<table>
<thead>
<tr>
<th>Title</th>
<th>Sobolev inequalities with symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Cho, Yonggeun; Ozawa, Tohru</td>
</tr>
<tr>
<td>Citation</td>
<td>Hokkaido University Preprint Series in Mathematics, 859, 1-8</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2007</td>
</tr>
<tr>
<td>DOI</td>
<td>10.14943/84009</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/69668">http://hdl.handle.net/2115/69668</a></td>
</tr>
<tr>
<td>Type</td>
<td>bulletin (article)</td>
</tr>
</tbody>
</table>

File Information:
pre859.pdf

Hokkaido University Collection of Scholarly and Academic Papers: HUSCAP
SOBOLEV INEQUALITIES WITH SYMMETRY

YONGGEUN CHO AND TOHRU OZAWA

Abstract. In this paper we derive some Sobolev inequalities for radially symmetric functions in $\dot{H}^s$ with $\frac{1}{2} < s < \frac{n}{2}$. We show the end point case $s = \frac{1}{2}$ on the homogeneous Besov space $\dot{B}_{2,1}^{\frac{1}{2}}$. These results are extensions of the well-known Strauss’ inequality [11]. Also non-radial extensions are given, which show that compact embeddings are possible in some $L^p$ spaces if a suitable angular regularity is imposed.

1. Introduction

In this paper we derive Sobolev inequalities with symmetry. We first consider several Sobolev inequalities for radially symmetric functions in $\dot{H}^s(\mathbb{R}^n)$ with $\frac{1}{2} < s < \frac{n}{2}$. There is a sharp result by Sickel and Skrzypczak [8], although the argument below is much simpler and direct and a constant in the inequality in Proposition 1 below is given explicitly in terms of $s$.

Definition 1.

$$\dot{H}^{s}_{\text{rad}} = \{ u \in \dot{H}^s(\mathbb{R}^n) : u \text{ is radially symmetric} \}, \quad s \geq 0.$$  

$$\dot{B}^{s}_{p,q,\text{rad}} = \{ u \in \dot{B}^s_{p,q}(\mathbb{R}^n) : u \text{ is radially symmetric} \}, \quad s \geq 0, \quad 1 \leq p, q \leq \infty.$$  

The inhomogeneous spaces of radially symmetric functions are defined by the same way with spaces $H^s$ and $B^s_{p,q}$.

Proposition 1. Let $n \geq 2$ and let $s$ satisfy $1/2 < s < n/2$.

Then

$$\sup_{x \in \mathbb{R}^n \setminus \{0\}} |x|^{n/2-s} |u(x)| \leq C(n, s) \|(-\Delta)^{s/2} u\|_{L^2}$$  

for all $u \in \dot{H}^{s}_{\text{rad}}$, where

$$C(n, s) = \left( \frac{\Gamma(2s-1)\Gamma(n/2-s)\Gamma(n/2)}{2^{2s-n/2}\Gamma(s)^2\Gamma(n/2-1+s)} \right)^{1/2}$$

and $\Gamma$ is the gamma function.

Remark 1. For $s = 1$ with $n \geq 3$, the inequality (1) reduces to Ni’s inequality [6, 7].

Remark 2. The restriction $1/2 < s < n/2$ is necessary for $C(n, s)$ to be finite.

2000 Mathematics Subject Classification. Primary 46E35; Secondary 42C15.

Keywords and phrases. Sobolev inequality, function space with radial symmetry, angular regularity.
Remark 3. The inequality (1) fails for $s = n/2$. Indeed, $u(x) = \mathcal{F}^{-1}\left(\frac{1}{(1+|\xi|)^{n/(1+\log(1+|\xi|))}}\right)$ satisfies $u \in H_{\text{rad}}^{n/2}$, and $u \not\in L^\infty$ where $\mathcal{F}$ is the Fourier transform [12] and $\mathcal{F}^{-1}$ is its inverse.

