1) with  $u(0) = u_\varepsilon$ .*

Our second main result is the following.

**Theorem 2.** *Let  $\gamma \leq 0$ ,  $\omega > \gamma^2/4$  and  $p \geq 5$ . Then the standing wave  $e^{i\omega t}\varphi_{\omega,\gamma}(x)$  is strongly unstable in  $H^1(\mathbb{R})$ .*

Whether the perturbed standing wave blows-up or not depends on the perturbation. Indeed, in Remark 30 we define an invariant set of solutions and show that if we consider an initial data in this set, then the solution exists globally even when the standing wave  $e^{i\omega t}\varphi_{\omega,\gamma}(x)$  is strongly unstable.

We also point out that when  $1 < p < 5$ , it is easy to prove using the conservation laws and Gagliardo-Nirenberg inequality that the Cauchy problem in  $H^1(\mathbb{R})$  associated with (1) is globally well posed. Accordingly, even if the standing wave may be unstable when  $1 < p < 5$  (see Theorem 1), a strong instability cannot occur.

As in [2, 38], which deal with the classical case  $\gamma = 0$ , we use the virial identity for the proof of Theorem 2. However, even if the formal calculations are similar to those of the case  $\gamma = 0$ , a rigorous proof of the virial theorem does not immediately follow from the approximation by regular solutions (e.g. see [6, Proposition 6.4.2], or [17]). Indeed, the argument in [6] relies on the  $H^2(\mathbb{R})$  regularity of the solutions of (1). Because of the defect term, we do not know if this  $H^2(\mathbb{R})$  regularity still holds when  $\gamma \neq 0$ . Thus we need another approach. We approximate the solutions of (1) by solutions of the same equation where the defect is approximated by a Gaussian potential for which it is easy to have the virial theorem. Then we pass to the limit in the virial identity to obtain :

**Proposition 10.** *Let  $u_0 \in H^1(\mathbb{R})$  such that  $xu_0 \in L^2(\mathbb{R})$  and  $u(t)$  be the solution of (1). Then the function  $f : t \mapsto \|xu(t)\|_2^2$  is  $\mathcal{C}^2$  and*

$$(6) \quad \partial_t f(t) = 4\text{Im} \int_{\mathbb{R}} \bar{u}x \partial_x u dx,$$

$$(7) \quad \partial_t^2 f(t) = 8Q_\gamma(u(t)),$$

where  $Q_\gamma$  is defined for  $v \in H^1(\mathbb{R})$  by

$$Q_\gamma(v) = \|\partial_x v\|_2^2 - \frac{\gamma}{2}|v(0)|^2 - \frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1}.$$

Even if we benefit from the virial identity, the proofs given in [2, 38] for the case  $\gamma = 0$  do not apply to the case  $\gamma < 0$ . For example, the method of Weinstein [38] in the case  $p = 5$  requires in a crucial way an equality between  $2E$  and  $Q$  which does not hold anymore when  $\gamma < 0$ . Moreover, the heart of the proof of [2] consists in minimizing the functional  $S_{\omega,\gamma}$  on the constraint  $Q_\gamma(v) = 0$ , but the standard variational methods to prove such results are not so easily applied to the case of  $\gamma \neq 0$ . To get over these difficulties we introduce an approach based on a minimization problem involving two constraints. Using this minimization problem, we identify some invariant properties under the flow of (1). The combination with these invariant properties and the conservation of energy and charge allows us to prove strong instability. We mention that some related techniques have been introduced in [28, 29, 30, 32, 40]. In conclusion, we can give a simpler method to prove Theorem 2 than that of [2] even though we have a term of delta potential.

**Remark 11.** The case  $\gamma < 0$ ,  $\omega = \omega_2$  and  $3 < p < 5$  cannot be treated with our approach and is left open (see Remark 15). In light of Theorem 1, we believe that the standing wave is unstable in this case, at least in  $H^1(\mathbb{R})$  (see also [12, Remark 12]). When  $\gamma > 0$ , the case  $\omega = \omega_1$  and  $p > 5$  is also open (see [13, Remark 1.5]).

Let us summarize the previously known and our new rigorous results on stability in (1):

- (i) For both positive and negative  $\gamma$ , there is always only one negative eigenvalue of linearized operator in  $H_{\text{rad}}^1(\mathbb{R})$  ([12], subsection 2.5). Hence, the standing wave is stable in  $H_{\text{rad}}^1(\mathbb{R})$  if the slope is positive, and unstable if the slope is negative.
- (ii)  $\gamma > 0$ . In this case the number of the negative eigenvalues of linearized operator is always one in  $H^1(\mathbb{R})$ . Stability is determined by the slope condition, and the standing wave is stable in  $H_{\text{rad}}^1(\mathbb{R})$  if and only if it is stable in  $H^1(\mathbb{R})$ . Specifically ([12, 13], subsection 2.4),
  - (a)  $1 < p \leq 5$ : Stability in  $H^1(\mathbb{R})$  for any  $\omega > \gamma^2/4$ .
  - (b)  $5 < p$ : Stability in  $H^1(\mathbb{R})$  for  $\gamma^2/4 < \omega < \omega_1$ , instability in  $H_{\text{rad}}^1(\mathbb{R})$  for  $\omega > \omega_1$ .
- (iii)  $\gamma < 0$ . In this case the number of negative eigenvalues is always two (Lemma 18) and all standing waves are unstable in  $H^1(\mathbb{R})$  (Theorem 1

and Theorem 2). Stability in  $H_{\text{rad}}^1(\mathbb{R})$  is determined by the slope condition and is as follows ([12]):

- (a)  $1 < p \leq 3$ : Stability in  $H_{\text{rad}}^1(\mathbb{R})$  for any  $\omega > \gamma^2/4$ .
- (b)  $3 < p < 5$ : Stability in  $H_{\text{rad}}^1(\mathbb{R})$  for  $\omega > \omega_2$ , instability in  $H_{\text{rad}}^1(\mathbb{R})$  for  $\gamma^2/4 < \omega < \omega_2$ .
- (c)  $5 \leq p$ : Strong instability in  $H_{\text{rad}}^1(\mathbb{R})$  (and in  $H^1(\mathbb{R})$ ) for any  $\gamma^2/4 < \omega$  (Theorem 2).

There are, however, several important questions which are still open, and which we explore using numerical simulations. Our simulations suggest the following:

- (i) Although an attractive defect ( $\gamma > 0$ ) stabilizes the standing waves in the critical case ( $p = 5$ ), their stability is weaker than in the subcritical case, in particular for  $0 < \gamma \ll 1$ .
- (ii) Theorem 2 shows that instability occurs by blow-up when  $\gamma < 0$  and  $p \geq 5$ . In all other cases, however, it remains to understand the nature of instability. Our simulations suggest the following:
  - (a) When  $\gamma > 0$ ,  $p > 5$ , and  $\omega > \omega_1$ , instability can occur by blow-up.
  - (b) When  $\gamma < 0$ ,  $3 < p < 5$ , and  $\gamma^2/4 < \omega < \omega_2$ , the instability in  $H_{\text{rad}}^1(\mathbb{R})$  is a *finite-width instability*, i.e., the solution initially narrows down along a curve  $\phi_{\omega^*(t), \gamma}$ , where  $\omega^*(t)$  can be defined by the relation

$$\max_x \phi_{\omega^*(t), \gamma}(x) = \max_x |u(x, t)|.$$

As the solution narrows down,  $\omega^*(t)$  increases and crosses from the unstable region  $\omega < \omega_2$  to the stable region  $\omega > \omega_2$ . Subsequently, collapse is arrested at some finite width.

- (c) When  $\gamma < 0$ , the standing waves undergo a *drift instability*, away from the (repulsive) defect, sometimes in combination with finite-width or blowup instability. Specifically,
  - (c.i) When  $1 < p \leq 3$  and when  $3 < p < 5$  and  $\omega > \omega_2$  (i.e., when the standing waves are stable in  $H_{\text{rad}}^1(\mathbb{R})$ ), the standing waves undergo a *drift instability*.
  - (c.ii) When  $3 < p < 5$  and  $\gamma^2/4 < \omega < \omega_2$ , the instability in  $H^1(\mathbb{R})$  is a combination of a drift instability and a finite-width instability.
  - (c.iii) When  $p \geq 5$ , the instability in  $H^1(\mathbb{R})$  is a combination of a drift instability and a blowup instability.
- (iii) Although when  $p = 5$  and  $\gamma > 0$ , and when  $p > 5$ ,  $\gamma > 0$ , and  $\gamma^2/4 < \omega < \omega_1$  the standing wave is stable, it can collapse under a sufficiently large perturbation.

We note that all of the above holds, more generally, for NLS equations with an inhomogeneous nonlinearity [9] and with a linear potential [36].

The paper is organized as follows. In Section 2, we prove Theorem 1 and explain how our method allows us to recover the results of [12, 13]. In Section 3, we establish Theorem 2 and in Section 4 we prove Proposition 10. Numerical results are given in Section 5.

Throughout the paper the letter  $C$  will denote various positive constants whose exact values may change from line to line but are not essential to the analysis of the problem.

**Acknowledgements.** The authors are grateful to Louis Jeanjean for fruitful discussions and helpful advice. Reika Fukuizumi would like to thank Shin-ichi Shirai and Clément Gallo for useful discussions particularly about Section 2. Stefan Le Coz wishes to thank Mariana Hărăguș for fruitful discussions.

## 2. INSTABILITY WITH RESPECT TO NON-RADIAL PERTURBATIONS

We use the general theory of Grillakis, Shatah and Strauss [21] to prove Theorem 1.

First, we explain how we derive a criterion for stability or instability for our case from the theory of Grillakis, Shatah and Strauss. In our case, it is clear that *Assumption 1* and *Assumption 2* of [21] are satisfied. The last assumption, *Assumption 3*, will be checked in subsection 2.2. We consider the sesquilinear form  $S''_{\omega,\gamma}(\varphi_{\omega,\gamma}) : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \rightarrow \mathbb{C}$  as a linear operator  $H''_{\omega} : H^1(\mathbb{R}) \rightarrow H^{-1}(\mathbb{R})$ . The spectrum of  $H''_{\omega}$  is the set  $\{\lambda \in \mathbb{C} \text{ such that } H''_{\omega} - \lambda I \text{ is not invertible}\}$ , where  $I$  denote the usual  $H^1(\mathbb{R}) - H^{-1}(\mathbb{R})$  isomorphism, and we denote by  $n(H''_{\omega})$  the number of negative eigenvalues of  $H''_{\omega}$ . Having established the assumptions of [21], the next proposition follows from [21, Instability Theorem and Stability Theorem].

**Proposition 12.** (1) *The standing wave  $e^{i\omega_0 t} \varphi_{\omega_0,\gamma}(x)$  is unstable if the integer  $(n(H''_{\omega_0}) - p(d''(\omega_0)))$  is odd, where*

$$p(d''(\omega_0)) = \begin{cases} 1 & \text{if } \partial_{\omega} \|\varphi_{\omega,\gamma}\|_2^2 > 0 \text{ at } \omega = \omega_0, \\ 0 & \text{if } \partial_{\omega} \|\varphi_{\omega,\gamma}\|_2^2 < 0 \text{ at } \omega = \omega_0. \end{cases}$$

(2) *The standing wave  $e^{i\omega_0 t} \varphi_{\omega_0,\gamma}(x)$  is stable if  $(n(H''_{\omega_0}) - p(d''(\omega_0))) = 0$ .*

Let us now consider the case  $\gamma < 0$ . It was proved in [12] that

**Lemma 13.** *Let  $\gamma < 0$  and  $\omega > \gamma^2/4$ . We have :*

- (i) *If  $1 < p \leq 3$  and  $\omega > \gamma^2/4$  then  $\partial_{\omega} \|\varphi_{\omega,\gamma}\|_2^2 > 0$ ,*

- (ii) If  $3 < p < 5$  and  $\omega > \omega_2$  then  $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 > 0$ ,
- (iii) If  $3 < p < 5$  and  $\gamma^2/4 < \omega < \omega_2$  then  $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 < 0$ ,
- (iv) If  $p \geq 5$  and  $\omega > \gamma^2/4$  then  $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 < 0$ .

Thus Theorem 1 follows from Proposition 12, Lemma 13 and

**Lemma 14.** *If  $\gamma < 0$ , then  $n(H_\omega^\gamma) = 2$ .*

- Remark 15.** (1) Let  $\gamma < 0$ . In the cases  $3 < p < 5$  and  $\omega < \omega_2$  or  $p \geq 5$  it was proved in [12] that  $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 < 0$ . From Lemma 14, we know that the number of negative eigenvalues of  $H_\omega^\gamma$  is  $n(H_\omega^\gamma) = 2$  when  $H_\omega^\gamma$  is considered on the whole space  $H^1(\mathbb{R})$ . Therefore  $n(H_\omega^\gamma) - p(d''(\omega)) = 2$  and this corresponds to a case where the assumption of [21] may not be applied. However, if we consider  $H_\omega^\gamma$  in  $H_{\text{rad}}^1(\mathbb{R})$ , then it follows from [12] that  $n(H_\omega^\gamma) = 1$ , thus  $n(H_\omega^\gamma) - p(d''(\omega)) = 1$ . Then, we can apply Proposition 12 to this case and it allows us to conclude instability in  $H_{\text{rad}}^1(\mathbb{R})$  (as it was done in [12]). But, with Remark 6, we can conclude that instability holds on the whole space  $H^1(\mathbb{R})$ .
- (2) Note that the case  $\omega = \omega_2$  corresponds to  $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 = 0$  ( $3 < p < 5$ ) and will not be treated here. In view of Theorem 1, we believe that the standing wave is unstable in this case, at least in  $H^1(\mathbb{R})$ .

