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INEQUALITIES ASSOCIATED WITH DILATIONS

TOHRU OZAWA AND HIRONOBU Sasaki∗

Abstract. Some properties of distributions \( f \) satisfying \( x \cdot \nabla f \in L^p(\mathbb{R}^n) \), \( 1 \leq p < \infty \), are studied. The operator \( x \cdot \nabla \) is the generator of a semi-group of dilations. We first give Sobolev type inequalities with respect to the operator \( x \cdot \nabla \). Using the inequalities, we also show that if \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \), \( x \cdot \nabla f \in L^p(\mathbb{R}^n) \) and \( |x|^{n/p} |f(x)| \) vanishes at infinity, then \( f \) belongs to \( L^p(\mathbb{R}^n) \). One of the Sobolev type inequalities is shown to be equivalent to the Hardy inequality in \( L^2(\mathbb{R}^n) \).

1. Introduction

In this paper, we study some properties of distributions \( f \in \mathcal{D}'(\Omega) \) satisfying \( x \cdot \nabla f \in L^p(\Omega) \). Here, \( \Omega \subset \mathbb{R}^n \) is an open set, \( \mathcal{D}'(\Omega) \) is the set of all distributions on \( \Omega \), \( x \cdot \nabla = \sum_{j=1}^n x_j \partial_j \), \( x = (x_1, \cdots, x_n) \in \Omega \) and \( \partial_j f \) is a weak derivative of \( f \) with respect to \( x_j \). The operator \( x \cdot \nabla \) is well-known as the generator of a semi-group of dilations \( \{ T(t) \}_{t \geq 0} \) defined by

\[
(T(t)g)(x) = g(e^t x), \quad g : \mathbb{R}^n \to \mathbb{C}, \quad x \in \mathbb{R}^n.
\]

Let us recall the Sobolev inequality. For a Banach space \( A \), we denote the norm of \( A \) by \( \| \cdot \|_A \). It is well-known that if \( 1 < p, p^* < \infty \) and

\[
\frac{1}{p} = \frac{1}{p^*} - \frac{1}{n}, \quad \text{(1.1)}
\]

then we have the Sobolev inequality:

\[
\|g|L^p(\mathbb{R}^n)\| \leq C(p)\|\nabla g|L^{p^*}(\mathbb{R}^n)\|. \quad \text{(1.2)}
\]

Remark that the constant \( C(p) \) in (1.2) is independent of \( g \). For any \( \lambda > 0 \), we obtain

\[
\lambda^{-n/p}\|h|L^p(\mathbb{R}^n)\| \leq \lambda^{-n/p^*+1} C(p)\|\nabla h|L^{p^*}(\mathbb{R}^n)\|
\]

by substituting \( g(x) = h(\lambda x) \) into (1.2). Therefore, we observe that (1.1) is a necessary condition.

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Throughout this paper, we consider the following Sobolev type inequality with respect to the operator $x \cdot \nabla$ instead of $\nabla$:

$$\|g| L^p(\mathbb{R}^n)\| \leq C'(p)\|x \cdot \nabla g| L^p(\mathbb{R}^n)\|. \quad (1.3)$$

Substituting $g(x) = h(\lambda x)$ into (1.3), we observe that $p = q$ is a necessary condition to obtain (1.3). Later, we shall prove that (1.3) holds if $1 \leq p = q < \infty$, $f \in L^p$ and $x \cdot \nabla f \in L^p$.

To state our results, we list some notation which will be used later. For $1 \leq p \leq \infty$, we put $L^p = L^p(\mathbb{R}^n)$ and $\| \cdot \|^p = \| \cdot | L^p\|$. For $k = 0, 1, \cdots$ and for an open set $\Omega \subset \mathbb{R}^n$, let $W^{1,p}_c(\Omega)$ be the completion of $C^\infty_c(\Omega)$ with respect to $\|g| W^{1,p}(\Omega)\| = \|g| L^p(\Omega)\| + \|\nabla g| L^p(\Omega)\|$.

Let $\zeta : \mathbb{R} \to \mathbb{R}$ be an even, $C^\infty$-function satisfying
- $0 \leq \zeta \leq 1$,
- $\zeta(r) = 1$ if $|r| \leq 1$,
- $\zeta(r) = 0$ if $|r| \geq 2$.

We are ready to state our first result.

