INEQUALITIES ASSOCIATED WITH DILATIONS

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Abstract. Some properties of distributions f satisfying $x \cdot \nabla f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, are studied. The operator $x \cdot \nabla$ is the generator of a semi-group of dilations. We first give Sobolev type inequalities with respect to the operator $x \cdot \nabla$. Using the inequalities, we also show that if $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $x \cdot \nabla f \in L^p(\mathbb{R}^n)$ and $|x|^{n/p}|f(x)|$ vanishes at infinity, then f belongs to $L^p(\mathbb{R}^n)$. One of the Sobolev type inequalities is shown to be equivalent to the Hardy inequality in $L^2(\mathbb{R}^n)$.

1. Introduction

In this paper, we study some properties of distributions $f \in \mathcal{D}'(\Omega)$ satisfying $x \cdot \nabla f \in L^p(\Omega)$. Here, $\Omega \subset \mathbb{R}^n$ is an open set, $\mathcal{D}'(\Omega)$ is the set of all distributions on Ω, $x \cdot \nabla = \sum_{j=1}^n x_j \partial_j$, $x = (x_1, \cdots, x_n) \in \Omega$ and $\partial_j f$ is a weak derivative of f with respect to x_j. The operator $x \cdot \nabla$ is well-known as the generator of a semi-group of dilations $\{T(t)\}_{t \geq 0}$ defined by

$$\big(T(t)g\big)(x) = g(e^t x), \quad g : \mathbb{R}^n \to \mathbb{C}, \quad x \in \mathbb{R}^n.$$

Let us recall the Sobolev inequality. For a Banach space A, we denote the norm of A by $\| \cdot | A\|$. It is well-known that if $1 < p, p^* < \infty$ and

$$\frac{1}{p} = \frac{1}{p^*} - \frac{1}{n}, \tag{1.1}$$

then we have the Sobolev inequality:

$$\|g | L^p(\mathbb{R}^n)\| \leq C(p) \|\nabla g | L^{p^*}(\mathbb{R}^n)\|. \tag{1.2}$$

Remark that the constant $C(p)$ in (1.2) is independent of g. For any $\lambda > 0$, we obtain

$$\lambda^{-n/p} \|h | L^p(\mathbb{R}^n)\| \leq \lambda^{-n/p^*+1} C(p) \|\nabla h | L^{p^*}(\mathbb{R}^n)\|$$

by substituting $g(x) = h(\lambda x)$ into (1.2). Therefore, we observe that (1.1) is a necessary condition.

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Throughout this paper, we consider the following Sobolev type inequality with respect to the operator $x \cdot \nabla$ instead of ∇:

$$\|g|L^p(\mathbb{R}^n)\| \leq C'(p)\|x \cdot \nabla g|L^q(\mathbb{R}^n)\|. \tag{1.3}$$

Substituting $g(x) = h(\lambda x)$ into (1.3), we observe that $p = q$ is a necessary condition to obtain (1.3). Later, we shall prove that (1.3) holds if $1 \leq p = q < \infty$, $f \in L^p$ and $x \cdot \nabla f \in L^p$.

To state our results, we list some notation which will be used later. For $1 \leq p \leq \infty$, we put $L^p = L^p(\mathbb{R}^n)$ and $\| \cdot \|_p = \| \cdot |L^p\|$. For $k = 0, 1, \cdots$ and for an open set $\Omega \subset \mathbb{R}^n$, We denote by $C_c^\infty(\Omega)$ the set of all C^∞-functions with compact support in Ω. For $1 \leq p \leq \infty$ and for an open set $\Omega \subset \mathbb{R}^n$, let $W_0^{1,p}(\Omega)$ be the completion of $C_c^\infty(\Omega)$ with respect to

$$\|g|W^{1,p}(\Omega)\| = \|g|L^p(\Omega)\| + \|\nabla g|L^p(\Omega)\|.$$

Let $\zeta : \mathbb{R} \to \mathbb{R}$ be an even, C^∞-function satisfying

- $0 \leq \zeta \leq 1$,
- $\zeta(r) = 1$ if $|r| \leq 1$,
- $\zeta(r) = 0$ if $|r| \geq 2$.

We are ready to state our first result.