Remark 4. The inequality (1) fails if $0 \leq s < 1/2$ and $n \geq 3$. Indeed, $u = \mathcal{F}^{-1}(J_{\frac{n}{2}-1}(|\xi|)|\xi|^{-n/2})$ satisfies $u \in \dot{H}_{\text{rad}}^s$ and $u(x) = \infty$ for all $x \in S^{n-1}$, where we note that $u \in \dot{H}_{\text{rad}}^s$ if and only if $1 - n/2 < s < 1/2$, since

$$\|(-\Delta)^{s/2}u\|_{L^2}^2 = c_n \int_0^\infty |J_{\frac{n}{2}-1}(\rho)|^2 \rho^{2s-1} d\rho$$

and that by the asymptotic behavior of Bessel function (10)

$$u(x) = \int_0^\infty |J_{\frac{n}{2}-1}(\rho)|^2 d\rho = \infty, \ x \in S^{n-1}.$$ 

See also the proof of Proposition 1 below.

In the endpoint case $s = 1/2$, we have the following proposition.

**Proposition 2.** Let $n \geq 2$. Then there exists a constant $C$ such that

$$\sup_{x \in \mathbb{R}^n \setminus \{0\}} |x|^{(n-1)/2} |u(x)| \leq C \|u\|_{\dot{B}_{2,1}^{1/2}},$$  

(2)

for all $u \in \dot{B}_{2,1,\text{rad}}^{1/2}$.

**Remark 5.** The inhomogeneous version of (2) has been given in [8] whose proof is based on the atomic decomposition.

**Proposition 3.** Let $n \geq 2$ and let $s$ satisfy $1/2 \leq s < 1$. Then there exists $C$ such that for all $u \in H_{\text{rad}}^1$

$$\sup_{x \in \mathbb{R}^n \setminus \{0\}} |x|^{n/2-s} |u(x)| \leq C(n, s) \|u\|_{L^2}^{1-s} \|\nabla u\|_2^s.$$  

(3)

**Remark 6.** For $s = 1/2$, the inequality (3) reduces to Strauss’ inequality [11].

**Remark 7.** For $s = 0$, the inequality (3) holds for nonincreasing functions in $|x|$ [2]. For $s = 1$, the inequality (3) holds for $n \geq 3$ and fails for $n = 2$. See Proposition 1 and Remark 2.

Now we extend the results above on radial functions to the non-radial functions with additional angular regularity. For details, let us define function spaces $H_{\omega}^{s,m}$ and $B_{2,1,\omega}^{s,m}$, $s \geq 0, m \geq 0$ as follows.
Definition 2.

\( H^{s,m}_\omega = \left\{ u \in H^s : \|u\|_{H^{s,m}_\omega} \equiv \|(1 - \Delta_\omega)^{m/2} u\|_{H^s} \leq \infty \right\} \),

\( B^{s,m}_{2,1,\omega} = \left\{ u \in B^{\frac{3}{2}}_{2,1} : \|u\|_{B^{s,m}_{2,1,\omega}} \equiv \|(1 - \Delta_\omega)^{m/2} u\|_{B^{2,1}} \leq \infty \right\}, \)

where \( \Delta_\omega \) is the Laplace-Beltrami operator on \( S^{n-1} \).

The homogeneous spaces \( \dot{H}^{s,m}_\omega \) and \( \dot{B}^{s,m}_{2,1,\omega} \) is similarly defined by the definition of \( \dot{H}^s \) and \( \dot{B}^s \). Then we have the following.

**Proposition 4.**

1. If \( 1/2 < s < n/2 \) and \( m > n - 1 - s \), then there exists a constant \( C \) such that for any \( u \in H^{s,m}_\omega \)

\[
\sup_{\mathbb{R}^n \setminus \{0\}} |x|^{n/2-s} |u(x)| \leq C \|u\|_{H^{s,m}_\omega}. \tag{4}
\]

2. If \( m > n - \frac{3}{2} \), then there exists a constant \( C \) such that for any \( u \in B^{s,m}_{2,1,\omega} \)

\[
\sup_{\mathbb{R}^n \setminus \{0\}} |x|^{(n-1)/2} |u(x)| \leq C \|u\|_{B^{s,m}_{2,1,\omega}}. \tag{5}
\]

**Remark 8.** \( H^{s,m}_\omega \) and \( B^{s,m}_{2,1,\omega} \) are closed subspaces of \( H^s \) and \( B^{s}_{2,1,\omega} \), respectively and they contain \( H^{s}_{\text{rad}} \) and \( B^{\frac{3}{2}}_{2,1,\text{rad}} \), respectively. We can identify the spaces \( H^s \) with 
\( (1 - \Delta_\omega)^{m/2} H^{s,m}_\omega \) and also \( B^{\frac{3}{2}}_{2,1,\omega} \) with \( (1 - \Delta_\omega)^{m/2} B^{s,m}_{2,1,\omega} \).