**Theorem 1.1.** Let $n \geq 1$.

(i) Assume that $\Omega \subset \mathbb{R}^n$ is an open set. If $1 \leq p < \infty$, $f \in W^{1,p}_c(\Omega)$ and $x \cdot \nabla f \in L^p(\Omega)$, then we have

$$\|f| L^p(\Omega)\| \leq \frac{p}{n}\|x \cdot \nabla f| L^p(\Omega)\|. \quad (1.4)$$

(ii) If $1 \leq p < \infty$, $f \in L^p$ and $x \cdot \nabla f \in L^p$, then (1.4) holds. That is, we have

$$\|f\|_p \leq \frac{p}{n}\|x \cdot \nabla f\|_p. \quad (1.5)$$

(iii) If $f \in C^1(\mathbb{R}^n)$ and if there exist positive numbers $\varepsilon$ and $R$ such that

$$\text{supp} f \subset \{ x \in \mathbb{R}^n; \varepsilon \leq |x| \leq R \}, \quad (1.6)$$

then we have

$$\|f\|_\infty \leq \ln \left( \frac{R}{\varepsilon} \right)\|x \cdot \nabla f\|_\infty. \quad (1.7)$$

The proof of Theorem 1.1 will be given in Section 2. We now list three remarks on Theorem 1.1.

**Remark 1.** A constant function $f_C(x) := C \neq 0$ is a typical example of a function which does not satisfy (1.4). Remark that

- $f_C \in W^{1,p}(\Omega) \setminus W^{1,p}_c(\Omega)$ if $1 \leq p < \infty$ and $\Omega$ has a finite measure,
- $f_C \in L^p(\Omega) \setminus L^p(\Omega)_{loc}$ if $1 \leq p < \infty$. 

...
Remark 2. The condition (1.6) is a necessary condition in some sense. For $0 < \delta < 1/2$ and $0 < \varepsilon < R < \infty$, we define functions $f_{1,\delta,R}$, $f_{2,\delta,\varepsilon}$ and $f_{3,\delta}$ by

\[
f_{1,\delta,R}(x) = \begin{cases} 
(\ln \delta)^{-1} \zeta \left( \frac{|x|}{R} \right) \ln \sqrt{|x|^2 + \delta^2} & \text{if } 0 \leq |x| \leq R, \\
0 & \text{otherwise,}
\end{cases}
\]

\[
f_{2,\delta,\varepsilon}(x) = \begin{cases} 
0 & \text{if } 0 \leq |x| \leq \varepsilon, \\
(\ln \delta)^{-1} \zeta \left( \frac{\varepsilon}{|x|} \right)^{-\delta} \ln \sqrt{\frac{\varepsilon^2}{|x|^2} + \delta^2} & \text{otherwise,}
\end{cases}
\]

\[
f_{3,\delta}(x) = \langle x \rangle^{-\delta},
\]

respectively. Here, $\langle x \rangle = (1 + |x|^2)^{1/2}$. Then we see that $f_{1,\delta,R}$, $f_{2,\delta,\varepsilon}$ and $f_{3,\delta}$ are $C^\infty$-functions vanishing at infinity and satisfy

\[
\text{supp} f_{1,\delta,R} = \{ x \in \mathbb{R}^n; 0 \leq |x| \leq R \},
\]

\[
\text{supp} f_{2,\delta,\varepsilon} = \{ x \in \mathbb{R}^n; |x| \geq \varepsilon \},
\]

\[
\text{supp} f_{3,\delta} = \mathbb{R}^n.
\]

Furthermore, we observe that

\[
\lim_{\delta \to 0^+} \frac{\| g_\delta \|_{\infty}}{\| x \cdot \nabla g_\delta \|_{\infty}} = \infty, \quad g_\delta = f_{1,\delta,R}, f_{2,\delta,\varepsilon}, f_{3,\delta}.
\]

Remark 3. The number $p/n$ appearing in (1.5) is the best constant. In fact, if we take

\[
f_\varepsilon(x) = \begin{cases} 
|x|^{-n/p+\varepsilon} & \text{for } |x| < 1, \\
|x|^{-n/p-\varepsilon} & \text{for } |x| \geq 1,
\end{cases}
\]

then we have

\[
x \cdot \nabla f_\varepsilon(x) = \begin{cases} 
(-n/p + \varepsilon)|x|^{-n/p+\varepsilon} & \text{if } |x| < 1, \\
(-n/p - \varepsilon)|x|^{-n/p-\varepsilon} & \text{if } |x| > 1
\end{cases}
\]
in the $L^p$-sense. Hence we see that

\[
\frac{\| f_\varepsilon \|_p}{\| x \cdot \nabla f_\varepsilon \|_p} \geq 2^p \left\{ \left( \frac{n}{p} - \varepsilon \right)^p + \left( \frac{n}{p} + \varepsilon \right)^p \right\}^{-\frac{1}{p}} \to \frac{p}{n}
\]
as $\varepsilon \to 0$. Moreover, if we take $g(x) = e^{-|x|^2}$, then we have

\[
\frac{\| g \|_1}{\| x \cdot \nabla g \|_1} = \frac{1}{n}.
\]

