SOBOLEV INEQUALITIES WITH SYMMETRY

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Abstract. In this paper we derive some Sobolev inequalities for radially symmetric functions in $\dot{H}^s$ with $\frac{1}{2} < s < \frac{n}{2}$. We show the end point case $s = \frac{1}{2}$ on the homogeneous Besov space $\dot{B}^{\frac{1}{2}}_{2,1}$. These results are extensions of the well-known Strauss' inequality [11]. Also non-radial extensions are given, which show that compact embeddings are possible in some L^p spaces if a suitable angular regularity is imposed.

1. Introduction

In this paper we derive Sobolev inequalities with symmetry. We first consider several Sobolev inequalities for radially symmetric functions in $\dot{H}^s(\mathbb{R}^n)$ with $\frac{1}{2} < s < \frac{n}{2}$. There is a sharp result by Sickel and Skrzypczak [8], although the argument below is much simpler and direct and a constant in the inequality in Proposition 1 below is given explicitly in terms of s.

Definition 1.

$$\dot{H}^s_{\mathrm{rad}} = \{u \in \dot{H}^s(\mathbb{R}^n) \ : \ u \text{ is radially symmetric } \}, \ s \geq 0.$$
$$\dot{B}^s_{p,q,\mathrm{rad}} = \{u \in \dot{B}^s_{p,q}(\mathbb{R}^n) \ : \ u \text{ is radially symmetric}\}, \ s \geq 0, \ 1 \leq p, q \leq \infty.$$

The inhomogeneous spaces of radially symmetric functions are defined by the same way with spaces H^s and $B^s_{p,q}$.

Proposition 1. *Let $n \geq 2$ and let s satisfy $1/2 < s < n/2$.*

Then

$$\sup_{x \in \mathbb{R}^n \setminus \{0\}} |x|^{n/2-s}|u(x)| \leq C(n,s)\|(-\Delta)^{s/2}u\|_{L^2} \tag{1}$$

for all $u \in \dot{H}^s_{\mathrm{rad}}$, where

$$C(n,s) = \left(\frac{\Gamma(2s-1)\Gamma(\frac{n}{2}-s)\Gamma(\frac{n}{2})}{2^{2s}\pi^{n/2}\Gamma(s)^2\Gamma(\frac{n}{2}-1+s)}\right)^{1/2}$$

and Γ is the gamma function.

Remark 1. For $s = 1$ with $n \geq 3$, the inequality (1) reduces to Ni's inequality [6, 7].

Remark 2. The restriction $1/2 < s < n/2$ is necessary for $C(n,s)$ to be finite.

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Remark 3. The inequality (1) fails for $s = n/2$. Indeed, $u(x) = \mathcal{F}^{-1}\left(\frac{1}{(1+|\xi|)^n(1+\log(1+|\xi|))}\right)$ satisfies $u \in H^{n/2}_{\rm rad}$, and $u \notin L^\infty$ where $\mathcal{F}$ is the Fourier transform [12] and $\mathcal{F}^{-1}$ is its inverse.

Remark 4. The inequality (1) fails if $0 \le s < 1/2$ and $n \ge 3$. Indeed, $u = \mathcal{F}^{-1}(J_{\frac{n}{2}-1}(|\xi|)|\xi|^{-n/2})$ satisfies $u \in \dot H^s_{\rm rad}$ and $u(x) = \infty$ for all $x \in S^{n-1}$, where we note that $u \in \dot H^s_{\rm rad}$ if and only if $1 - n/2 < s < 1/2$, since

$$\|(-\Delta)^{s/2}u\|_{L^2}^2 = c_n \int_0^\infty |J_{\frac{n}{2}-1}(\rho)|^2 \rho^{2s-1} d\rho$$

and that by the asymptotic behavior of Bessel function (10)

$$u(x) = \int_0^\infty |J_{\frac{n}{2}-1}(\rho)|^2 d\rho = \infty, \quad x \in S^{n-1}.$$

See also the proof of Proposition 1 below.

In the endpoint case $s = 1/2$, we have the following propostion.

