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<thead>
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<th>Title</th>
<th>Sobolev inequalities with symmetry</th>
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</thead>
<tbody>
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SOBOLEV INEQUALITIES WITH SYMMETRY

YONGGEUN CHO AND TOHRU OZAWA

Abstract. In this paper we derive some Sobolev inequalities for radially symmetric functions in \( \dot{H}^s \) with \( \frac{1}{2} < s < \frac{n}{2} \). We show the end point case \( s = \frac{1}{2} \) on the homogeneous Besov space \( \dot{B}^\frac{1}{2}_{2,1} \). These results are extensions of the well-known Strauss’ inequality [11]. Also non-radial extensions are given, which show that compact embeddings are possible in some \( L^p \) spaces if a suitable angular regularity is imposed.

1. Introduction

In this paper we derive Sobolev inequalities with symmetry. We first consider several Sobolev inequalities for radially symmetric functions in \( \dot{H}^s(\mathbb{R}^n) \) with \( \frac{1}{2} < s < \frac{n}{2} \). There is a sharp result by Sickel and Skrzypczak [8], although the argument below is much simpler and direct and a constant in the inequality in Proposition 1 below is given explicitly in terms of \( s \).

Definition 1.

\[ \dot{H}^s_{\text{rad}} = \{ u \in \dot{H}^s(\mathbb{R}^n) : u \text{ is radially symmetric} \}, \quad s \geq 0. \]

\[ \dot{B}^s_{p,q,rad} = \{ u \in \dot{B}^s_{p,q}(\mathbb{R}^n) : u \text{ is radially symmetric} \}, \quad s \geq 0, \quad 1 \leq p, q \leq \infty. \]

The inhomogeneous spaces of radially symmetric functions are defined by the same way with spaces \( H^s \) and \( B^s_{p,q} \).

Proposition 1. Let \( n \geq 2 \) and let \( s \) satisfy \( 1/2 < s < n/2 \).

Then

\[ \sup_{x \in \mathbb{R}^n \setminus \{0\}} |x|^{n/2-s} |u(x)| \leq C(n, s) \| (-\Delta)^{s/2} u \|_{L^2} \tag{1} \]

for all \( u \in \dot{H}^s_{\text{rad}} \), where

\[ C(n, s) = \left( \frac{\Gamma(2s-1)\Gamma(n/2-s)\Gamma(n/2)}{2^{2s-n/2}\Gamma(s)^2\Gamma(n/2-1+s)} \right)^{1/2} \]

and \( \Gamma \) is the gamma function.

Remark 1. For \( s = 1 \) with \( n \geq 3 \), the inequality (1) reduces to Ni’s inequality [6, 7].

Remark 2. The restriction \( 1/2 < s < n/2 \) is necessary for \( C(n, s) \) to be finite.

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Remark 3. The inequality (1) fails for $s = n/2$. Indeed, $u(x) = \mathcal{F}^{-1}\left(\frac{1}{(1+|\xi|)^{n/2}(1+\log(1+|\xi|))}\right)$ satisfies $u \in H_{rad}^{n/2}$, and $u \notin L^\infty$ where $\mathcal{F}$ is the Fourier transform [12] and $\mathcal{F}^{-1}$ is its inverse.

Remark 4. The inequality (1) fails if $0 \leq s < 1/2$ and $n \geq 3$. Indeed, $u = \mathcal{F}^{-1}(J_{n/2}^2(|\xi|)|\xi|^{-n/2})$ satisfies $u \in \dot{H}_{rad}^s$ and $u(x) = \infty$ for all $x \in S^{n-1}$, where we note that $u \in \dot{H}_{rad}^s$ if and only if $1 - n/2 < s < 1/2$, since
\[
\|(-\Delta)^{s/2}u\|_{L^2}^2 = c_n \int_0^\infty |J_{n/2}^2(\rho)|^2 \rho^{2s-1}d\rho
\]
and that by the asymptotic behavior of Bessel function (10)
\[u(x) = \int_0^\infty |J_{n/2}^2(\rho)|^2 d\rho = \infty, \quad x \in S^{n-1}.
\]
See also the proof of Proposition 1 below.