**Remark 9.** \( H^{s,m}_\omega \) is a Hilbert space with the same inner product as \( L^2 \) space and its dual space is given by \( H^{-s,-m}_\omega \).

From the decay at infinity we deduce compact embeddings of \( H^{s,m}_\omega \) and \( B^{s,m}_{2,1,\omega} \) into some \( L^p \) spaces as follows. See [2, 3, 4, 8, 9] for the radial case.

**Corollary 1.** The embedding \( H^{s,m}_\omega \hookrightarrow L^p \) is compact for \( 1/2 < s < n/2, m > n - 1 - s \) and \( 2 < p < 2n/(n-2s) \).

**Corollary 2.** The embedding \( B^{s,m}_{2,1,\omega} \hookrightarrow L^p \) is compact for \( m > n - 3/2 \) and \( 2 < p < 2n/(n-1) \).

2. Proofs

2.1. Proof of Proposition 1. We use the following Fourier representation for radially symmetric functions as

\[
u(x) = |x|^{n/2} \int_0^\infty J_{\frac{n}{2} - 1}(|x|\rho)\hat{u}(\rho)\rho^{\frac{n}{2}} d\rho, \tag{6}
\]

where \( J_\nu \) is the Bessel function of order \( \nu \), \( \hat{u} \) is the Fourier transform normalized as

\[
\hat{u}(\xi) = (2\pi)^{-n/2} \int e^{-i\xi x} u(x) dx,
\]
and we have identified radially symmetric functions on $\mathbb{R}^n$ with the corresponding functions on $(0, \infty)$.

By the Cauchy-Schwarz inequality and the Plancherel formula, we have

$$
|x|^{\frac{n}{2}-s}|u(x)|
\leq |x|^{1-s} \left( \int_0^\infty |J_{\frac{n}{2}-1}(|x|\rho)|^2 \rho^{1-2s}d\rho \right)^{1/2} \left( \int_0^\infty |\hat{u}(\rho)|^2 \rho^{2s+n-1}d\rho \right)^{1/2}
= \left( \int_0^\infty |J_{\frac{n}{2}-1}(\rho)|^2 \rho^{1-2s}d\rho \right)^{1/2} \left( \frac{\Gamma(\frac{n}{2})}{2\pi^{n/2}} \int |\xi|^{2s} |\hat{u}(\xi)|^2d\xi \right)^{1/2}
= C(n, s)\|(-\Delta)^{s/2}u\|_{L^2},
$$

as required.

2.2. Proof of Proposition 2. We use the following estimates on Bessel functions:

$$
\sup_{r \geq 0} |J_{\frac{n}{2}-1}(r)| \leq 1.
$$

(8)

$$
\sup_{r \geq 0} r^{1/2}|J_{\frac{n}{2}-1}(r)| \leq C.
$$

(9)

The first inequality (8) follows from the integral representation (see [13])

$$
J_{\frac{n}{2}-1}(r)^2 = \frac{2}{\pi} \int_0^{\pi/2} J_{n-2}(2r \cos \theta)d\theta,
$$

$$
J_m(t) = \frac{1}{2\pi} \int_0^{2\pi} \cos(m\theta - t \sin \theta)d\theta, \ m \in \mathbb{Z}.
$$

The second inequality (9) follows from the first and the well-known asymptotics (see [10])

$$
J_\nu(r) \sim \sqrt{\frac{2}{\pi r}} \cos \left( r - \frac{(2\nu + 1)\pi}{4} \right) \text{ as } r \to \infty.
$$

(10)

We apply the Littlewood-Paley decomposition \( \{\varphi_j\}_{j \in \mathbb{Z}} \) on $\mathbb{R}^n \setminus \{0\}$ to (4) to obtain

$$
u(x) = \left| x \right|^{1-\frac{n}{2}} \sum_{j \in \mathbb{Z}} \int_0^\infty J_{\frac{n}{2}-1}(|x|\rho)\varphi_j(\rho)\varphi_j(\hat{u}(\rho))\rho^\frac{n}{2}d\rho,
$$

(11)

where $\varphi_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}$ and supp $\varphi_j \subset \{2^j-1 \leq \rho \leq 2^j+1\}$.