Proposition 2. *Let $n \ge 2$. Then there exists a constant C such that*

$$\sup_{x \in \mathbb{R}^n \setminus \{0\}} |x|^{(n-1)/2}|u(x)| \le C\|u\|_{\dot B^{1/2}_{2,1}} \tag{2}$$

for all $u \in \dot B^{1/2}_{2,1,{\rm rad}}$.

Remark 5. The inhomogeneous version of (2) has been given in [8] whose proof is based on the atomic decomposition.

Proposition 3. *Let $n \ge 2$ and let s satisfy $1/2 \le s < 1$. Then there exists C such that for all $u \in H^1_{\rm rad}$*

$$\sup_{x \in \mathbb{R}^n \setminus \{0\}} |x|^{n/2-s}|u(x)| \le C(n,s)\|u\|_{L^2}^{1-s}\|\nabla u\|_{L^2}^s. \tag{3}$$

Remark 6. For $s = 1/2$, the inequality (3) reduces to Strauss' inequality [11].

Remark 7. For $s = 0$, the inequality (3) holds for nonincreasing functions in $|x|$ [2]. For $s = 1$, the inequality (3) holds for $n \ge 3$ and fails for $n = 2$. See Proposition 1 and Remark 2.

Now we extend the results above on radial functions to the non-radial functions with additional angular regularity. For details, let us define function spaces $H^{s,m}_\omega$ and $B^{s,m}_{2,1,\omega}$, $s \ge 0, m \ge 0$ as follows.

Definition 2.

$$H^{s,m}_\omega = \left\{ u \in H^s : \|u\|_{H^{s,m}_\omega} \equiv \|(1-\Delta_\omega)^{\frac{m}{2}} u\|_{H^s} < \infty \right\},$$

$$B^{s,m}_{2,1,\omega} = \left\{ u \in B^{\frac{1}{2}}_{2,1} : \|u\|^2_{B^{s,m}_{2,1,\omega}} \equiv \|(1-\Delta_\omega)^{\frac{m}{2}} u\|_{B^s_{2,1}} < \infty \right\},$$

where Δ_ω is the Laplace-Beltrami operator on S^{n-1}.

The homogeneous spaces $\dot{H}^{s,m}_\omega$ and $\dot{B}^{s,m}_{2,1,\omega}$ is similarly defined by the definition of $\dot{H}^s$ and $\dot{B}^s_{2,1}$. Then we have the following.

Proposition 4. *(1) If $1/2 < s < n/2$ and $m > n-1-s$, then there exists a constant C such that for any $u \in H^{s,m}_\omega$*

$$\sup_{\mathbb{R}^n \setminus \{0\}} |x|^{n/2-s} |u(x)| \le C \|u\|_{\dot{H}^{s,m}_\omega}. \tag{4}$$

(2) If $m > n - \frac{3}{2}$, then there exists a constant C such that for any $u \in B^{\frac{1}{2},m}_{2,1,\omega}$

$$\sup_{\mathbb{R}^n \setminus \{0\}} |x|^{(n-1)/2} |u(x)| \le C \|u\|_{\dot{B}^{\frac{1}{2},m}_{2,1,\omega}}. \tag{5}$$

Remark 8. $H^{s,m}_\omega$ and $B^{\frac{1}{2},m}_{2,1,\omega}$ are closed subspaces of H^s and $B^s_{2,1}$, respectively and they contain H^s_{rad} and $B^{\frac{1}{2}}_{2,1,\mathrm{rad}}$ naturally, respectively. We can identify the spaces H^s with $(1-\Delta_\omega)^{m/2} H^{s,m}_\omega$ and also $B^{\frac{1}{2}}_{2,1}$ with $(1-\Delta_\omega)^{m/2} B^{\frac{1}{2},m}_{2,1,\omega}$.

Remark 9. $H^{s,m}_\omega$ is a Hilbert space with the same inner product as L^2 space and its dual space is given by $H^{-s,-m}_\omega$.

From the decay at infinity we deduce compact embeddings of $H^{s,m}_\omega$ and $B^{\frac{1}{2},m}_{2,1,\omega}$ into some L^p spaces as follows. See [2, 3, 4, 8, 9] for the radial case.