In the endpoint case $s = 1/2$, we have the following proposition.

**Proposition 2.** Let $n \geq 2$. Then there exists a constant $C$ such that
\[
\sup_{x \in \mathbb{R}^n \setminus \{0\}} |x|^{(n-1)/2}|u(x)| \leq C\|u\|_{\dot{B}_{2,1}^{1/2}}
\]
for all $u \in \dot{B}_{2,1,rad}^{1/2}$.

**Remark 5.** The inhomogeneous version of (2) has been given in [8] whose proof is based on the atomic decomposition.

**Proposition 3.** Let $n \geq 2$ and let $s$ satisfy $1/2 \leq s < 1$. Then there exists $C$ such that for all $u \in H_{rad}^1$
\[
\sup_{x \in \mathbb{R}^n \setminus \{0\}} |x|^{n/2-s}|u(x)| \leq C(n, s)\|u\|_{L^2}^{1-s}\|\nabla u\|_{L^2}^s.
\]

**Remark 6.** For $s = 1/2$, the inequality (3) reduces to Strauss’ inequality [11].

**Remark 7.** For $s = 0$, the inequality (3) holds for nonincreasing functions in $|x|$ [2]. For $s = 1$, the inequality (3) holds for $n \geq 3$ and fails for $n = 2$. See Proposition 1 and Remark 2.

Now we extend the results above on radial functions to the non-radial functions with additional angular regularity. For details, let us define function spaces $H_{\omega}^{s,m}$ and $B_{2,1,\omega}^{s,m}$, $s \geq 0, m \geq 0$ as follows.
Definition 2.

\[ H^{s,m}_\omega = \left\{ u \in H^s : \|u\|_{H^{s,m}_\omega} \equiv \|(1 - \Delta_\omega)^{m/2} u\|_{H^s} < \infty \right\} , \]

\[ B^{s,m}_{2,1,\omega} = \left\{ u \in B^{2,1}_{2,1} : \|u\|_{B^{s,m}_{2,1,\omega}} \equiv \|(1 - \Delta_\omega)^{m/2} u\|_{B^{s}_{2,1}} < \infty \right\} , \]

where \( \Delta_\omega \) is the Laplace-Beltrami operator on \( S^{n-1} \).

The homogeneous spaces \( \dot{H}^{s,m}_\omega \) and \( \dot{B}^{s,m}_{2,1,\omega} \) is similarly defined by the definition of \( \dot{H}^s \) and \( \dot{B}^s_{2,1} \). Then we have the following.

**Proposition 4.**

1. If \( 1/2 < s < n/2 \) and \( m > n - 1 - s \), then there exists a constant \( C \) such that for any \( u \in H^{s,m}_\omega \)

\[
\sup_{\mathbb{R}^n \setminus \{0\}} |x|^{n/2 - s} |u(x)| \leq C \|u\|_{\dot{H}^{s,m}_\omega} . \tag{4}
\]

2. If \( m > n - \frac{3}{2} \), then there exists a constant \( C \) such that for any \( u \in B^{s,m}_{2,1,\omega} \)

\[
\sup_{\mathbb{R}^n \setminus \{0\}} |x|^{(n-1)/2} |u(x)| \leq C \|u\|_{B^{s,m}_{2,1,\omega}} . \tag{5}
\]

**Remark 8.** \( H^{s,m}_\omega \) and \( B^{s,m}_{2,1,\omega} \) are closed subspaces of \( H^s \) and \( B^s_{2,1} \), respectively and they contain \( H^{s}_{\text{rad}} \) and \( B^{s}_{2,1,\text{rad}} \) naturally, respectively. We can identify the spaces \( H^s \) with \( (1 - \Delta_\omega)^{m/2} H^{s,m}_\omega \) and also \( B^{s}_{2,1} \) with \( (1 - \Delta_\omega)^{m/2} B^{s,m}_{2,1,\omega} \).