Remark 4. If $\Omega \subset \mathbb{R}^n$ is bounded, then (1.4) implies the usual Poincaré inequality. See [5, 8] for related results.
As the application of the inequality (1.5), we give the following three propositions: The first proposition is concerned with a function space \( L^p := \{ f \in L^p; x \cdot \nabla f \in L^p \} \) equipped with a semi-norm
\[
\| f \|_{L^p} := \| x \cdot \nabla f \|_p.
\]
We shall prove that \( L^p \) becomes a Banach space. Proposition 1.3, which is the second application, indicates that if a function \( f \in L^p_{\text{loc}} \) with \( x \cdot \nabla f \in L^p \) satisfies some decreasing condition, then \( f \) belongs to \( L^p \). The third application Proposition 1.4 means that if \( n \geq 3 \) and \( p = 2 \), then (1.5) is equivalent to the Hardy inequality.

**Proposition 1.2.** Let \( n \geq 1 \) and \( 1 \leq p < \infty \). Then the semi-normed space \( (L^p, \| \cdot \|_{L^p}) \) becomes a Banach space. Furthermore, the embedding operator \( \iota: L^p \rightarrow L^p \) has the norm
\[
\| \iota \|_{L^p \rightarrow L^p} = \frac{p}{n}. \tag{1.9}
\]

**Proof.** By (1.5), we find that \( f = 0 \) if and only if \( \| f \|_{L^p} = 0 \). Thus, \( \| \cdot \|_{L^p} \) is a norm of \( L^p \). Let \( \{ f_m \}_{m=1}^\infty \) be a Cauchy sequence in \( L^p \). From (1.5), there exist \( f \in L^p \) and \( g \in L^p \) such that
\[
\lim_{m \to \infty} f_m = f, \quad \lim_{m \to \infty} x \cdot \nabla f_m = g
\]
in \( L^p \). For any \( \varphi \in C_c^\infty(\mathbb{R}^n) \), it follows that
\[
\int_{\mathbb{R}^n} (x \cdot \nabla f) \varphi dx = -\int_{\mathbb{R}^n} f \text{ div}(x \varphi) dx
\]
\[
= -\lim_{m \to \infty} \int_{\mathbb{R}^n} f_m \text{ div}(x \varphi) dx
\]
\[
= \lim_{m \to \infty} \int_{\mathbb{R}^n} (x \cdot \nabla f_m) \varphi dx
\]
\[
= \int_{\mathbb{R}^n} g \varphi dx.
\]
Therefore, we have \( x \cdot \nabla f = g \in L^p \) and \( L^p \) is complete. By Remark 3, (1.9) holds. □

In Theorem 1.1, one of the main assumptions is that \( f \in L^p \). As we see in Remark 1 above, the last space is not generalized to \( L^p_{\text{loc}} \). The following proposition shows a sufficient condition for the condition that \( f \in L^p \).

**Proposition 1.3.** Let \( n \geq 1 \) and \( 1 \leq p < \infty \). Assume that \( f \in \mathcal{D}'(\mathbb{R}^n) \) is measurable on \( \mathbb{R}^n \) and that \( x \cdot \nabla f \in L^p \). If there exists \( \{ \phi_i \}_{i=1}^\infty \subset C_c^\infty(\mathbb{R}^n) \) such that
\[
(i) \sup_{i \geq 1} \| \phi_i \|_\infty < \infty,
(ii) \lim_{i \to \infty} \phi_i(x) = 1 \quad \text{for a.e. } x,
(iii) \phi_i f \in L^p,
\]
(iv) $$\lim inf_{l \to \infty} \left\| (x \cdot \nabla \phi_l) f \right\|_p = 0,$$
then $$f \in L^p.$$

Remark 5. In Section 3 below, we show some corollaries of the above proposition. In particular, we shall show that if $$f \in L^p_{\text{loc}}$$ satisfies $$x \cdot \nabla f \in L^p$$ and if $$|x|^{n/p} |f(x)|$$ vanishes at infinity, then $$f$$ belongs to $$L^p$$.