Corollary 1. *The embedding $H^{s,m}_\omega \hookrightarrow L^p$ is compact for $1/2 < s < n/2$, $m > n-1-s$ and $2 < p < 2n/(n-2s)$.*

Corollary 2. *The embedding $B^{\frac{1}{2},m}_{2,1,\omega} \hookrightarrow L^p$ is compact for $m > n - 3/2$ and $2 < p < 2n/(n-1)$.*

2. Proofs

2.1. **Proof of Proposition 1.** We use the following Fourier representation for radially symmetric functions as

$$u(x) = |x|^{1-\frac{n}{2}} \int_0^\infty J_{\frac{n}{2}-1}(|x|\rho) \widehat{u}(\rho) \rho^{\frac{n}{2}} d\rho, \tag{6}$$

where J_ν is the Bessel function of order ν, $\widehat{u}$ is the Fourier transform normalized as

$$\widehat{u}(\xi) = (2\pi)^{-n/2} \int e^{-ix\cdot\xi} u(x) dx,$$

and we have identified radially symmetric functions on $\mathbb{R}^n$ with the corresponding functions on $(0, \infty)$.

By the Cauchy-Schwarz inequality and the Plancherel formula, we have

$$
\begin{aligned}
&|x|^{\frac{n}{2}-s}|u(x)| \\
&\le |x|^{1-s}\left(\int_0^\infty |J_{\frac{n}{2}-1}(|x|\rho)|^2\rho^{1-2s}d\rho\right)^{1/2}\left(\int_0^\infty |\widehat{u}(\rho)|^2\rho^{2s+n-1}d\rho\right)^{1/2} \\
&= \left(\int_0^\infty |J_{\frac{n}{2}-1}(\rho)|^2\rho^{1-2s}d\rho\right)^{1/2}\left(\frac{\Gamma(\frac{n}{2})}{2\pi^{n/2}}\int |\xi|^{2s}|\widehat{u}(\xi)|^2 d\xi\right)^{1/2} \\
&= C(n,s)\|(-\Delta)^{s/2}u\|_{L^2},
\end{aligned}
\tag{7}
$$

as required.

2.2. **Proof of Proposition 2.** We use the following estimates on Bessel functions:

$$\sup_{r\ge 0}|J_{\frac{n}{2}-1}(r)| \le 1. \tag{8}$$

$$\sup_{r\ge 0} r^{1/2}|J_{\frac{n}{2}-1}(r)| \le C. \tag{9}$$

The first inequality (8) follows from the integral representation (see [13])

$$
\begin{aligned}
J_{\frac{n}{2}-1}(r)^2 &= \frac{2}{\pi}\int_0^{\pi/2} J_{n-2}(2r\cos\theta)d\theta, \\
J_m(t) &= \frac{1}{2\pi}\int_0^{2\pi}\cos(m\theta - t\sin\theta)d\theta,\ \ m\in\mathbb{Z}.
\end{aligned}
$$

The second inequality (9) follows from the first and the well-known asymptotics (see [10])

$$J_\nu(r) \sim \sqrt{\frac{2}{\pi r}}\cos\left(r - \frac{(2\nu+1)\pi}{4}\right) \quad \text{as} \quad r\to\infty. \tag{10}$$

We apply the Littlewood-Paley decomposition $\{\varphi_j\}_{j\in\mathbb{Z}}$ on $\mathbb{R}^n\setminus\{0\}$ to (4) to obtain

$$u(x) = |x|^{1-\frac{n}{2}}\sum_{j\in\mathbb{Z}}\int_0^\infty J_{\frac{n}{2}-1}(|x|\rho)\psi_j(\rho)\varphi_j(\rho)\widehat{u}(\rho)\rho^{\frac{n}{2}}d\rho, \tag{11}$$

where $\psi_j = \varphi_{j-1}+\varphi_j+\varphi_{j+1}$ and supp $\varphi_j \subset \{2^{j-1}\le\rho\le 2^{j+1}\}$.