We divide the rest of this section into five parts. In subsection 2.1 we introduce the general setting to perform our proof. In subsection 2.2, we study the spectrum of  $H_\omega^\gamma$  and prove that *Assumption 3* of [21] is satisfied. Lemma 14 will be proved in subsection 2.3. Finally, we discuss the positive case and the radial case in subsections 2.4 and 2.5.

### 2.1. Setting for the spectral problem.

To express  $H_\omega^\gamma$ , it is convenient to split  $u$  in real and imaginary part : for  $u \in H^1(\mathbb{R}, \mathbb{C})$  we write  $u = u_1 + iu_2$  where  $u_1 = \text{Re}(u) \in H^1(\mathbb{R}, \mathbb{R})$  and  $u_2 = \text{Im}(u) \in H^1(\mathbb{R}, \mathbb{R})$ . Now we set

$$H_\omega^\gamma u = L_{1,\omega}^\gamma u_1 + iL_{2,\omega}^\gamma u_2$$

where the operators  $L_{1,\omega}^\gamma, L_{2,\omega}^\gamma : H^1(\mathbb{R}, \mathbb{R}) \rightarrow H^{-1}(\mathbb{R})$  are defined for  $v \in H^1(\mathbb{R})$  by

$$\begin{aligned} L_{1,\omega}^\gamma v &= -\partial_x^2 v + \omega v - \gamma v \delta - p\varphi_{\omega,\gamma}^{p-1} v, \\ L_{2,\omega}^\gamma v &= -\partial_x^2 v + \omega v - \gamma v \delta - \varphi_{\omega,\gamma}^{p-1} v. \end{aligned}$$

When we will work with  $L_{1,\omega}^\gamma, L_{2,\omega}^\gamma$ , the functions considered will be understood to be real valued.

For the spectral study of  $H_\omega^\gamma$ , it is convenient to view  $H_\omega^\gamma$  as an unbounded operator on  $L^2(\mathbb{R})$ , thus we rewrite our spectral problem in this setting. First, we redefine the two operators  $L_{1,\omega}^\gamma$  and  $L_{2,\omega}^\gamma$  as unbounded operators on  $L^2(\mathbb{R})$ . We begin by considering the bilinear forms on  $H^1(\mathbb{R})$  associated with  $L_{1,\omega}^\gamma$  and  $L_{2,\omega}^\gamma$  by setting for  $v, w \in H^1(\mathbb{R})$

$$B_{1,\omega}^\gamma(v, w) := \langle L_{1,\omega}^\gamma v, w \rangle \text{ and } B_{2,\omega}^\gamma(v, w) := \langle L_{2,\omega}^\gamma v, w \rangle,$$

which are explicitly given by

$$(8) \quad \begin{aligned} B_{1,\omega}^\gamma(v, w) &= \int_{\mathbb{R}} \partial_x v \partial_x w dx + \omega \int_{\mathbb{R}} v w dx - \gamma v(0)w(0) - \int_{\mathbb{R}} p \varphi_{\omega,\gamma}^{p-1} v w dx, \\ B_{2,\omega}^\gamma(v, w) &= \int_{\mathbb{R}} \partial_x v \partial_x w dx + \omega \int_{\mathbb{R}} v w dx - \gamma v(0)w(0) - \int_{\mathbb{R}} \varphi_{\omega,\gamma}^{p-1} v w dx. \end{aligned}$$

Let us now consider  $B_{1,\omega}^\gamma$  and  $B_{2,\omega}^\gamma$  as bilinear forms on  $L^2(\mathbb{R})$  with domain  $D(B_{1,\omega}^\gamma) = D(B_{2,\omega}^\gamma) := H^1(\mathbb{R})$ . It is clear that these forms are bounded from below and closed. Then the theory of representation of forms by operators (see [27, VI.§2.1]) implies that we define two self-adjoint operators  $\widetilde{L}_{1,\omega}^\gamma : D(\widetilde{L}_{1,\omega}^\gamma) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  and  $\widetilde{L}_{2,\omega}^\gamma : D(\widetilde{L}_{2,\omega}^\gamma) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  by setting

$$\begin{aligned} D(\widetilde{L}_{1,\omega}^\gamma) &:= \{v \in H^1(\mathbb{R}) \mid \exists w \in L^2(\mathbb{R}) \text{ s.t. } \forall z \in H^1(\mathbb{R}), B_{1,\omega}^\gamma(v, z) = (w, z)_2\}, \\ D(\widetilde{L}_{2,\omega}^\gamma) &:= \{v \in H^1(\mathbb{R}) \mid \exists w \in L^2(\mathbb{R}) \text{ s.t. } \forall z \in H^1(\mathbb{R}), B_{2,\omega}^\gamma(v, z) = (w, z)_2\}. \end{aligned}$$

and setting for  $v \in D(\widetilde{L}_{1,\omega}^\gamma)$  (resp.  $v \in D(\widetilde{L}_{2,\omega}^\gamma)$ ) that  $\widetilde{L}_{1,\omega}^\gamma v := w$  (resp.  $\widetilde{L}_{2,\omega}^\gamma v := w$ ), where  $w$  is the (unique) function of  $L^2(\mathbb{R})$  which satisfies  $B_{1,\omega}^\gamma(v, z) = (w, z)_2$  (resp.  $B_{2,\omega}^\gamma(v, z) = (w, z)_2$ ) for all  $z \in H^1(\mathbb{R})$ .

For notational simplicity, we drop the tilde over  $\widetilde{L}_{1,\omega}^\gamma$  and  $\widetilde{L}_{2,\omega}^\gamma$ .

It turns out that we are able to describe explicitly  $L_{1,\omega}^\gamma$  and  $L_{2,\omega}^\gamma$ .

**Lemma 16.** *The domain of  $L_{1,\omega}^\gamma$  and of  $L_{2,\omega}^\gamma$  in  $L^2(\mathbb{R})$  is*

$$D_\gamma = \{v \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}); \partial_x v(0^+) - \partial_x v(0^-) = -\gamma v(0)\}$$

and for  $v \in D_\gamma$  the operators are given by

$$(9) \quad \begin{aligned} L_{1,\omega}^\gamma v &= -\partial_x^2 v + \omega v - p \varphi_{\omega,\gamma}^{p-1} v, \\ L_{2,\omega}^\gamma v &= -\partial_x^2 v + \omega v - \varphi_{\omega,\gamma}^{p-1} v. \end{aligned}$$

*Proof.* The proof for  $L_{2,\omega}^\gamma$  being similar to the one of  $L_{1,\omega}^\gamma$  we only deal with  $L_{1,\omega}^\gamma$ . The form  $B_{1,\omega}^\gamma$  can be decomposed into  $B_{1,\omega}^\gamma = B_{1,1}^\gamma + B_{1,2,\omega}^\gamma$  with  $B_{1,1}^\gamma : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \rightarrow \mathbb{R}$  and  $B_{1,2,\omega}^\gamma : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow \mathbb{R}$  defined by

$$(10) \quad \begin{aligned} B_{1,1}^\gamma(v, z) &= \int_{\mathbb{R}} \partial_x v \partial_x z dx - \gamma v(0)z(0), \\ B_{1,2,\omega}^\gamma(v, z) &= \omega \int_{\mathbb{R}} v z dx - \int_{\mathbb{R}} p \varphi_{\omega,\gamma}^{p-1} v z dx. \end{aligned}$$

If we denote by  $T_1$  (resp.  $T_2$ ) the self-adjoint operator on  $L^2(\mathbb{R})$  associated with  $B_{1,1}^\gamma$  (resp  $B_{1,2,\omega}^\gamma$ ), it is clear that  $D(T_2) = L^2(\mathbb{R})$  and

$$D(L_{1,\omega}^\gamma) = D(T_1).$$

If we take  $v \in H^2(\mathbb{R})$  such that  $v(0) = 0$ , and put  $w = -\partial_x^2 v \in L^2(\mathbb{R})$ , it follows that for any  $z \in H^1(\mathbb{R})$  we have

$$B_{1,1}^\gamma(v, z) = \int_{\mathbb{R}} \partial_x v \partial_x z dx = (w, z)_2.$$

Thus  $v \in D(T_1)$ , and we can deduce that  $T_1$  is a self-adjoint extension of the operator  $T$  defined by

$$T = -\partial_x^2, \quad D(T) = \{v \in H^2(\mathbb{R}); v(0) = 0\}.$$

On the other hand, using the theory of self-adjoint extensions of symmetric operators, one can see (see [1, Theorem I-3.1.1]) that there exists  $\alpha \in \mathbb{R}$  such that

$$D(T_1) = \{v \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}); \partial_x v(0^+) - \partial_x v(0^-) = -\alpha v(0)\}.$$

Now, take  $v \in D(T_1)$  with  $v(0) \neq 0$ . Then

$$\begin{aligned} (T_1 v, v)_2 &= \int_{-\infty}^0 (-\partial_x^2 v) v dx + \int_0^{+\infty} (-\partial_x^2 v) v dx \\ &= -v(0) \partial_x v(0^-) + \int_{-\infty}^0 |\partial_x v|^2 dx + v(0) \partial_x v(0^+) + \int_0^{+\infty} |\partial_x v|^2 dx \\ &= \int_{\mathbb{R}} |\partial_x v|^2 dx - \alpha v(0)^2 \end{aligned}$$

which should be equal to

$$B_{1,1}^\gamma(v, v) = \int_{\mathbb{R}} |\partial_x v|^2 dx - \gamma v(0)^2.$$

Thus  $\gamma = \alpha$ , and the lemma is proved.  $\square$

## 2.2. Verification of *Assumption 3*.

To check [21, *Assumption 3*] is equivalent to check that the following lemma hold.

**Lemma 17.** *Let  $\gamma \in \mathbb{R} \setminus \{0\}$  and  $\omega > \gamma^2/4$ .*

- (i) *The operator  $H_\omega^\gamma$  has only a finite number of negative eigenvalues,*
- (ii) *The kernel of  $H_\omega^\gamma$  is  $\text{span}\{i\varphi_{\omega,\gamma}\}$ ,*
- (iii) *The rest of the spectrum of  $H_\omega^\gamma$  is positive and bounded away from 0.*

Our proof of Lemma 17 borrows some elements of [12]. In particular, (ii) in Lemma 17 corresponds to [12, Lemma 28 and Lemma 31].

*Proof of Lemma 17.* We start by showing that (i) and (iii) are satisfied. We work on  $L_{1,\omega}^\gamma$  and  $L_{2,\omega}^\gamma$ . The essential spectrum of  $T_1$  (see the proof of Lemma 16) is  $\sigma_{\text{ess}}(T_1) = [0, +\infty)$ . This is standard when  $\gamma = 0$  and a proof for  $\gamma \neq 0$  can be found in [1, Theorem I-3.1.4]. From Weyl's theorem (see [27, Theorem IV-5.35]), the essential spectrum of both operators  $L_{1,\omega}^\gamma$  and  $L_{2,\omega}^\gamma$  is  $[\omega, +\infty)$ . Since both operators are bounded from below, there can be only finitely many isolated eigenvalues (of finite multiplicity) in  $(-\infty, \omega')$  for any  $\omega' < \omega$ . Then (i) and (iii) follow easily.

Next, we consider (ii). Since  $\varphi_{\omega,\gamma}$  satisfies  $L_{2,\omega}^\gamma \varphi_{\omega,\gamma} = 0$  and  $\varphi_{\omega,\gamma} > 0$ , the first eigenvalue of  $L_{2,\omega}^\gamma$  is 0 and the rest of the spectrum is positive. This is classical for  $\gamma = 0$  and can be easily proved for  $\gamma \neq 0$ , see [4, Chapter 2, Section 2.3, Paragraph 3]. Thus to ensure that the kernel of  $H_\omega^\gamma$  is reduced to  $\text{span}\{i\varphi_{\omega,\gamma}\}$  it is enough to prove that the kernel of  $L_{1,\omega}^\gamma$  is  $\{0\}$ . It is equivalent to prove that 0 is the unique solution of

$$(11) \quad L_{1,\omega}^\gamma u = 0, \quad u \in D(L_{1,\omega}^\gamma).$$

To be more precise, the solutions of (11) satisfy

$$(12) \quad u \in H^2(\mathbb{R} \setminus \{0\}) \cap H^1(\mathbb{R}),$$

$$(13) \quad -\partial_x^2 u + \omega u - p\varphi_{\omega,\gamma}^{p-1} u = 0,$$

$$(14) \quad \partial_x u(0+) - \partial_x u(0-) = -\gamma u(0).$$

Consider first (13) on  $(0, +\infty)$ . If we look at (2) only on  $(0, +\infty)$ , we see that  $\varphi_{\omega,\gamma}$  satisfies

$$(15) \quad -\partial_x^2 \varphi_{\omega,\gamma} + \omega \varphi_{\omega,\gamma} - \varphi_{\omega,\gamma}^p = 0 \text{ on } (0, +\infty).$$

If we differentiate (15) with respect to  $x$  (which is possible because  $\varphi_{\omega,\gamma}$  is smooth on  $(0, +\infty)$ ), we see that  $\partial_x \varphi_{\omega,\gamma}$  satisfies (13) on  $(0, +\infty)$ . Since we look for solutions in  $L^2(\mathbb{R})$  (in fact solutions going to 0 at infinity), it is standard that every solution of (13) in  $(0, +\infty)$  is of the form  $\mu \partial_x \varphi_{\omega,\gamma}$ ,  $\mu \in \mathbb{R}$  (see, for example, [4, Chapter 2, Theorem 3.3]). A similar argument can be applied to (13) on  $(-\infty, 0)$ , thus every solution of (13) in  $(-\infty, 0)$  is of the form  $\nu \partial_x \varphi_{\omega,\gamma}$ ,  $\nu \in \mathbb{R}$ .