Proof of Proposition 1.3. By (iii) and (1.5), we have

$$\| \phi_l f \|_p \leq \frac{p}{n} \left\{ \| (x \cdot \nabla \phi_l) f \|_p + \| \phi_l (x \cdot \nabla f) \|_p \right\}$$

for any $$l \geq 1$$. Since $$x \cdot \nabla f \in L^p$$, we see from (i), (ii) and the Lebesgue dominated theorem that

$$\lim_{l \to \infty} \| \phi_l (x \cdot \nabla f) \|_p = \| x \cdot \nabla f \|_p.$$

By (iv), we hence obtain

$$\lim inf_{l \to \infty} \| \phi_l f \|_p \leq \frac{p}{n} \| x \cdot \nabla f \|_p.$$

It follows from (ii) and the Fatou lemma that

$$\| f \|_p = \int_{\mathbb{R}^n} \lim inf_{l \to \infty} \left| (\phi_l f) (x) \right|^p dx$$

$$\leq \lim inf_{l \to \infty} \int_{\mathbb{R}^n} \left| (\phi_l f) (x) \right|^p dx$$

$$= \lim inf_{l \to \infty} \| \phi_l f \|_p^p.$$

Thus, we see that $$f \in L^p$$. \hfill \Box

The inequality (1.5) is equivalent to Hardy’s inequality if $$p = 2$$. To be more specific, we have:

**Proposition 1.4.** Let $$n \geq 3$$. Then the following two statements are equivalent:

(a) For any $$f \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$$, we have

$$\| f \|_2 \leq \frac{2}{n} \| x \cdot \nabla f \|_2.$$  \hspace{1cm} (1.10)

(b) For any $$g \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$$, we have

$$\left\| \frac{g}{|x|} \right\|_2 \leq \frac{2}{n - 2} \left\| \frac{x}{|x|} \cdot \nabla g \right\|_2.$$  \hspace{1cm} (1.11)
Remark 6. The original Hardy inequality was given by [6]. Later, a lot of generalized Hardy inequalities are studied (see, e.g., [1, 2, 3, 4, 7, 9]). In particular, it was shown that
\[
\left\| \frac{g}{|x|} \right\|_p \leq \frac{p}{n-p} \| \nabla g \|_p, \quad g \in W^{1,p}(\mathbb{R}^n), \quad p < n.
\] (1.12)
The constant \( p/(n-p) \) is optimal. Since \( C_\infty^\infty(\mathbb{R}^n \setminus \{0\}) \) is dense in \( W^{1,2}(\mathbb{R}^n) \) for \( n \geq 3 \), by using Theorem 1.1,(ii), the above proposition and (1.12), we can see that (1.11) holds for any \( W^{1,2}(\mathbb{R}^n) \). If we put \( g_\varepsilon = |x| f_\varepsilon \), where \( f_\varepsilon \) have been given in Remark 3, then we have
\[
\left\| g_\varepsilon/|x| \right\|_2 \geq 2^{\frac{1}{2}} \left\{ \left( \frac{n}{2} - 1 - \varepsilon \right)^2 + \left( \frac{n}{2} - 1 + \varepsilon \right)^2 \right\}^{-\frac{1}{2}} \to \frac{2}{n-2}
\]
as \( \varepsilon \to 0 \). Therefore, the constant \( 2/(n-2) \) is optimal.

Proof of Proposition 1.4. We first prove that the statement (a) implies (b). Put \( g = |x| f \). Then we obtain
\[
\left\| x \cdot \nabla f \right\|_2 = \left\| -\frac{g}{|x|} + \frac{x}{|x|} \cdot \nabla g \right\|_2 = \left\| \frac{g}{|x|} \right\|_2^2 - 2 \Re \left\langle \frac{g}{|x|}, \frac{x}{|x|} \cdot \nabla g \right\rangle + \left\| \frac{x}{|x|} \cdot \nabla g \right\|_2^2,
\] (1.13)
where \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( L^2 \). Since
\[
-2 \Re \left\langle \frac{g}{|x|}, \frac{x}{|x|} \cdot \nabla g \right\rangle = -\left\langle \frac{x}{|x|^2}, \nabla |g|^2 \right\rangle = \left\langle \text{div} \left( \frac{x}{|x|^2} \right), |g|^2 \right\rangle = \left\langle \frac{n-2}{|x|^2}, |g|^2 \right\rangle,
\]
It follows from the statement (a) and (1.13) that
\[
\left\| \frac{g}{|x|} \right\|_2^2 \leq \frac{4}{n^2} \left( n-1 \right) \left\| \frac{g}{|x|} \right\|_2^2 + \left\| \frac{x}{|x|} \cdot \nabla g \right\|_2^2,
\]
which implies (1.11).

