SOBOLEV INEQUALITIES WITH SYMMETRY

YONGGEUN CHO AND TOHRU OZAWA

Abstract. In this paper we derive some Sobolev inequalities for radially symmetric functions in $\dot{H}^s$ with $\frac{1}{2} < s < \frac{n}{2}$. We show the end point case $s = \frac{1}{2}$ on the homogeneous Besov space $\dot{B}^{\frac{1}{2}}_{\infty,1}$. These results are extensions of the well-known Strauss’ inequality [11]. Also non-radial extensions are given, which show that compact embeddings are possible in some $L^p$ spaces if a suitable angular regularity is imposed.

1. Introduction

In this paper we derive Sobolev inequalities with symmetry. We first consider several Sobolev inequalities for radially symmetric functions in $\dot{H}^s(\mathbb{R}^n)$ with $\frac{1}{2} < s < \frac{n}{2}$. There is a sharp result by Sickel and Skrzypczak [8], although the argument below is much simpler and direct and a constant in the inequality in Proposition 1 below is given explicitly in terms of $s$.

Definition 1.

$$\dot{H}^s_{\text{rad}} = \{ u \in \dot{H}^s(\mathbb{R}^n) : u \text{ is radially symmetric} \} , \ s \geq 0.$$ 

$$\dot{B}^s_{p,q,\text{rad}} = \{ u \in \dot{B}^s_{p,q}(\mathbb{R}^n) : u \text{ is radially symmetric} \} , \ s \geq 0, \ 1 \leq p,q \leq \infty.$$ 

The inhomogeneous spaces of radially symmetric functions are defined by the same way with spaces $H^s$ and $B^s_{p,q}$.

Proposition 1. Let $n \geq 2$ and let $s$ satisfy $1/2 < s < n/2$.

Then

$$\sup_{x \in \mathbb{R}^n \setminus \{0\}} |x|^{n/2-s} |u(x)| \leq C(n,s) \| (-Delta)^{s/2} u \|_{L^2}$$

for all $u \in \dot{H}^s_{\text{rad}}$, where

$$C(n,s) = \left( \frac{\Gamma(2s-1)\Gamma(\frac{n}{2}-s)\Gamma(\frac{n}{2})}{2^{2s-n/2}\Gamma(s)^2\Gamma(\frac{n}{2}-1+s)} \right)^{1/2},$$

and $\Gamma$ is the gamma function.

Remark 1. For $s = 1$ with $n \geq 3$, the inequality (1) reduces to Ni’s inequality [6, 7].

Remark 2. The restriction $1/2 < s < n/2$ is necessary for $C(n,s)$ to be finite.
Remark 3. The inequality (1) fails for $s = n/2$. Indeed, $u(x) = \mathcal{F}^{-1}\left(\frac{1}{(1 + |\xi|)^{n/2}(1 + \log(1 + |\xi|))}\right)$ satisfies $u \in H_{\text{rad}}^{n/2}$, and $u \notin L^\infty$ where $\mathcal{F}$ is the Fourier transform [12] and $\mathcal{F}^{-1}$ is its inverse.

Remark 4. The inequality (1) fails if $0 \leq s < 1/2$ and $n \geq 3$. Indeed, $u = \mathcal{F}^{-1}(J_{n/2-1}(|\xi|)|\xi|^{-n/2})$ satisfies $u \in \dot{H}_{\text{rad}}^s$ and $u(x) = \infty$ for all $x \in S^{n-1}$, where we note that $u \in \dot{H}_{\text{rad}}^s$ if and only if $1 - n/2 < s < 1/2$, since

$$\|(-\Delta)^{s/2}u\|_{L^2}^2 = c_n \int_0^\infty |J_{n/2-1}(\rho)|^2 \rho^{2s-1}d\rho$$

and that by the asymptotic behavior of Bessel function (10)

$$u(x) = \int_0^\infty |J_{n/2-1}(\rho)|^2 \rho^2 d\rho = \infty, \quad x \in S^{n-1}.$$ See also the proof of Proposition 1 below.