**Remark 9.** \( H^{s,m}_\omega \) is a Hilbert space with the same inner product as \( L^2 \) space and its dual space is given by \( H^{-s,-m}_\omega \).

From the decay at infinity we deduce compact embeddings of \( H^{s,m}_\omega \) and \( B^{s,m}_{2,1,\omega} \) into some \( L^p \) spaces as follows. See [2, 3, 4, 8, 9] for the radial case.

**Corollary 1.** The embedding \( H^{s,m}_\omega \hookrightarrow L^p \) is compact for \( 1/2 < s < n/2, m > n - 1 - s \) and \( 2 < p < 2n/(n-2s) \).

**Corollary 2.** The embedding \( B^{s,m}_{2,1,\omega} \hookrightarrow L^p \) is compact for \( m > n - 3/2 \) and \( 2 < p < 2n/(n-1) \).

2. Proofs

2.1. Proof of Proposition 1. We use the following Fourier representation for radially symmetric functions as

\[
u(x) = |x|^{1 - \frac{\nu}{2}} \int_0^{\infty} J_{\frac{\nu}{2} - 1}(|x|\rho) \hat{u}(\rho) \rho^{\frac{\nu}{2}} d\rho, \tag{6}
\]

where \( J_\nu \) is the Bessel function of order \( \nu \), \( \hat{u} \) is the Fourier transform normalized as

\[
\hat{u}(\xi) = (2\pi)^{-n/2} \int e^{-ix\cdot\xi} u(x) dx,
\]
and we have identified radially symmetric functions on $\mathbb{R}^n$ with the corresponding functions on $(0, \infty)$.

By the Cauchy-Schwarz inequality and the Plancherel formula, we have

$$
|x|^{\frac{n}{2} - s}|u(x)|
\leq |x|^{1-s} \left( \int_0^\infty |J_{\frac{n}{2} - 1}(|x|\rho)|^2 \rho^{1-2s} d\rho \right)^{1/2} \left( \int_0^\infty |\hat{u}(\rho)|^2 \rho^{2s+n-1} d\rho \right)^{1/2} 
= \left( \int_0^\infty |J_{\frac{n}{2} - 1}(\rho)|^2 \rho^{1-2s} d\rho \right)^{1/2} \left( \frac{\Gamma(n/2)}{2\pi^{n/2}} \int |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{1/2} 
= C(n, s)\|(-\Delta)^{s/2} u\|_{L^2},
$$

as required.

2.2. Proof of Proposition 2. We use the following estimates on Bessel functions:

$$
\sup_{r \geq 0} |J_{\frac{n}{2} - 1}(r)| \leq 1, \quad \sup_{r \geq 0} r^{1/2} |J_{\frac{n}{2} - 1}(r)| \leq C.
$$

The first inequality (8) follows from the integral representation (see [13])

$$
J_{\frac{n}{2} - 1}(r)^2 = \frac{2}{\pi} \int_0^{\pi/2} J_{n-2}(2r \cos \theta) d\theta,
$$

$$
J_m(t) = \frac{1}{2\pi} \int_0^{2\pi} \cos(m \theta - t \sin \theta) d\theta, \quad m \in \mathbb{Z}.
$$

The second inequality (9) follows from the first and the well-known asymptotics (see [10])

$$
J_{\nu}(r) \sim \sqrt{\frac{2}{\pi r}} \cos \left( r - \frac{(2\nu + 1)\pi}{4} \right) \quad \text{as} \quad r \to \infty.
$$

We apply the Littlewood-Paley decomposition $\{\varphi_j\}_{j \in \mathbb{Z}}$ on $\mathbb{R}^n \setminus \{0\}$ to (4) to obtain

$$
u(x) = |x|^{1-n/2} \sum_{j \in \mathbb{Z}} \int_0^\infty J_{\frac{n}{2} - 1}(|x|\rho) \psi_j(\rho) \varphi_j(\rho) \hat{u}(\rho) \rho^{n/2} d\rho,
$$

where $\psi_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}$ and supp $\varphi_j \subset \{2^{j-1} \leq \rho \leq 2^{j+1}\}$.