As in (7), we have

$$
\begin{aligned}
&|x|^{(n-1)/2}|u(x)| \\
&\le |x|^{1/2}\sup_{j\in\mathbb{Z}}\left(\int_0^\infty |J_{\frac{n}{2}-1}(|x|\rho)|^2\psi_j(\rho)^2 d\rho\right)^{1/2} \\
&\qquad \cdot\sum_{j\in\mathbb{Z}}\left(\int_0^\infty |\varphi_j(\rho)\widehat{u}(\rho)|^2\rho^n d\rho\right)^{1/2}.
\end{aligned}
$$

By (9), we estimate

$$\begin{aligned}
&|x|^{1/2} \sup_{j\in\mathbb{Z}} \left(\int_0^\infty |J_{\frac{n}{2}-1}(|x|\rho)|^2 \psi_j(\rho)^2 d\rho \right)^{1/2} \\
&\leq C \sup_{j\in\mathbb{Z}} \left(\int_0^\infty \frac{1}{\rho} \psi_j(\rho)^2 d\rho \right)^{1/2} \\
&\leq C \sup_{j\in\mathbb{Z}} \left(\int_{2^{j-2}}^{2^{j+2}} \frac{1}{\rho} d\rho \right)^{1/2} \leq C.
\end{aligned} \tag{12}$$

This proves (2) since

$$\sum_{j\in\mathbb{Z}} \left(\int_0^\infty |\varphi_j(\rho)\widehat{u}(\rho)|^2 \rho^n d\rho \right)^{1/2}$$

is equivalent to the seminorm on $\dot{B}^{1/2}_{2,1,\mathrm{rad}}$.

2.3. **Proof of Proposition 3.** If we use Cauchy-Schwartz inequality as in (7), we have for any $M > 0$

$$\begin{aligned}
&|x|^{n/2-s}|u(x)| \\
&\leq |x|^{1-s} \left(\int_0^{M|x|} |J_{\frac{n}{2}-1}(|x|\rho)|^2 \rho d\rho \right)^{1/2} \left(\int_0^{M|x|} |\widehat{u}(\rho)|^2 \rho^{n-1} d\rho \right)^{1/2} \\
&+|x|^{1-s} \left(\int_{M|x|}^\infty |J_{\frac{n}{2}-1}(|x|\rho)|^2 \rho^{-1} d\rho \right)^{1/2} \left(\int_{M|x|}^\infty |\widehat{u}(\rho)|^2 \rho^{n+1} d\rho \right)^{1/2} \\
&\leq |x|^{-s} \left(\int_0^M |J_{\frac{n}{2}-1}(r)|^2 r dr \right)^{1/2} \|u\|_{L^2} \\
&+|x|^{1-s} \left(\int_M^\infty |J_{\frac{n}{2}-1}(r)|^2 r^{-1} dr \right)^{1/2} \|\nabla u\|_{L^2}.
\end{aligned}$$

From (8) and (9) we deduce that $\sup_{r\geq 0} |r^{1-s} J_{\frac{n}{2}-1}(r)| \leq C$ for any $\frac{1}{2} \leq s < 1$. Hence we have for any $M > 0$

$$\begin{aligned}
&|x|^{n/2-s}|u(x)| \\
&\leq |x|^{-s} \left(\int_0^M |J_{\frac{n}{2}-1}(r)|^2 r\, dr \right)^{\frac{1}{2}} \|u\|_{L^2} \\
&+ |x|^{1-s} \left(\int_M^\infty |J_{\frac{n}{2}-1}(r)|^2 r^{-1}\, dr \right)^{\frac{1}{2}} \|\nabla u\|_{L^2} \\
&\leq C|x|^{-s} M^s \|u\|_{L^2} + |x|^{1-s} M^{-(1-s)} \|\nabla u\|_{L^2}.
\end{aligned}$$

The minimization of the RHS of the last inequality with respect to M yields Proposition 3.

