ELLIPTIC DEDEKIND-RADEMACHER SUMS AND TRANSFORMATION FORMULAE OF CERTAIN INFINITE SERIES

TOMOYA MACHIDE

Abstract. We give a transformation formula for certain infinite series in which some elliptic Dedekind-Rademacher sums arise. In the course of its proof, we also obtain a transformation formula for elliptic Dedekind-Rademacher sums. When a complex parameter τ tends to $i\infty$, these represent some classical results which include the reciprocity formula for Apostol-Dedekind sums.

1. Introduction

Let $\tilde{B}_1(x)$ be the first Bernoulli function defined by

$$\tilde{B}_1(x) := \begin{cases} \{x\} - 1/2 & (\text{if } x \text{ is not an integer}), \\ 0 & (\text{if } x \text{ is an integer}). \end{cases}$$

If p and q are relatively prime integers with $q \neq 0$, the Dedekind sum $s(p,q)$ is

$$s(p,q) := \operatorname{sign} q \sum_{j(|q|)} \tilde{B}_1(\frac{j}{q})\tilde{B}_1(\frac{pj}{q}). \tag{1.1}$$

Here $\{x\}$ means the fractional part of a real number x, sign q equals $q/|q|$, and the summation runs through a complete residue system modulo $|q|$. R. Dedekind [De] introduced this sum in connection with the transformation formula for the Dedekind η-function under the group $\mathrm{SL}_2(\mathbb{Z})$ of the two by two matrices with integer entries and determinant one, and deduced from this his reciprocity formula which is a special case of the following (transformation) formula studied in [Ca2, Eq. (4,5)], [Ha, Theorem 2], [HH, Eq. (26)] and [RG]. If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is in $\mathrm{SL}_2(\mathbb{Z})$ with $c \neq 0$ and $cp + dq \neq 0$,

$$\begin{aligned} &s(p,q) - s(ap+bq, cp+dq) + s(d,c) \\ &\qquad = -\frac{1}{4}\operatorname{sign} cq(cp+dq) + \frac{1}{12}\Big(\frac{c}{q(cp+dq)} + \frac{cp+dq}{cq} + \frac{q}{(cp+dq)c}\Big). \end{aligned} \tag{1.2}$$

Dedekind's reciprocity formula is the case that $a = d = 0$ and $-b = c = 1$.

The Dedekind sum has been generalized by many authors (see [Bec] and [RG] for good pictures of their generalizations). For any nonnegative integer m with $m \neq 1$, let $\tilde{B}_m(x) := B_m(\{x\})$ denote the m-th Bernoulli function where $B_m(x)$ is the m-th Bernoulli polynomial defined by $\sum_{n=1}^{\infty} \frac{B_n(x)}{n!} X^n = X\mathrm{e}^{xX}/(\mathrm{e}^X - 1)$. If a, b and c are integers with $c \geq 1$, the generalized Dedekind-Rademacher sum is

$$S_{m,n}\begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix} := \sum_{j(c)} \tilde{B}_m\Big(a\frac{j+z}{c} - x\Big)\tilde{B}_n\Big(b\frac{j+z}{c} - y\Big). \tag{1.3}$$

T.M. Apostol [Ap1] showed that the sums $S_{m,n}\left(\begin{smallmatrix}1 & p & q\\ 0 & 0 & 0\end{smallmatrix}\right)$ arise in the transformation formula of certain Lambert series, and L. Carlitz [Ca1] deduced from this a reciprocity relation for the sums $S_{m,n}\left(\begin{smallmatrix}1 & p & q\\ 0 & 0 & 0\end{smallmatrix}\right)$ which includes (1.2) and the reciprocity formula for so-called Apostol-Dedekind sums $S_{1,n}\left(\begin{smallmatrix}1 & p & q\\ 0 & 0 & 0\end{smallmatrix}\right)$. U. Halbritter [Ha] gave a transformation formula for the sums $S_{m,n}\left(\begin{smallmatrix}1 & p & q\\ 0 & x & y\end{smallmatrix}\right)$ which includes them too. It is noted that there are many proofs of the reciprocity formulas for Dedekind sums and Apostol-Dedekind sums (for example, see [RG] for those of Dedekind sums, and see [Ap2, Bec, Ber3, Fu, Ha, HWZ, Mi, NOS, Ta] for those of Apostol-Dedekind sums), and that a special case of Apostol's transformation formula is Ramanujan's formula (cf. [Ber4, p.275, Entry 21(i)]). The monograph [RG] of H. Rademacher and E. Grosswald is the standard reference for Dedekind sums.

T. Arakawa [Ar1, Ar2], B.C. Berndt [Ber1, Ber2, Ber3], S. Iseki [Is] and J. Lewittes [Le] also gave transformation formulas which are considered as generalizations or analogues of Apostol's transformation formula. Arakawa's [Ar1, Ar2] which is deduced from Berndt's [Ber1] uses the infinite series

$$\eta(\alpha, s, y, x) := \sum_{m=1}^{\infty} \frac{\mathrm{e}(mx)}{m^{1-s}} \frac{\mathrm{e}(ym\alpha)}{1-\mathrm{e}(m\alpha)}. \tag{1.4}$$

In their transformation formulas, Berndt let α to be a complex number with positive imaginary part, but Arakawa to be an irrational real algebraic number. If s is an integer with $s \leq -1$, then Arakawa's transformation formula with $x = y = 0$ becomes the same form of Apostol's and consequently yields Carlitz's reciprocity relation.

On the other hand, various elliptic analogues of Dedekind sums have been studied by many people ([As, Ba1, Ba2, Eg, FY, It1, It2, Sc]). Recently the author [Ma] has introduced elliptic analogue of generalized Dedekind-Rademacher sums which we call *elliptic Dedekind-Rademacher sums*, and has obtained their reciprocity formula. His elliptic analogue and the others have had no transformation formula in which themselves appear.

The purpose of this paper is to give a transformation formula for certain infinite series in which not all but some elliptic Dedekind-Rademacher sums arise. Our transformation formula is considered as an elliptic analogue of Arakawa's with $s \in \mathbb{Z}$ and $s \leq -2$, because it reproduces Arakawa's when a parameter τ in the upper half plane tends to $i\infty$, and the infinite series in it satisfy a modular property on the parameter τ under $\mathrm{SL}_2(\mathbb{Z})$. In the course of its proof, we also obtain elliptic generalizations of Carlitz's reciprocity relation and a part of Halbritter's transformation formula.

The paper is organized as follows: We introduce elliptic Dedekind-Rademacher sums and certain infinite series in Section 2 and 3 respectively. In Section 4, we give our transformation formula. Section 5, 6 and 7 devote the proofs of propositions in Section 2, 3 and 4. Section 6 also includes a transformation formula for elliptic Dedekind-Rademacher sums which reproduce a part of Halbritter's transformation formula.

2. Elliptic Dedekind-Rademacher sums

In order to define the elliptic Dedekind-Rademacher sums which arise in our transformation formula, we introduce Kronecker's double series and their generating function. Let τ be in the upper half plane, and $\mathrm{e}(x) := \mathrm{e}^{2\pi i x}$. If $q = \mathrm{e}(\tau)$, Jacobi's theta function $\theta(x;\tau)$ is defined by

$$\theta(x;\tau) := \sum_{m\in\mathbb{Z}} \mathrm{e}(\frac{1}{2}(m+\frac{1}{2})^2\tau + (m+\frac{1}{2})(x+\frac{1}{2})) \tag{2.1}$$
$$= iq^{1/8}(\mathrm{e}(\frac{x}{2}) - \mathrm{e}(-\frac{x}{2})) \prod_{m=1}^{\infty}(1-\mathrm{e}(-x)q^m)(1-\mathrm{e}(x)q^m)(1-q^m)$$

which is an odd and quasi periodic entire function:

$$\theta(-x;\tau) = -\theta(x;\tau), \quad \theta(x+1;\tau) = -\theta(x;\tau), \quad \theta(x+\tau;\tau) = -\mathrm{e}(-\frac{\tau}{2}-x)\theta(x;\tau). \tag{2.2}$$

Let $\theta'(x;\tau)$ denote the derivative of $\theta(x;\tau)$ with respect to x. For any vector $\vec{x} = (x', x)$ in $\mathbb{R}^2 \setminus \mathbb{Z}^2$, Kronecker's double series $B_m(\vec{x};\tau)$ and their generating function $\underline{F}(\vec{x}; X;\tau)$ are defined by

$$\underline{F}(\vec{x}; X;\tau) := \mathrm{e}(xX)\frac{\theta'(0;\tau)\theta(-x'+x\tau+X;\tau)}{\theta(-x'+x\tau;\tau)\theta(X;\tau)} = \sum_{m=0}^{\infty}\frac{B_m(\vec{x};\tau)}{m!}(2\pi i)^m X^{m-1}. \tag{2.3}$$