In the endpoint case $s = 1/2$, we have the following proposition.

**Proposition 2.** Let $n \geq 2$. Then there exists a constant $C$ such that

$$\sup_{x \in \mathbb{R}^n \setminus \{0\}} |x|^{(n-1)/2} |u(x)| \leq C \|u\|_{\dot{B}^{1/2}_{2,1}}^1$$

for all $u \in \dot{B}^{1/2}_{2,1,\text{rad}}$.

**Remark 5.** The inhomogeneous version of (2) has been given in [8] whose proof is based on the atomic decomposition.

**Proposition 3.** Let $n \geq 2$ and let $s$ satisfy $1/2 \leq s < 1$. Then there exists $C$ such that for all $u \in H^1_{\text{rad}}$

$$\sup_{x \in \mathbb{R}^n \setminus \{0\}} |x|^{n/2-s} |u(x)| \leq C(n, s) \|u\|_{L^2}^{1-s} \|
abla u\|_{L^2}^s.$$ (3)

**Remark 6.** For $s = 1/2$, the inequality (3) reduces to Strauss’ inequality [11].

**Remark 7.** For $s = 0$, the inequality (3) holds for nonincreasing functions in $|x|$ [2]. For $s = 1$, the inequality (3) holds for $n \geq 3$ and fails for $n = 2$. See Proposition 1 and Remark 2.

Now we extend the results above on radial functions to the non-radial functions with additional angular regularity. For details, let us define function spaces $H^{s,m}_\omega$ and $B^{s,m}_{2,1,\omega}$, $s \geq 0, m \geq 0$ as follows.
Definition 2.  
\[ H^{s,m}_\omega = \left\{ u \in H^s : \|u\|_{H^{s,m}_\omega} = \|(1 - \Delta_\omega)^{\frac{m}{2}} u\|_{H^s} < \infty \right\}, \]
\[ B^{s,m}_{2,1,\omega} = \left\{ u \in B^{\frac{3}{2}}_{2,1} : \|u\|_{B^{s,m}_{2,1,\omega}} = \|(1 - \Delta_\omega)^{\frac{m}{2}} u\|_{B^{2}_{2,1}} < \infty \right\}, \]
where \( \Delta_\omega \) is the Laplace-Beltrami operator on \( S^{n-1} \).

The homogeneous spaces \( \hat{H}^{s,m}_\omega \) and \( \hat{B}^{s,m}_{2,1,\omega} \) is similarly defined by the definition of \( \hat{H}^s \) and \( \hat{B}^s_{2,1} \). Then we have the following.

Proposition 4.  
(1) If \( 1/2 < s < n/2 \) and \( m > n - 1 - s \), then there exists a constant \( C \) such that for any \( u \in H^{s,m}_\omega \)
\[ \sup_{\mathbb{R}^n \setminus \{0\}} |x|^{n/2-s}|u(x)| \leq C \|u\|_{\hat{H}^{s,m}_\omega}. \tag{4} \]
(2) If \( m > n - \frac{3}{2} \), then there exists a constant \( C \) such that for any \( u \in B^{\frac{1}{2},m}_{2,1,\omega} \)
\[ \sup_{\mathbb{R}^n \setminus \{0\}} |x|^{(n-1)/2}|u(x)| \leq C \|u\|_{B^{\frac{1}{2},m}_{2,1,\omega}}. \tag{5} \]

Remark 8. \( H^{s,m}_\omega \) and \( B^{\frac{1}{2},m}_{2,1,\omega} \) are closed subspaces of \( H^s \) and \( B^{\frac{3}{2}}_{2,1,\omega} \), respectively and they contain \( H^s_{\text{rad}} \) and \( B^{\frac{3}{2}}_{2,1,\text{rad}} \) naturally, respectively. We can identify the spaces \( H^s \) with \( (1 - \Delta_\omega)^{m/2} H^{s,m}_\omega \) and also \( B^{\frac{3}{2}}_{2,1} \) with \( (1 - \Delta_\omega)^{m/2} B^{\frac{1}{2},m}_{2,1,\omega} \).