Theorem 1.1. *Let $n \geq 1$.*

(i) Assume that $\Omega \subset \mathbb{R}^n$ is an open set. If $1 \leq p < \infty$, $f \in W_0^{1,p}(\Omega)$ and $x \cdot \nabla f \in L^p(\Omega)$, then we have

$$\|f|L^p(\Omega)\| \leq \frac{p}{n}\|x \cdot \nabla f|L^p(\Omega)\|. \tag{1.4}$$

(ii) If $1 \leq p < \infty$, $f \in L^p$ and $x \cdot \nabla f \in L^p$, then (1.4) holds. That is, we have

$$\|f\|_p \leq \frac{p}{n}\|x \cdot \nabla f\|_p. \tag{1.5}$$

(iii) If $f \in C^1(\mathbb{R}^n)$ and if there exist positive numbers ε and R such that

$$\mathrm{supp} f \subset \{x \in \mathbb{R}^n;\ \varepsilon \leq |x| \leq R\}, \tag{1.6}$$

then we have

$$\|f\|_\infty \leq \ln\left(\frac{R}{\varepsilon}\right)\|x \cdot \nabla f\|_\infty. \tag{1.7}$$

The proof of Theorem 1.1 will be given in Section 2. We now list three remarks on Theorem 1.1.

Remark 1. A constant function $f_C(x) := C \neq 0$ is a typical example of a function which dose not satisfy (1.4). Remark that

$$f_C \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega) \quad \text{if } 1 \leq p < \infty \text{ and } \Omega \text{ has a finite measure,}$$
$$f_C \in L^p_{loc} \setminus L^p \quad \text{if } 1 \leq p < \infty.$$

Remark 2. The condition (1.6) is a necessary condition in some sense. For $0 < \delta < 1/2$ and $0 < \varepsilon < R < \infty$, we define functions $f^1_{\delta,R}$, $f^2_{\delta,\varepsilon}$ and f^3_δ by

$$f^1_{\delta,R}(x) = \begin{cases} (\ln\delta)^{-1}\zeta\left(\dfrac{|x|}{R}\right)\ln\sqrt{\dfrac{|x|^2}{R^2}+\delta^2} & \text{if } 0 \le |x| \le R, \\ 0 & \text{otherwise}, \end{cases}$$

$$f^2_{\delta,\varepsilon}(x) = \begin{cases} 0 & \text{if } 0 \le |x| \le \varepsilon, \\ (\ln\delta)^{-1}\zeta\left(\dfrac{\varepsilon}{|x|}\right)\left\langle\dfrac{x}{\varepsilon}\right\rangle^{-\delta}\ln\sqrt{\dfrac{\varepsilon^2}{|x|^2}+\delta^2} & \text{otherwise}, \end{cases}$$

$$f^3_\delta(x) = \langle x\rangle^{-\delta},$$

respectively. Here, $\langle x\rangle = (1+|x|^2)^{1/2}$. Then we see that $f^1_{\delta,R}$, $f^2_{\delta,\varepsilon}$ and f^3_δ are C^∞-functions vanishing at infinity and satisfy

$$\begin{aligned} \mathrm{supp} f^1_{\delta,R} &= \{x \in \mathbb{R}^n;\ 0 \le |x| \le R\}, \\ \mathrm{supp} f^2_{\delta,\varepsilon} &= \{x \in \mathbb{R}^n;\ |x| \ge \varepsilon\}, \\ \mathrm{supp} f^3_\delta &= \mathbb{R}^n. \end{aligned}$$

Furthermore, we observe that

$$\lim_{\delta\to 0+}\frac{\|g_\delta\|_\infty}{\|x\cdot\nabla g_\delta\|_\infty} = \infty, \quad g_\delta = f^1_{\delta,R}, f^2_{\delta,\varepsilon}, f^3_\delta.$$

Remark 3. The number p/n appearing in (1.5) is the best constant. In fact, if we take

$$f_\varepsilon(x) = \begin{cases} |x|^{-n/p+\varepsilon} & \text{for } |x| < 1, \\ |x|^{-n/p-\varepsilon} & \text{for } |x| \ge 1, \end{cases} \tag{1.8}$$

then we have

$$x\cdot\nabla f_\varepsilon(x) = \begin{cases} \left(-\frac{n}{p}+\varepsilon\right)|x|^{-n/p+\varepsilon} & \text{if } |x| < 1, \\ \left(-\frac{n}{p}-\varepsilon\right)|x|^{-n/p-\varepsilon} & \text{if } |x| > 1 \end{cases}$$

in the L^p-sense. Hence we see that

$$\frac{\|f_\varepsilon\|_p}{\left\|x\cdot\nabla f_\varepsilon\right\|_p} \ge 2^{\frac{1}{p}}\left\{\left(\frac{n}{p}-\varepsilon\right)^p + \left(\frac{n}{p}+\varepsilon\right)^p\right\}^{-\frac{1}{p}} \to \frac{p}{n}$$

as $\varepsilon \to 0$. Moreover, if we take $g(x) = e^{-|x|^2}$, then we have

$$\frac{\|g\|_1}{\|x\cdot\nabla g\|_1} = \frac{1}{n}.$$

Remark 4. If $\Omega \subset \mathbb{R}^n$ is bounded, then (1.4) implies the usual Poincaré inequality. See [5, 8] for related results.