Let $\vec{a}$ be a vector in $\mathbb{Z}^2$. We find from (2.1) that the function $\underline{F}(\vec{x}; X;\tau)$ with respect to X is meromorphic with only simple poles on the lattice $\mathbb{Z}+\tau\mathbb{Z}$, and from (2.2) that

$$\underline{F}(\vec{x}; X+1;\tau) = \mathrm{e}(x)\underline{F}(\vec{x}; X;\tau), \quad \underline{F}(\vec{x}; X+\tau;\tau) = \mathrm{e}(x')\underline{F}(\vec{x}; X;\tau), \tag{2.4}$$
$$\underline{F}(-\vec{x}; -X;\tau) = -\underline{F}(\vec{x}; X;\tau), \quad \underline{F}(\vec{x}+\vec{a}; X;\tau) = \underline{F}(\vec{x}; X;\tau). \tag{2.5}$$

We deduce from (2.3) and (2.5)

$$B_m(-\vec{x};\tau) = (-1)^m B_m(\vec{x};\tau), \quad B_m(\vec{x}+\vec{a};\tau) = B_m(\vec{x};\tau). \tag{2.6}$$

Put $q = \mathrm{e}(\tau)$. We also have explicit expressions of Kronecker's double series (see [Ma, Eq. (9)]):

$$B_m(\vec{x};\tau) = m\Bigg(\sum_{j=1}^{\infty}(x-j)^{m-1}\frac{\mathrm{e}(-x\tau)q^j}{\mathrm{e}(-x')-\mathrm{e}(-x\tau)q^j} \tag{2.7}$$
$$-\sum_{j=1}^{\infty}(x+j)^{m-1}\frac{\mathrm{e}(x\tau)q^j}{\mathrm{e}(x')-\mathrm{e}(x\tau)q^j} + x^{m-1}\frac{\mathrm{e}(-x'+x\tau)}{\mathrm{e}(-x'+x\tau)-1}\Bigg) + B_m(x).$$

So $B_1(\vec{x};\tau)$ and $B_2(\vec{x};\tau)$ are continuous at $\vec{x} \in \mathbb{R}^2\setminus\mathbb{Z}^2$ and discontinuous at $\vec{x} \in \mathbb{Z}^2$, but the others are continuous at every $\vec{x} \in \mathbb{R}^2$.

Let us define the elliptic Dedekind-Rademacher sums. Let r be a rational number. There is a unique pare $(n(r), d(r))$ of integers with $\gcd(n(r), d(r)) = 1, d(r) \geq 1$ and $r = n(r)/d(r)$. Let $\mathrm{M}_2(\mathbb{R})$ be the set of the two by two matrices with real number entries, and $\mathrm{SM}_2(r)$ be its subset defined as follows:

$$\mathrm{SM}_2(r) := \left\{ \begin{pmatrix} x' & x \\ y' & y \end{pmatrix} = \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \in \mathrm{M}_2(\mathbb{R}) \middle| \vec{y}, n(r)\vec{y} - d(r)\vec{x} \notin \mathbb{Z}^2 \right\}, \tag{2.8}$$

where $\vec{x}$ and $\vec{y}$ denote the vectors (x', x) and (y', y) respectively. If $\Xi = \left(\begin{smallmatrix} \vec{x} \\ \vec{y} \end{smallmatrix}\right)$ is in $\mathrm{SM}_2(r)$, then the elliptic Dedekind-Rademacher sum is defined by

$$S^{\tau}_{m,n}(\Xi; r) := \frac{1}{d(r)} \sum_{j', j(d(r))} B_m\Big(\frac{\vec{j} + \vec{y}}{d(r)}; \tau\Big) B_n\Big(n(r)\frac{\vec{j} + \vec{y}}{d(r)} - \vec{x}; \tau\Big), \tag{2.9}$$

where $\vec{j}$ denotes the vector (j', j). By continuity of Kronecker's double series, the elliptic Dedekind-Rademacher sum $S^{\tau}_{m,n}(\Xi; r)$ is continuous at $\Xi \in \mathrm{SM}_2(r)$. In particular, if $m, n \geq 3$, $S^{\tau}_{m,n}(\Xi; r)$ is continuous at every $\Xi \in \mathrm{M}_2(\mathbb{R})$. The relation between this sum and the sum $S^{\tau}_{m,n}\left(\begin{smallmatrix} (a',a) & (b',b) & (c',c) \\ (x',x) & (y',y) & (z',z) \end{smallmatrix}\right)$ defined in [Ma] is

$$S^{\tau}_{m,n}(\Xi; r) = S^{\tau}_{m,n}\begin{pmatrix} (1,1) & (n(r), n(r)) & (d(r), d(r)) \\ (0,0) & \vec{x} & \vec{y} \end{pmatrix}. \tag{2.10}$$

It is noted that the sums $S^{\tau}_{m,n}\left(\begin{smallmatrix} (1,1) & (b',b) & (c',c) \\ (0,0) & \vec{x} & \vec{y} \end{smallmatrix}\right)$ with $b' \neq b$ or $c' \neq c$ can not be written in terms of $S^{\tau}_{m,n}(\Xi; r)$, or do not appear in our transformation formula.

To give some properties of the elliptic Dedekind-Rademacher sums, the cotangent sum studied by U. Dieter [Di] is introduced:

$$\mathfrak{c}(a, b, c; x, y, z) := \frac{1}{c} \sum_{j(c)}{}' \cot\Big(\pi(a\frac{j+z}{c} - x)\Big) \cot\Big(\pi(b\frac{j+z}{c} - y)\Big),$$

where the prime of the summation means excluding j such that $a\frac{j+z}{c} - x \in \mathbb{Z}$ or $b\frac{j+z}{c} - y \in \mathbb{Z}$. For convenience, we shall use

$$S_{m,n}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}; r\right) = S_{m,n}\left(\begin{smallmatrix} 1 & n(r) & d(r) \\ 0 & x & y \end{smallmatrix}\right), \quad \mathfrak{c}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}; r\right) = \mathfrak{c}(1, n(r), d(r); 0, x, y).$$

If $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a two by two matrix, let $V\tau$ and $j(V; \tau)$ denote $(a\tau + b)/(c\tau + d)$ and $c\tau + d$ respectively. The elliptic Dedekind-Rademacher sum $S^{\tau}_{m,n}(\Xi; r)$ has a modular property, and reproduces the sum $S_{m,n}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}; r\right)$ when τ tends to $i\infty$. We shall prove these properties in Section 7.

PROPOSITION 2.1. *Let r be a rational number, and $\Xi = \begin{pmatrix} x' & x \\ y' & y \end{pmatrix}$ be a matrix in $SM_2(r)$.*
(i) *If V is in $SL_2(\mathbb{Z})$, then*

$$S^{\tau}_{m,n}(\Xi; r) = \frac{1}{j(V; \tau)^{m+n}} S^{V\tau}_{m,n}(\Xi\, {}^tV; r), \tag{2.11}$$

where ${}^tV$ means the transpose of the matrix V. In particular, $S^{\tau}_{m,n}\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}; r\right)$ is a modular form of weight $m + n$ if $m, n \geq 3$.
(ii) *Let* Re z *denote the real part of a complex number z. Then we have*

$$\mathrm{Re} \lim_{\tau \to i\infty} S^{\tau}_{m,n}(\Xi; r) = \begin{cases} S_{m,n}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}; r\right) - \dfrac{1}{4}\mathfrak{c}\left(\begin{smallmatrix} x' \\ y' \end{smallmatrix}; r\right) & (m = n = 1, x, y \in \mathbb{Z}), \\ S_{m,n}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}; r\right) & (\textit{otherwise}). \end{cases} \tag{2.12}$$

3. Certain infinite series

Let α be an irrational real algebraic number and l an integer with $l \geq 3$. If $\Xi = \left(\begin{smallmatrix} \vec{x} \\ \vec{y} \end{smallmatrix}\right) = \left(\begin{smallmatrix} x' & x \\ y' & y \end{smallmatrix}\right)$ is in $\mathrm{M}_2(\mathbb{R})$ with $\vec{y} \notin \mathbb{Z}^2$, the certain infinite series which appears in our transformation formula is defined by

$$H_l^\tau(\Xi;\alpha) := (-1)^l \frac{l!}{(2\pi i)^{l+1}} \sum_{m',m}{}' \frac{\mathrm{e}(\vec{m}\cdot\vec{x})}{(\tau m'+m)^l} \underline{F}(-\vec{y};\alpha(\tau m'+m);\tau). \tag{3.1}$$

Here the summation ranges over all elements in $\mathbb{Z}^2$ with $(m',m)\neq(0,0)$, $\vec{m}$ denotes the vector (m',m), and $\vec{m}\cdot\vec{x}$ means the inner product of $\vec{m}$ and $\vec{x}$, or $m'x'+mx$. By the following lemma, the series in (3.1) absolutely converges, that is, $H_l^\tau(\Xi;\alpha)$ is a continuous function at $\vec{x}\in\mathbb{R}^2$ and $\vec{y}\in\mathbb{R}^2\setminus\mathbb{Z}^2$.