2.4. **Proofs of Proposition 4 and Corollaries.** The proof for (4) follows from the one of Proposition 1 and the spherical harmonic expansion of functions in $H_\omega^{s,m}$ [5, 10]. In fact, if we write $u(r\omega) = \sum_{k\ge 0}\sum_{1\le l\le d(k)} f_{k,l}(r)Y_{k,l}(\omega)$, where $d(k)$ is the dimension of space of spherical harmonic functions of degree k and

$$d(k) \le Ck^{n-2} \ \text{ for large } \quad k. \tag{13}$$

Then we have

$$|x|^{\frac{n}{2}-s}u(|x|\omega) = c_n \sum_{k,l} |x|^{1-s}\int_0^\infty J_{\nu(k)}(|x|\rho)\rho^{\frac{n}{2}}g_{k,l}(\rho)d\rho Y_{k,l}(\omega), \tag{14}$$

where $\omega \in S^{n-1}$, $\nu(k) = \frac{n+2k-2}{2}$ and $\widehat{f_{k,l}Y_{k,l}}(\rho\omega) = g_{k,l}(\rho)Y_{k,l}(\omega)$. Here

$$g_{k,l}(\rho) = c_{n,k}\int_0^\infty f_{k,l}(r)r^{n-1}(r\rho)^{-\frac{n-2}{2}}J_{\nu(k)}(r\rho)\, dr.$$

The absolute value of $c_{n,k}$ is bounded by a constant depending only on n. See [10] for this.

Using the Cauchy-Schwarz inequality as in (7), we have

$$\begin{aligned}
&|x|^{\frac{n}{2}-s}|u(|x|\omega)| \\
&\le C\sum_{k,l}\|Y_{k,l}\|_{L^\infty(S^{n-1})}\left(\int_0^\infty |J_{\nu(k)}(|x|\rho)|^2\rho^{1-2s}\right)^{\frac12}\left(\int_0^\infty |g_{k,l}(\rho)|^2\rho^{2s+n-1}d\rho\right)^{\frac12} \\
&\le C\sum_{k,l} k^{\frac{n-2}{2}}\left(\frac{\Gamma(\nu(k)+1-s)}{\Gamma(\nu(k)+s)}\right)^{\frac12}\left(\int_0^\infty |g_{k,l}(\rho)|^2\rho^{2s+n-1}d\rho\right)^{\frac12}\|Y_{k,l}\|_{L^2(S^{n-1})}.
\end{aligned}$$

Here we used the inequality that $\|Y_{k,l}\|_{L^\infty} \le Ck^{\frac{n-2}{2}}\|Y_{k,l}\|_{L^2}$ (see for instance [10]). Using the Stirling's formula for gamma function that $\Gamma(t) \approx t^{t-\frac12}e^{-(t-1)}$ for large t (for instance, see [1]) and the fact $-\Delta_\omega Y_{k,\,l} = k(k+n-2)Y_{k,\,l}$, we have from (13)

$$\begin{aligned}
&|x|^{\frac{n}{2}-s}|u(|x|\omega)| \\
&\le C\sum_k k^{\frac{n-2}{2}}d(k)^{\frac12}\left(\frac{\Gamma(\nu(k)+1-s)}{\Gamma(\nu(k)+s)}\right)^{\frac12} \\
&\qquad \cdot\left(\sum_{1\le l\le d(k)}\int_0^\infty |g_{k,l}(\rho)|^2\rho^{2s+n-1}d\rho\|Y_{k,l}\|_{L^2(S^{n-1})}^2\right)^{\frac12} \\
&\le C\left(\sum_k k^{2(n-\frac32-s-m)}\right)^{\frac12}\left(\sum_{k,l} k^{2m}\int_0^\infty |g_{k,l}(\rho)|^2\rho^{2s+n-1}d\rho\|Y_{k,l}\|_{L^2(S^{n-1})}^2\right)^{\frac12} \\
&\le C\left(\sum_{k,l} k^{2m}\int_0^\infty\int_{S^{n-1}} |\mathcal{F}(f_{k,l}Y_{k,l})(\rho\omega)|^2\rho^{2s+n-1}d\rho d\omega\right)^{\frac12} \\
&\le C\left(\sum_{k,l}\int_0^\infty\int_{S^{n-1}} \left|\mathcal{F}((1-\Delta_\omega)^{\frac{m}{2}}(f_{k,l}Y_{k,l}))(\rho\omega)\right|^2\rho^{2s+n-1}d\rho d\omega\right)^{\frac12} \\
&\le C\|u\|_{\dot H_\omega^{s,m}},
\end{aligned}$$

where $\mathcal{F}$ is the Fourier transform. This proves part (1).