As in (7), we have

$$
\left| x \right|^{(n-1)/2}|u(x)|
\leq \left| x \right|^{1/2} \sup_{j \in \mathbb{Z}} \left( \int_0^\infty |J_{\frac{n}{2}-1}(|x|\rho)|^2 \varphi_j(\rho)^2d\rho \right)^{1/2}
\cdot \sum_{j \in \mathbb{Z}} \left( \int_0^\infty |\varphi_j(\rho)\hat{u}(\rho)|^2 \rho^n d\rho \right)^{1/2}.
$$
By (9), we estimate
\[
|x|^{1/2} \sup_{j \in \mathbb{Z}} \left( \int_0^\infty |J_{\frac{n}{2}-1}(|x|\rho)|^2 \psi_j(\rho)^2 d\rho \right)^{1/2} \\
\leq C \sup_{j \in \mathbb{Z}} \left( \int_0^\infty \frac{1}{\rho} \psi_j(\rho)^2 d\rho \right)^{1/2} \\
\leq C \sup_{j \in \mathbb{Z}} \left( \int_{2j-2}^{2j+2} \frac{1}{\rho} d\rho \right)^{1/2} \leq C.
\]

This proves (2) since
\[
\sum_{j \in \mathbb{Z}} \left( \int_0^\infty |\varphi_j(\rho) \hat{u}(\rho)|^2 \rho^n d\rho \right)^{1/2}
\]
is equivalent to the seminorm on \( B^{1/2}_{2,1, \text{rad}} \).

2.3. **Proof of Proposition 3.** If we use Cauchy-Schwartz inequality as in (7), we have for any \( M > 0 \)
\[
|x|^{n/2-s} |u(x)| \\
\leq |x|^{-s} \left( \int_0^M |J_{\frac{n}{2}-1}(|x|\rho)|^2 \rho d\rho \right)^{1/2} \left( \int_0^M \hat{u}(\rho)^2 \rho^{n-1} d\rho \right)^{1/2} \\
+ |x|^{-s} \left( \int_M^\infty |J_{\frac{n}{2}-1}(|x|\rho)|^2 \rho^{-1} d\rho \right)^{1/2} \left( \int_M^\infty \hat{u}(\rho)^2 \rho^{n+1} d\rho \right)^{1/2} \\
\leq |x|^{-s} \left( \int_0^M |J_{\frac{n}{2}-1}(r)|^2 r dr \right)^{1/2} \|u\|_{L^2} \\
+ |x|^{-s} \left( \int_M^\infty |J_{\frac{n}{2}-1}(r)|^2 r^{-1} dr \right)^{1/2} \|\nabla u\|_{L^2}.
\]
From (8) and (9) we deduce that \( \sup_{r \geq 0} |r^{1-s} J_{\frac{n}{2}-1}(r)| \leq C \) for any \( \frac{1}{2} \leq s < 1 \). Hence we have for any \( M > 0 \)
\[
|x|^{n/2-s} |u(x)| \\
\leq |x|^{-s} \left( \int_0^M |J_{\frac{n}{2}-1}(r)|^2 r dr \right)^{\frac{1}{2}} \|u\|_{L^2} \\
+ |x|^{-s} \left( \int_M^\infty |J_{\frac{n}{2}-1}(r)|^2 r^{-1} dr \right)^{\frac{1}{2}} \|\nabla u\|_{L^2} \\
\leq C|x|^{-s} M^n \|u\|_{L^2} + |x|^{1-s} M^{-(1-s)} \|\nabla u\|_{L^2}.
\]
The minimization of the RHS of the last inequality with respect to \( M \) yields Proposition 3.
2.4. Proofs of Proposition 4 and Corollaries. The proof for (4) follows from the one of Proposition 1 and the spherical harmonic expansion of functions in $H^{s,m}_\omega$ [5, 10]. In fact, if we write $u(r\omega) = \sum_{k \geq 0} \sum_{1 \leq l \leq d(k)} f_{k,l}(r) Y_{k,l}(\omega)$, where $d(k)$ is the dimension of space of spherical harmonic functions of degree $k$ and

$$d(k) \leq Ck^{n-2} \text{ for large } k.$$ (13)