Now, let  $u$  be a solution of (12)-(14). Then there exists  $\mu \in \mathbb{R}$  and  $\nu \in \mathbb{R}$  such that

$$\begin{aligned} u &= \nu \partial_x \varphi_{\omega,\gamma} && \text{on } (-\infty, 0), \\ u &= \mu \partial_x \varphi_{\omega,\gamma} && \text{on } (0, +\infty). \end{aligned}$$

Since  $u \in H^1(\mathbb{R})$ ,  $u$  is continuous at 0, thus we must have  $\mu = -\nu$ , that is  $u$  is of the form

$$\begin{aligned} u &= -\mu \partial_x \varphi_{\omega, \gamma} \text{ on } (-\infty, 0), \\ u &= \mu \partial_x \varphi_{\omega, \gamma} \text{ on } (0, +\infty), \\ u(0) &= -\mu \partial_x \varphi_{\omega, \gamma}(0-) = \mu \partial_x \varphi_{\omega, \gamma}(0+) = \frac{-\mu}{2} \gamma \varphi_{\omega, \gamma}(0). \end{aligned}$$

Furthermore,  $u$  should satisfies the jump condition (14). Since  $\varphi_{\omega, \gamma}$  satisfies

$$\partial_x^2 \varphi_{\omega, \gamma}(0-) = \partial_x^2 \varphi_{\omega, \gamma}(0+) = \omega \varphi_{\omega, \gamma}(0) - \varphi_{\omega, \gamma}^p(0),$$

if we suppose  $\mu \neq 0$  then (14) reduces to

$$\varphi_{\omega, \gamma}^{p-1}(0) = \frac{4\omega - \gamma^2}{4}.$$

But from (3) we know that

$$\varphi_{\omega, \gamma}^{p-1}(0) = \frac{p+1}{8}(4\omega - \gamma^2).$$

It is a contradiction, therefore  $\mu = 0$ . In conclusion,  $u \equiv 0$  on  $\mathbb{R}$ , and the lemma is proved.  $\square$

### 2.3. Count of the number of negative eigenvalues.

In this subsection, we prove Lemma 14. First, we remark that, as it was shown in the proof of Lemma 17, 0 is the first eigenvalue of  $L_{2, \omega}^\gamma$ . Thus  $n(H_\omega^\gamma) = n(L_{1, \omega}^\gamma)$ , where  $n(L_{1, \omega}^\gamma)$  is the number of negative eigenvalues of  $L_{1, \omega}^\gamma$ . Therefore, Lemma 14 follows from

**Lemma 18.** *Let  $\gamma < 0$  and  $\omega > \gamma^2/4$ . Then  $n(L_{1, \omega}^\gamma) = 2$ .*

Our proof of Lemma 18 is divided in two steps. First, we use a perturbative approach to prove that, if  $\gamma$  is close to 0 and negative,  $L_{1, \omega}^\gamma$  has two negative eigenvalues (Lemma 23). To do this, we have to ensure that the eigenvalues and the eigenvectors are regular enough with respect to  $\gamma$  (Lemma 22) to make use of Taylor formula. It follows from the use of the analytic perturbation theory of operators (see [27, 33]). The second step consists in extending the result of the first step to any values of  $\gamma < 0$ . Our argument relies on the continuity of the spectral projections with respect to  $\gamma$  and it is crucial, as it was proved in Lemma 17, that 0 can not be an eigenvalue of  $L_{1, \omega}^\gamma$  (see [14, 15] for related arguments).

We fix  $\omega > \gamma^2/4$ . For the sake of simplicity we denote  $L_{1, \omega}^\gamma$  by  $L_1^\gamma$  and  $\varphi_{\omega, \gamma}$  by  $\varphi_\gamma$ , and so on in this section 2.

**Lemma 19.** *As a function of  $\gamma$ ,  $(L_1^\gamma)$  is a real-holomorphic family of self-adjoint operators (of type (B) in the sense of Kato).*

*Proof.* We recall that  $L_1^\gamma$  is defined with the help of a bilinear form  $B_1^\gamma$  (see (8)). To prove the holomorphicity of  $(L_1^\gamma)$  it is enough to prove that  $(B_1^\gamma)$  is bounded from below and closed, and that for any  $v \in H^1(\mathbb{R})$  the function  $B_1^\gamma(v) : \gamma \mapsto B_1^\gamma(v, v)$  is holomorphic (see [27, Theorem VII-4.2]). It is clear that  $B_1^\gamma$  is bounded from below and closed on the same domain  $H^1(\mathbb{R})$  for all  $\gamma$ , thus we just have to check the holomorphicity of  $B_1^\gamma(v) : \gamma \mapsto B_1^\gamma(v, v)$  for any  $v \in H^1(\mathbb{R})$ . We recall the decomposition of  $B_1^\gamma$  into  $B_{1,1}^\gamma$  and  $B_{1,2}^\gamma$  (see (10)). We see that  $B_{1,1}^\gamma(v)$  is clearly holomorphic in  $\gamma$ . From the explicit form of  $\varphi_\gamma$  (see (3)) it is clear that  $\gamma \mapsto \varphi_\gamma^{p-1}(x)$  is holomorphic in  $\gamma$  for any  $x \in \mathbb{R}$ . It then also follows that  $\gamma \mapsto B_{1,2}^\gamma(v)$  is holomorphic.  $\square$

**Remark 20.** There exists another way to show that  $(L_1^\gamma)$  is a real-holomorphic family with respect to  $\gamma \in \mathbb{R}$ . We can use the explicit resolvent formula in [1],

$$(T_1 - k^2)^{-1} = (-\partial_x^2 - k^2)^{-1} + 2\gamma k(-i\gamma + 2k)^{-1}(\overline{G_k(\cdot)}, \cdot)G_k(\cdot),$$

where  $k^2 \in \rho(T_1)$ ,  $\text{Im}k > 0$ ,  $G_k(x) = (i/2k)e^{ik|x|}$ , to verify the holomorphicity.

The following classical result of Weinstein [39] gives a precise description of the spectrum of the operator we want to perturb.

**Lemma 21.** *The operator  $L_1^0$  has exactly one negative simple isolated first eigenvalue. The second eigenvalue is 0, and it is simple and isolated. The nullspace is  $\text{span}\{\partial_x \varphi_0\}$ , and the rest of the spectrum is positive.*

Combining Lemma 19 and Lemma 21, we can apply the theory of analytic perturbations for linear operators (see [27, VII.§1.3]) to get the following lemma. Actually, the perturbed eigenvalues are holomorphic since they are simple.

**Lemma 22.** *There exist  $\gamma_0 > 0$  and two functions  $\lambda : (-\gamma_0, \gamma_0) \mapsto \mathbb{R}$  and  $f : (-\gamma_0, \gamma_0) \mapsto L^2(\mathbb{R})$  such that*

- (i)  $\lambda(0) = 0$  and  $f(0) = \partial_x \varphi_0$ ,
- (ii) For all  $\gamma \in (-\gamma_0, \gamma_0)$ ,  $\lambda(\gamma)$  is the simple isolated second eigenvalue of  $L_1^\gamma$  and  $f(\gamma)$  is an associated eigenvector,
- (iii)  $\lambda(\gamma)$  and  $f(\gamma)$  are holomorphic in  $(-\gamma_0, \gamma_0)$ .

Furthermore,  $\gamma_0 > 0$  can be chosen small enough to ensure that, except the two first eigenvalues, the spectrum of  $L_1^\gamma$  is positive.

Now we investigate how the perturbed second eigenvalue moves depending on the sign of  $\gamma$ .

**Lemma 23.** *There exists  $0 < \gamma_1 < \gamma_0$  such that  $\lambda(\gamma) < 0$  for any  $-\gamma_1 < \gamma < 0$  and  $\lambda(\gamma) > 0$  for any  $0 < \gamma < \gamma_1$ .*

*Proof of Lemma 23.* We develop the functions  $\lambda(\gamma)$  and  $f(\gamma)$  of Lemma 22. There exist  $\lambda_0 \in \mathbb{R}$  and  $f_0 \in L^2(\mathbb{R})$  such that for  $\gamma$  close to 0 we have

$$(16) \quad \lambda(\gamma) = \gamma\lambda_0 + O(\gamma^2),$$

$$(17) \quad f(\gamma) = \partial_x \varphi_0 + \gamma f_0 + O(\gamma^2).$$

From the explicit expression (3) of  $\varphi_\gamma$ , we deduce that there exists  $g_0 \in H^1(\mathbb{R})$  such that for  $\gamma$  close to 0 we have

$$(18) \quad \varphi_\gamma = \varphi_0 + \gamma g_0 + O(\gamma^2).$$

Furthermore, using (18) to substitute into (2) and differentiating (2) with respect to  $\gamma$ , we obtain

$$(19) \quad \langle L_1^0 g_0, \psi \rangle = \varphi_0(0)\psi(0) + O(\gamma),$$

for any  $\psi \in H^1(\mathbb{R})$ .

To develop  $\lambda_0$  with respect to  $\gamma$ , we compute  $(L_1^\gamma f(\gamma), \partial_x \varphi_0)_2$  in two different ways.

On one hand, using  $L_1^\gamma f(\gamma) = \lambda(\gamma)f(\gamma)$ , (16) and (17) lead us to

$$(20) \quad (L_1^\gamma f(\gamma), \partial_x \varphi_0)_2 = \lambda_0 \gamma \|\partial_x \varphi_0\|_2^2 + O(\gamma^2).$$

On the other hand, since  $L_1^\gamma$  is self-adjoint, we get

$$(21) \quad (L_1^\gamma f(\gamma), \partial_x \varphi_0)_2 = (f(\gamma), L_1^\gamma \partial_x \varphi_0)_2.$$

Here we note that  $\partial_x \varphi_0 \in D(L_1^\gamma)$ : indeed,  $\partial_x \varphi_0 \in H^2(\mathbb{R})$  and  $\partial_x \varphi_0(0) = 0$ . We compute the right hand side of (21). We use (9),  $L_1^0 \partial_x \varphi_0 = 0$ , and (18) to obtain

$$(22) \quad \begin{aligned} L_1^\gamma \partial_x \varphi_0 &= p(\varphi_0^{p-1} - \varphi_\gamma^{p-1})\partial_x \varphi_0, \\ &= -\gamma p(p-1)\varphi_0^{p-2}g_0\partial_x \varphi_0 + O(\gamma^2). \end{aligned}$$

Hence, it follows from (17) that

$$(23) \quad (L_1^\gamma f(\gamma), \partial_x \varphi_0)_2 = -(\partial_x \varphi_0, \gamma g_0 p(p-1)\varphi_0^{p-2}\partial_x \varphi_0)_2 + O(\gamma^2).$$

Now, as it was remarked in [9, Lemma 28], it is easy to see that using (2) with  $\gamma = 0$  we get

$$(24) \quad L_1^0(\varphi_0 - \varphi_0^{p-1}) = p(p-1)\varphi_0^{p-2}\partial_x \varphi_0^2,$$

which combined with (23) gives

$$(25) \quad (L_1^\gamma f(\gamma), \partial_x \varphi_0)_2 = -\gamma \langle L_1^0 g_0, \varphi_0 - \varphi_0^p \rangle + O(\gamma^2).$$

Finally, with (19) we obtain from (25)

$$(26) \quad (L_1^\gamma f(\gamma), \partial_x \varphi_0)_2 = -\gamma(\varphi_0(0)^2 - \varphi_0(0)^{p+1}) + O(\gamma^2).$$

Combining (26) and (20) we obtain

$$\lambda_0 = -\frac{\varphi_0(0)^2 - \varphi_0(0)^{p+1}}{\|\partial_x \varphi_0\|_2^2} + O(\gamma).$$

It follows that  $\lambda_0$  is positive for sufficiently small  $|\gamma|$ , which in view of (16) ends the proof.  $\square$

We are now in position to prove Lemma 18.

*Proof of Lemma 18.* Let  $\gamma_\infty$  be defined by

$$\gamma_\infty = \inf\{\tilde{\gamma} < 0; L_1^{\tilde{\gamma}} \text{ has exactly two negative eigenvalues for all } \gamma \in (\tilde{\gamma}, 0]\}.$$

From Lemma 23, we know that  $\gamma_\infty$  is well defined and  $\gamma_\infty \in [-\infty, 0)$ . Arguing by contradiction, we suppose  $\gamma_\infty > -\infty$ .