Remark 9. \( H^{s,m}_\omega \) is a Hilbert space with the same inner product as \( L^2 \) space and its dual space is given by \( H^{-s,-m}_\omega \).

From the decay at infinity we deduce compact embeddings of \( H^{s,m}_\omega \) and \( B^{\frac{1}{2},m}_{2,1,\omega} \) into some \( L^p \) spaces as follows. See [2, 3, 4, 8, 9] for the radial case.

Corollary 1. The embedding \( H^{s,m}_\omega \hookrightarrow L^p \) is compact for \( 1/2 < s < n/2 \), \( m > n - 1 - s \) and \( 2 < p < 2n/(n-2s) \).

Corollary 2. The embedding \( B^{\frac{1}{2},m}_{2,1,\omega} \hookrightarrow L^p \) is compact for \( m > n - 3/2 \) and \( 2 < p < 2n/(n-1) \).

2. Proofs

2.1. Proof of Proposition 1. We use the following Fourier representation for radially symmetric functions as
\[ u(x) = |x|^{1-n/2} \int_0^\infty J_{\frac{n}{2}-1}(|x|\rho)\hat{u}(\rho)\rho^{\frac{n}{2}} d\rho, \tag{6} \]
where \( J_\nu \) is the Bessel function of order \( \nu \), \( \hat{u} \) is the Fourier transform normalized as
\[ \hat{u}(\xi) = (2\pi)^{-n/2} \int e^{-i\xi \cdot x} u(x) dx, \]
and we have identified radially symmetric functions on $\mathbb{R}^n$ with the corresponding functions on $(0, \infty)$.

By the Cauchy-Schwarz inequality and the Plancherel formula, we have

$$
|x|^s |u(x)|
\leq |x|^{1-s} \left( \int_0^\infty |J_{n/2-1}(|x|\rho)|^2 \rho^{1-2s} d\rho \right)^{1/2} \left( \int_0^\infty |\widehat{u}(\rho)|^2 \rho^{2s+n-1} d\rho \right)^{1/2}
$$

$$
= \left( \int_0^\infty |J_{n/2-1}(\rho)|^2 \rho^{1-2s} d\rho \right)^{1/2} \left( \frac{\Gamma(n/2)}{2\pi^{n/2}} \int |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi \right)^{1/2}
$$

$$
= C(n, s) \| (-\Delta)^{s/2} u \|_{L^2},
$$

as required.

2.2. Proof of Proposition 2. We use the following estimates on Bessel functions:

$$
\sup_{r \geq 0} |J_{n/2-1}(r)| \leq 1. \quad (8)
$$

$$
\sup_{r \geq 0} r^{1/2} |J_{n/2-1}(r)| \leq C. \quad (9)
$$

The first inequality (8) follows from the integral representation (see [13])

$$
J_{n/2-1}(r)^2 = \frac{2}{\pi} \int_0^{\pi/2} J_{n-2}(2r \cos \theta) d\theta,
$$

$$
J_m(t) = \frac{1}{2\pi} \int_0^{2\pi} \cos(m\theta - t \sin \theta) d\theta, \ m \in \mathbb{Z}.
$$

The second inequality (9) follows from the first and the well-known asymptotics (see [10])

$$
J_\nu(r) \sim \sqrt{\frac{2}{\pi r}} \cos \left( r - \frac{(2\nu + 1)\pi}{4} \right) \quad \text{as} \quad r \to \infty.
$$

(10)

We apply the Littlewood-Paley decomposition $\{ \varphi_j \}_{j \in \mathbb{Z}}$ on $\mathbb{R}^n \setminus \{0\}$ to (4) to obtain

$$
u(x) = |x|^{1-n/2} \sum_{j \in \mathbb{Z}} \int_0^\infty J_{n/2-1}(|x|\rho) \psi_j(\rho) \varphi_j(\rho) \rho^n d\rho,
$$

(11)

where $\psi_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}$ and supp $\varphi_j \subset \{2^{j-1} \leq \rho \leq 2^{j+1} \}$.