As the application of the inequality (1.5), we give the following three propositions: The first proposition is concerned with a function space $\mathfrak{L}^p := \{f \in L^p; x \cdot \nabla f \in L^p\}$ equipped with a semi-norm

$$\|f|\mathfrak{L}^p\| := \|x \cdot \nabla f\|_p.$$

We shal prove that $\mathfrak{L}^p$ becomes a Banach space. Proposition 1.3, which is the second application, indicates that if a function $f \in L^p_{\mathrm{loc}}$ with $x \cdot \nabla f \in L^p$ satisfies some decreasing condition, then f belongs to L^p. The third application Proposition 1.4 means that if $n \geq 3$ and $p = 2$, then (1.5) is equivalent to the Hardy inequality.

Proposition 1.2. *Let $n \geq 1$ and $1 \leq p < \infty$. Then the semi-normed space $(\mathfrak{L}^p, \| \cdot |\mathfrak{L}^p\|)$ becomes a Banach space. Furthermore, the embedding operator $\iota : \mathfrak{L}^p \hookrightarrow L^p$ has the norm*

$$\|\iota|\mathfrak{L}^p \to L^p\| = \frac{p}{n}. \tag{1.9}$$

Proof. By (1.5), we find that $f = 0$ if and only if $\|f|\mathfrak{L}^p\| = 0$. Thus, $\| \cdot |\mathfrak{L}^p\|$ is a norm of $\mathfrak{L}^p$. Let $\{f_m\}_{m=1}^\infty$ be a Cauchy sequence in $\mathfrak{L}^p$. From (1.5), there exist $f \in L^p$ and $g \in L^p$ such that

$$\lim_{m\to\infty} f_m = f, \quad \lim_{m\to\infty} x \cdot \nabla f_m = g$$

in L^p. For any $\varphi \in C_c^\infty(\mathbb{R}^n)$, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} \big(x \cdot \nabla f\big)\varphi dx &= -\int_{\mathbb{R}^n} f \operatorname{div}(x\varphi) dx \\
&= -\lim_{m\to\infty} \int_{\mathbb{R}^n} f_m \operatorname{div}(x\varphi) dx \\
&= \lim_{m\to\infty} \int_{\mathbb{R}^n} \big(x \cdot \nabla f_m\big)\varphi dx \\
&= \int_{\mathbb{R}^n} g\varphi dx.
\end{aligned}$$

Therefore, we have $x \cdot \nabla f = g \in L^p$ and $\mathfrak{L}^p$ is complete. By Remark 3, (1.9) holds. $\square$

In Theorem 1.1, one of the main assumptions is that $f \in L^p$. As we see in Remark 1 above, the last space is not generalized to L^p_{loc}. The following proposition shows a sufficient condition for the condition that $f \in L^p$.

Proposition 1.3. *Let $n \geq 1$ and $1 \leq p < \infty$. Assume that $f \in \mathcal{D}'(\mathbb{R}^n)$ is measurable on $\mathbb{R}^n$ and that $x \cdot \nabla f \in L^p$. If there exists $\{\phi_l\}_{l=1}^\infty \subset C_c^\infty(\mathbb{R}^n)$ such that*

(i) $\sup_{l\geq 1} \|\phi_l\|_\infty < \infty$,
(ii) $\lim_{l\to\infty} \phi_l(x) = 1$ *for a.e. x,*
(iii) $\phi_l f \in L^p$,

(iv) $\liminf_{l\to\infty}\left\|\left(x\cdot\nabla\phi_l\right)f\right\|_p = 0,$

then $f\in L^p$.

Remark 5. In Section 3 below, we show some corollaries of the above proposition. In particular, we shall show that if $f\in L^p_{\mathrm{loc}}$ satisfies $x\cdot\nabla f\in L^p$ and if $|x|^{n/p}|f(x)|$ vanishes at infinity, then f belongs to L^p.

Proof of Proposition 1.3. By (iii) and (1.5), we have

$$\|\phi_l f\|_p \le \frac{p}{n}\Big\{\big\|(x\cdot\nabla\phi_l)f\big\|_p + \big\|\phi_l(x\cdot\nabla f)\big\|_p\Big\}$$

for any $l\ge 1$. Since $x\cdot\nabla f\in L^p$, we see from (i), (ii) and the Lebesgue dominated theorem that

$$\lim_{l\to\infty}\big\|\phi_l(x\cdot\nabla f)\big\|_p = \|x\cdot\nabla f\|_p.$$

By (iv), we hence obtain

$$\liminf_{l\to\infty}\|\phi_l f\|_p \le \frac{p}{n}\|x\cdot\nabla f\|_p.$$

It follows from (ii) and the Fatou lemma that

$$\begin{aligned}\|f\|_p^p &= \int_{\mathbb{R}^n}\liminf_{l\to\infty}\big|(\phi_l f)(x)\big|^p dx\\ &\le \liminf_{l\to\infty}\int_{\mathbb{R}^n}\big|(\phi_l f)(x)\big|^p dx\\ &= \liminf_{l\to\infty}\|\phi_l f\|_p^p.\end{aligned}$$