For the part (2), if we use the Littlewood-Paley decomposition $\{\varphi_j\}_{j\in\mathbb{Z}}$ as in the proof of Proposition 2, the we have

$$|x|^{\frac{n-1}{2}}u(|x|\omega) = c_n\sum_{j\in\mathbb{Z}}\sum_{k,l}|x|^{\frac{1}{2}}\int_0^\infty J_{\nu(k)}(|x|\rho)\rho^{\frac{n}{2}}\psi_j(\rho)\varphi_j(\rho)g_{k,l}(\rho)d\rho Y_{k,l}(\omega), \tag{15}$$

Since $m > n-3/2$, by (12) we deduce that

$$\begin{aligned}
&|x|^{\frac{n-1}{2}}|u(|x|\omega)| \\
&\le C\sum_{j\in\mathbb{Z}}\sum_{k,l}\left(\int_0^\infty|\varphi_j(\rho)g_{k,l}(\rho)|^2\rho^n\,d\rho\right)^{\frac{1}{2}}\|Y_{k,l}\|_{L^\infty(S^{n-1})} \\
&\le C\sum_{j\in\mathbb{Z}}\left(\sum_k k^{2(n-2-m)}\right)^{\frac{1}{2}}\left(\sum_{k,l}k^{2m}\int_0^\infty|\varphi_j(\rho)g_{k,l}(\rho)|^2\rho^n\,d\rho\|Y_{k,l}\|^2_{L^2(S^{n-1})}\right)^{\frac{1}{2}} \\
&\le C\sum_{j\in\mathbb{Z}}2^{\frac{j}{2}}\|\varphi_j\mathcal{F}((1-\Delta_\omega)^{\frac{m}{2}}u)\|_{L^2} = C\|u\|_{\dot{B}^{\frac{1}{2}}_{2,1,\omega}}.
\end{aligned}$$

To show Corollary 1 we use the fact that $H^{s,m}_\omega$ is a Hilbert space. Hence any bounded sequence $\{u_j\}$ in $H^{s,m}_\omega$ satisfies $u_j(x)\to 0$ as $|x|\to\infty$ uniformly and has a subsequence converges to u in $H^{s,m}_\omega$ weakly. Let us denote the subsequence by u_{j_k}.

Now choose a smooth function φ supported in the ball of radius $R+1$ and with value 1 in the ball of radius R. By the standard argument one can easily show that for each R the mapping $u\mapsto\varphi u$ is compact from H^t to $H^{t'}$ if $t'<t$. By the compactness above and Sobolev embedding we may assume that the sequence φu_{j_k} satisfies that for $2\le q<\frac{2n}{n-2s}$

$$\|\varphi u_{j_k}-\varphi u\|_{L^q}\to 0 \ \text{ as } \ k\to\infty. \tag{16}$$

Thus we have

$$\|u_{j_k}-u\|_{L^p}\le\|\varphi(u_{j_k}-u)\|_{L^p}+\|(1-\varphi)(u_{j_k}-u)\|_{L^p}\equiv I_k+II_k$$

with $I_k\to 0$ as $k\to\infty$ by (16) since $2<p<\frac{2n}{n-2s}$. From the uniform convergence that $|u_{j_k}(x)|+|u(x)|\to 0$ as $|x|\to\infty$ it follows that

$$\limsup_{k\to\infty} II_k\le\sup_k\|u_{j_k}-u\|^{\frac{p-2}{p}}_{L^\infty(|x|>R)}\to 0 \ \text{ as } \ R\to\infty.$$

This proves the compactness of the embedding $H^{s,m}_\omega\hookrightarrow L^p$.

Since $B^{\frac{1}{2},m}_{2,1,\omega}\hookrightarrow H^{\frac{1}{2},m}_\omega$, one can adapt the same arguments (compactness of cut-off mapping and uniform convergence at infinity) as above except for weak-$*$ convergence of u_{j_k} to u in $B^{\frac{1}{2},m}_{2,1,\omega}$ for the compactness of embedding $B^{\frac{1}{2},m}_{2,1,\omega}$ to L^p. This completes the proof.

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