As in (7), we have

$$
|x|^{(n-1)/2} \|u(x)\|
\leq |x|^{1/2} \sup_{j \in \mathbb{Z}} \left( \int_0^\infty |J_{\frac{n}{2} - 1}(|x|\rho)|^2 \psi_j(\rho)^2 d\rho \right)^{1/2} 
\cdot \sum_{j \in \mathbb{Z}} \left( \int_0^\infty |\varphi_j(\rho)\hat{u}(\rho)|^2 \rho^n d\rho \right)^{1/2}.
$$
By (9), we estimate
\[
|x|^{1/2} \sup_{j \in \mathbb{Z}} \left( \int_{0}^{\infty} |J_{\frac{n}{2}-1}(|x|\rho)|^2 |\psi_j(\rho)|^2 d\rho \right)^{1/2} \leq C \sup_{j \in \mathbb{Z}} \left( \int_{0}^{\infty} \frac{1}{\rho} |\psi_j(\rho)|^2 d\rho \right)^{1/2} \leq C \sup_{j \in \mathbb{Z}} \left( \int_{j^{1/2}}^{j+\frac{1}{2}} \frac{1}{\rho} d\rho \right)^{1/2} \leq C. \tag{12}
\]
This proves (2) since
\[
\sum_{j \in \mathbb{Z}} \left( \int_{0}^{\infty} |\varphi_j(\rho)\hat{u}(\rho)|^2 \rho^{n-1} d\rho \right)^{1/2}
\]
is equivalent to the seminorm on $\dot{B}^{1/2}_{2,1,\text{rad}}$.

2.3. **Proof of Proposition 3.** If we use Cauchy-Schwartz inequality as in (7), we have for any $M > 0$
\[
|x|^{n/2-s}|u(x)| \\
\leq |x|^{-s} \left( \int_{0}^{M|x|} |J_{\frac{n}{2}-1}(|x|\rho)|^2 \rho d\rho \right)^{1/2} \left( \int_{0}^{M|x|} |\hat{u}(\rho)|^2 \rho^{n-1} d\rho \right)^{1/2} \\
+ |x|^{-s} \left( \int_{M|x|}^{\infty} |J_{\frac{n}{2}-1}(|x|\rho)|^2 \rho d\rho \right)^{1/2} \left( \int_{M|x|}^{\infty} |\hat{u}(\rho)|^2 \rho^{n+1} d\rho \right)^{1/2} \\
\leq |x|^{-s} \left( \int_{0}^{M} |J_{\frac{n}{2}-1}(r)|^2 r dr \right)^{1/2} \|u\|_{L^2} \\
+ |x|^{-s} \left( \int_{M}^{\infty} |J_{\frac{n}{2}-1}(r)|^2 r^{-1} dr \right)^{1/2} \|\nabla u\|_{L^2}.
\]
From (8) and (9) we deduce that $\sup_{r \geq 0} |r^{1-s}J_{\frac{n}{2}-1}(r)| \leq C$ for any $\frac{1}{2} \leq s < 1$. Hence we have for any $M > 0$
\[
|x|^{n/2-s}|u(x)| \\
\leq |x|^{-s} \left( \int_{0}^{M} |J_{\frac{n}{2}-1}(r)|^2 r dr \right)^{1/2} \|u\|_{L^2} \\
+ |x|^{-s} \left( \int_{M}^{\infty} |J_{\frac{n}{2}-1}(r)|^2 r^{-1} dr \right)^{1/2} \|\nabla u\|_{L^2} \\
\leq C|x|^{-s} M^{s} \|u\|_{L^2} + |x|^{-s} M^{-(1-s)} \|\nabla u\|_{L^2}.
\]
The minimization of the RHS of the last inequality with respect to $M$ yields Proposition 3.
2.4. Proofs of Proposition 4 and Corollaries. The proof for (4) follows from the one of Proposition 1 and the spherical harmonic expansion of functions in $H^s_w$ [5, 10]. In fact, if we write $u(r\omega) = \sum_{k,l} f_{k,l}(r) Y_{k,l}(\omega)$, where $d(k)$ is the dimension of space of spherical harmonic functions of degree $k$ and