Thus, we see that $f\in L^p$. $\square$

The inequality (1.5) is equivalent to Hardy's inequality if $p=2$. To be more specific, we have:

Proposition 1.4. *Let $n\ge 3$. Then the following two statements are equivalent:*
(a) For any $f\in C_c^\infty(\mathbb{R}^n\setminus\{0\})$, we have

$$\|f\|_2 \le \frac{2}{n}\|x\cdot\nabla f\|_2. \tag{1.10}$$

(b) For any $g\in C_c^\infty(\mathbb{R}^n\setminus\{0\})$, we have

$$\left\|\frac{g}{|x|}\right\|_2 \le \frac{2}{n-2}\left\|\frac{x}{|x|}\cdot\nabla g\right\|_2. \tag{1.11}$$

Remark 6. The original Hardy inequality was given by [6]. Later, a lot of generalized Hardy inequalities are studied (see, e.g., [1, 2, 3, 4, 7, 9]). In particular, it was shown that

$$\left\| \frac{g}{|x|} \right\|_p \leq \frac{p}{n-p} \|\nabla g\|_p, \quad g \in W^{1,p}(\mathbb{R}^n), \quad p < n. \tag{1.12}$$

The constant $p/(n-p)$ is optimal. Since $C_c^\infty(\mathbb{R}^n \setminus \{0\})$ is dense in $W^{1,2}(\mathbb{R}^n)$ for $n \geq 3$, by using Theorem 1.1,(ii), the above proposition and (1.12), we can see that (1.11) holds for any $W^{1,2}(\mathbb{R}^n)$. If we put $g_\varepsilon = |x| f_\varepsilon$, where f_ε have been given in Remark 3, then we have

$$\frac{\big\| g_\varepsilon/|x| \big\|_2}{\big\| (x/|x|) \cdot \nabla g_\varepsilon \big\|_2} \geq 2^{\frac{1}{2}} \left\{ \left(\frac{n}{2} - 1 - \varepsilon \right)^2 + \left(\frac{n}{2} - 1 + \varepsilon \right)^2 \right\}^{-\frac{1}{2}} \to \frac{2}{n-2}$$

as $\varepsilon \to 0$. Therefore, the constant $2/(n-2)$ is optimal.

Proof of Proposition 1.4. We first prove that the statement (a) implies (b). Put $g = |x| f$. Then we obtain

$$\|x \cdot \nabla f\|_2 = \left\| -\frac{g}{|x|} + \frac{x}{|x|} \cdot \nabla g \right\|_2^2 = \left\| \frac{g}{|x|} \right\|_2^2 - 2\Re \left\langle \frac{g}{|x|}, \frac{x}{|x|} \cdot \nabla g \right\rangle + \left\| \frac{x}{|x|} \cdot \nabla g \right\|_2^2, \tag{1.13}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in L^2. Since

$$-2\Re \left\langle \frac{g}{|x|}, \frac{x}{|x|} \cdot \nabla g \right\rangle = -\left\langle \frac{x}{|x|^2}, \nabla |g|^2 \right\rangle = \left\langle \operatorname{div} \left(\frac{x}{|x|^2} \right), |g|^2 \right\rangle = \left\langle \frac{n-2}{|x|^2}, |g|^2 \right\rangle,$$

It follows from the statement (a) and (1.13) that

$$\left\| \frac{g}{|x|} \right\|_2^2 \leq \frac{4}{n^2} \left((n-1) \left\| \frac{g}{|x|} \right\|_2^2 + \left\| \frac{x}{|x|} \cdot \nabla g \right\|_2^2 \right),$$

which implies (1.11).

Conversely, we assume that (b) holds. We put $f = g/|x|$. Then we have

$$\left\| \frac{x}{|x|} \cdot \nabla(|x| f) \right\|_2^2 = \|f + x \cdot \nabla f\|_2^2 = \|f\|_2^2 + 2\Re \langle xf, \nabla f \rangle + \|x \cdot \nabla f\|_2^2$$

Since $2\Re \langle xf, \nabla f \rangle = -n \|f\|_2^2$, we see from the statement (b) that

$$\|f\|_2^2 \leq \frac{4}{(n-2)^2} \left(\|x \cdot \nabla f\|_2^2 - (n-1) \|f\|_2^2 \right),$$

which implies (1.10).