Conversely, we assume that (b) holds. We put \( f = g/|x| \). Then we have
\[
\left\| \frac{x}{|x|} \cdot \nabla (|x|f) \right\|_2^2 = \| f + x \cdot \nabla f \|_2^2 = \| f \|_2^2 + 2 \Re \langle xf, \nabla f \rangle + \| x \cdot \nabla f \|_2^2
\]
Since \( 2 \Re \langle xf, \nabla f \rangle = -n \| f \|_2^2 \), we see from the statement (b) that
\[
\| f \|_2^2 \leq \frac{4}{(n-2)^2} \left( \| x \cdot \nabla f \|_2^2 - (n-1) \| f \|_2^2 \right),
\]
which implies (1.10). \( \square \)
2. Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. For this purpose, we prepare some notation. For \( \lambda > 0 \) and for \( g: \mathbb{R}^n \to \mathbb{C} \), \((\delta_\lambda g)(x) = g(\lambda x)\). For \( \lambda > 0 \) and for \( \Omega \subset \mathbb{R}^n \), \( \lambda \Omega = \{ \lambda \omega; \omega \in \Omega \} \). Let \( \varphi \in C^\infty_c(\mathbb{R}^n) \) be a positive function such that \( \int \varphi = 1 \) and \( \text{supp} \varphi \subset \{ x \in \mathbb{R}^n; |x| \leq 1 \} \). For \( \lambda > 0 \), we set \( \zeta_\lambda(x) = \varphi(\lambda |x|) \) and \( \varphi_\lambda = \lambda^{-n} \delta_{\lambda^{-1}} \varphi \).

**Proposition 2.1.** Let \( n \geq 1 \) and \( 1 \leq p < \infty \). Assume that \( \Omega \subset \mathbb{R}^n \) is an open set. If \( f \in C^\infty_c(\Omega) \), then we have (1.4).

**Proof.** For \( \phi: \Omega \to \mathbb{C} \), we put
\[
\tilde{\phi}(x) = \begin{cases} 
\phi(x) & \text{if } x \in \Omega, \\
0 & \text{otherwise}.
\end{cases}
\]
Then we have
\[
\|\delta_\lambda \tilde{\phi}\|_{L^p(\Omega)}^p = \int_\Omega |\tilde{\phi}(\lambda x)|^p dx = \lambda^{-n} \int_{\lambda \Omega} |\tilde{\phi}(x)|^p dx 
\leq \lambda^{-n} \int_{\Omega} |\tilde{\phi}(x)|^p dx = \lambda^{-n} \|\phi\|_{L^p(\Omega)}^p.
\] (2.1)

For any \( \lambda > 0 \) and \( x \in \Omega \),
\[
(\delta_\lambda \tilde{f} - \tilde{f})(x) = \tilde{f}(\lambda x) - \tilde{f}(x) = \int_1^\lambda \frac{\partial}{\partial \alpha} \tilde{f}(\alpha x) d\alpha = \int_1^\lambda \alpha^{-1} (\delta_\alpha x \cdot \nabla f)(x) d\alpha.
\] (2.2)

For any \( \lambda > 1 \), we see from (2.1) and (2.2) that
\[
(1 - \lambda^{-n/p}) \|f\|_{L^p(\Omega)} \leq \|\delta_\lambda \tilde{f} - \tilde{f}\|_{L^p(\Omega)} \leq \int_1^\lambda \alpha^{-1} \|\delta_\alpha x \cdot \nabla f\|_{L^p(\Omega)} d\alpha \leq \int_1^\lambda \alpha^{-1-n/p} \|x \cdot \nabla f\|_{L^p(\Omega)} d\alpha = \frac{p}{n} (1 - \lambda^{-n/p}) \|x \cdot \nabla f\|_{L^p(\Omega)},
\]
which implies (1.5) by letting \( \lambda \to \infty \). \( \square \)
Proof of Theorem 1.1. We first show (i). There exists some \( \{ f_l \}_{l=1}^{\infty} \subset C^\infty_c(\Omega) \) such that 
\( f_l \to f \) in \( W^{1,p}(\Omega) \) as \( l \to \infty \). Let \( \lambda > 0 \). We observe from Proposition 2.1 that 
\[
\| \zeta_\lambda f_l \|_{L^p(\Omega)} \leq \frac{p}{n} \left\{ \| (x \cdot \nabla \zeta_\lambda) f_l \|_{L^p(\Omega)} + \| \zeta_\lambda (x \cdot \nabla f_l) \|_{L^p(\Omega)} \right\}
\]
for all \( l \geq 1 \). We easily see that 
\[
\lim_{l \to \infty} \zeta_\lambda f_l = \zeta_\lambda f, \quad \lim_{l \to \infty} (x \cdot \nabla \zeta_\lambda) f_l = (x \cdot \nabla \zeta_\lambda) f, \quad \lim_{l \to \infty} \zeta_\lambda (x \cdot \nabla f_l) = \lim_{l \to \infty} (\zeta_\lambda x) \cdot (\nabla f_l) = \zeta_\lambda (x \cdot \nabla f)
\]
in \( L^p(\Omega) \). Hence we obtain 
\[
\| \zeta_\lambda f \|_{L^p(\Omega)} \leq \frac{p}{n} \left\{ \| (x \cdot \nabla \zeta_\lambda) f \|_{L^p(\Omega)} + \| \zeta_\lambda (x \cdot \nabla f) \|_{L^p(\Omega)} \right\}.
\]
Since 
\[
| (x \cdot \nabla \zeta_\lambda)(x) | \leq \begin{cases} \| x \cdot \nabla \zeta \|_{\infty} & \text{if } \lambda^{-1} < |x| < 2\lambda^{-1}, \\ 0 & \text{otherwise}, \end{cases}
\]
(1.5) holds as \( \lambda \to 0 \).