As in (7), we have

$$
\left| x \right|^{(n-1)/2} |u(x)|
\leq |x|^1 \sup_{j \in \mathbb{Z}} \left( \int_0^\infty |J_{n/2-1}(|x|\rho)|^2 \psi_j(\rho)^2 d\rho \right)^{1/2}
\cdot \sum_{j \in \mathbb{Z}} \left( \int_0^\infty |\varphi_j(\rho) \widehat{u}(\rho)|^2 \rho^n d\rho \right)^{1/2}.
$$
By (9), we estimate
\[
|x|^{1/2} \sup_{j \in \mathbb{Z}} \left( \int_0^{\infty} |J_{2^{-1}}(\rho)|^2 \psi_j(\rho)^2 d\rho \right)^{1/2}
\leq C \sup_{j \in \mathbb{Z}} \left( \int_0^{\infty} \frac{1}{\rho} \psi_j(\rho)^2 d\rho \right)^{1/2}
\leq C \sup_{j \in \mathbb{Z}} \left( \int_{2^{j-2}}^{2^{j+2}} \frac{1}{\rho} d\rho \right)^{1/2} \leq C.
\]
(12)

This proves (2) since
\[
\sum_{j \in \mathbb{Z}} \left( \int_0^{\infty} |\varphi_j(\rho) \hat{u}(\rho)|^2 \rho^n d\rho \right)^{1/2}
\]
is equivalent to the seminorm on \( \dot{B}^{1/2}_{2,1,\text{rad}} \).

2.3. Proof of Proposition 3. If we use Cauchy-Schwartz inequality as in (7), we have for any \( M > 0 \)
\[
|x|^{n/2-s}|u(x)|
\leq |x|^{1-s} \left( \int_0^{M|x|} |J_{2^{-1}}(\rho)|^2 \rho d\rho \right)^{1/2} \left( \int_0^{M|x|} |\hat{u}(\rho)|^2 \rho^{n-1} d\rho \right)^{1/2}
+ |x|^{1-s} \left( \int_{M|x|}^{\infty} |J_{2^{-1}}(\rho)|^2 \rho^{n-1} d\rho \right)^{1/2} \left( \int_{M|x|}^{\infty} |\hat{u}(\rho)|^2 \rho^{n+1} d\rho \right)^{1/2}
\leq |x|^{-s} \left( \int_0^{M} |J_{2^{-1}}(r)|^2 r dr \right)^{1/2} \|u\|_{L^2}
+ |x|^{-s} \left( \int_{M}^{\infty} |J_{2^{-1}}(r)|^2 r^{-1} dr \right)^{1/2} \|\nabla u\|_{L^2}.
\]
From (8) and (9) we deduce that \( \sup_{r \geq 0} |r^{1-s}J_{2^{-1}}(r)| \leq C \) for any \( \frac{1}{2} \leq s < 1 \). Hence we have for any \( M > 0 \)
\[
|x|^{n/2-s}|u(x)|
\leq |x|^{-s} \left( \int_0^{M} |J_{2^{-1}}(r)|^2 r dr \right)^{1/2} \|u\|_{L^2}
+ |x|^{1-s} \left( \int_{M}^{\infty} |J_{2^{-1}}(r)|^2 r^{-1} dr \right)^{1/2} \|\nabla u\|_{L^2}
\leq C|x|^{-s}M^n \|u\|_{L^2} + |x|^{1-s}M^{n-(1-s)} \|\nabla u\|_{L^2}.
\]
The minimization of the RHS of the last inequality with respect to \( M \) yields Proposition 3.
Then we have
\[ |x|^{\frac{n}{2} - s}|u(x,\omega)| = c_n \sum_{k,l} |x|^1 |\int_0^\infty J_{\nu(k)}(|x|\rho)^2 g_{k,l}(\rho) d\rho| Y_{k,l}(\omega), \]  
(14)
where \( \omega \in S^{n-1}, \nu(k) = \frac{n + 2k - 2}{2} \) and \( \tilde{f}_{k,l}(\rho \omega) = g_{k,l}(\rho) Y_{k,l}(\omega) \). Here
\[ g_{k,l}(\rho) = c_{n,k} \int_0^\infty f_{k,l}(r) r^{-n-1}(r\rho)^{-\frac{n+2}{2}} Y_{k,l}(r\rho) dr. \]