Let  $N$  be the number of negative eigenvalues of  $L_1^{\gamma_\infty}$ . Denote the first eigenvalue of  $L_1^{\gamma_\infty}$  by  $\Lambda_{\gamma_\infty}$ . Let  $\Gamma$  be defined by

$$\Gamma = \{z \in \mathbb{C}; z = z_1 + iz_2, (z_1, z_2) \in [-b, 0] \times [-a, a], \text{ for some } a > 0, b > |\Lambda_{\gamma_\infty}|\}.$$

From Lemma 17, we know that  $L_1^{\gamma_\infty}$  does not admit zero as eigenvalue. Thus  $\Gamma$  define a contour in  $\mathbb{C}$  of the segment  $[\Lambda_{\gamma_\infty}, 0]$  containing no positive part of the spectrum of  $L_1^{\gamma_\infty}$ , and without any intersection with the spectrum of  $L_1^{\gamma_\infty}$ . It is easily seen (for example, along the lines of the proof of [27, Theorem VII-1.7]) that there exists a small  $\gamma_* > 0$  such that for any  $\gamma \in [\gamma_\infty - \gamma_*, \gamma_\infty + \gamma_*]$ , we can define a holomorphic projection on the negative part of the spectrum of  $L_1^\gamma$  contained in  $\Gamma$  by

$$\Pi(\gamma) = \frac{-1}{2\pi i} \int_{\Gamma} (L_1^\gamma - z)^{-1} dz.$$

Let us insist on the fact that we can choose  $\Gamma$  independently of the parameter  $\gamma$  because 0 is not an eigenvalue of  $L_1^\gamma$  for all  $\gamma$ .

Since  $\Pi$  is holomorphic,  $\Pi$  is continuous in  $\gamma$ , then by a classical connectedness argument (for example, see [27, Lemma I-4.10]), we know that  $\dim(\text{Ran } \Pi(\gamma)) = N$  for any  $\gamma \in [\gamma_\infty - \gamma_*, \gamma_\infty + \gamma_*]$ . Furthermore,  $N$  is exactly the number of negative eigenvalues of  $L_1^\gamma$  when  $\gamma \in [\gamma_\infty - \gamma_*, \gamma_\infty + \gamma_*]$ : indeed, if  $L_1^\gamma$  has a negative eigenvalue outside of  $\Gamma$  it suffice to enlarge  $\Gamma$  (i.e., enlarge  $b$ ) until it contains this eigenvalue to raise a contradiction since then  $L_1^{\gamma_\infty}$  would

have, at least,  $N + 1$  eigenvalues. Now by the definition of  $\gamma_\infty$ ,  $L_1^{\gamma_\infty + \gamma^*}$  has two negative eigenvalues and thus we see that  $L_1^\gamma$  has two negative eigenvalues for all  $\gamma \in [\gamma_\infty - \gamma^*, 0[$  contradicting the definition of  $\gamma_\infty$ .

Therefore  $\gamma_\infty = -\infty$ .  $\square$

**Remark 24.** In [12, Lemma 32], the authors proved that there are *at most* two negative eigenvalues of  $L_1^\gamma$  in  $H^1(\mathbb{R})$  using variational methods. In our present proof, we can directly show that there are exactly two negative eigenvalues without such variational techniques.

#### 2.4. The case $\gamma > 0$ .

The proof of Lemma 18 can be easily adapted to the case  $\gamma > 0$ , and with Lemma 23 we can infer that  $L_1^\gamma$  has only one simple negative eigenvalue when  $\gamma > 0$ . Since  $n(H^\gamma) = n(L_1^\gamma)$ , it follows that (in the following Lemmas 25, 26 and Proposition 27, there is no omission of parameter  $\omega$  to understand the dependence clearly)

**Lemma 25.** *Let  $\gamma > 0$  and  $\omega > \gamma^2/4$ . Then the operator  $H_\omega^\gamma$  has only one negative eigenvalue, that is  $n(H_\omega^\gamma) = 1$ .*

When  $\gamma > 0$ , the sign of  $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2$  was computed in [13]. Precisely :

**Lemma 26.** *Let  $\gamma > 0$  and  $\omega > \gamma^2/4$ . We have :*

- (i) *If  $1 < p \leq 5$  and  $\omega > \gamma^2/4$  then  $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 > 0$ ,*
- (ii) *If  $p > 5$  and  $\gamma^2/4 < \omega < \omega_1$  then  $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 > 0$ ,*
- (iii) *If  $p > 5$  and  $\omega > \omega_1$  then  $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 < 0$ .*

Here  $\omega_1$  is defined as follows:

$$\frac{p-5}{p-1}J(\omega_1) = \frac{\gamma}{2\sqrt{\omega_1}} \left(1 - \frac{\gamma^2}{4\omega_1}\right)^{-(p-3)/(p-1)},$$

$$J(\omega_1) = \int_{A(\omega_1,\gamma)}^\infty \operatorname{sech}^{4/(p-1)} y dy, \quad A(\omega_1,\gamma) = \tanh^{-1} \left( \frac{\gamma}{2\sqrt{\omega_1}} \right).$$

Then, using Lemma 25, Lemma 26 and Proposition 12, we can give an alternative proof of [13, Theorem 1] (see also [12, Remark 33]). Precisely, we obtain :

**Proposition 27.** *Let  $\gamma > 0$  and  $\omega > \gamma^2/4$ .*

- (i) *Let  $1 < p \leq 5$ . Then  $e^{i\omega t} \varphi_{\omega,\gamma}(x)$  is stable in  $H^1(\mathbb{R})$  for any  $\omega \in (\gamma^2/4, +\infty)$ .*
- (ii) *Let  $p > 5$ . Then  $e^{i\omega t} \varphi_{\omega,\gamma}(x)$  is stable in  $H^1(\mathbb{R})$  for any  $\omega \in (\gamma^2/4, \omega_1)$ , and unstable in  $H^1(\mathbb{R})$  for any  $\omega \in (\omega_1, +\infty)$ .*

### 2.5. The radial case.

Before we start to discuss the stability in the radial case, we mention the following remarkable fact.

**Lemma 28.** *The function  $f(\gamma)$  defined in Lemma 22 and corresponding to the second negative eigenvalue of  $L_1^\gamma$  can be extended to  $(-\infty, +\infty)$ . Furthermore,  $f(\gamma) \in H^1(\mathbb{R})$  is an odd function, for each  $\gamma \in (-\infty, +\infty)$ .*

*Proof.* First, we indicate how the extension of  $f$  to  $(-\infty, +\infty)$  can be done. We see by the proof in [33, Theorem XII.8] that the functions  $f(\gamma)$  and  $\lambda(\gamma)$  defined in Lemma 22 exist, are holomorphic and represent an eigenvector and an eigenvalue for all  $\gamma \in \mathbb{R}$ , since  $(L_1^\gamma)$  is a real-holomorphic family in  $\gamma \in \mathbb{R}$ . Namely we can repeat the argument of Lemma 22 at each point  $\gamma$  and on each neighborhood of  $\gamma$ . This is possible because the set  $\{(\gamma, \lambda); \gamma \in \mathbb{R}, \lambda \in \rho(L_1^\gamma)\}$  is open and the function  $(L_1^\gamma - \lambda)^{-1}$  defined on this set is a holomorphic function of two variables ([33, Theorem XII.7]).

Secondly, as it was observed in [12, 9], the eigenvectors of  $L_1^\gamma$  are even or odd. Indeed, let  $\xi$  be an eigenvalue of  $L_1^\gamma$  with eigenvector  $v \in D(L_1^\gamma)$ . Then clearly  $\tilde{v}$  with  $\tilde{v}(x) = v(-x)$  is also an eigenvector associated to  $\xi$ . In particular,  $v$  and  $\tilde{v}$  satisfy both

$$-\partial_x^2 v + (\omega - \xi)v - p\varphi_\gamma^{p-1}v = 0 \text{ on } [0, +\infty),$$

thus there exists  $\eta \in \mathbb{R}$  such that  $v = \eta\tilde{v}$  on  $[0, +\infty)$  (this is standard, see, for example, [4, Chapter 2, Theorem 3.3]). If  $v(0) \neq 0$ , it is immediate that  $\eta = 1$ . If  $v(0) = 0$ , then  $\partial_x v(0+) \neq 0$  (otherwise the Cauchy-Lipschitz Theorem leads to  $v \equiv 0$ ), and it is also immediate that  $\eta = -1$ . Arguing in a same way on  $(-\infty, 0]$ , we conclude that  $v$  is even or odd, and in particular  $v$  is even if and only if  $v(0) \neq 0$ .

Finally, we prove the last statement only for the case  $\gamma < 0$  since the case  $\gamma > 0$  is similar. We remark that  $\partial_x \varphi_0$  is odd. Since  $\lim_{\gamma \rightarrow 0} (f(\gamma), \partial_x \varphi_0)_2 = \|\partial_x \varphi_0\|_2^2 \neq 0$ , we have  $(f(\gamma), \partial_x \varphi_0)_2 \neq 0$  for  $\gamma$  close to 0, thus  $f(\gamma)$  cannot be even, and therefore  $f(\gamma)$  is odd. Let  $\tilde{\gamma}_\infty$  be

$$\tilde{\gamma}_\infty = \inf\{\tilde{\gamma} < 0; f(\gamma) \text{ is odd for any } \gamma \in (\tilde{\gamma}, 0]\}.$$

We suppose that  $\tilde{\gamma}_\infty > -\infty$ . If  $f(\tilde{\gamma}_\infty)$  is odd, by continuity in  $\gamma$  of  $f(\gamma)$  with  $L^2$  value, there exists  $\varepsilon > 0$  such that  $f(\tilde{\gamma}_\infty - \varepsilon)$  is odd which is a contradiction with the definition of  $\tilde{\gamma}_\infty$ , thus  $f(\tilde{\gamma}_\infty)$  is even. Now,  $f(\tilde{\gamma}_\infty)$  is the limit of odd functions, thus is odd. The only possibility to have  $f(\tilde{\gamma}_\infty)$  both even and odd is  $f(\tilde{\gamma}_\infty) \equiv 0$ , which is impossible because  $f(\tilde{\gamma}_\infty)$  is an eigenvector.  $\square$

We can deduce the number of negative eigenvalues of  $L_1^\gamma$  in the radial case from the result on the eigenvalues of  $L_1^\gamma$  considered in the whole space  $L^2(\mathbb{R})$ .

Indeed, Lemma 28 ensures that the second eigenvalue of  $L_1^\gamma$  considered in the whole space  $L^2(\mathbb{R})$  is associated with an odd eigenvector, and thus disappears when the problem is restricted to subspace of radial functions. Furthermore, since  $\varphi_\gamma \in H_{\text{rad}}^1(\mathbb{R})$  and  $\langle L_1^\gamma \varphi_\gamma, \varphi_\gamma \rangle < 0$ , we can infer that the first negative eigenvalue of  $L_1^\gamma$  is still present when the problem is restricted to sets of radial functions. Recalling that  $n(H^\gamma) = n(L_1^\gamma)$ , we obtain.

**Lemma 29.** *Let  $\gamma < 0$ . Then the operator  $H^\gamma$  considered on  $H_{\text{rad}}^1(\mathbb{R})$  has only one negative eigenvalue, that is  $n(H^\gamma) = 1$ .*

Combining Lemma 29, Lemma 13 and Proposition 12, we recover the results of [12] recalled in Proposition 7.

Alternatively, subsection 2.3 can be adapted to the radial case. All the function spaces should be reduced to spaces of even functions, and Lemma 29 can also be proved in this way.

### 3. STRONG INSTABILITY

This section is devoted to the proof of Theorem 2.

We begin by introducing some notations

$$\begin{aligned} \mathbb{M} &= \{v \in H_{\text{rad}}^1(\mathbb{R}) \setminus \{0\}; Q_\gamma(v) = 0, I_{\omega,\gamma}(v) \leq 0\}, \\ d_{\mathbb{M}} &= \inf\{S_{\omega,\gamma}(v); v \in \mathbb{M}\}, \end{aligned}$$

where  $S_{\omega,\gamma}$  and  $I_{\omega,\gamma}$  are defined in Proposition 1 and  $Q_\gamma$  in Proposition 10.

Our proof is divided in three steps.