$$d(k) \leq C k^{n-2} \text{ for large } k. \quad (13)$$

Then we have

$$|x|^{\frac{n}{2} - s} u(|x|\omega) = c_n \sum_{k,l} |x|^{1-s} \int_0^\infty J_{\nu(k)}(|x|\rho) \rho^s g_{k,l}(\rho) d\rho Y_{k,l}(\omega),$$

where $\omega \in S^{n-1}$, $\nu(k) = \frac{n+2k-2}{2}$ and $\int f_{k,l}(r\omega) Y_{k,l}(\omega) = g_{k,l}(\rho) Y_{k,l}(\omega)$. Here

$$g_{k,l}(\rho) = c_{n,k} \int_0^\infty f_{k,l}(r) \rho^{-n-1} (r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(r\rho) \, dr.$$

The absolute value of $c_{n,k}$ is bounded by a constant depending only on $n$. See [10] for this.

Using the Cauchy-Schwarz inequality as in (7), we have

$$|x|^{\frac{n}{2} - s} |u(|x|\omega)| \leq C \sum_{k,l} \|Y_{k,l}\|_{L^\infty(S^{n-1})} \left( \int_0^\infty |J_{\nu(k)}(|x|\rho)|^2 \rho^{1-2s} \right)^{\frac{1}{2}} \left( \int_0^\infty |g_{k,l}(\rho)|^2 \rho^{2s+n-1} d\rho \right)^{\frac{1}{2}}$$

$$\leq C \sum_{k,l} k^{\frac{n-2}{2}} \left( \frac{\Gamma(\nu(k) + 1 - s)}{\Gamma(\nu(k) + s)} \right)^{\frac{1}{2}} \left( \int_0^\infty |g_{k,l}(\rho)|^2 \rho^{2s+n-1} d\rho \right)^{\frac{1}{2}} \|Y_{k,l}\|_{L^2(S^{n-1})}.$$ 

Here we used the inequality that $\|Y_{k,l}\|_{L^\infty} \leq C k^{-\frac{n-2}{2}} \|Y_{k,l}\|_{L^2}$ (see for instance [10]). Using the Stirling’s formula for gamma function that $\Gamma(t) \approx \sqrt{\pi} t^{-\frac{1}{2}} e^{-(t-\frac{1}{2})}$ for large $t$ (for instance, see [1]) and the fact $-\Delta_\omega Y_{k,l} = (k+n-2) Y_{k,l}$, we have from (13)

$$|x|^{\frac{n}{2} - s} |u(|x|\omega)| \leq C \sum_{k} c_{n} \sum_{1 \leq l \leq d(k)} \int_0^\infty |g_{k,l}(\rho)|^2 \rho^{2s+n-1} d\rho \|Y_{k,l}\|_{L^2(S^{n-1})}^{\frac{1}{2}}$$

$$\leq C \left( \sum_{k} k^{2(n-\frac{1}{2} - s-m)} \right)^{\frac{1}{2}} \left( \sum_{k,l} k^{2m} \int_0^\infty |g_{k,l}(\rho)|^2 \rho^{2s+n-1} d\rho \|Y_{k,l}\|_{L^2(S^{n-1})}^{2} \right)^{\frac{1}{2}}$$

$$\leq C \left( \sum_{k,l} \int_0^\infty \int_{S^{n-1}} |F(f_{k,l} Y_{k,l})(\rho\omega)|^2 \rho^{2s+n-1} d\rho d\omega \right)^{\frac{1}{2}}$$

$$\leq C \left( \sum_{k,l} \int_0^\infty \int_{S^{n-1}} |F((1 - \Delta_\omega)^m (f_{k,l} Y_{k,l}))(\rho\omega)|^2 \rho^{2s+n-1} d\rho d\omega \right)^{\frac{1}{2}}$$

$$\leq C \|u\|_{H^s_w},$$
where \( \mathcal{F} \) is the Fourier transform. This proves part (1).