We next prove (ii). We have only to prove that 
\[
\lim_{\lambda \to 0} \zeta_\lambda (f * \varphi_\lambda) = f, \quad \lim_{\lambda \to 0} x \cdot \nabla (\zeta_\lambda (f * \varphi_\lambda)) = x \cdot \nabla f
\]
in \( L^p \) because \( f_\lambda \equiv \zeta_\lambda (f * \varphi_\lambda) \in C^\infty_c(\mathbb{R}^n) \). We obviously have (2.4).

It follows that 
\[
x \cdot \nabla (\zeta_\lambda (f * \varphi_\lambda)) = (x \cdot \nabla \zeta_\lambda)(f * \varphi_\lambda) + \zeta_\lambda (x \cdot \nabla)(f * \varphi_\lambda) = (x \cdot \nabla \zeta_\lambda)(f * \varphi_\lambda) + \zeta_\lambda ((x \cdot \nabla f) * \varphi_\lambda) + \zeta_\lambda (f * \text{div}(x \varphi_\lambda)) \equiv I_\lambda + II_\lambda + III_\lambda.
\]

By (2.3), we have \( \lim_{\lambda \to 0} I_\lambda = 0 \). Repeating the same argument of the proof of (2.4), we obtain \( \lim_{\lambda \to 0} II_\lambda = 0 \). Since \( \int \text{div}(x \varphi_\lambda) = 0 \), we see that 
\[
(f * \text{div}(x \varphi_\lambda))(x) = \int_{\mathbb{R}^n} (f(x - \lambda y) - f(x) \text{div}(y \varphi(y))) dy.
\]

Hence we obtain 
\[
\| f * \text{div}(x \varphi_\lambda) \|_p \leq \sup_{|z| \leq \lambda} \| f(\cdot - z) - f \|_p \| \text{div}(x \varphi) \|_1 \to 0
\]
as \( \lambda \to 0 \). Thus, we have \( \lim_{\lambda \to 0} III_\lambda = 0 \).
Finally, we show (iii). Since $f \in C^1(\mathbb{R}^n)$, we have
\[
f(e^\theta x) - f(x) = \int_0^\theta \frac{\partial}{\partial \alpha} f(e^\alpha x) d\alpha = \int_0^\theta e^\alpha x \cdot (\nabla f)(e^\alpha x) d\alpha \tag{2.6}
\]
for any $x \in \mathbb{R}^n$ and $\theta > 0$. Let $\theta_0 = \ln(R/\varepsilon)$ and let $|x| \geq \varepsilon$. Since $|e^{\theta_0}x| \geq R$, we see that $f(e^{\theta_0}x) = 0$. By (2.6), we have
\[
f(x) = -\int_0^{\theta_0} e^\alpha x \cdot (\nabla f)(e^\alpha x) d\alpha
\]
for any $|x| \geq \varepsilon$. Thus (1.7) holds. \qed

3. Application

In this section, we show some corollaries of Proposition 1.3.

**Corollary 3.1.** Let $n \geq 1$ and $1 \leq p < \infty$. Assume that $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ satisfies $x \cdot \nabla f \in L^p$. If there exist sequences $\{\rho_l\}_{l=1}^\infty \subset (0, \infty)$ and $\{R_l\}_{l=1}^\infty \subset (0, \infty)$ such that
\[
(i) \quad R_l \leq R_k \text{ if } 1 \leq l \leq k,
(ii) \quad \lim_{l \to \infty} R_l = \infty,
(iii) \quad \liminf_{l \to \infty} \left( \frac{R_l + \rho_l}{\rho_l} \right)^p \int_{R_l < |x| < R_l + \rho_l} |f(x)|^p dx = 0, \tag{3.1}
\]
then $f \in L^p$.