**STEP 1.** We prove that  $\varphi_{\omega,\gamma}$  is also a minimizer of  $d_{\mathbb{M}}$ .

Because of Pohozaev identity  $Q_\gamma(\varphi_{\omega,\gamma}) = 0$  (see [3]), it is clear that  $d_{\mathbb{M}} \leq d(\omega)$ , thus we only have to show  $d_{\mathbb{M}} \geq d(\omega)$ . Let  $v \in \mathbb{M}$ . If  $I_{\omega,\gamma}(v) = 0$ , we have  $S_{\omega,\gamma}(v) \geq d(\omega)$ , therefore we suppose  $I_{\omega,\gamma}(v) < 0$ . For  $\alpha > 0$ , let  $v^\alpha$  be such that  $v^\alpha(x) = \alpha^{1/2}v(\alpha x)$ . We have

$$I_{\omega,\gamma}(v^\alpha) = \alpha^2 \|\partial_x v\|_2^2 + \omega \|v\|_2^2 - \gamma \alpha |v(0)|^2 - \alpha^{(p-1)/2} \|v\|_{p+1}^{p+1},$$

thus  $\lim_{\alpha \rightarrow 0} I_{\omega,\gamma}(v^\alpha) = \omega \|v\|_2^2 > 0$ , and by continuity there exists  $0 < \alpha_0 < 1$  such that  $I_{\omega,\gamma}(v^{\alpha_0}) = 0$ . Therefore

$$(27) \quad S_{\omega,\gamma}(v^{\alpha_0}) \geq d(\omega).$$

Consider now  $\frac{\partial}{\partial \alpha} S_{\omega,\gamma}(v^\alpha) = \alpha \|\partial_x v\|_2^2 - \frac{\gamma}{2} |v(0)|^2 - \frac{p-1}{2(p+1)} \alpha^{(p-3)/2} \|v\|_{p+1}^{p+1}$ . Since  $p \geq 5$  and  $Q_\gamma(v) = 0$ , we have for  $\alpha \in [0, 1]$

$$\frac{\partial}{\partial \alpha} S_{\omega,\gamma}(v^\alpha) \geq \alpha Q_\gamma(v) - \frac{\gamma}{2} (1-\alpha) |v(0)|^2 = -\frac{\gamma}{2} (1-\alpha) |v(0)|^2$$

and thus  $\frac{\partial}{\partial \alpha} S_{\omega, \gamma}(v^\alpha) \geq 0$  for all  $\alpha \in [0, 1]$ , which leads to  $S_{\omega, \gamma}(v) \geq S_{\omega, \gamma}(v^{\alpha_0})$ . It follows by (27) that  $S_{\omega, \gamma}(v) \geq d(\omega)$ , which concludes  $d_M = d(\omega)$ .

STEP 2. We construct a sequence of initial data  $\varphi_\omega^\alpha$  satisfying the following properties :

$$S_{\omega, \gamma}(\varphi_\omega^\alpha) < d(\omega), I_{\omega, \gamma}(\varphi_\omega^\alpha) < 0 \text{ and } Q_\gamma(\varphi_\omega^\alpha) < 0.$$

These properties are invariant under the flow of (1).

For  $\alpha > 0$ , we define  $\varphi_\omega^\alpha$  by  $\varphi_\omega^\alpha(x) = \alpha^{1/2} \varphi_{\omega, \gamma}(\alpha x)$ . Since  $p \geq 5$ ,  $\gamma < 0$  and  $Q_\gamma(\varphi_{\omega, \gamma}) = 0$ , easy computations permit to obtain

$$\frac{\partial^2}{\partial \alpha^2} S_{\omega, \gamma}(\varphi_\omega^\alpha)|_{\alpha=1} < 0, \frac{\partial}{\partial \alpha} I_{\omega, \gamma}(\varphi_\omega^\alpha)|_{\alpha=1} < 0 \text{ and } \frac{\partial}{\partial \alpha} Q_\gamma(\varphi_\omega^\alpha)|_{\alpha=1} < 0,$$

and thus for any  $\alpha > 1$  close enough to 1 we have

$$(28) \quad S_{\omega, \gamma}(\varphi_\omega^\alpha) < S_{\omega, \gamma}(\varphi_{\omega, \gamma}), I_{\omega, \gamma}(\varphi_\omega^\alpha) < 0 \text{ and } Q_\gamma(\varphi_\omega^\alpha) < 0.$$

Now fix a  $\alpha > 1$  such that (28) is satisfied, and let  $u^\alpha(t, x)$  be the solution of (1) with  $u^\alpha(0) = \varphi_\omega^\alpha$ . Since  $\varphi_\omega^\alpha$  is radial,  $u^\alpha(t)$  is also radial for all  $t > 0$  (see Remark 4). We claim that the properties described in (28) are invariant under the flow of (1). Indeed, since from (4) and (5) we have for all  $t > 0$

$$(29) \quad S_{\omega, \gamma}(u^\alpha(t)) = S_{\omega, \gamma}(\varphi_\omega^\alpha) < S_{\omega, \gamma}(\varphi_{\omega, \gamma}),$$

we infer that  $I_{\omega, \gamma}(u^\alpha(t)) \neq 0$  for any  $t \geq 0$ , and by continuity we have  $I_{\omega, \gamma}(u^\alpha(t)) < 0$  for all  $t \geq 0$ . It follows that  $Q_\gamma(u^\alpha(t)) \neq 0$  for any  $t \geq 0$  (if not  $u^\alpha(t) \in M$  and thus  $S_{\omega, \gamma}(u^\alpha(t)) > S_{\omega, \gamma}(\varphi_{\omega, \gamma})$  which contradicts (29)), and by continuity we have  $Q_\gamma(u^\alpha(t)) < 0$  for all  $t \geq 0$ .

STEP 3. We prove that  $Q_\gamma(u^\alpha)$  stays negative and away from 0 for all  $t \geq 0$ .

Let  $t > 0$  be arbitrary chosen, define  $v = u^\alpha(t)$  and for  $\beta > 0$  let  $v^\beta$  be such that  $v^\beta(x) = v(\beta x)$ . Then we have

$$Q_\gamma(v^\beta) = \beta \|\partial_x v\|_2^2 - \frac{\gamma}{2} |v(0)|^2 - \beta^{-1} \frac{p-1}{2(p+1)} \|v\|_{p+1}^{p+1},$$

thus  $\lim_{\beta \rightarrow +\infty} Q_\gamma(v^\beta) = +\infty$ , and by continuity there exists  $\beta_0$  such that  $Q_\gamma(v^{\beta_0}) = 0$ . If  $I_{\omega, \gamma}(v^{\beta_0}) \leq 0$ , we keep  $\beta_0$  unchanged; otherwise, we replace it by  $\tilde{\beta}_0$  such that  $1 < \tilde{\beta}_0 < \beta_0$ ,  $I_{\omega, \gamma}(v^{\tilde{\beta}_0}) = 0$  and  $Q_\gamma(v^{\tilde{\beta}_0}) \leq 0$ . Thus in any case we have  $S_{\omega, \gamma}(v^{\beta_0}) \geq d(\omega)$ . Now, we have

$$S_{\omega, \gamma}(v) - S_{\omega, \gamma}(v^{\beta_0}) = \frac{1 - \beta_0}{2} \|\partial_x v\|_2^2 + (1 - \beta_0^{-1}) \left( \frac{\omega}{2} \|v\|_2^2 - \frac{1}{p+1} \|v\|_{p+1}^{p+1} \right),$$

from the expression of  $Q_\gamma$  and  $\beta_0 > 1$  it follows that

$$(30) \quad S_{\omega,\gamma}(v) - S_{\omega,\gamma}(v^{\beta_0}) \geq \frac{1}{2}(Q_\gamma(v) - Q_\gamma(v^{\beta_0})).$$

Therefore, from (30),  $Q_\gamma(v^{\beta_0}) \leq 0$  and  $S_{\omega,\gamma}(v^{\beta_0}) \geq d(\omega)$  we have

$$(31) \quad Q_\gamma(v) \leq -m = 2(S_{\omega,\gamma}(v) - d(\omega)) < 0$$

where  $m$  is independent of  $t$  since  $S_{\omega,\gamma}$  is a conserved quantity.

CONCLUSION. Finally, thanks to (31) and Proposition 10, we have

$$(32) \quad \|xu^\alpha(t)\|_2^2 \leq -mt^2 + Ct + \|x\varphi_\omega^\alpha\|_2^2.$$

For  $t$  large, the right member of (32) becomes negative, thus there exists  $T^\alpha < +\infty$  such that

$$\lim_{t \rightarrow T^\alpha} \|\partial_x u^\alpha(t)\|_2^2 = +\infty.$$

Since it is clear that  $\varphi_\omega^\alpha \rightarrow \varphi_{\omega,\gamma}$  in  $H^1(\mathbb{R})$  when  $\alpha \rightarrow 1$ , Theorem 2 is proved.

**Remark 30.** It is not hard to see that the set

$$\mathcal{I} = \{v \in H^1(\mathbb{R}); S_{\omega,\gamma}(v) < d(\omega), I_{\omega,\gamma}(v) > 0\}$$

is invariant under the flow of (1), and that a solution with initial data belonging to  $\mathcal{I}$  is global. Thus using the minimizing character of  $\varphi_{\omega,\gamma}$  and performing an analysis in the same way than in [20], it is possible to find a family of initial data in  $\mathcal{I}$  approaching  $\varphi_{\omega,\gamma}$  in  $H^1(\mathbb{R})$  and such that the associated solution of (1) exists globally but escaped in finite time from a tubular neighborhood of  $\varphi_{\omega,\gamma}$  (see also [11, 18] for an illustration of this approach on a related problem).

#### 4. THE VIRIAL THEOREM

This section is devoted to the proof of Proposition 10.

For  $a \in \mathbb{N} \setminus \{0\}$ , we define  $V^a(x) = \gamma a e^{-\pi a^2 x^2}$ . It is clear that  $\int_{\mathbb{R}} V^a(x) = \gamma$  and  $V^a \rightharpoonup \gamma \delta$  weak- $\star$  in  $H^{-1}(\mathbb{R})$  when  $a \rightarrow +\infty$ .

We begin by the construction of approximate solutions : for

$$(33) \quad \begin{cases} i\partial_t u &= -\partial_x^2 u - V^a u - |u|^{p-1}u, \\ u(0) &= u_0, \end{cases}$$

and thanks to [6, Proposition 6.4.1], for every  $a \in \mathbb{N} \setminus \{0\}$  there exists  $T^a > 0$  and a unique maximal solution  $u^a \in \mathcal{C}([0, T^a], H^1(\mathbb{R})) \cap \mathcal{C}^1([0, T^a], H^{-1}(\mathbb{R}))$  of (33) which satisfies for all  $t \in [0, T^a)$

$$(34) \quad E^a(u^a(t)) = E^a(u_0),$$

$$(35) \quad \|u^a(t)\|_2 = \|u_0\|_2,$$

where  $E^a(v) = \frac{1}{2}\|\partial_x v\|_2^2 - \frac{1}{2}\int_{\mathbb{R}} V^a|v|^2 dx - \frac{1}{p+1}\|v\|_{p+1}^{p+1}$ . Moreover, the function  $f^a : t \mapsto \int_{\mathbb{R}} x^2|u^a(t, x)|^2 dx$  is  $\mathcal{C}^2$  by [6, Proposition 6.4.2], and

$$(36) \quad \partial_t f^a = 4\text{Im} \int_{\mathbb{R}} \overline{u^a} x \partial_x u^a dx,$$

$$(37) \quad \partial_t^2 f^a = 8Q_\gamma^a(u^a)$$

where  $Q_\gamma^a$  is defined for  $v \in H^1(\mathbb{R})$  by

$$Q_\gamma^a(v) = \|\partial_x v\|_2^2 + \frac{1}{2}\int_{\mathbb{R}} x(\partial_x V^a)|v|^2 dx - \frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1}.$$

Then, we find estimates on  $(u^a)$ . Let  $M \geq \|u_0\|_{H^1(\mathbb{R})}$  (an exact value of  $M$  will be precise later). We define

$$(38) \quad t^a = \sup\{t > 0; \|u^a(s)\|_{H^1(\mathbb{R})} \leq 2M \text{ for all } s \in [0, t]\}.$$

Since  $u^a$  satisfies (33), we have

$$\sup_{a \in \mathbb{N} \setminus \{0\}} \|\partial_t u^a\|_{L^\infty([0, t^a], H^{-1}(\mathbb{R}))} \leq C,$$

and thus for all  $t \in [0, t^a)$  and for all  $a \in \mathbb{N} \setminus \{0\}$  we get

$$(39) \quad \|u^a(t) - u_0\|_2^2 = 2 \int_0^t (u^a(s) - u_0, \partial_t u^a(s))_2 ds \leq Ct$$

where  $C$  depends only on  $M$ . Now we have

$$\frac{1}{p+1}(\|u^a\|_{p+1}^{p+1} - \|u_0\|_{p+1}^{p+1}) = \int_0^1 \int_{\mathbb{R}} (u^a - u_0)|su^a + (1-s)u_0|^p dx ds$$

which combined with Hölder inequality, Sobolev embeddings, (38) and (39) gives

$$(40) \quad \frac{1}{p+1}(\|u^a\|_{p+1}^{p+1} - \|u_0\|_{p+1}^{p+1}) \leq Ct^{1/2}.$$

Moreover, using (38), Sobolev embeddings, Gagliardo-Nirenberg inequality and (39) we obtain

$$(41) \quad \left| \int_{\mathbb{R}} V^a(|u^a|^2 - |u_0|^2) \right| \leq Ct^{1/4}.$$

Combining (34), (35), (40) and (41) leads to

$$\|u^a(t)\|_{H^1(\mathbb{R})}^2 \leq M^2 + C(t^{1/4} + t^{1/2}) \text{ for all } t \in [0, t^a) \text{ and for all } a \in \mathbb{N} \setminus \{0\},$$

and choosing  $T_M$  (depending only on  $M$ ) such that  $C(T_M^{1/4} + T_M^{1/2}) = 3M^2$  we obtain for all  $a \in \mathbb{N} \setminus \{0\}$  the estimates

$$(42) \quad \begin{aligned} \|u^a\|_{L^\infty([0, T_M], H^1(\mathbb{R}))} &\leq 2M, \\ \|\partial_t u^a\|_{L^\infty([0, T_M], H^{-1}(\mathbb{R}))} &\leq C. \end{aligned}$$

In particular, it follows from (42) that  $T_M \leq t^a$  for all  $a \in \mathbb{N} \setminus \{0\}$ .