LEMMA 3.1. *Let s be a complex number, and λ',λ nonnegative real numbers. The series $\sum_{m',m}' \frac{\mathrm{e}(\vec{m}\cdot\vec{x}) m'^{\lambda'} m^\lambda}{(\tau m'+m)^s} \underline{F}(-\vec{y};\alpha(\tau m'+m);\tau)$ absolutely converges if* Re $s>\lambda'+\lambda+2$.

Proof. Let $I = \sum_{m',m}' \left| \frac{\mathrm{e}(\vec{m}\cdot\vec{x}) m'^{\lambda'} m^\lambda}{(\tau m'+m)^s} \underline{F}(-\vec{y};\alpha(\tau m'+m);\tau)\right|$, and $\vec{x},\vec{y},\tau$ be fixed. Since the function $\underline{F}(\vec{x};X;\tau)$ with respect to X is meromorphic with only simple poles on the lattice $\mathbb{Z}+\tau\mathbb{Z}$, there is a positive real number $C=C(\vec{y},\tau)$ such that if $X=\tau\xi'+\xi$ is a complex number with $-1/2\le\xi',\xi\le 1/2$, then

$$|X\underline{F}(\vec{y};X;\tau)|\le C.$$

For any real number x, let $[[x]]$ and $\langle\langle x\rangle\rangle$ respectively denote the integer and the real number with $-1/2<\langle\langle x\rangle\rangle\le 1/2$, $x=[[x]]+\langle\langle x\rangle\rangle$. By (2.4),

$$\underline{F}(-\vec{y};\alpha(\tau m'+m);\tau) = \mathrm{e}(-y'[[\alpha m']]-y[[\alpha m]])\underline{F}(-\vec{y};\tau\langle\langle\alpha m'\rangle\rangle+\langle\langle\alpha m\rangle\rangle;\tau),$$

thus we have

$$I\le C\sum_{m',m}{}' \frac{|m'|^{\lambda'}|m|^\lambda}{|\tau m'+m|^s} \frac{1}{|\tau\langle\langle\alpha m'\rangle\rangle+\langle\langle\alpha m\rangle\rangle|}.$$

If X' and X are real numbers, some calculations show that

$$\begin{aligned} &|\tau X'+X|\ge\frac{\mathrm{Im}\ \tau}{|\tau|}|X|,\quad |\tau X'+X|\ge|\tau|^2\mathrm{Im}\ (-\frac{1}{\tau})\ |X'|,\\ &|\tau X'+X|^2\ge 2(|\tau|-|\mathrm{Re}\ \tau|)|X'X|. \end{aligned} \tag{3.2}$$

Hence, there is a positive real number $D=D(\vec{y},\tau)$ such that the following inequality holds.

$$\begin{aligned} I\le D\Big(&\sum_{m'}{}'\sum_{m}{}' \frac{1}{|m'|^{(s-\lambda'-\lambda)/2}|m|^{(s-\lambda'-\lambda)/2}|\langle\langle\alpha m'\rangle\rangle|^{1/2}|\langle\langle\alpha m\rangle\rangle|^{1/2}}\\ &+\frac{1}{|\tau|^{s-\lambda'-\lambda+1}}\sum_{m'}{}'\frac{1}{|m'|^{s-\lambda'-\lambda}|\langle\langle\alpha m'\rangle\rangle|}+\sum_{m}{}'\frac{1}{|m|^{s-\lambda'-\lambda}|\langle\langle\alpha m\rangle\rangle|}\Big), \end{aligned}$$

where the summation $\sum_m'$ ranges over all integers except zero. Since it was obtained in the proof of [Ar1, Lemma 1] that the series $\sum_{m=1}^{\infty}\frac{1}{m^{1+\epsilon}|\langle\langle\alpha m\rangle\rangle|}$ converges if $\epsilon>0$, we complete the proof. □

Let us give some properties of the infinite series $H_l^\tau(\vec{x},\vec{y};\alpha)$. For any real number x, let $\langle x\rangle$ be $\{x\}$ if x is not an integer, and 1 otherwise. By using Berndt's results

[Ber1, Ber3], Arakawa [Ar1, Ar2] gave his transformation formula for the infinite series defined by

$$H(\alpha, s, y, x) := \eta(\alpha, s, \langle y\rangle, x) + \mathrm{e}(s/2)\eta(\alpha, s, \langle -y\rangle, -x). \tag{3.3}$$

In particular, if l is an integer with $l \geq 2$,

$$H(\alpha, -l+1, y, x) = \begin{cases} \dfrac{i}{2}\Big(\xi_l(x;\alpha) + i\sum\limits_{m=1}^{\infty} \dfrac{\mathrm{e}(mx) + (-1)^{l-1}\mathrm{e}(-mx)}{m^l}\Big) & (y \in \mathbb{Z}), \\ \dfrac{i}{2}\Big(\xi_l(x + \{y\}\alpha;\alpha) - i\sum\limits_{m}{}' \dfrac{\mathrm{e}(m(x+\{y\}\alpha))}{m^l}\Big) & (y \notin \mathbb{Z}), \end{cases} \tag{3.4}$$

where $\xi_l(x;\alpha) := \sum\limits_{m}{}' \dfrac{\mathrm{e}(mx)}{m^l} \cot \pi m\alpha$. In analogy to Proposition 2.1, the infinite series $H_l^\tau(\Xi;\alpha)$ has a modular property, and reproduces $H(\alpha, -l+1, -y, x)$ when τ tends to $i\infty$. We shall prove these properties in Section 7 too.

PROPOSITION 3.2. *Let α be an irrational real algebraic number, l an integer with $l \geq 3$, and $\Xi = \begin{pmatrix} x' & x \\ y' & y \end{pmatrix}$ be a matrix in $M_2(\mathbb{R})$ with $(y', y) \notin \mathbb{Z}^2$.*

(i) *If V is in $SL_2(\mathbb{Z})$, then*

$$H_l^\tau(\Xi;\alpha) = \frac{1}{j(V;\tau)^{l+1}} H_l^{V\tau}(\Xi\ {}^tV;\alpha). \tag{3.5}$$

(ii) *Set $CL(x) = \dfrac{1}{2}\sum\limits_{m=1}^{\infty} \dfrac{\mathrm{e}(mx) + (-1)^{l-1}\mathrm{e}(-mx)}{m^l}$. Then we have*

$$\mathrm{Re} \lim_{\tau \to i\infty} H_l^\tau(\Xi;\alpha) = \begin{cases} (-1)^{l+1}\dfrac{l!}{(2\pi i)^l}\Big(H(\alpha, -l+1, 0, x) + CL(x)\Big) & (y \in \mathbb{Z}), \\ (-1)^{l+1}\dfrac{l!}{(2\pi i)^l} H(\alpha, -l+1, -y, x) & (y \notin \mathbb{Z}). \end{cases} \tag{3.6}$$

4. Transformation formula

In this section, we give a transformation formula for the infinite series $H_l^\tau(\Xi;\alpha)$ in which the elliptic Dedekind-Rademacher sums $S_{m,n}^\tau(\Xi;\alpha)$ arise. For any matrix $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{Z})$, let $\mathrm{SM}_2(V)$ be the subset of $\mathrm{M}_2(\mathbb{R})$ such that

$$\mathrm{SM}_2(V) := \left\{ \begin{pmatrix} x' & x \\ y' & y \end{pmatrix} = \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \in \mathrm{M}_2(\mathbb{R}) \,\middle|\, \vec{y}, d\vec{y} + c\vec{x} \notin \mathbb{Z}^2 \right\}. \tag{4.1}$$

It is noted that we use the same symbol SM_2 for a rational number r and for a matrix V in $\mathrm{SL}_2(\mathbb{Z})$ because $\mathrm{SM}_2(r)$ and $\mathrm{SM}_2(V)$ have a close relation, i.e., $\mathrm{SM}_2(V) = \mathrm{SM}_2(-d/c)$ when $c \neq 0$. If $\Xi = \left(\begin{smallmatrix} \vec{x} \\ \vec{y} \end{smallmatrix}\right)$ is in $\mathrm{SM}_2(V)$, c is positive, and z is a complex number, then a sum of elliptic Dedekind-Rademacher sums is defined by

$$R_l^\tau(\Xi; z; V) := \begin{cases} \dfrac{(-1)^l}{l+1}\sum\limits_{k=-1}^{l} \dbinom{l+1}{k+1}\big(-j(V,z)\big)^k S_{k+1,l-k}^\tau\left(\begin{matrix} -\vec{x} \\ \vec{y} \end{matrix}; \dfrac{d}{c}\right) & (c > 0), \\ 0 & (c = 0). \end{cases} \tag{4.2}$$

When c is negative, $R_l^\tau(\Xi; z; V)$ also denotes $R_l^\tau(\Xi; z; -V)$. It is a continuous function at $\Xi \in \mathrm{SM}_2(V)$ by virtue of continuity of elliptic Dedekind-Rademacher sums. The transformation formula is

THEOREM 4.1. *Let α be an irrational real algebraic number, l an integer with $l \geq 3$, and V a matrix in $SL_2(\mathbb{Z})$. If Ξ is a matrix in $SM_2(V)$, then*

$$H_l^\tau(\Xi; \alpha) - j(V; \alpha)^{l-1} H_l^\tau(V\Xi; V\alpha) = R_l^\tau(\Xi; \alpha; V). \tag{4.3}$$

To give its proof, we need the following two lemmas which shall be proved in Section 5 and 6.