Then we have

$$|x|^\frac{n}{2-s} u(|x|\omega) = c_n \sum_{k,l} |x|^{1-s} \int_0^{\infty} J_{\nu(k)}(|x|\rho) \rho^s g_{k,l}(\rho) d\rho Y_{k,l}(\omega),$$ (14)

where $\omega \in S^{n-1}$, $\nu(k) = \frac{n+2k-2}{2}$ and $\int_{S^{n-1}} f_{k,l}(\rho\omega) = g_{k,l}(\rho) Y_{k,l}(\omega)$. Here

$$g_{k,l}(\rho) = c_{n,k} \int_0^\infty f_{k,l}(r)r^{n-1}(r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(r\rho) dr.$$ The absolute value of $c_{n,k}$ is bounded by a constant depending only on $n$. See [10] for this.

Using the Cauchy-Schwarz inequality as in (7), we have

$$|x|^\frac{n}{2-s}|u(|x|\omega)| \leq C \sum_{k,l} \|Y_{k,l}\|_{L^\infty(S^{n-1})} \left( \int_0^{\infty} |J_{\nu(k)}(|x|\rho)|^2 \rho^{1-2s} \right)^{1/2} \left( \int_0^{\infty} |g_{k,l}(\rho)|^2 \rho^{2s+n-1} d\rho \right)^{1/2} \|Y_{k,l}\|_{L^2(S^{n-1})}.$$

Here we used the inequality that $\|Y_{k,l}\|_{L^\infty} \leq C k^{n-2} \|Y_{k,l}\|_{L^2}$ (see for instance [10]). Using the Stirling’s formula for gamma function that $\Gamma(t) \approx t^{\frac{1}{2}} e^{-(t-1)}$ for large $t$ (for instance, see [1]) and the fact $-\Delta_\omega Y_{k,l} = k(k + n - 2) Y_{k,l}$, we have from (13)

$$|x|^\frac{n}{2-s}|u(|x|\omega)| \leq C \sum_{k} k^{\frac{n-2}{2} d(k)^{1/2}} \left( \sum_{1 \leq l \leq d(k)} \int_0^{\infty} |g_{k,l}(\rho)|^2 \rho^{2s+n-1} d\rho \right)^{1/2} \left( \sum_{k,l} k^{2m} \int_0^{\infty} |g_{k,l}(\rho)|^2 \rho^{2s+n-1} d\rho \right)^{1/2} \|Y_{k,l}\|_{L^2(S^{n-1})}$$

$$\leq C \left( \sum_{k} k^{2(n-s-m)} \right)^{1/2} \left( \sum_{k,l} k^{2m} \int_0^{\infty} |g_{k,l}(\rho)|^2 \rho^{2s+n-1} d\rho \right)^{1/2} \left( \sum_{k,l} \int_{S^{n-1}} \mathcal{F}(f_{k,l}Y_{k,l})(\rho\omega)|^2 \rho^{2s+n-1} d\rho \right)^{1/2}$$

$$\leq C \left( \sum_{k,l} \int_{S^{n-1}} \mathcal{F}((1 - \Delta_\omega) \frac{m}{\pi}(f_{k,l}Y_{k,l}))(\rho\omega)|^2 \rho^{2s+n-1} d\rho \right)^{1/2}$$

$$\leq C \|u\|_{H^{s,m}_\omega}.$$
where $F$ is the Fourier transform. This proves part (1).