Now we can pass to the limit : thanks to (42) there exists  $u \in L^\infty([0, T_M], H^1(\mathbb{R}))$  such that for all  $t \in [0, T_M)$  we have

$$(43) \quad u^a(t) \rightharpoonup u(t) \text{ weakly in } H^1(\mathbb{R}) \text{ when } a \rightarrow +\infty,$$

which immediately induces that when  $a \rightarrow +\infty$ ,

$$(44) \quad |u^a(t)|^{p-1}u^a(t) \rightharpoonup |u(t)|^{p-1}u(t) \text{ weakly in } H^{-1}(\mathbb{R}).$$

In particular, thanks to Sobolev embeddings, we have

$$u^a(t, x) \rightarrow u(t, x) \text{ a.e. and uniformly on the compact sets of } \mathbb{R},$$

and it is not hard to see that it permit to show

$$(45) \quad V^a u^a \rightharpoonup u \gamma \delta \text{ weak-}\star \text{ in } H^{-1}(\mathbb{R}).$$

Since  $u^a$  satisfies (33), it follows from (43), (44) and (45) that  $u$  satisfies (1). Finally, by (5) and (35), we have

$$u^a \rightarrow u \text{ in } \mathcal{C}([0, T_M], L^2(\mathbb{R})),$$

thus, from Gagliardo-Nirenberg inequality and (42), we have

$$u^a \rightarrow u \text{ in } \mathcal{C}([0, T_M], L^{p+1}(\mathbb{R})),$$

and by (4) and (34) it follows that

$$(46) \quad u^a \rightarrow u \text{ in } \mathcal{C}([0, T_M], H^1(\mathbb{R})).$$

We have to prove that the time interval  $[0, T_M)$  can be extended as large as we need. Let  $0 < T < T_{u_0}$  and

$$M = \sup\{\|u(t)\|_{H^1(\mathbb{R})}, t \in [0, T]\}.$$

If  $T_M \geq T$ , there is nothing left to do, thus we suppose  $T_M < T$ . From (46) we have  $\|u^a(T_M)\|_{H^1(\mathbb{R})} \leq M$  for  $a$  large enough. By performing a shift of time of length  $T_M$  in (1) and (33) and repeating the first steps of the proof we obtain

$$u^a \rightarrow u \text{ in } \mathcal{C}([T_M, 2T_M], H^1(\mathbb{R})).$$

Now we reiterate this procedure a finite number of times until we covered the interval  $[0, T]$  to obtain

$$(47) \quad u^a \rightarrow u \text{ in } \mathcal{C}([0, T], H^1(\mathbb{R})).$$

To conclude, we remark that (6) follows from the same proof than [6, Lemma 6.4.3] (computing with  $\|e^{\varepsilon|x|^2}xu(t)\|_2^2$  and passing to the limit  $\varepsilon \rightarrow 0$ ), thus we have

$$(48) \quad \|xu(t)\|_2^2 = \|xu_0\|_2^2 + 4 \int_0^t \operatorname{Im} \int_{\mathbb{R}} \overline{u(s)} x \partial_x u(s) dx ds.$$

From (36), Cauchy-Schwartz inequality and (42) we have

$$\partial_t (\|xu^a(t)\|_2^2) \leq C \|xu^a(t)\|_2,$$

which implies that

$$\|xu^a(t)\|_2 \leq \|xu_0\|_2 + Ct.$$

Since in addition we have

$$xu^a(t, x) \rightarrow xu(t, x) \text{ a.e.},$$

we infer that

$$xu^a(t, x) \rightharpoonup xu(t, x) \text{ weakly in } L^2(\mathbb{R}).$$

Recalling that

$$\partial_x u^a \rightarrow \partial_x u \text{ strongly in } L^2(\mathbb{R})$$

we can pass to the limit in (48) to have

$$\|xu^a(t)\|_2 \rightarrow \|xu(t)\|_2.$$

On the other side, since we have (37) and (47), we get (7).

**Remark 31.** Our method of approximation is inspired of the one developed in [8] by Cazenave and Weissler to prove the local well-posedness of the Cauchy problem for nonlinear Schrödinger equations. Actually, slight modifications in our proof of Proposition 10 would permit to give an alternative proof of Proposition 3.

## 5. NUMERICAL RESULTS

In this Section, we use numerical simulations to complement the rigorous theory on stability and instability of the standing waves of (1). Our approach here is similar to the one in [9]. In order to study stability under radial perturbations, we use the initial condition

$$(49) \quad u_0(x) = (1 + \delta_p)\varphi_{\omega,\gamma}(x).$$

In order to study stability under non-radial (asymmetric) perturbations, we use the initial condition

$$(50) \quad u_0(x) = \varphi_{\omega,\gamma}(x - \delta_c),$$

when  $\delta_c$  is the lateral shift of the initial condition. In some cases (when the standing wave has a negative slope and the linearized problem has two negative eigenvalues), we use the initial condition

$$(51) \quad u_0(x) = (1 + \delta_p)\varphi_{\omega,\gamma}(x - \delta_c).$$

### 5.1. Stability in $H_{\text{rad}}^1(\mathbb{R})$ .

5.1.1. *Strength of radial stability.* When  $\gamma > 0$ , the standing waves are known to be stable in  $H_{\text{rad}}^1(\mathbb{R})$  for  $1 < p \leq 5$ . The rigorous theory, however, does not address the issue of the *strength of radial stability*. This issue is of most interest in the case  $p = 5$ , which is unstable when  $\gamma = 0$ .

For  $\delta_p > 0$ , it is useful to define

$$(52) \quad F(\delta_p) = \max_{t \geq 0} \left\{ \frac{\max_x |u(x, t)| - \max_x \varphi_{\omega,\gamma}}{\max_x \varphi_{\omega,\gamma}} \right\}$$

as a measure of the strength of radial stability. Figure 2 shows the normalized values  $\max_x |u|/\max_x \varphi_{\omega,\gamma}$  as a function of  $t$ , for the initial condition (49) with  $\omega = 4$  and  $\gamma = 1$ . When  $p = 3$ , a perturbation of  $\delta_p = 0.01$  induces small oscillations and  $F(0.01) = 1.9\%$ . Therefore, roughly speaking, a 1% perturbation of the initial condition leads to a maximal deviation of 2%. A larger perturbation of  $\delta_p = 0.08$  causes the magnitude of the oscillations to increase approximately by the same ratio, so that  $F(0.08) = 15\%$ . Using the same perturbations with  $p = 5$ , however, leads to significantly larger deviations. Thus,  $F(0.01) = 8.8\%$ , i.e., more than 4 times bigger than for  $p = 3$ , and  $F(0.08) = 122\%$ , i.e., more than 8 times than for  $p = 3$ .

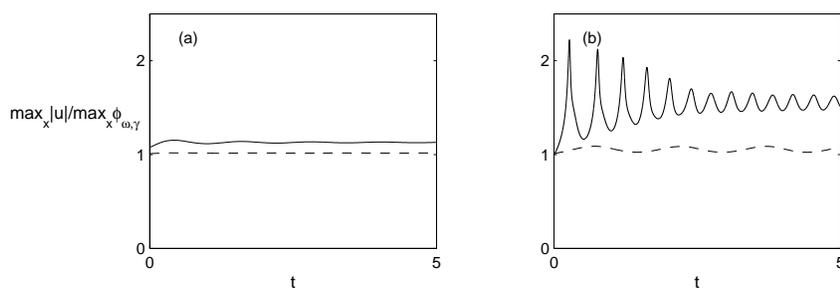


FIGURE 2.  $\max_x |u|/\max_x \varphi_{\omega,\gamma}$  as a function of  $t$  for  $\omega = 4, \gamma = 1, \delta_p = 0.01$  (dashed line) and  $\delta_p = 0.08$  (solid line). (a)  $p = 3$  (b)  $p = 5$ .

In [9, 10], Fibich, Sivan and Weinstein observed that the strength of radial stability is related to the magnitude of slope  $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2$ , so that the larger

$\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2$ , the "more stable" the solution is. Indeed, numerically we found that when  $\omega = 4$ ,  $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2$  is equal to 1.0 for  $p = 3$  and 0.056 for  $p = 5$ .

Since when  $\gamma = 0$ , the slope is positive for  $p < 5$  but zero for  $p = 5$ , for  $\gamma > 0$  the slope is smaller in the critical case than in the subcritical case. Therefore, we make the following informal observation:

**Observation 32.** *Radial stability of the standing waves of (1) with  $\gamma > 0$  is "weaker" in the critical case  $p = 5$  than in the subcritical case  $p < 5$ .*

Clearly, this difference would be more dramatic at smaller (positive) values of  $\gamma$ . Indeed, if in the simulation of Figure 2 with  $\delta_p = 0.01$  we reduce  $\gamma$  from 1 to 0.5 and then to 0.1, this has almost no effect when  $p = 3$ , where the value of  $F$  slightly increases from 1.9% to 2.1% and to 2.5%, respectively, see Figure 3a. However, if we repeat the same simulations with  $p = 5$ , then reducing the value of  $\gamma$  has a much larger effect, see Figure 3b, where  $F$  increases from 8.9% for  $\gamma = 1$  to 24% for  $\gamma = 0.5$ . Moreover, when we further reduced  $\gamma$  to 0.1, the solution seems to undergo collapse.<sup>1</sup> This implies that when  $p = 5$  and  $\gamma > 0$ , the standing wave is stable, yet it can collapse under a sufficiently large perturbation.

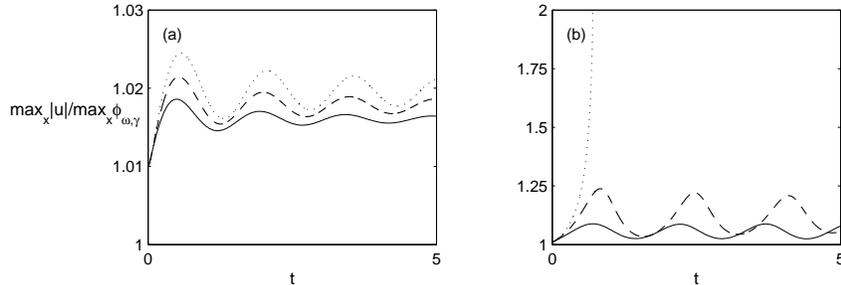


FIGURE 3.  $\max_x |u| / \max_x \varphi_{\omega,\gamma}$  as a function of  $t$  for  $\omega = 4$ ,  $\delta_p = 0.01$ , and  $\gamma = 1$  (solid line),  $\gamma = 0.5$  (dashed line) and  $\gamma = 0.1$  (dots). (a)  $p = 3$  (b)  $p = 5$ .

5.1.2. *Characterization of radial instability for  $3 < p < 5$  and  $\gamma < 0$ .* We consider the subcritical repulsive case  $p = 4$  and  $\gamma = -1$ . In this case, there is threshold  $\omega_2$  such that  $\varphi_{\omega,\gamma}$  is stable for  $\omega > \omega_2$  and unstable for  $\omega < \omega_2$ . By numerical calculation we found that  $\omega_2(p = 4, \gamma = -1) \approx 0.82$ . Accordingly, we chose two representative values of  $\omega$ :  $\omega = 0.5$  in the unstable regime, and  $\omega = 2$  in the stable regime.

<sup>1</sup>Clearly, one cannot use numerics to determine that a solution becomes singular, as it is always possible that collapse would be arrested at some higher focusing levels.

Figure 4a demonstrates the stability for  $\omega = 2$ . Indeed, reducing the perturbation from  $\delta_p = 0.005$  to 0.001 results in reduction of the relative magnitude of the oscillations by roughly five times, from  $F(0.005) \approx 10\%$  to  $F(0.001) \approx 2\%$ . The dynamics in the unstable case  $\omega = 0.5$  is also oscillatory, see Figure 4b. However, in this case  $F(0.005) = 79\%$ , i.e., eight times larger than for  $\omega = 2$ . More importantly, unlike the stable case, a perturbation of  $\delta_p = 0.001$  does not result in a reduction of the relative magnitude of the oscillations by  $\approx 5$ . In fact, the relative magnitude of the oscillations decreases only to  $F(0.001) = 66\%$ .

In the homogeneous NLS, unstable standing waves perturbed with  $\delta_p > 0$  always undergo collapse. Since, however, for  $p = 4$  it is impossible to have collapse, an interesting question is the nature of the instability in the unstable region  $\omega < \omega_2$ . In Figure 4b we already saw that  $\max|u(x, t)|$  undergoes oscillations. In order to better understand the nature of this unstable oscillatory dynamics, we plot in Figure 5 the spatial profile of  $|u(x, t)|$  at various values of  $t$ . In addition, at each  $t$  we plot  $\phi_{\omega^*(t), \gamma}(x)$ , where  $\omega^*(t)$  is determined from the relation

$$\max_x \phi_{\omega^*(t), \gamma}(x) = \max_x |u(x, t)|.$$

Since the two curves are nearly indistinguishable (especially in the central region), this shows that the unstable dynamics corresponds to "movement along the curve  $\phi_{\omega^*(t)}$ ".