LEMMA 4.2. *Let T and S denote the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ respectively. If $V \in \{T^{\pm}, S\}$ and $\Xi \in SM_2(V)$, then (4.3) holds.*

LEMMA 4.3. *Let z be a complex number, l a positive integer, and V, V_1, V_2 matrices in $SL_2(\mathbb{Z})$ with $V = V_2V_1$. If Ξ is in $SM_2(V) \bigcap SM_2(V_1)$, then*

$$R_l^\tau(\Xi; z; V_1) + j(V_1, z)^{l-1} R_l^\tau(V_1\Xi; V_1 z; V_2) = R_l^\tau(\Xi; z; V). \tag{4.4}$$

Proof of Theorem 4.1. Put $G = \{T^{\pm}, S\}$. For any positive integer n, let U_n be a subset of $\mathrm{SL}_2(\mathbb{Z})$ such that if $V \in U_n$, there are n matrices $V_1, \ldots, V_n$ in G with $V = V_n V_{n-1} \cdots V_1$. If V is a matrix in $\mathrm{SL}_2(\mathbb{Z})$, there is a positive integer n with $V \in U_n$ by [Ap3, Theorem 2.1]. We will prove (4.3) by induction on n. If n is one, the claim has been shown in Lemma 4.2. Suppose that it is true when $V \in U_{n-1}$. Let V be a matrix in U_n. Then there are two matrices V_1 and V_2 such that $V_1 \in G, V_2 \in U_{n-1}$ and $V = V_2V_1$. Since $j(V; \alpha) = j(V_1; \alpha) j(V_2; V_1\alpha)$, the left hand side of (4.3) equals

$$H_l^\tau(\Xi; \alpha) - j(V_1; \alpha)^{l-1} H_l^\tau(V_1\Xi; V_1\alpha) + \\ j(V_1; \alpha)^{l-1} \Big(H_l^\tau(V_1\Xi; V_1\alpha) - j(V_2; V_1\alpha)^{l-1} H_l^\tau(V_2(V_1\Xi); V_2(V_1\alpha)) \Big).$$

Thus we obtain (4.3) with $\Xi \in \mathrm{SM}_2(V) \bigcap \mathrm{SM}_2(V_1)$ by virtue of Lemma 4.3 and the induction hypothesis. The set $\mathrm{SM}_2(V) \bigcap \mathrm{SM}_2(V_1)$ is dense in $\mathrm{M}_2(\mathbb{R})$, so we also get (4.3) with $\Xi \in \mathrm{SM}_2(V)$ by continuity of $H_l^\tau(\Xi; \alpha)$ and $R_l^\tau(\Xi; \alpha; V)$, which completes the proof. $\square$

REMARK 4.4. When τ tends to $i\infty$, $R_l^\tau(\Xi; z; V)$ represents $f_{l+1}(-d, c; z)$ in [Ca2, Eq. (1.2)], that is, (4.4) is an elliptic generalization of Carlitz's reciprocity relation [Ca2, Eq. (1.11)].

By (2.12) and (3.6), we can reproduce a part of Arakawa's transformation formula [Ar1, Ar2]. Since it is not a new result, we do not give the formula here.

Motivated by Knopp's identity [Kn], J.A. Parson and K.H. Rosen [PR] gave a formula for the sums $S_{m,n}\left(\begin{smallmatrix} 1 & p & q \\ 0 & 0 & 0 \end{smallmatrix}\right)$ by looking at the action of the Hecke operators on certain Lambert series studied by Apostol [Ap1] together with the transformation formula for these series. Aarakawa [Ar2] also obtained an identity of Knopp type in a different way from theirs. For the identity, he not only considered the Hecke operators on the Dirichlet series $\xi(s, \alpha) = \sum\limits_{m=1}^{\infty} \dfrac{\cot \pi m\alpha}{m^s}$ with the transformation formula for this, but also used the fact that the function $\xi(s, \alpha)$ for a real quadratic number α can be analytically continued to a meromorphic function of s in the whole complex plane. It is noted that the Dirichlet series $\xi(s,\alpha)$ relates to the infinite series $H(\alpha, 1-s, 1, 0)$ (see [Ar2, Eq. (2.2)]).

$$\xi(s;\alpha) = -2i\Big(\frac{1}{1+\mathrm{e}((1-s)/2)}H(\alpha,1-s,1,0) + \frac{1}{2}\zeta(s)\Big), \tag{4.5}$$

where $\zeta(s)$ denotes the Riemann zeta function. We will end the section by posing three questions.

Questions (i) Can we construct Hecke operators on the series $H_l^\tau(\Xi;\alpha)$, and give elliptic analogues of identities of Knopp, Parson and Rosen by using the action of the Hecke operators together with the transformation formula for the series?

(ii) If α is a real quadratic number, can we analytically continue the series

$$\sum_{m',m}{}' \frac{\mathrm{e}(\vec{m}\cdot\vec{x})}{(\tau m'+m)^s}\underline{F}(-\vec{y};\alpha(\tau m'+m);\tau)$$

to a meromorphic function of s in the whole complex plane, and produce an elliptic analogue of Arakawa's identity in a similar way of his?

(iii) Not all elliptic Dedekind-Rademacher sums appear in our transformation formula. Is there a transformation formula in which the others arise?

5. Proof of Lemma 4.2

We give a proof of Lemma 4.2. Its method is based on Siegel's idea [Si] (see [Ka]). Set

$$\sum_{m',m}{}_e = \lim_{M\to\infty}\sum_{m'=-M}^{M}\sum_{m=-M}^{M}, \qquad \sum_{m',m}{}'_e = \lim_{M\to\infty}\sum_{m'=-M}^{M}\sum_{m=-M}^{M}{}_{(m',m)\neq(0,0)}.$$

In order to prove Lemma 4.2, we need the following equation.

PROPOSITION 5.1. *Let α be an irrational real algebraic number, and s be a complex number with* Re $s>3$. *If $x',x,y',y\in\mathbb{R}\setminus\mathbb{Z}$, then*

$$\Big(\sum_{m',m}{}'_e\sum_{n',n}{}'_e - \sum_{n',n}{}'_e\sum_{m',m}{}'_e\Big)\frac{\mathrm{e}(\vec{m}\cdot\vec{x})}{(\tau m'+m)^s}\frac{\mathrm{e}(\vec{n}\cdot\vec{y})}{\alpha(\tau m'+m)+\tau n'+n} = 0, \tag{5.1}$$

or the order of (m',m) and (n',n) in the sum can be changed. Here $\vec{x},\vec{y},\vec{m}$ and $\vec{n}$ denote the vectors $(x',x),(y',y),(m',m)$ and (n',n) respectively.

We give the proof of Lemma 4.2 before showing (5.1).