The absolute value of \( c_{n,k} \) is bounded by a constant depending only on \( n \). See [10] for this.

Using the Cauchy-Schwarz inequality as in (7), we have
\[ |x|^{\frac{n}{2} - s}|u(x,\omega)| \]
\[ \leq C \sum_{k,l} \|Y_{k,l}\|_{L^\infty(S^{n-1})} \left( \int_0^\infty |J_{\nu(k)}(|x|\rho)^2\rho^{1-2s}|^2 \right)^{\frac{1}{2}} \left( \int_0^\infty |g_{k,l}(\rho)^2\rho^{2s+n-1} d\rho| \right)^{\frac{1}{2}} \]
\[ \leq C \sum_{k,l} k^{\frac{2s}{2}} \left( \frac{\Gamma(\nu(k) + 1 - s)}{\Gamma(\nu(k) + s)} \right)^{\frac{1}{2}} \left( \int_0^\infty |g_{k,l}(\rho)^2\rho^{2s+n-1} d\rho| \right)^{\frac{1}{2}} \|Y_{k,l}\|_{L^2(S^{n-1})}. \]
Here we used the inequality that \( \|Y_{k,l}\|_{L^\infty} \leq C \frac{n^2}{\pi} \|Y_{k,l}\|_{L^2} \) (see for instance [10]). Using the Stirling’s formula for gamma function that \( \Gamma(t) \approx t^{-\frac{1}{2}} e^{-t} \) for large \( t \) (for instance, see [1]) and the fact \( -\Delta_\omega Y_{k,l} = k(k + n - 2)Y_{k,l} \), we have from (13)
\[ |x|^{\frac{n}{2} - s}|u(x,\omega)| \]
\[ \leq C \sum_{k} k^{2s} \left( \sum_{1 \leq d \leq k} \int_0^\infty |g_{k,l}(\rho)^2\rho^{2s+n-1} d\rho| \right)^{\frac{1}{2}} \left( \sum_{1 \leq d \leq k} \int_0^\infty \|Y_{k,l}\|_{L^2(S^{n-1})}^2 \right)^{\frac{1}{2}} \]
\[ \leq C \left( \sum_{k} k^{2(n-s-m)} \right)^{\frac{1}{2}} \left( \sum_{k,l} \int_0^\infty |\mathcal{F}(f_{k,l} Y_{k,l})(\rho \omega)|^2\rho^{2s+n-1} d\rho d\omega \right)^{\frac{1}{2}} \]
\[ \leq C \left( \sum_{k,l} \int_0^\infty \int_{S^{n-1}} |\mathcal{F}(f_{k,l} Y_{k,l})(\omega)|^2\rho^{2s+n-1} d\rho d\omega \right)^{\frac{1}{2}} \]
\[ \leq C \|u\|_{H^s_{\omega,m}}. \]
where $\mathcal{F}$ is the Fourier transform. This proves part (1).