In Figure 6 we see that  $\omega^*(t)$  undergoes oscillations, in accordance with the oscillations of  $\max_x |u|$ . Furthermore, as one may expect, collapse is arrested only when  $\omega^*(t)$  reaches a value ( $\approx 2.86$ ) which is in the stability region (i.e., above  $\omega_2$ ).

**Observation 33.** *When  $\gamma < 0$  and  $3 < p < 5$ , the instability in  $H_{\text{rad}}^1(\mathbb{R})$  is a "finite width instability", i.e., the solution narrows down along the curve  $\phi_{\omega^*(t), \gamma}$  until it "reaches" a finite width in the stable region  $\omega > \omega_2$ , at which point collapse is arrested.*

Note that this behavior was already observed in [9], Fig 19. Therefore, more generally, we conjecture that

**Observation 34.** *When the slope is negative (i.e.,  $\partial_\omega \|\varphi_{\omega, \gamma}\|_2^2 < 0$ ), then the symmetric perturbation (49) with  $0 < \delta_p \ll 1$  leads to a finite-width instability in the subcritical case, and to a finite-time collapse in the critical and supercritical cases.*

5.1.3. *Supercritical case ( $p > 5$ ).* We recall that when  $\gamma > 0$  and  $p > 5$ , the standing wave is stable for  $\gamma^2/4 < \omega < \omega_1$  and unstable for  $\omega_1 < \omega$ . When  $\gamma < 0$  and  $p > 5$  the standing wave is strongly unstable under radial

perturbations for any  $\omega$ , i.e., an infinitesimal perturbation can lead to collapse. Figure 7 shows the behavior of perturbed solutions for  $p = 6$  and  $\omega = 1$ . As predicted by the theory, when  $\delta_p = 0.001$ , the solution blows up for  $\gamma = -1$  and  $\gamma = 0$ , but undergoes small oscillations (i.e., is stable) for  $\gamma = 1$ . Indeed, we found numerically that  $\omega_1(p = 6, \gamma = 1) \approx 2.9$ , so that the standing wave is stable for  $\omega = 1$ . However, when we increase the perturbation to  $\delta_p = 0.1$ , the solution with  $\gamma = 1$  also seems to undergo collapse. This implies that when  $p > 5, \gamma > 0$  and  $\omega < \omega_1$  the standing wave is stable, yet it can collapse under a sufficiently large perturbation. In order to find the type of instability for  $\gamma > 0$  and  $\omega > \omega_1$ , we solve the NLS (1) with  $p = 6, \gamma = 1$  and  $\omega = 4$ . In this case,  $\delta_p = 0.001$  seems to lead to collapse, see Figure 8, suggesting a strong instability for  $p > 5, \gamma > 0$  and  $\omega > \omega_1$ . Therefore, we make the following informal observation:

**Observation 35.** *If a standing wave of (1) with  $p > 5$  is unstable in  $H_{\text{rad}}^1(\mathbb{R})$ , then the instability is strong.*

## 5.2. Stability under non-radial perturbations.

5.2.1. *Stability for  $1 < p < 5$  and  $\gamma > 0$ .* Figure 9 shows the evolution of the solution when  $p = 3, \gamma = 1, \omega = 1$  and  $\delta_c = 0.1$ . The peak of the solution moves back towards  $x = 0$  very quickly (around  $t \approx 0.003$ ) and stays there at later times. Subsequently, the solution converges to the bound state  $\phi_{\omega^*=0.995}$ . This convergence starts near  $x = 0$  and spreads sideways, accompanied by radiation of the excess power  $\|u_0\|_2^2 - \|\phi_{\omega^*=0.995}\|_2^2 \cong 2.00 - 1.99 = 0.01$ . In Fig 10 we repeat this simulation with a larger shift of  $\delta_c = 0.5$ . The overall dynamics is similar: The solution peak moves back to  $x = 0$ , and the solution converges (from the center outwards) to  $\phi_{\omega^*=0.905}$ . In this case, it takes longer for the maximum to return to  $x = 0$  (at  $t \approx 0.11$ ), and more power is radiated in the process ( $\|u_0\|_2^2 - \|\phi_{\omega^*=0.905}\|_2^2 \cong 2.00 - 1.81 = 0.19$ ). We verified that the "non-smooth" profiles (e.g., at  $t = 0.2$ ) are not numerical artifacts.

5.2.2. *Drift instability for  $1 < p \leq 3$  and  $\gamma < 0$ .* Figure 11 shows the evolution of the solution for  $p = 3, \gamma = -1, \omega = 1$  and  $\delta_c = 0.1$ . Unlike the attractive case with the same parameters (Figure 9), as a result of this small initial shift to the right, nearly all the power flows from the left side of the defect ( $x < 0$ ) to the right side ( $x > 0$ ), see Figure 12a, so that by  $t \approx 3, \approx 90\%$  of the power is in the right side. Subsequently, the right component moves to the right at a constant speed (see Fig 12b) while assuming the *sech* profile of the homogeneous NLS bound state (see Fig 11 at  $t=8$ ); the left component also drifts away from the defect.

We thus see that

**Observation 36.** *When  $1 < p \leq 3$ , the standing waves are stable under shifts in the attractive case, but undergo a drift instability away from the defect in the repulsive case.*

We note that a similar behavior was observed in the subcritical NLS with a periodic nonlinearity, see [9], Section 5.1.

5.2.3. *Drift and finite-width instability for  $3 < p < 5$  and  $\gamma < 0$ .* In Figure 4b, Figure 5, and Figure 6 we saw that when  $p = 4$ ,  $\gamma = -1$ ,  $\omega = 0.5$ , and  $\delta_p = 0.005$ , the solution undergoes a finite-width instability in  $H^1_{\text{rad}}(\mathbb{R})$ . In Figures 13 and 14 we show the dynamics (in  $H^1(\mathbb{R})$ ) when we add a small shift of  $\delta_c = 0.1$ . In this case, the (larger) right component undergoes a combination of a drift instability and a finite-width instability, whereas the (smaller) left component undergoes a drift instability. Therefore, we make the following observation

**Observation 37.** *When  $3 < p < 5$ ,  $\gamma^2/4 < \omega < \omega_2$  and  $\gamma < 0$ , the standing waves undergo a combined drift and finite-width instability.*

5.2.4. *Drift and strong instability for  $5 \leq p$  and  $\gamma < 0$ .* In Figures 15 and 16 we show the solution of the NLS (1) with  $p = 6$ ,  $\gamma = -1$  and  $\omega = 1$ , for the initial condition (51) with  $\delta_c = 0.2$  and  $\delta_p = 0.001$ . As predicted by the theory, this strongly unstable solution undergoes collapse. Note, however, that, in parallel, the solution also undergoes a drift instability. We thus see that

**Observation 38.** *In the critical and supercritical repulsive case, the standing waves collapse while undergoing a drift instability away from the defect.*

Note that a similar behavior was observed in [9], Section 5.2.

5.3. **Numerical Methods.** We solve the NLS (1) using fourth-order finite differences in  $x$  and second-order implicit Crank-Nicholson scheme in time. Clearly, the main question is how to discretize the delta potential at  $x = 0$ . Recall that in continuous case

$$\lim_{x \rightarrow 0^+} \partial_x u(x) - \lim_{x \rightarrow 0^-} \partial_x u(x) = -\gamma u(0).$$

Discretizing this relation with  $\mathcal{O}(h^2)$  accuracy gives

$$\frac{u(2h) - 4u(h) + 3u(0)}{2h} - \frac{-u(-2h) + 4u(-h) - 3u(0)}{2h} = -\gamma u(0),$$

when  $h$  is the spatial grid size. By rearrangement of the terms we get the equation

$$(53) \quad -u(2h) + 4u(h) + [2h\gamma - 6]u(0) + 4u(-h) - u(-2h) = 0.$$

When we simulate symmetric perturbations (section 5.1), we enforce symmetry by solving only on half space  $[0, +\infty)$ . In this case, because of the symmetry condition  $u(-x) = u(x)$ , (53) becomes

$$[2h\gamma - 6]u(0) + 8u(h) - 2u(2h) = 0.$$

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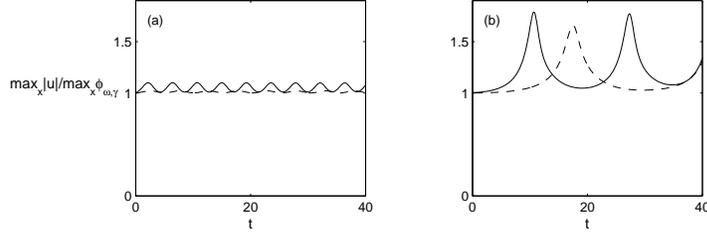


FIGURE 4.  $\max_x |u| / \max_x \phi_{\omega, \gamma}$  as a function of  $t$  for  $p = 4, \gamma = -1, \delta_p = 0.001$  (dashed line) and  $\delta_p = 0.005$  (solid line). (a)  $\omega = 2$ ; (b)  $\omega = 0.5$ .

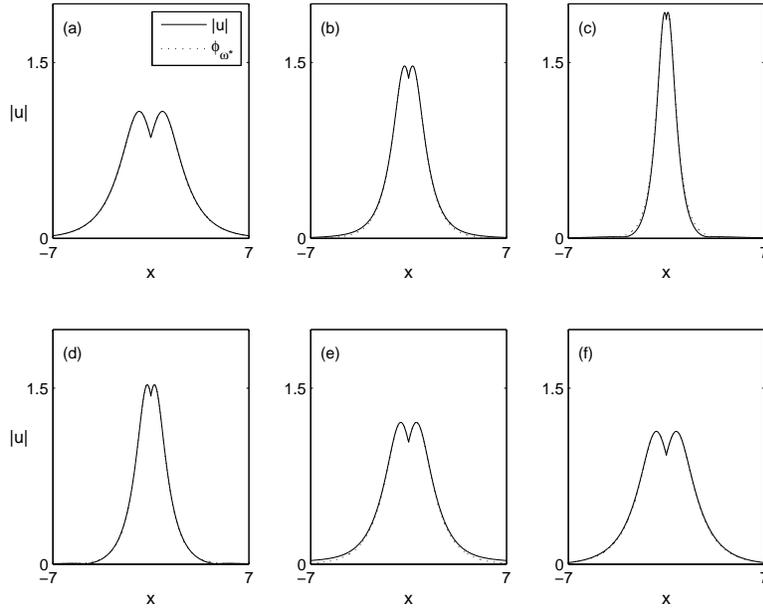


FIGURE 5.  $|u(x, t)|$  (solid line) and  $\phi_{\omega^*(t)}(x)$  (dots) as a function of  $x$  for the simulation of Fig. 4b with  $\delta_p = 0.005$ . (a)  $t = 0$  ( $\omega^* = 0.508$ ) (b)  $t = 9$  ( $\omega^* = 1.27$ ) (c)  $t = 10.69$  ( $\omega^* = 2.86$ ) (d)  $t = 12$  ( $\omega^* = 1.43$ ) (e)  $t = 15$  ( $\omega^* = 0.706$ ) (f)  $t = 20$  ( $\omega^* = 0.58$ ).

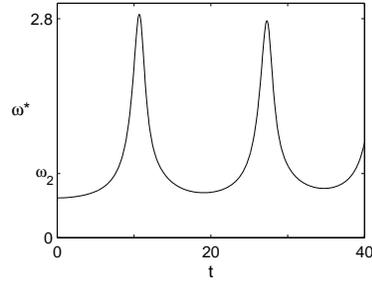


FIGURE 6.  $\omega^*$  as a function of  $t$  for the simulation of Fig 5.

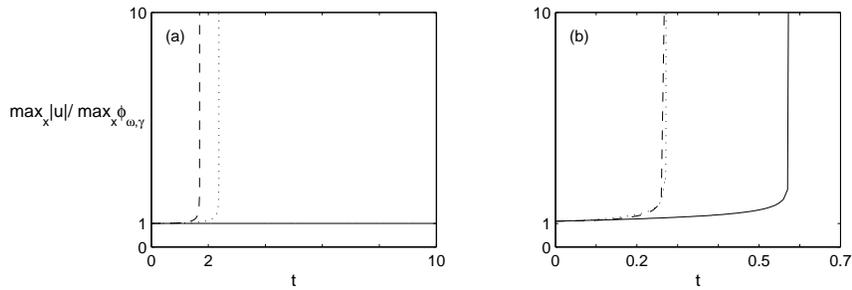


FIGURE 7.  $\max_x |u(x, t)| / \max_x \varphi_{\omega, \gamma}$  as a function of  $t$  for  $p = 6, \omega = 1$  and  $\gamma = -1$  (dashed line),  $\gamma = 0$  (dots),  $\gamma = +1$  (solid line). (a)  $\delta_p = 0.001$  (b)  $\delta_p = 0.1$ .