Proof of Lemma 4.2. Let V be in $\{T^{\pm},S\}$ and $\left(\begin{smallmatrix}x' & x\\ y' & y\end{smallmatrix}\right)=\left(\begin{smallmatrix}\vec{x}\\ \vec{y}\end{smallmatrix}\right)$ in $\mathrm{SM}_2(V)$. It follows from (2.4) that

$$H_l^\tau\left(\begin{smallmatrix}\vec{x}\\ \vec{y}\end{smallmatrix};\alpha\right) = H_l^\tau\left(\begin{smallmatrix}\vec{x}+\vec{y}\\ \vec{y}\end{smallmatrix};\alpha+1\right),$$

thus we obtain (4.3) when $V=T^{\pm}$. We will show (4.3) with $V=S$. Let l be an integer with $l\geq 4$, and $x',x,y',y\in\mathbb{R}\backslash\mathbb{Z}$. Since $Y^l-X^l=(Y-X)\sum_{k=0}^{l-1}X^kY^{l-1-k}$, we have

$$\frac{1}{X^l}\frac{1}{\alpha X+Y} - \alpha^{l-1}\frac{1}{(-Y)^l}\frac{1}{\frac{1}{\alpha}Y+X} = \sum_{k=0}^{l-1}(-\alpha)^k\frac{1}{Y^{k+1}}\frac{1}{X^{l-k}}.$$

By substituting $\tau m' + m$ and $\tau n' + n$ for X and Y respectively, we get

$$\sum_{n',n}{}'_e \sum_{m',m}{}'_e \Big(\frac{\mathrm{e}(\vec{m}\cdot\vec{x})}{(\tau m'+m)^l} \frac{\mathrm{e}(\vec{n}\cdot\vec{y})}{\alpha(\tau m'+m)+\tau n'+n} - \alpha^{l-1} \frac{\mathrm{e}(\vec{n}\cdot\vec{y})}{(-\tau n'-n)^l} \tag{5.2}$$
$$\times \frac{\mathrm{e}(\vec{m}\cdot\vec{x})}{\frac{1}{\alpha}(\tau n'+n)+\tau m'+m} \Big) = \sum_{k=0}^{l-1} (-\alpha)^k \sum_{n',n}{}'_e \frac{\mathrm{e}(\vec{n}\cdot\vec{y})}{(\tau n'+n)^{k+1}} \sum_{m',m}{}'_e \frac{\mathrm{e}(\vec{m}\cdot\vec{x})}{(\tau m'+m)^{l-k}}.$$

On the other hand, L. Kronecker [Kr] showed that

$$\underline{F}(\vec{x};X;\tau) = \sum_{m',m}{}_e \frac{\mathrm{e}(-\vec{m}\cdot\vec{x})}{X+\tau m'+m}, \quad B_k(\vec{x};\tau) = -\frac{k!}{(2\pi i)^k} \sum_{m',m}{}'_e \frac{\mathrm{e}(\vec{m}\cdot\vec{x})}{(\tau m'+m)^k}.$$

Thus, by (5.1), the left hand side of (5.2) equals

$$\sum_{m',m}{}' \frac{\mathrm{e}(\vec{m}\cdot\vec{x})}{(\tau m'+m)^l} \underline{F}(-\vec{y};\alpha(\tau m'+m);\tau) + \frac{1}{\alpha}\frac{(2\pi i)^{l+1}}{(l+1)!} B_{l+1}(\vec{x};\tau)$$
$$- \alpha^{l-1} \Big(\sum_{n',n}{}'_e \frac{\mathrm{e}(-\vec{n}\cdot\vec{y})}{(\tau n'+n)^l} \underline{F}(-\vec{x};-\frac{1}{\alpha}(\tau n'+n);\tau) + (-1)^l \alpha \frac{(2\pi i)^{l+1}}{(l+1)!} B_{l+1}(\vec{y};\tau) \Big),$$

and the right hand side of (5.2) equals

$$\frac{(2\pi i)^{l+1}}{(l+1)!} \sum_{k=0}^{l-1} (-\alpha)^k \binom{l+1}{k+1} B_{k+1}(\vec{y};\tau) B_{l-k}(\vec{x};\tau).$$

These yield (4.3) with $l \geq 4$ and $V = S$ since the set $\{ \left(\begin{smallmatrix} x' & x \\ y' & y \end{smallmatrix}\right) | x', x, y', y \in \mathbb{R} \setminus \mathbb{Z} \}$ is dense in $\mathrm{M}_2(\mathbb{R})$. In order to get (4.3) with $l = 3$ and $V = S$, we introduce differential equations for Kronecker's double series and their generating function.

$$(\tau \frac{\partial}{\partial x'} + \frac{\partial}{\partial x}) \underline{F}(\vec{x};X;\tau) = 2\pi i X \underline{F}(\vec{x};X;\tau),$$
$$(\tau \frac{\partial}{\partial x'} + \frac{\partial}{\partial x}) B_m(\vec{x};\tau) = m B_{m-1}(\vec{x};\tau).$$

If Re $s > 3$ and $z \in \{x', x, y', y\}$, we see from Lemma 3.1 that the series

$$\sum_{m',m}{}' \frac{\partial}{\partial z} \Big(\frac{\mathrm{e}(\vec{m}\cdot\vec{x})}{(\tau m'+m)^s} \underline{F}(-\vec{y};\alpha(\tau m'+m);\tau) \Big)$$

absolutely converges and its term wise differentiation is possible. So, by applying the differential operator $\tau \frac{\partial}{\partial x'} + \frac{\partial}{\partial x}$ to (4.3) with $l = 4$, we obtain (4.3) with $l = 3$. □

To give a proof of Proposition 5.1, the following lemma is introduced.

LEMMA 5.2. *Let y', y, λ be real numbers, and $N_1', N_1, N_2', N_2, m', m$ integers. If $y', y \notin \mathbb{Z}, \lambda > 0, N_1' \leq N_2', N_1 \leq N_2$ and $(m', m) \neq (0,0)$, then there is a positive real number $D = D(y', y, \alpha, \lambda, \tau)$ such that*

$$\Big| \sum_{n'=N_1'}^{N_2'} \sum_{n=N_1}^{N_2} \frac{\mathrm{e}(\vec{n}\cdot\vec{y})}{\alpha(\tau m'+m)+\tau n'+n} \Big| \leq D\Big(|\tau m'+m|^{\lambda+1} + 1 \Big). \tag{5.3}$$

Proof. First, we verify (5.3) when $-\alpha m' \notin [N_1', N_2']$ and $-\alpha m \notin [N_1, N_2]$. Set $z = \alpha(\tau m' + m)$ and

$$a'_{n'} = \frac{\mathrm{e}(n'y')}{\mathrm{e}(y') - 1}, \quad a_n = \frac{\mathrm{e}(ny)}{\mathrm{e}(y) - 1}, \quad f_{n',n} = z + \tau n' + n.$$

Since $\mathrm{e}(n'y') = a'_{n'+1} - a'_{n'}$ and $\mathrm{e}(ny) = a_{n+1} - a_n$, we find that

$$\begin{aligned}(5.4) \quad \sum_{n'=N_1'}^{N_2'} \sum_{n=N_1}^{N_2} \frac{\mathrm{e}(\vec{n}\cdot\vec{y})}{z + \tau n' + n} &= \sum_{n'=N_1'+1}^{N_2'+1} \sum_{n=N_1+1}^{N_2+1} \frac{a'_{n'} a_n}{f_{n'-1,n-1}} \\ &- \sum_{n'=N_1'+1}^{N_2'+1} \sum_{n=N_1}^{N_2} \frac{a'_{n'} a_n}{f_{n'-1,n}} - \sum_{n'=N_1'}^{N_2'} \sum_{n=N_1+1}^{N_2+1} \frac{a'_{n'} a_n}{f_{n',n-1}} + \sum_{n'=N_1'}^{N_2'} \sum_{n=N_1}^{N_2} \frac{a'_{n'} a_n}{f_{n',n}}.\end{aligned}$$

If $\xi', \xi, \xi_1', \xi_1, \xi_2'$ and ξ_2 are real numbers with $\xi_1' \le \xi_2', \xi_1 \le \xi_2, -\alpha m' \notin [\xi_1', \xi_2']$ and $-\alpha m \notin [\xi_1, \xi_2]$, then we have

(5.5)

$$\frac{1}{f_{\xi_1',\xi}} - \frac{1}{f_{\xi_2',\xi}} = \tau \int_{\xi_1'}^{\xi_2'} \frac{d\mu'}{(z + \tau\mu' + \xi)^2}, \quad \frac{1}{f_{\xi',\xi_1}} - \frac{1}{f_{\xi',\xi_2}} = \int_{\xi_1}^{\xi_2} \frac{d\mu}{(z + \tau\xi' + \mu)^2},$$

$$\frac{1}{f_{\xi_1',\xi_1}} - \frac{1}{f_{\xi_1',\xi_2}} - \frac{1}{f_{\xi_2',\xi_1}} + \frac{1}{f_{\xi_2',\xi_2}} = 2\tau \int_{\xi_1'}^{\xi_2'} \int_{\xi_1}^{\xi_2} \frac{d\mu' d\mu}{(z + \tau\mu' + \mu)^3}.$$

Let $A = A(\vec{y}, \tau)$ denote $(1 + 2|\tau|)/(|\mathrm{e}(y') - 1||\mathrm{e}(y) - 1|)$, and I be the left hand side of (5.3). It follows from (5.4) and (5.5) that

$$\begin{aligned}(5.6) \quad I \le A\Big(&\int_{N_1'}^{N_2'} \int_{N_1}^{N_2} \frac{d\mu' d\mu}{|z + \tau\mu' + \mu|^3} + \int_{N_1'}^{N_2'} \frac{d\mu'}{|z + \tau\mu' + N_1|^2} \\ &+ \int_{N_1'}^{N_2'} \frac{d\mu'}{|z + \tau\mu' + N_2|^2} + \int_{N_1}^{N_2} \frac{d\mu}{|z + \tau N_1' + \mu|^2} + \int_{N_1}^{N_2} \frac{d\mu}{|z + \tau N_2' + \mu|^2} \\ &+ \frac{1}{|f_{N_1',N_1}|} + \frac{1}{|f_{N_1',N_2}|} + \frac{1}{|f_{N_2',N_1}|} + \frac{1}{|f_{N_2',N_2}|}\Big).\end{aligned}$$