$\square$

2. Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. For this purpose, we prepare some notation. For $\lambda > 0$ and for $g : \mathbb{R}^n \to \mathbb{C}$, $(\delta_\lambda g)(x) = g(\lambda x)$. For $\lambda > 0$ and for $\Omega \subset \mathbb{R}^n$, $\lambda\Omega = \{\lambda\omega; \omega \in \Omega\}$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a positive function such that $\int \varphi = 1$ and $\operatorname{supp}\varphi \subset \{x \in \mathbb{R}^n; |x| \leq 1\}$. For $\lambda > 0$, we set $\zeta_\lambda(x) = \zeta(\lambda|x|)$ and $\varphi_\lambda = \lambda^{-n}\delta_{\lambda^{-1}}\varphi$.

Proposition 2.1. *Let $n \geq 1$ and $1 \leq p < \infty$. Assume that $\Omega \subset \mathbb{R}^n$ is an open set. If $f \in C_c^\infty(\Omega)$, then we have (1.4).*

Proof. For $\phi : \Omega \to \mathbb{C}$, we put

$$\widetilde{\phi}(x) = \begin{cases} \phi(x) & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} \big\|\delta_\lambda\widetilde{\phi}\big|L^p(\Omega)\big\|^p &= \int_\Omega \big|\widetilde{\phi}(\lambda x)\big|^p dx \\ &= \lambda^{-n}\int_{\lambda\Omega} \big|\widetilde{\phi}(x)\big|^p dx \\ &\leq \lambda^{-n}\int_{\Omega} \big|\widetilde{\phi}(x)\big|^p dx \\ &= \lambda^{-n}\|\phi|L^p(\Omega)\|^p. \end{aligned} \tag{2.1}$$

For any $\lambda > 0$ and $x \in \Omega$,

$$\begin{aligned} \big(\delta_\lambda\widetilde{f} - \widetilde{f}\big)(x) &= \widetilde{f}(\lambda x) - \widetilde{f}(x) \\ &= \int_1^\lambda \frac{\partial}{\partial\alpha}\widetilde{f}(\alpha x)d\alpha \\ &= \int_1^\lambda \alpha^{-1}\Big(\delta_\alpha\widetilde{x\cdot\nabla f}\Big)(x)d\alpha. \end{aligned} \tag{2.2}$$

For any $\lambda > 1$, we see from (2.1) and (2.2) that

$$\begin{aligned} \big(1 - \lambda^{-n/p}\big)\|f|L^p(\Omega)\| &\leq \big\|\delta_\lambda\widetilde{f} - \widetilde{f}\big|L^p(\Omega)\big\| \\ &\leq \int_1^\lambda \alpha^{-1}\Big\|\delta_\alpha\widetilde{x\cdot\nabla f}\Big|L^p(\Omega)\Big\|d\alpha \\ &\leq \int_1^\lambda \alpha^{-1-n/p}\|x\cdot\nabla f|L^p(\Omega)\|d\alpha \\ &= \frac{p}{n}\big(1 - \lambda^{-n/p}\big)\|x\cdot\nabla f|L^p(\Omega)\|, \end{aligned}$$

which implies (1.5) by letting $\lambda \to \infty$. □

Proof of Theorem 1.1. We first show (i). There exists some $\{f_l\}_{l=1}^{\infty} \subset C_c^{\infty}(\Omega)$ such that $f_l \to f$ in $W^{1,p}(\Omega)$ as $l \to \infty$. Let $\lambda > 0$. We observe from Proposition 2.1 that

$$\|\zeta_\lambda f_l | L^p(\Omega)\| \leq \frac{p}{n}\Big\{\big\|(x\cdot\nabla\zeta_\lambda)f_l\big|L^p(\Omega)\big\|+\big\|\zeta_\lambda(x\cdot\nabla f_l)\big|L^p(\Omega)\big\|\Big\}$$

for all $l \geq 1$. We easily see that

$$\begin{aligned}\lim_{l\to\infty}\zeta_\lambda f_l &= \zeta_\lambda f,\\ \lim_{l\to\infty}(x\cdot\nabla\zeta_\lambda)f_l &= (x\cdot\nabla\zeta_\lambda)f,\\ \lim_{l\to\infty}\zeta_\lambda(x\cdot\nabla f_l) &= \lim_{l\to\infty}(\zeta_\lambda x)\cdot(\nabla f_l) = \zeta_\lambda(x\cdot\nabla f)\end{aligned}$$

in $L^p(\Omega)$. Hence we obtain

$$\|\zeta_\lambda f | L^p(\Omega)\| \leq \frac{p}{n}\Big\{\big\|(x\cdot\nabla\zeta_\lambda)f\big|L^p(\Omega)\big\|+\big\|\zeta_\lambda(x\cdot\nabla f)\big|L^p(\Omega)\big\|\Big\}.$$

Since

$$\big|(x\cdot\nabla\zeta_\lambda)(x)\big| \leq \begin{cases}\|x\cdot\nabla\zeta\|_\infty & \text{if } \lambda^{-1}<|x|<2\lambda^{-1},\\ 0 & \text{otherwise,}\end{cases} \tag{2.3}$$

(1.5) holds as $\lambda \to 0$.