For the part (2), if we use the Littlewood-Paley decomposition \( \{ \varphi_j \}_{j \in \mathbb{Z}} \) as in the proof of Proposition 2, the we have

\[
|x|^{n-1} u(|x| \omega) = c_n \sum_{j \in \mathbb{Z}} \sum_{k,l} |x|^\frac{1}{2} \int_0^\infty J_{\nu(k)}(|x| \rho \psi_j(\rho) \varphi_j(\rho) g_{k,l}(\rho) d\rho Y_{k,l}(\omega),
\]

(15)

Since \( m > n - 3/2 \), by (12) we deduce that

\[
|x|^{n-1} |u(|x| \omega)|
\]

\[
\leq C \sum_{j \in \mathbb{Z}} \sum_{k,l} \left( \int_0^\infty |\varphi_j(\rho) g_{k,l}(\rho)|^2 \rho^{n-} d\rho \right)^{\frac{1}{2}} \| Y_{k,l} \|_{L^\infty(S^{n-1})}^2
\]

\[
\leq C \sum_{j \in \mathbb{Z}} \left( \sum_{k} k^{2(n-2-m)} \right)^{\frac{1}{2}} \left( \sum_{k,l} \int_0^\infty |\varphi_j(\rho) g_{k,l}(\rho)|^2 \rho^{n-} d\rho \| Y_{k,l} \|_{L^2(S^{n-1})}^2 \right)^{\frac{1}{2}}
\]

\[
\leq C \sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \| \varphi_j \mathcal{F}(1 - \Delta_\omega)^{\frac{m}{2}} u \|_{L^2} = C \| u \|_{B^{\frac{1}{2}}_{2,1,\omega}}.
\]

To show Corollary 1 we use the fact that \( H_\omega^{s,m} \) is a Hilbert space. Hence any bounded sequence \( \{ u_j \} \) in \( H_\omega^{s,m} \) satisfies \( u_j(x) \to 0 \) as \( |x| \to \infty \) uniformly and has a subsequence converges to \( u \) in \( H_\omega^{s,m} \) weakly. Let us denote the subsequence by \( u_{j_k} \).

Now choose a smooth function \( \varphi \) supported in the ball of radius \( R + 1 \) and with value 1 in the ball of radius \( R \). By the standard argument one can easily show that for each \( R \) the mapping \( u \mapsto \varphi u \) is compact from \( H^t \) to \( H^t' \) if \( t' < t \). By the compactness above and Sobolev embedding we may assume that the sequence \( \varphi u_{j_k} \) satisfies that for \( 2 \leq q < \frac{2n}{n-2} \)

\[
\| \varphi u_{j_k} - \varphi u \|_{L^q} \to 0 \quad \text{as} \quad k \to \infty.
\]

(16)

Thus we have

\[
\| u_{j_k} - u \|_{L^p} \leq \| \varphi (u_{j_k} - u) \|_{L^p} + \| (1 - \varphi)(u_{j_k} - u) \|_{L^p} \equiv I_k + II_k
\]

with \( I_k \to 0 \) as \( k \to \infty \) by (16) since \( 2 < p < \frac{2n}{n-2} \). From the uniform convergence that \( |u_{j_k}(x)| + |u(x)| \to 0 \) as \( |x| \to \infty \) it follows that

\[
\limsup_{k \to \infty} II_k \leq \sup_k \| u_{j_k} - u \|_{L^\infty(|x| > R)} \to 0 \quad \text{as} \quad R \to \infty.
\]

This proves the compactness of the embedding \( H_\omega^{s,m} \hookrightarrow L^p \).

Since \( B^{\frac{1}{2}}_{2,1,\omega} \hookrightarrow H_\omega^{s,m} \), one can adapt the same arguments (compactness of cut-off mapping and uniform convergence at infinity) as above except for weak-* convergence of \( u_{j_k} \) to \( u \) in \( B^{\frac{1}{2}}_{2,1,\omega} \) for the compactness of embedding \( B^{\frac{1}{2}}_{2,1,\omega} \hookrightarrow L^p \). This completes the proof.
References


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