Put $t' = \alpha m'$ and $t = \alpha m$. Since

$$\int_{N_1'}^{N_2'} \frac{d\mu'}{|z + \tau\mu' + N|^2} = \frac{1}{|\tau|^2 |\mathrm{Im}\ \frac{1}{\tau}||t + N|} \left[\arctan \frac{t' + \mu' + \mathrm{Re}\ (\frac{1}{\tau})(t + N)}{|\mathrm{Im}\ \frac{1}{\tau}||t + N|}\right]_{\mu'=N_1'}^{\mu'=N_2'},$$

$$\int_{N_1}^{N_2} \frac{d\mu}{|z + \tau N' + \mu|^2} = \frac{1}{|\mathrm{Im}\ \tau||t' + N'|} \left[\arctan \frac{t + \mu + \mathrm{Re}\ (\tau)(t' + N')}{|\mathrm{Im}\ \tau||t' + N'|}\right]_{\mu=N_1}^{\mu=N_2},$$

we can find from (3.2) and (5.6) that there is a positive real number $B = B(\vec{y}, \tau)$ with

$$\begin{aligned}I \le B\Big(&\big(\frac{1}{|\alpha m' + N_1'|^{1/2}} + \frac{1}{|\alpha m' + N_2'|^{1/2}}\big)\big(\frac{1}{|\alpha m + N_1|^{1/2}} + \frac{1}{|\alpha m + N_2|^{1/2}}\big) \\ &+ \frac{1}{|\alpha m' + N_1'|} + \frac{1}{|\alpha m' + N_2'|} + \frac{1}{|\alpha m + N_1|} + \frac{1}{|\alpha m + N_2|}\Big).\end{aligned}$$

On the other hand, by the theorem of Thue-Siegel-Roth in the diophantine approximation theory, there is a positive real number $C = C(\alpha, \lambda)$ such that

$$\Big|\alpha - \frac{k}{l}\Big| > \frac{1}{Cl^{\lambda+2}} \quad (k, l \in \mathbb{Z},\ l > 0).$$

It can be assumed that C is more than one. We deduce from this

$$\frac{1}{|\alpha l + k|} < C(|l|^{\lambda+1} + 1) \le C(|l|^{(\lambda+1)/2} + 1)^2 \quad (k, l \in \mathbb{Z},\ (k, l) \neq (0, 0)).$$

Thus we have

$$I \le 8BC\Big(|m'|^{(\lambda+1)/2}|m|^{(\lambda+1)/2} + |m'|^{\lambda+1} + |m|^{\lambda+1} + 1\Big)$$

which together with (3.2) implies (5.3) with $-\alpha m' \notin [N_1', N_2']$ and $-\alpha m \notin [N_1, N_2]$. It is remarked that if $m' = N_1' = N_2' = 0$ or $m = N_1 = N_2 = 0$, one can also prove (5.3) in a similar way.

In order to complete the proof, we consider the case when $-\alpha m' \in [N_1', N_2']$ and $-\alpha m \notin [N_1, N_2]$. Let $[x]$ denote the integer part of a real number x, i.e., $x = [x] + \{x\}$. If $m' \neq 0$, then $-\alpha m' \notin [N_1', [-\alpha m']]$ and $-\alpha m' \notin [[-\alpha m'] + 1, N_2']$, and if $m' = 0$, then $-\alpha m' \notin [N_1', -1], -\alpha m' \notin [1, N_2']$ and $-\alpha m' \in [0, 0]$. Thus this case is reduced to the case studied above, and verified. The other cases are also proved the same way. $\square$

We are in a position to prove Proposition 5.1 now.

Proof of Proposition 5.1. Since Re $s > 3$, there is positive real numbers λ, s_0 with Re $s - \lambda - 1 > s_0 > 2$. Let ϵ be an arbitrary positive real number. Since the series $\displaystyle\sum_{m',m}{}' \frac{1}{|\tau m' + m|^{s_0}}$ and $\displaystyle\sum_{n',n}{}'_e \frac{\mathrm{e}(\vec{n} \cdot \vec{y})}{\xi + \tau n' + n}$ converge, there are positive integers L and M such that if $|m_0'|, |m_0| \le M, (m_0', m_0) \neq (0, 0)$ and $N \ge L$, then

$$\sum_{\substack{m' \ge M \\ or\ m \ge M}} \frac{1}{|\tau m' + m|^{s_0}} < \epsilon, \quad \Big|\sum_{\substack{|n'| \ge N \\ or\ |n| \ge N}} \frac{\mathrm{e}(\vec{n} \cdot \vec{y})}{\alpha(\tau m_0' + m_0) + \tau n' + n}\Big| < \epsilon.$$

A calculation shows that

$$\Big|\Big(\sum_{m',m}{}'_e \sum_{n',n}{}'_e - \sum_{\substack{|n'|<N \\ and\ |n|<N}}{}' \sum_{m',m}{}'_e\Big) \frac{\mathrm{e}(\vec{m} \cdot \vec{x})}{(\tau m' + m)^s} \frac{\mathrm{e}(\vec{n} \cdot \vec{y})}{\alpha(\tau m' + m) + \tau n' + n}\Big|$$

$$= \Big|\Big(\sum_{\substack{|m'| \ge M \\ or\ |m| \ge M}} + \sum_{\substack{|m'|<M \\ and\ |m|<M}}{}'\Big) \sum_{\substack{|n'| \ge N \\ or\ |n| \ge N}} \frac{\mathrm{e}(\vec{m} \cdot \vec{x})}{(\tau m' + m)^s} \frac{\mathrm{e}(\vec{n} \cdot \vec{y})}{\alpha(\tau m' + m) + \tau n' + n}\Big|.$$

Let I be the left hand side of the above equation. By (5.3), we get

$$I \le D \sum_{\substack{|m'| \ge M \\ or\ |m| \ge M}} \frac{1}{|\tau m' + m|^s}(|\tau m' + m|^{\lambda+1} + 1) + \epsilon \sum_{\substack{|m'|<M \\ and\ |m|<M}}{}' \frac{1}{|\tau m' + m|^s}$$

$$\le \epsilon\Big(2D + \sum_{m',m}{}' \frac{1}{|\tau m' + m|^s}\Big),$$

which completes the proof. $\square$

6. Proof of Lemma 4.3 and transformation formulae of elliptic Dedekind-Rademacher sums

In order to prove Lemma 4.3, we give the following transformation formula for the elliptic Dedekind-Rademacher sums $S^{\tau}_{1,l}(\Xi; r)$ which reproduces a part of Halbritter's result [Ha, Theorem 2].

THEOREM 6.1. *Let $r = n(r)/d(r)$ be a rational number, and l a positive integer. If $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $SL_2(\mathbb{Z})$ with $j(V; r) \neq 0$ and Ξ is in $SM_2(V) \bigcap SM_2(r)$, then we have*

$$S^{\tau}_{1,l}(\Xi; r) - j(V; r)^{l-1} S^{\tau}_{1,l}(V\Xi; Vr) = R^{\tau}_{l}(\Xi; r; V) + \frac{lcB_{l+1}(n(r)\vec{y} - d(r)\vec{x}; \tau)}{(l+1)d(r)^{l+1} j(V; r)}, \tag{6.1}$$

For any two by two matrix $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, put $\tilde{j}(V; r) = \dfrac{c}{j(V; r)}$. First we prove Lemma 4.3 by using (6.1).