We next prove (ii). We have only to prove that

$$\lim_{\lambda\to 0}\zeta_\lambda(f*\varphi_\lambda) = f, \tag{2.4}$$

$$\lim_{\lambda\to 0} x\cdot\nabla\big(\zeta_\lambda(f*\varphi_\lambda)\big) = x\cdot\nabla f \tag{2.5}$$

in L^p because $f_\lambda \equiv \zeta_\lambda(f*\varphi_\lambda) \in C_c^\infty(\mathbb{R}^n)$. We obviously have (2.4).

It follows that

$$\begin{aligned} x\cdot\nabla\big(\zeta_\lambda(f*\varphi_\lambda)\big) &= (x\cdot\nabla\zeta_\lambda)(f*\varphi_\lambda) + \zeta_\lambda(x\cdot\nabla)(f*\varphi_\lambda)\\ &= (x\cdot\nabla\zeta_\lambda)(f*\varphi_\lambda) + \zeta_\lambda\big((x\cdot\nabla f)*\varphi_\lambda\big) + \zeta_\lambda\big(f*\operatorname{div}(x\varphi_\lambda)\big)\\ &\equiv I_\lambda + II_\lambda + III_\lambda.\end{aligned}$$

By (2.3), we have $\lim_{\lambda\to 0} I_\lambda = 0$. Repeating the same argument of the proof of (2.4), we obtain $\lim_{\lambda\to 0} II_\lambda = 0$. Since $\int \operatorname{div}(x\varphi_\lambda) = 0$, we see that

$$\big(f*\operatorname{div}(x\varphi_\lambda)\big)(x) = \int_{\mathbb{R}^n}\big(f(x-\lambda y) - f(x)\operatorname{div}(y\varphi(y))\big)dy.$$

Hence we obtain

$$\|f*\operatorname{div}(x\varphi_\lambda)\|_p \leq \sup_{|z|\leq\lambda}\|f(\cdot - z) - f\|_p \|\operatorname{div}(x\varphi)\|_1 \to 0$$

as $\lambda \to 0$. Thus, we have $\lim_{\lambda\to 0} III_\lambda = 0$.

Finally, we show (iii). Since $f \in C^1(\mathbb{R}^n)$, we have

$$f(e^\theta x) - f(x) = \int_0^\theta \frac{\partial}{\partial \alpha} f(e^\alpha x) d\alpha = \int_0^\theta e^\alpha x \cdot (\nabla f)(e^\alpha x) d\alpha \tag{2.6}$$

for any $x \in \mathbb{R}^n$ and $\theta > 0$. Let $\theta_0 = \ln(R/\varepsilon)$ and let $|x| \geq \varepsilon$. Since $|e^{\theta_0} x| \geq R$, we see that $f(e^{\theta_0} x) = 0$. By (2.6), we have

$$f(x) = -\int_0^{\theta_0} e^\alpha x \cdot (\nabla f)(e^\alpha x) d\alpha$$

for any $|x| \geq \varepsilon$. Thus (1.7) holds. $\square$

3. Application

In this section, we show some corollaries of Proposition 1.3.

Corollary 3.1. *Let $n \geq 1$ and $1 \leq p < \infty$. Assume that $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies $x \cdot \nabla f \in L^p$. If there exist sequences $\{\rho_l\}_{l=1}^\infty \subset (0, \infty)$ and $\{R_l\}_{l=1}^\infty \subset (0, \infty)$ such that*

(i) $R_l \leq R_k$ *if* $1 \leq l \leq k$,
(ii) $\lim_{l \to \infty} R_l = \infty$,
(iii)

$$\liminf_{l \to \infty} \left(\frac{R_l + \rho_l}{\rho_l} \right)^p \int_{R_l < |x| < R_l + \rho_l} |f(x)|^p dx = 0, \tag{3.1}$$

then $f \in L^p$.

Remark 7. Taking $\rho_l = R_l$, we see from the above Corollary that if we have $x \cdot \nabla f \in L^p$ and

$$\liminf_{R \to \infty} \int_{R < |x| < 2R} |f(x)|^p dx = 0, \tag{3.2}$$

then $f \in L^p$. If the left hand side of (3.2) is positive, then $f \notin L^p$ (even if $x \cdot \nabla f \in L^p$).