**Remark 7.** Taking $\rho_l = R_l$, we see from the above Corollary that if we have $x \cdot \nabla f \in L^p$ and
\[
\liminf_{R \to \infty} \int_{R < |x| < 2R} |f(x)|^p dx = 0, \tag{3.2}
\]
then $f \in L^p$. If the left hand side of (3.2) is positive, then $f \notin L^p$ (even if $x \cdot \nabla f \in L^p$).

**Proof of Corollary 3.1.** Set
\[
\phi_l(x) = \left\{ \begin{array}{ll}
1 & \text{for } |x| \leq R_l, \\
\zeta \left( \frac{|x| - R_l + \rho_l}{\rho_l} \right) & \text{for } |x| > R_l,
\end{array} \right.
\]
We immediately see that
\[
\sup_{l \geq 1} \|\phi_l\|_{\infty} = 1 \quad \text{and} \quad \lim_{l \to \infty} \phi_l(x) = 1 \text{ for a.e. } x.
\]
Since
\[ x \cdot \nabla \phi_l(x) = \begin{cases} \frac{|x|}{\rho_l} \frac{\zeta_l}{|x - R_l + \rho_l|} & \text{if } R_l < |x| < R_l + \rho_l, \\ 0 & \text{otherwise,} \end{cases} \]
it follows that
\[ \liminf_{l \to \infty} \left\| (x \cdot \nabla \phi_l) f \right\|^p_p \leq \liminf_{l \to \infty} \left( \frac{R_l + \rho_l}{\rho_l} \right)^p \int_{R_l < |x| < R_l + \rho_l} |f(x)|^p \, dx = 0. \]
Thus, we see from Proposition 1.3 that \( f \in L^p \).

**Corollary 3.2.** Let \( n \geq 1, 1 \leq p < \infty \). Assume that \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \) satisfies \( x \cdot \nabla f \in L^p \). If we have either of the following three conditions, then \( f \in L^p \):

(i) It follows that
\[ \lim \sup_{|x| < 2R} |x|^{n/p} |f(x)| = 0. \]

(ii) \( p = 1 \) and there exists some spherically symmetric, unbounded open set \( A \) such that
\[ \lim_{|x| \to \infty, x \in A} |x|^n |f(x)| = 0. \]

(iii) \( n = 1, p = 1 \) and there exists some unbounded open set \( A \) such that \( -\inf A = \sup A = \infty \) and (3.4) holds.

**Remark 8.** If
\[ \liminf_{|x| \to \infty} |x|^{n/p} |f(x)| > 0, \]
then \( f \notin L^p \) even if \( x \cdot \nabla f \in L^p \).

**Proof of Corollary 3.2.** We first prove (i). It follows that
\[ \int_{R < |x| < 2R} |f(x)|^p \, dx = \int_{R < |x| < 2R} \left( |x|^{n/p} |f(x)| \right)^p |x|^{-n} \, dx \]
\[ \leq \left( \sup_{R < |x| < 2R} |x|^{n/p} |f(x)| \right)^p \int_{R < |x| < 2R} |x|^{-n} \, dx \]
\[ \leq \ln(2) \left( \sup_{R < |x| < 2R} |x|^{n/p} |f(x)| \right)^p \left( \sum_{\mathbb{S}^{n-1}} \right) \]
\[ \to 0 \quad \text{as } R \to \infty. \]
From Remark 7, we have \( f \in L^p \).

We next show (ii). There exist \( \{\rho_l\}_{l=1}^\infty \subset (0,1) \) and \( \{R_l\}_{l=1}^\infty \subset (0,\infty) \) such that \( \lim_{l \to \infty} \rho_l = 0, R_l \leq R_k \) if \( 1 \leq l \leq k \), \( \lim_{l \to \infty} R_l = \infty \) and
\[ A_l \equiv \{x \in \mathbb{R}^n; R_l < |x| < R_l + \rho_l\} \subset A. \]
Then we obtain
\[
\frac{R_l + \rho_l}{\rho_l} \int_{R_l < |x| < R_l + \rho_l} |f(x)| \, dx \\
\leq \frac{R_l + \rho_l}{\rho_l} \left( \sup_{R_l < |x| < R_l + \rho_l} |x|^n |f(x)| \right) \left( \int_{\mathbb{S}^{n-1}} d\omega \right) \int_{R_l}^{R_l + \rho_l} r^{-1} \, dr \\
\leq \left( \sup_{|x| > R_l, x \in A} |x|^n |f(x)| \right) \left( \int_{\mathbb{S}^{n-1}} d\omega \right) \frac{R_l + \rho_l}{\rho_l} \cdot \frac{1}{R_l} \\
\to 0 \quad \text{as } l \to \infty.
\]
Applying Corollary 3.1, we see that \( f \in L^1 \).