Proof of Lemma 4.3. Suppose that $\Xi \in \mathrm{SM}_2(V) \bigcap \mathrm{SM}_2(V_1) \bigcap \mathrm{SM}_2(r)$. It is easily seen that $\Xi \in \mathrm{SM}_2(V) \bigcap \mathrm{SM}_2(V_1)$ yields $V_1\Xi \in \mathrm{SM}_2(V_2)$. Since the determinant of V is one, we get $\gcd(an(r) + bd(r), cn(r) + dd(r)) = 1$ and

$$(n(Vr), d(Vr)) = \begin{cases} (an(r) + bd(r), cn(r) + dd(r)) & (j(V; r) > 0), \\ -(an(r) + bd(r), cn(r) + dd(r)) & (j(V; r) < 0). \end{cases} \tag{6.2}$$

If ξ and η are variables, a calculation shows that

$$(an(r) + bd(r))(c\xi + d\eta) - (cn(r) + dd(r))(a\xi + b\eta) = n(r)\eta - d(r)\xi, \tag{6.3}$$

which together with $\Xi \in \mathrm{SM}_2(r) \bigcap \mathrm{SM}_2(V_1)$ gives $V_1\Xi \in \mathrm{SM}_2(V_1 r)$. Thus we find from (6.1) that

$$\begin{aligned} &R^{\tau}_{l}(\Xi; r; V) - R^{\tau}_{l}(\Xi; r; V_1) - j(V_1, r)^{l-1} R^{\tau}_{l}(V_1\Xi; V_1 r; V_2) \\ &\quad = -\frac{lB_{l+1}(n(r)\vec{y} - d(r)\vec{x}; \tau)}{(l+1)d(r)^{l+1}} \Big(\tilde{j}(V; r) - \tilde{j}(V_1; r) - \frac{1}{j(V_1; r)^2}\tilde{j}(V_2; V_1 r)\Big) = 0. \end{aligned}$$

Since the set $\mathrm{SM}_2(r)$ is dense in $\mathrm{M}_2(\mathbb{R})$, (4.4) holds if $z = r$ is a rational number and $\Xi \in \mathrm{SM}_2(V) \bigcap \mathrm{SM}_2(V_1)$. The function $R^{\tau}_{l}(\Xi; z; V)$ with respect to z is meromorphic, thus we can remove the condition that z is rational. □

To verify (6.1), we introduce a further transformation formula. Let $r = n(r)/d(r)$ be a rational number, l a positive integer, $\Xi = \left(\begin{smallmatrix} \vec{x} \\ \vec{y} \end{smallmatrix}\right)$ be a matrix in $\mathrm{M}_2(\mathbb{R})$, and X, Y be complex variables. If Ξ is in $\mathrm{SM}_2(r)$, then a function appeared in the transformation formula is defined by

$$\begin{aligned} S^{\tau}_{l}(\Xi; {\textstyle {X \atop Y}}; r) := \frac{1}{d(r)} \sum_{j', j(d(r))} &\underline{F}\Big(\frac{\vec{j} + \vec{y}}{d(r)}; n(r)Y - d(r)X; \tau\Big) \\ &\times \underline{F}^{(l-1)}\Big(n(r)\frac{\vec{j} + \vec{y}}{d(r)} - \vec{x}; -Y; \tau\Big), \end{aligned} \tag{6.4}$$

where $\underline{F}^{(m)}(\vec{x}; X; \tau)$ denotes $\dfrac{1}{(2\pi i)^m}\Big(\dfrac{\partial}{\partial X}\Big)^m \underline{F}(\vec{x}; X; \tau)$. This corresponds to the elliptic Dedekind-Rademacher sum $S^{\tau}_{1,l}(\Xi; r)$ in (6.1). If $\vec{x} = (x', x)$ and $\vec{y} = (y', y)$, we see from (2.4) that

$$
\begin{aligned}
S_l^\tau(\Xi; \tfrac{X+1}{Y}; r) &= \mathrm{e}(-y) S_l^\tau(\Xi; \tfrac{X}{Y}; r), \quad S_l^\tau(\Xi; \tfrac{X}{Y+1}; r) = \mathrm{e}(x) S_l^\tau(\Xi; \tfrac{X}{Y}; r), \\
S_l^\tau(\Xi; \tfrac{X+\tau}{Y}; r) &= \mathrm{e}(-y') S_l^\tau(\Xi; \tfrac{X}{Y}; r), \quad S_l^\tau(\Xi; \tfrac{X}{Y+\tau}; r) = \mathrm{e}(x') S_l^\tau(\Xi; \tfrac{X}{Y}; r).
\end{aligned} \tag{6.5}
$$

We also introduce a function which corresponds to $R_l^\tau(\Xi; r; V)$ in (6.1). If $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is in $\mathrm{SL}_2(\mathbb{Z})$ and Ξ is in $\mathrm{SM}_2(V)$, then the function is defined by

(6.6)

$$
R_l^\tau(\Xi; \tfrac{X}{Y}; r; V) := \begin{cases} (-1)^l \displaystyle\sum_{k=0}^{l-1} \binom{l-1}{k} \big(-j(V,r)\big)^k S_{k+1,l-k}^\tau \begin{pmatrix} -\vec{x} & X & d \\ \vec{y} & Y & c \end{pmatrix} & (c \neq 0), \\ 0 & (c = 0), \end{cases}
$$

where $S_{k+1,l-k}^\tau\left(\begin{smallmatrix} -\vec{x} \\ \vec{y} \end{smallmatrix}; \begin{smallmatrix} X \\ Y \end{smallmatrix}; \begin{smallmatrix} d \\ c \end{smallmatrix}\right)$ denotes

$$
\frac{1}{c} \sum_{j',j(|c|)} \underline{F}^{(k)}\Big(\frac{\vec{j}+\vec{y}}{c}; -cX - dY; \tau\Big) \underline{F}^{(l-1-k)}\Big(d\frac{\vec{j}+\vec{y}}{c} + \vec{x}; Y; \tau\Big). \tag{6.7}
$$

We shall need the following two lemmas in a proof of the transformation formula for the functions $S_l^\tau(\Xi; \tfrac{X}{Y}; r)$ on the parameter r under $\mathrm{SL}_2(\mathbb{Z})$.

LEMMA 6.2. *Let $r = n(r)/d(r)$ be a rational number, and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ a matrix in $SL_2(\mathbb{Z})$. If $m', m, n', n, M', M, N', N$ are integers with $\left(\begin{smallmatrix} m' & n' \\ m & n \end{smallmatrix}\right) = \left(\begin{smallmatrix} M' & N' \\ M & N \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then we have*

$$
r(\tau m' + m) + \tau n' + n = (ar+b)(\tau M' + M) + (cr+d)(\tau N' + N). \tag{6.8}
$$

In particular,

$$
\frac{1}{d(r)}(\tau\mathbb{Z} + \mathbb{Z}) = r(\tau\mathbb{Z} + \mathbb{Z}) + \tau\mathbb{Z} + \mathbb{Z} = (ar+b)(\tau\mathbb{Z} + \mathbb{Z}) + (cr+d)(\tau\mathbb{Z} + \mathbb{Z}). \tag{6.9}
$$

Proof. A direct calculation verifies (6.8). The first equal sign in (6.9) follows from the fact that $\gcd(n(r), d(r)) = 1$, and the second from (6.8) and the fact that the determinant of the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is one. □

LEMMA 6.3. *Let c be a nonzero integer, l a nonnegative integer, and $\vec{i} = (i', i)$ a vector in $\mathbb{Z}^2$. Then we have*

$$
\underline{F}^{(l)}\Big(\vec{x}; X + \frac{\tau i' + i}{c}; \tau\Big) = c^{l-1} \sum_{j',j(|c|)} \mathrm{e}\Big(\vec{i} \cdot \frac{\vec{j}+\vec{x}}{c}\Big) \underline{F}^{(l)}\Big(\frac{\vec{j}+\vec{x}}{c}; cX; \tau\Big), \tag{6.10}
$$

$$
B_m(\vec{x}; \tau) = c^{m-2} \sum_{j',j(|c|)} B_m\Big(\frac{\vec{j}+\vec{x}}{c}; \tau\Big). \tag{6.11}
$$

Proof. Let $\vec{j}$ be the vector $(j', j) \in \mathbb{Z}^2$ and $X \notin \frac{\tau}{c}\mathbb{Z} + \frac{1}{c}\mathbb{Z}$. Set

$$
f(z) = \underline{F}(\vec{x}; -z + X; \tau) \underline{F}\Big(\frac{\vec{j}+\vec{x}}{c}; cz; \tau\Big).
$$

We find from (2.4) that the function $f(z)$ is a doubly periodic function with respect to 1 and τ, and from the pole situation of the function $\underline{F}(\vec{x}; X; \tau)$ that $f(z)$ has the only simple poles on the lattices $\dfrac{\tau}{c}\mathbb{Z}+\dfrac{1}{c}\mathbb{Z}$ and $X+\tau\mathbb{Z}+\mathbb{Z}$. Since the sum of the residues of $f(z)$ at its poles in any period parallelogram equals zero, we have

$$\frac{1}{c}\sum_{k',k(|c|)}\underline{F}(\vec{x};X+\frac{\tau k'+k}{c};\tau)\mathrm{e}(-k'\frac{j'+x'}{c}-k\frac{j+x}{c})=\underline{F}(\frac{\vec{j}+\vec{x}}{c};cX;\tau).$$

If we add each side of the above equation for $j',j=0,1,\ldots,|c|-1$, then

$$c\underline{F}(\vec{x};X;\tau)=\sum_{j',j(|c|)}\underline{F}(\frac{\vec{j}+\vec{x}}{c};cX;\tau).$$

Replacing X by $X+\dfrac{\tau i'+i}{c}$ and (2.4) imply (6.10). (6.11) follows from (6.10) with $i'=i=0$ and (2.3). $\square$

The transformation formula is the following. The method of the proof is to apply Liouville's theorem.