Proof of Corollary 3.1. Set

$$\phi_l(x) = \begin{cases} 1 & \text{for } |x| \leq R_l, \\ \zeta\left(\dfrac{|x| - R_l + \rho_l}{\rho_l} \right) & \text{for } |x| > R_l. \end{cases}$$

We immediately see that

$$\sup_{l \geq 1} \|\phi_l\|_\infty = 1 \quad \text{and} \quad \lim_{l \to \infty} \phi_l(x) = 1 \text{ for a.e. } x.$$

Since

$$x \cdot \nabla \phi_l(x) = \begin{cases} \dfrac{|x|}{\rho_l} \zeta'\left(\dfrac{|x| - R_l + \rho_l}{\rho_l} \right) & \text{if } R_l < |x| < R_l + \rho_l, \\ 0 & \text{otherwise,} \end{cases}$$

it follows that

$$\liminf_{l \to \infty} \left\| \left(x \cdot \nabla \phi_l \right) f \right\|_p^p \le \liminf_{l \to \infty} \left(\frac{R_l + \rho_l}{\rho_l} \right)^p \int_{R_l < |x| < R_l + \rho_l} |f(x)|^p dx = 0.$$

Thus, we see from Proposition 1.3 that $f \in L^p$. □

Corollary 3.2. *Let $n \ge 1$, $1 \le p < \infty$. Assume that $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies $x \cdot \nabla f \in L^p$. If we have either of the following three conditions, then $f \in L^p$:*

(i) It follows that

$$\lim_{R \to \infty} \sup_{R < |x| < 2R} |x|^{n/p} |f(x)| = 0. \tag{3.3}$$

(ii) $p = 1$ and there exists some spherically symmetric, unbounded open set A such that

$$\lim_{|x| \to \infty,\ x \in A} |x|^n |f(x)| = 0. \tag{3.4}$$

(iii) $n = 1$, $p = 1$ and there exists some unbounded open set A such that $-\inf A = \sup A = \infty$ and (3.4) holds.

Remark 8. If

$$\liminf_{|x| \to \infty} |x|^{n/p} |f(x)| > 0,$$

then $f \notin L^p$ even if $x \cdot \nabla f \in L^p$.

Proof of Corollary 3.2. We first prove (i). It follows that

$$\begin{aligned} \int_{R < |x| < 2R} |f(x)|^p dx &= \int_{R < |x| < 2R} \left(|x|^{n/p} |f(x)| \right)^p |x|^{-n} dx \\ &\le \left(\sup_{R < |x| < 2R} |x|^{n/p} |f(x)| \right)^p \int_{R < |x| < 2R} |x|^{-n} dx \\ &\le \ln 2 \left(\sup_{R < |x| < 2R} |x|^{n/p} |f(x)| \right)^p \left(\int_{\mathbb{S}^{n-1}} d\omega \right) \\ &\to 0 \quad \text{as } R \to \infty. \end{aligned}$$

From Remark 7, we have $f \in L^p$.

We next show (ii). There exist $\{\rho_l\}_{l=1}^\infty \subset (0, 1)$ and $\{R_l\}_{l=1}^\infty \subset (0, \infty)$ such that $\lim_{l \to \infty} \rho_l = 0$, $R_l \le R_k$ if $1 \le l \le k$, $\lim_{l \to \infty} R_l = \infty$ and

$$A_l \equiv \{x \in \mathbb{R}^n; R_l < |x| < R_l + \rho_l\} \subset A.$$

Then we obtain

$$\begin{aligned}
&\frac{R_l+\rho_l}{\rho_l}\int_{R_l<|x|<R_l+\rho_l}|f(x)|dx \\
&\le \frac{R_l+\rho_l}{\rho_l}\left(\sup_{R_l<|x|<R_l+\rho_l}|x|^n|f(x)|\right)\left(\int_{\mathbb{S}^{n-1}}d\omega\right)\int_{R_l}^{R_l+\rho_l}r^{-1}dr \\
&\le \left(\sup_{|x|>R_l,\ x\in A}|x|^n|f(x)|\right)\left(\int_{\mathbb{S}^{n-1}}d\omega\right)\frac{R_l+\rho_l}{\rho_l}\cdot\rho_l\cdot\frac{1}{R_l} \\
&\to 0 \quad \text{as } l\to\infty.
\end{aligned}$$

Applying Corollary 3.1, we see that $f\in L^1$.