For the part (2), if we use the Littlewood-Paley decomposition $\{\varphi_j\}_{j \in \mathbb{Z}}$ as in the proof of Proposition 2, the we have

\[
|x|^{\frac{n-1}{2}} u(|x|\omega) = c_n \sum_{j \in \mathbb{Z}} \sum_{k,l} |x|^j \int_0^\infty J_{\nu(k)}(|x|\rho)^{\frac{n}{2}} \psi_j(\rho) \varphi_j(\rho) g_{k,l}(\rho) d\rho Y_{k,l}(\omega), \tag{15}
\]

Since $m > n - 3/2$, by (12) we deduce that

\[
|x|^{\frac{n-1}{2}} |u(|x|\omega)| \leq C \sum_{j \in \mathbb{Z}} \sum_{k,l} \left( \int_0^\infty |\varphi_j(\rho) g_{k,l}(\rho)|^2 \rho^m d\rho \right)^{\frac{1}{2}} \|Y_{k,l}\|_{L^\infty(S^{n-1})} \leq C \sum_{j \in \mathbb{Z}} \left( \sum_{k} k^{2(n-2-m)} \right)^{\frac{1}{2}} \left( \sum_{k,l} \int_0^\infty |\varphi_j(\rho) g_{k,l}(\rho)|^2 \rho^m d\rho \|Y_{k,l}\|_{L^2(S^{n-1})}^2 \right)^{\frac{1}{2}} \leq C \sum_{j \in \mathbb{Z}} 2^{\frac{n}{2}} \|\varphi_j F((1 - \Delta_\omega)^{\frac{m}{2}} u)\|_{L^2} = C \|u\|_{B^{\frac{1}{2}}_{2,1,\omega}}.
\]

To show Corollary 1 we use the fact that $H^{s,m}_\omega$ is a Hilbert space. Hence any bounded sequence $\{u_j\}$ in $H^{s,m}_\omega$ satisfies $u_j(x) \to 0$ as $|x| \to \infty$ uniformly and has a subsequence converges to $u$ in $H^{s,m}_\omega$ weakly. Let us denote the subsequence by $u_{jk}$.

Now choose a smooth function $\varphi$ supported in the ball of radius $R + 1$ and with value 1 in the ball of radius $R$. By the standard argument one can easily show that for each $R$ the mapping $u \mapsto \varphi u$ is compact from $H^t$ to $H^{t'}$ if $t' < t$. By the compactness above and Sobolev embedding we may assume that the sequence $\varphi u_{jk}$ satisfies that for $2 \leq q < \frac{2n}{n-2s}$

\[
\|\varphi u_{jk} - \varphi u\|_{L^q} \to 0 \quad \text{as} \quad k \to \infty. \tag{16}
\]

Thus we have

\[
\|u_{jk} - u\|_{L^p} \leq \|\varphi (u_{jk} - u)\|_{L^p} + \|(1 - \varphi)(u_{jk} - u)\|_{L^p} \equiv I_k + II_k \leq I_k \to 0 \quad \text{as} \quad k \to \infty \quad \text{by (16)}
\]

since $2 < p < \frac{2n}{n-2s}$. From the uniform convergence that $|u_{jk}(x)| + |u(x)| \to 0$ as $|x| \to \infty$ it follows that

\[
\limsup_{k \to \infty} II_k \leq \sup_k \|u_{jk} - u\|_{L^\infty(|x|>R)} \to 0 \quad \text{as} \quad R \to \infty.
\]

This proves the compactness of the embedding $H^{s,m}_\omega \hookrightarrow L^p$.

Since $B^{\frac{1}{2},m}_{2,1,\omega} \hookrightarrow H^{s,m}_\omega$, one can adapt the same arguments (compactness of cut-off mapping and uniform convergence at infinity) as above except for weak-* convergence of $u_{jk}$ to $u$ in $B^{\frac{1}{2},m}_{2,1,\omega}$ for the compactness of embedding $B^{\frac{1}{2},m}_{2,1,\omega}$ to $L^p$. This completes the proof.
REFERENCES


Department of Mathematics, POSTECH, Pohang 790-784, Republic of Korea
E-mail address: changocho@postech.ac.kr

Department of Mathematics, Hokkaido University, Sapporo, 060-0810, Japan
E-mail address: ozawa@math.sci.hokudai.ac.jp