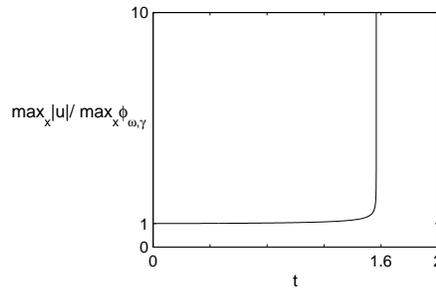


FIGURE 8.  $\max_x |u(x, t)| / \max_x \varphi_{\omega, \gamma}$  as a function of  $t$  for  $p = 6, \omega = 4, \gamma = 1$  and  $\delta_p = 0.001$ .

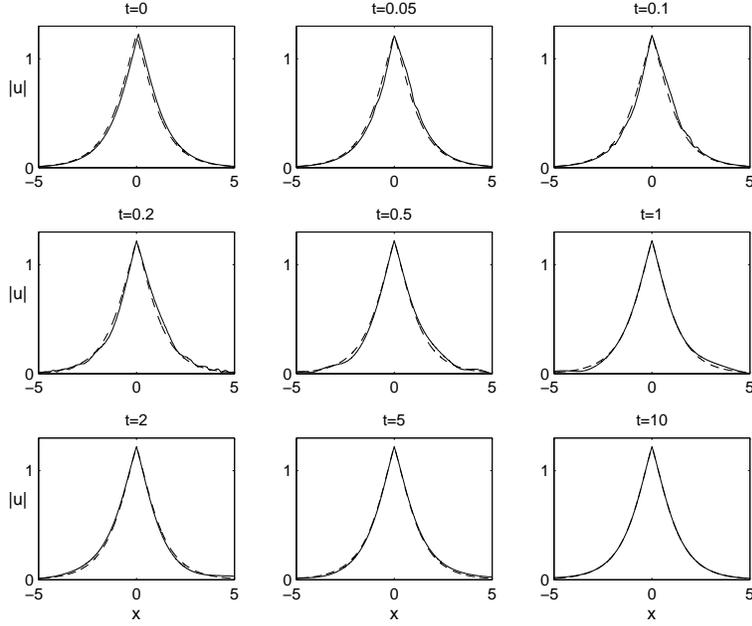


FIGURE 9.  $|u(x, t)|$  (solid line) and  $\phi_{\omega^*=0.995}(x)$  (dashed line) as a function of  $x$ . Here,  $p = 3, \omega = 1, \gamma = 1$  and  $\delta_c = 0.1$ .

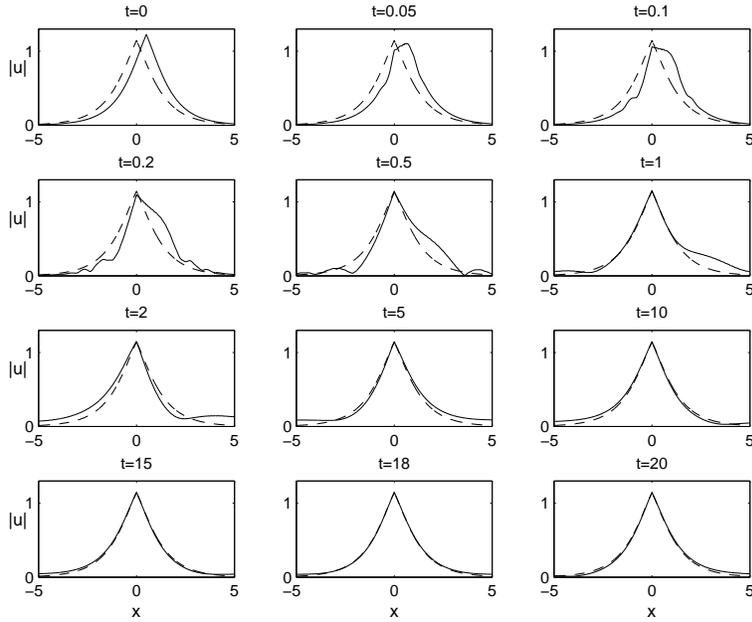


FIGURE 10. Same as Fig 9 with  $\delta_c = 0.5$  and  $\omega^* = 0.905$ .

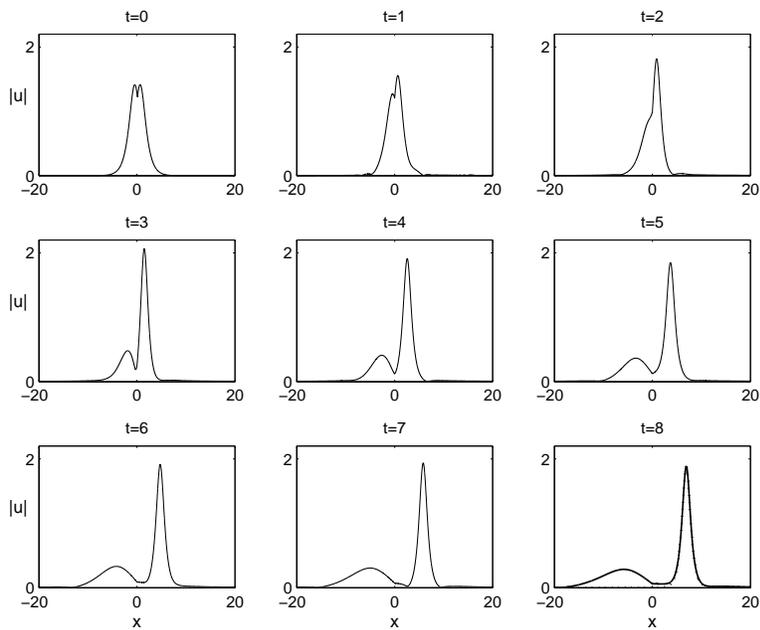


FIGURE 11.  $|u(x, t)|$  (solid line) as a function of  $x$ . Here  $p = 3, \gamma = -1, \omega = 1$  and  $\delta_c = 0.1$ . Dotted line at  $t = 8$  is  $\sqrt{2\omega^*} \operatorname{sech}(\sqrt{\omega^*}(x - x^*))$  with  $\omega^* = 1.768$  and  $x^* \approx 7$ .

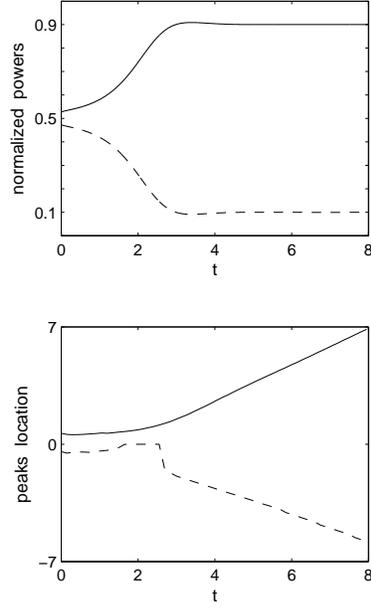


FIGURE 12. (a) The normalized powers  $\int_0^\infty |u|^2 dx / \int_{-\infty}^\infty |u_0|^2 dx$  (solid line) and  $\int_{-\infty}^0 |u|^2 dx / \int_{-\infty}^\infty |u_0|^2 dx$  (dashed line), and (b) location of  $\max_{0 \leq x} |u(x, t)|$  (solid line) and of  $\max_{x \leq 0} |u(x, t)|$  (dashed line), for the simulation of Figure 11.

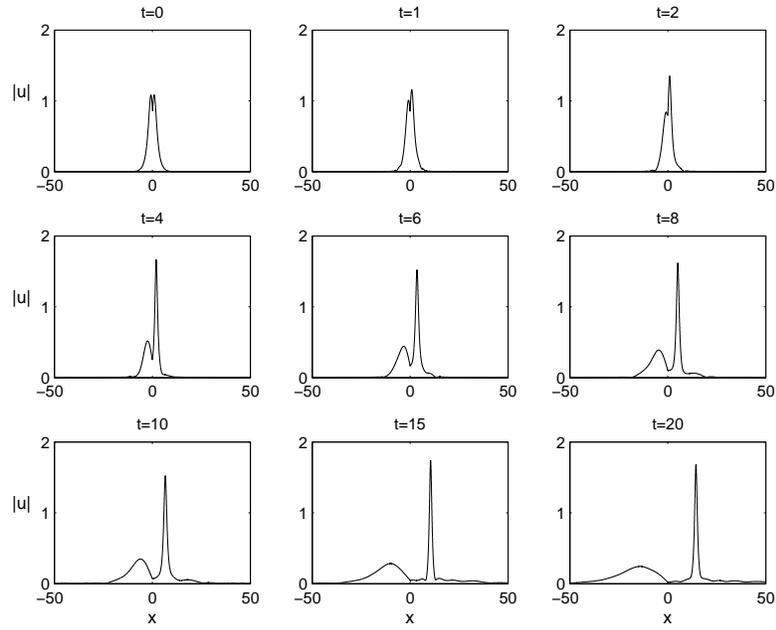


FIGURE 13.  $u(x, t)$  as a function of  $x$ . Here  $p = 4$ ,  $\gamma = -1$ ,  $\omega = 0.5$ ,  $\delta_p = 0.005$ , and  $\delta_c = 0.1$

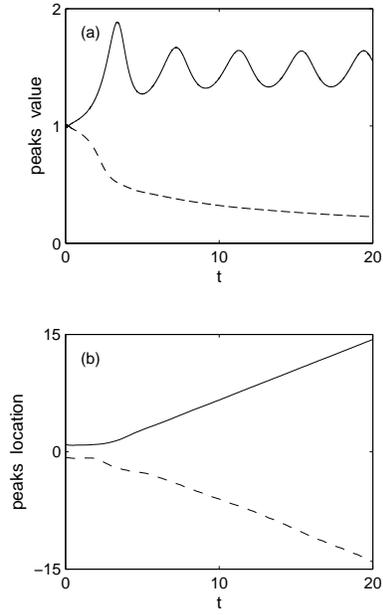


FIGURE 14. (a) The value, and (b) the location, of the right peak  $\max_{0 \leq x} |u(x, t)|$  (solid line) and left peak  $\max_{x \leq 0} |u(x, t)|$  (dashed line), for the simulation of Figure 13.

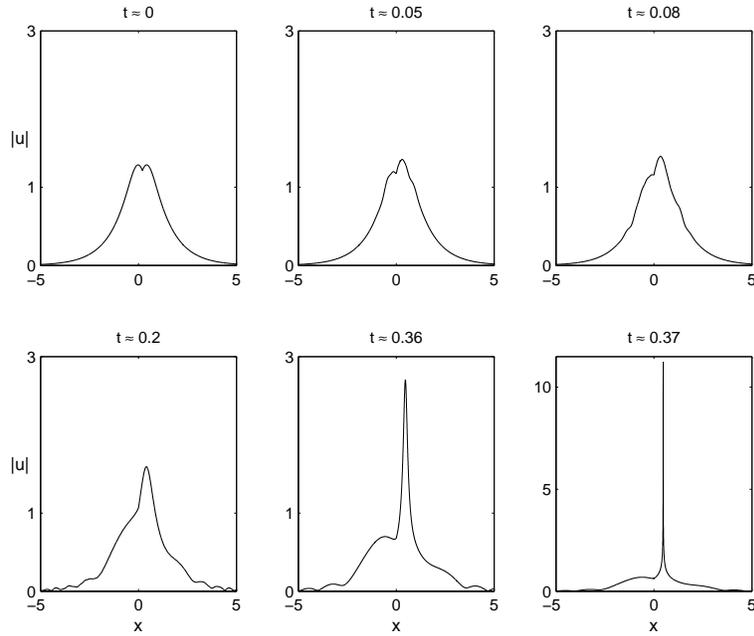


FIGURE 15.  $|u(x, t)|$  as a function of  $x$ , at various values of  $t$ . Here,  $p = 6$ ,  $\gamma = -1$ ,  $\omega = 1$ ,  $\delta_c = 0.2$  and  $\delta_p = 0.001$ .

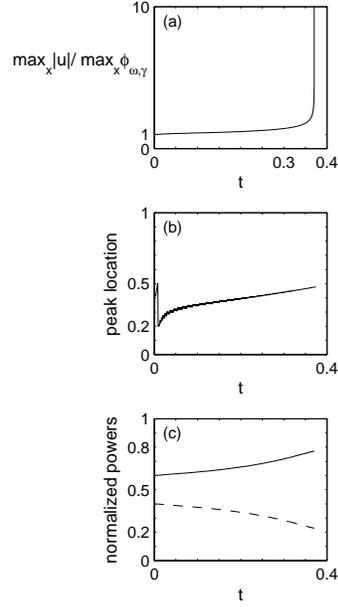


FIGURE 16. (a)  $\max_x |u(x, t)| / \max_x \varphi_{\omega, \gamma}$  (b) location of  $\max_x |u(x, t)|$  and (c) The normalized powers  $\int_0^\infty |u|^2 dx / \int_{-\infty}^\infty |u_0|^2 dx$  (solid line) and  $\int_{-\infty}^0 |u|^2 dx / \int_{-\infty}^\infty |u_0|^2 dx$  (dashed line), for the solution of Fig. 15.