Finally, we prove (iii). There exist \( \{x_l\}_{l=1}^\infty \subset \mathbb{R} \) and \( \{r_l\}_{l=1}^\infty \subset (0,1) \) such that
\[
1 < x_{2k-1} < x_{2k+1} \quad \text{for any } k \geq 1, \\
-1 > x_{2k} > x_{2k+2} \quad \text{for any } k \geq 1, \\
|r_l| \to 0 \quad \text{as } l \to \infty, \\
\{x \in \mathbb{R}; |x-x_l| < r_l\} \subset A \quad \text{for any } l \geq 1.
\]

Set
\[
\phi_1(x) = \begin{cases} 
\zeta(x) & \text{if } x < 0, \\
1 & \text{if } 0 \leq x \leq x_1, \\
\zeta \left( \frac{x-x_1 + r_1}{r_1} \right) & \text{if } x > x_1,
\end{cases}
\]
\[
\phi_{2k}(x) = \begin{cases} 
\zeta \left( \frac{x-x_{2k} - r_{2k}}{r_{2k}} \right) & \text{if } x < x_{2k}, \\
1 & \text{if } x_{2k} \leq x \leq 0, \\
\phi_{2k-1}(x) & \text{if } x > 0,
\end{cases}
\]
\[
\phi_{2k+1}(x) = \begin{cases} 
\phi_{2k}(x) & \text{if } x < 0, \\
1 & \text{if } 0 \leq x \leq x_{2k+1}, \\
\zeta \left( \frac{x-x_{2k+1} + r_{2k+1}}{r_{2k+1}} \right) & \text{if } x > x_{2k+1},
\end{cases}
\]
where \( k = 1, 2, \ldots \). Then we have
\[
\sup_{l \geq 1} \| \phi_l \|_\infty = 1, \\
\phi_l f \in L^p \quad \text{for any } l \geq 1, \\
\lim_{l \to \infty} \phi_l(x) = 1 \quad \text{for all } x \in \mathbb{R}.
\]
Furthermore, we see that
\[
x \cdot \nabla \phi_{2k+1} = \begin{cases} 
0 & \text{if } x < x_{2k} - r_{2k}, \\
\frac{x}{r_{2k}^{2k + 1}} \cdot \left( \frac{x + x_{2k - r_{2k}}}{r_{2k}} \right) & \text{if } x_{2k} - r_{2k} \leq x \leq x_{2k}, \\
\frac{x}{r_{2k+1}^{2k + 1}} \cdot \left( \frac{x - x_{2k+1} + r_{2k+1}}{r_{2k+1}} \right) & \text{if } x_{2k} < x < x_{2k+1}, \\
0 & \text{if } x_{2k+1} \leq x \leq x_{2k+1} + r_{2k+1}, \\
\frac{x}{r_{2k+1}^{2k + 1}} \cdot \left( \frac{x - x_{2k+1} + r_{2k+1}}{r_{2k+1}} \right) & \text{if } x > x_{2k+1} + r_{2k+1},
\end{cases}
\]
for any \( k \geq 1 \). Thus, we obtain
\[
\| (x \cdot \nabla \phi_{2k+1}) f \|_1 \leq -\frac{x_{2k} + r_{2k}}{r_{2k}^{2k + 1}} \int_{x_{2k} - r_{2k}}^{x_{2k}} |f(x)| dx + \frac{x_{2k+1} + r_{2k+1}}{r_{2k+1}^{2k + 1}} \int_{x_{2k+1}}^{x_{2k+1} + r_{2k+1}} |f(x)| dx
\]
\[
\leq -\frac{x_{2k} + r_{2k}}{r_{2k}^{2k + 1}} \cdot r_{2k} \cdot \frac{1}{x_{2k} - r_{2k}} \left( \sup_{x_{2k} - r_{2k} \leq x \leq x_{2k}} |x||f(x)| dx \right)
\]
\[
+ \frac{x_{2k+1} + r_{2k+1}}{r_{2k+1}^{2k + 1}} \cdot r_{2k+1} \cdot \frac{1}{x_{2k+1} + r_{2k+1}} \left( \sup_{x_{2k+1} \leq x \leq x_{2k+1} + r_{2k+1}} |x||f(x)| dx \right).
\]
Hence we observe that
\[
\liminf_{l \to \infty} \| (x \cdot \nabla \phi_l) f \|_1 = 0.
\]

From Proposition 1.3, we have \( f \in L^1 \).

\[\square\]

REFERENCES