For the part (2), if we use the Littlewood-Paley decomposition $\{\varphi_j\}_{j \in \mathbb{Z}}$ as in the proof of Proposition 2, we have

$$
|x|^{\frac{n-1}{2}} u(|x| \omega) = c_n \sum_{j \in \mathbb{Z}} \sum_{k,l} |x| \frac{1}{2} \int_0^\infty \lambda_{\nu(k)}(|x| |\omega| \rho^2 \psi_j(\rho) \varphi_j(\rho) g_{k,l}(\rho) d\rho Y_{k,l}(\omega),
$$

(15)

Since $m > n - 3/2$, by (12) we deduce that

$$
|x|^{\frac{n-1}{2}} |u(|x| \omega)|
\leq C \sum_{j \in \mathbb{Z}} \sum_{k,l} \left( \int_0^\infty |\varphi_j(\rho) g_{k,l}(\rho)|^2 \rho^m d\rho \right)^{\frac{1}{2}} \|Y_{k,l}\|_{L^\infty(S^{n-1})}
\leq C \sum_{j \in \mathbb{Z}} \left( \sum_{k} k^{2(n-2-m)} \right)^{\frac{1}{2}} \left( \sum_{k,l} \int_0^\infty |\varphi_j(\rho) g_{k,l}(\rho)|^2 \rho^m d\rho \|Y_{k,l}\|_{L^2(S^{n-1})}^2 \right)^{\frac{1}{2}}
\leq C \sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \|\varphi_j \mathcal{F}((1 - \Delta_\omega)^{\frac{m}{2}} u)\|_{L^2} = C \|u\|_{B_{2,1,\omega}^{\frac{1}{2}}}.
$$

To show Corollary 1 we use the fact that $H^{s,m}_\omega$ is a Hilbert space. Hence any bounded sequence $\{u_j\}$ in $H^{s,m}_\omega$ satisfies $u_j(x) \to 0$ as $|x| \to \infty$ uniformly and has a subsequence converges to $u$ in $H^{s,m}_\omega$ weakly. Let us denote the subsequence by $u_{j_k}$.

Now choose a smooth function $\varphi$ supported in the ball of radius $R + 1$ and with value $1$ in the ball of radius $R$. By the standard argument one can easily show that for each $R$ the mapping $u \mapsto \varphi u$ is compact from $H^t$ to $H^{t'}$ if $t' < t$. By the compactness above and Sobolev embedding we may assume that the sequence $\varphi u_{j_k}$ satisfies that for $2 \leq q < \frac{2n}{n-2s}$

$$
\|\varphi u_{j_k} - \varphi u\|_{L^q} \to 0 \text{ as } k \to \infty.
$$

(16)

Thus we have

$$
\|u_{j_k} - u\|_{L^p} \leq \|\varphi(u_{j_k} - u)\|_{L^p} + \|(1 - \varphi)(u_{j_k} - u)\|_{L^p} \equiv I_k + II_k
$$

with $I_k \to 0$ as $k \to \infty$ by (16) since $2 < p < \frac{2n}{n-2s}$. From the uniform convergence that $|u_{j_k}(x)| + |u(x)| \to 0$ as $|x| \to \infty$ it follows that

$$
\limsup_{k \to \infty} II_k \leq \sup_k \|u_{j_k} - u\|_{L^\infty(|x| > R)} \to 0 \text{ as } R \to \infty.
$$

This proves the compactness of the embedding $H^{s,m}_\omega \hookrightarrow L^p$.

Since $B^{\frac{1}{2},m}_{2,1,\omega} \to H^{s,m}_\omega$, one can adapt the same arguments (compactness of cut-off mapping and uniform convergence at infinity) as above except for weak-* convergence of $u_{j_k}$ to $u$ in $B^{\frac{1}{2},m}_{2,1,\omega}$ for the compactness of embedding $B^{\frac{1}{2},m}_{2,1,\omega}$ to $L^p$. This completes the proof.
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DEPARTMENT OF MATHEMATICS, POSTECH, POHANG 790-784, REPUBLIC OF KOREA
E-mail address: changocho@postech.ac.kr

DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY, SAPPORO, 060-0810, JAPAN
E-mail address: ozawa@math.sci.hokudai.ac.jp