Finally, we prove (iii). There exist $\{x_l\}_{l=1}^{\infty}\subset\mathbb{R}$ and $\{r_l\}_{l=1}^{\infty}\subset(0,1)$ such that

$$\begin{aligned}
1<x_{2k-1}<x_{2k+1} &\quad \text{for any } k\ge 1, \\
-1>x_{2k}>x_{2k+2} &\quad \text{for any } k\ge 1, \\
|r_l|\to 0 &\quad \text{as } l\to\infty, \\
\{x\in\mathbb{R};|x-x_l|<r_l\}\subset A &\quad \text{for any } l\ge 1.
\end{aligned}$$

Set

$$\phi_1(x)=\begin{cases}\zeta(x) & \text{if } x<0,\\ 1 & \text{if } 0\le x\le x_1,\\ \zeta\left(\dfrac{x-x_1+r_1}{r_1}\right) & \text{if } x>x_1,\end{cases}$$

$$\phi_{2k}(x)=\begin{cases}\zeta\left(\dfrac{x+x_{2k}-r_{2k}}{r_{2k}}\right) & \text{if } x<x_{2k},\\ 1 & \text{if } x_{2k}\le x\le 0,\\ \phi_{2k-1}(x) & \text{if } x>0,\end{cases}$$

$$\phi_{2k+1}(x)=\begin{cases}\phi_{2k}(x) & \text{if } x<0,\\ 1 & \text{if } 0\le x\le x_{2k+1},\\ \zeta\left(\dfrac{x-x_{2k+1}+r_{2k+1}}{r_{2k+1}}\right) & \text{if } x>x_{2k+1},\end{cases}$$

where $k=1,2,\cdots$. Then we have

$$\begin{aligned}
\sup_{l\ge 1}\|\phi_l\|_\infty &= 1, \\
\phi_l f\in L^p &\quad \text{for any } l\ge 1, \\
\lim_{l\to\infty}\phi_l(x)=1 &\quad \text{for all } x\in\mathbb{R}.
\end{aligned}$$

Furthermore, we see that

$$x \cdot \nabla \phi_{2k+1} = \begin{cases} 0 & \text{if } x < x_{2k} - r_{2k}, \\ \dfrac{x}{r_{2k}} \zeta' \Big(\dfrac{x + x_{2k} - r_{2k}}{r_{2k}} \Big) & \text{if } x_{2k} - r_{2k} \leq x \leq x_{2k}, \\ 0 & \text{if } x_{2k} < x < x_{2k+1}, \\ \dfrac{x}{r_{2k+1}} \zeta' \Big(\dfrac{x - x_{2k+1} + r_{2k+1}}{r_{2k+1}} \Big) & \text{if } x_{2k+1} \leq x \leq x_{2k+1} + r_{2k+1}, \\ 0 & \text{if } x > x_{2k+1} + r_{2k+1}, \end{cases}$$

for any $k \geq 1$. Thus, we obtain

$$\begin{aligned} & \big\| \big(x \cdot \nabla \phi_{2k+1} \big) f \big\|_1 \\ & \leq \frac{-x_{2k} + r_{2k}}{r_{2k}} \int_{x_{2k} - r_{2k}}^{x_{2k}} |f(x)| dx + \frac{x_{2k+1} + r_{2k+1}}{r_{2k+1}} \int_{x_{2k+1}}^{x_{2k+1} + r_{2k+1}} |f(x)| dx \\ & \leq \frac{-x_{2k} + r_{2k}}{r_{2k}} \cdot r_{2k} \cdot \frac{1}{-x_{2k}} \left(\sup_{x_{2k} - r_{2k} \leq x \leq x_{2k}} |x| |f(x)| dx \right) \\ & \quad + \frac{x_{2k+1} + r_{2k+1}}{r_{2k+1}} \cdot r_{2k+1} \cdot \frac{1}{x_{2k+1}} \left(\sup_{x_{2k+1} \leq x \leq x_{2k+1} + r_{2k+1}} |x| |f(x)| dx \right). \end{aligned}$$

Hence we observe that

$$\liminf_{l \to \infty} \big\| \big(x \cdot \nabla \phi_l \big) f \big\|_1 = 0.$$

From Proposition 1.3, we have $f \in L^1$. □

References

[1] Adimurthi, N. Chaudhuri and M. Ramaswamy, An improved Hardy-Sobolev inequality and its application. Proc. Amer. Math. Soc. 130 (2002), 489–505.
[2] H. Brezis and M. Marcus, Hardy's inequalities revisited, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 (1997), 217–237.
[3] T. Cazenave, Semilinear Schrödinger Equations, American Mathematical Society, 2003.
[4] J. García Azorero and I. Peral Alonso, Hardy inequalities and some critical elliptic and parabolic problems, J. Differential Equations 144 (1998), 441–476.
[5] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Grundlehren 224, Springer-Verlag, Berlin, 1983.
[6] G. H. Hardy, Notes on a theorem of Hilbert, Math. Z. 6 (1920), 314–317.
[7] B. Muckenhoupt, Weighted norm inequalities for the Fourier transform, Trans. Amer. Math. Soc. 276 (1983), 729–742.
[8] F. Planchon, Sur un inégalité de type Poincaré, C. R. Acad. Sci. Paris Sr. I Math. 330 (2000), 21–23.
[9] J. Zhang, Extensions of Hardy inequality, J. Ineq. Appl. 2006 (2006), Article ID 69379, 5pages.

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