PROPOSITION 6.4. *Let $r=n(r)/d(r)$ be a rational number, l a positive integer, and X,Y complex variables. If V is in $SL_2(\mathbb{Z})$ with $j(V;r)>0$ and Ξ is in $SM_2(V)\bigcap SM_2(r)$, then we have*

$$S_l^\tau(\Xi;{\textstyle\binom{X}{Y}};r)-j(V;r)^{l-1}S_l^\tau(V\Xi;V{\textstyle\binom{X}{Y}};Vr)=R_l^\tau(\Xi;{\textstyle\binom{X}{Y}};r;V). \tag{6.12}$$

Proof. Put $\Xi=\left(\begin{smallmatrix}\vec{x}\\ \vec{y}\end{smallmatrix}\right)=\left(\begin{smallmatrix}x' & x\\ y' & y\end{smallmatrix}\right)$ and $V=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$. Let Ξ and Y be fixed, and let $L(X)$ and $R(X)$ be the left hand and the right hand side of (6.12) respectively. The function $LR(X)$ is defined by $L(X)-R(X)$. We find from (2.4) and (6.5) that

$$LR(X+1)=\mathrm{e}(-y)LR(X),\quad LR(X+\tau)=\mathrm{e}(-y')LR(X). \tag{6.13}$$

Suppose that $Y\notin\frac{1}{d(Vr)}(\tau\mathbb{Z}+\mathbb{Z})$. In order to obtain $LR(X)=0$, we will show that the function $LR(X)$ is an entire function. Since $j(V;r)$ is positive, we see from (6.2) that $n(Vr)=an(r)+bd(r)$ and $d(Vr)=cn(r)+dd(r)$. By (6.3), we obtain

$$\begin{aligned}S_l^\tau(V\Xi;V{\textstyle\binom{X}{Y}};Vr)=\frac{1}{d(Vr)}\sum_{j',j(d(Vr))}&\underline{F}\Big(\frac{\vec{j}+c\vec{x}+d\vec{y}}{d(Vr)};n(r)Y-d(r)X;\tau\Big)\\ &\times\underline{F}^{(l-1)}\Big(\frac{n(Vr)\vec{j}+n(r)\vec{y}-d(r)\vec{x}}{d(Vr)};-cX-dY;\tau\Big).\end{aligned} \tag{6.14}$$

It is seen from (6.4), (6.6) and (6.14) that every possible pole of the function $LR(X)$ is on the lattices

$$\begin{cases}rY+\dfrac{1}{d(r)}(\tau\mathbb{Z}+\mathbb{Z}), & -\dfrac{d}{c}Y+\dfrac{1}{c}(\tau\mathbb{Z}+\mathbb{Z})\quad (c\neq 0),\\ rY+\dfrac{1}{d(r)}(\tau\mathbb{Z}+\mathbb{Z}) & \qquad\qquad\qquad\qquad\quad (c=0).\end{cases}$$

Note that $\big(rY+\frac{1}{d(r)}(\tau\mathbb{Z}+\mathbb{Z})\big)\bigcap\big(-\frac{d}{c}Y+\frac{1}{c}(\tau\mathbb{Z}+\mathbb{Z})\big)=\phi$ since $Y\notin\frac{1}{d(Vr)}(\tau\mathbb{Z}+\mathbb{Z})$.

Let us examine behaviors of the functions $L(X)$ and $R(X)$ at the lattices. Let z be a complex number on $\frac{1}{d(r)}(\tau\mathbb{Z}+\mathbb{Z})$. By (6.8) and (6.9), there are integers m',m, n',n, M',M, N',N such that $\left(\begin{smallmatrix}m' & n'\\ m & n\end{smallmatrix}\right)=\left(\begin{smallmatrix}M' & N'\\ M & N\end{smallmatrix}\right)\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ and

$$z=r(\tau m'+m)+\tau n'+n=(ar+b)(\tau M'+M)+(cr+d)(\tau N'+N).$$

Put $w = -\vec{m}\cdot\vec{x} - \vec{n}\cdot\vec{y}$ and $W = -\vec{M}\cdot(a\vec{x}+b\vec{y}) - \vec{N}\cdot(c\vec{x}+d\vec{y})$, where $\vec{m},\vec{n},\vec{M}$ and $\vec{N}$ denote the vectors $(m',m),(n',n),(M',M)$ and (N',N) respectively. Set $\vec{\xi} = d(r)\vec{x} - n(r)\vec{y}$. Then we see from (2.3) and (6.4) that

$$S_l^\tau(\Xi; {}^{X+rY+z}_{\;\;\;Y}; r) = \frac{(-1)^{l-1}\mathrm{e}(w)}{d(r)^2X}\sum_{j',j(d(r))}\mathrm{e}(\vec{m}\cdot\frac{-n(r)\vec{j}+\vec{\xi}}{d(r)})$$
$$\times\,\underline{F}^{(l-1)}\Big(\frac{-n(r)\vec{j}+\vec{\xi}}{d(r)};Y;\tau\Big)+O(1),$$

where $O(1)$ means a holomorphic function at $X=0$. Since it follows from (6.3) that

$$n(Vr)\vec{M}\cdot(\frac{\vec{j}+c\vec{x}+d\vec{y}}{d(Vr)}) = \vec{M}\cdot(a\vec{x}+b\vec{y}) + \vec{M}\cdot(\frac{n(Vr)\vec{j}-\vec{\xi}}{d(Vr)}),$$

we also see from (2.3) and (6.14) that

$$S_l^\tau(V\Xi; V\big({}^{X+rY+z}_{\;\;\;Y}\big); Vr) = \frac{(-1)^{l-1}\mathrm{e}(W)}{d(r)d(Vr)X}\sum_{j',j(d(Vr))}\mathrm{e}(\vec{M}\cdot\frac{-n(Vr)\vec{j}+\vec{\xi}}{d(Vr)})$$
$$\times\,\underline{F}^{(l-1)}\Big(\frac{-n(Vr)\vec{j}+\vec{\xi}}{d(Vr)};\frac{d(Vr)}{d(r)}Y+cz;\tau\Big)+O(1).$$

Because some calculations show that $w=W$ and

$$\frac{\tau m'+m}{d(r)} = \frac{cz}{d(Vr)} + \frac{\tau M'+M}{d(Vr)},$$

we find from (6.10) that $L(X+rY+z) = O(1)$, i.e., the function $LR(X)$ is holomorphic at $X \in rY + \frac{1}{d(r)}(\tau\mathbb{Z}+\mathbb{Z})$.

Suppose that c is not zero. Let $z = \frac{1}{c}(\tau m'+m)$ be in $\frac{1}{c}(\tau\mathbb{Z}+\mathbb{Z})$. Then one has

$$(6.15)\quad S_l^\tau(V\Xi; V\big({}^{X-(d/c)Y+z}_{\;\;\;\;\;Y}\big); Vr) = \frac{1}{d(Vr)}\sum_{j',j(d(Vr))}\mathrm{e}(-\vec{m}\cdot\frac{n(Vr)\vec{j}-\vec{\xi}}{d(Vr)})$$
$$\times\,\underline{F}\Big(\frac{\vec{j}+c\vec{x}+d\vec{y}}{d(Vr)};-d(r)(X+z)+\frac{d(Vr)Y}{c};\tau\Big)\underline{F}^{(l-1)}\Big(\frac{n(Vr)\vec{j}-\vec{\xi}}{d(Vr)};-cX;\tau\Big).$$

Since

$$(6.16)\qquad \underline{F}^{(n)}(\vec{x};X;\tau) = \frac{(-1)^n n!}{(2\pi i)^n X^{n+1}} + \sum_{m=0}^{\infty}\frac{B_{m+n+1}(\vec{x};\tau)}{(m+n+1)m!}(2\pi i)^{m+1}X^m,$$

the right hand side of (6.15) equals

$$\frac{(l-1)!}{d(Vr)(2\pi i)^{l-1}c^l}\mathrm{e}(\frac{\vec{m}\cdot\vec{\xi}+n(Vr)\vec{m}\cdot(c\vec{x}+d\vec{y})}{d(Vr)})\sum_{k=0}^{l-1}\frac{(2\pi i d(r))^k}{k!X^{l-k}}\sum_{j',j(d(Vr))}$$
$$\mathrm{e}(-n(Vr)\frac{\vec{m}\cdot(\vec{j}+c\vec{x}+d\vec{y})}{d(Vr)})\underline{F}^{(k)}\Big(-\frac{\vec{j}+c\vec{x}+d\vec{y}}{d(Vr)};-\frac{d(Vr)Y}{c}+d(r)z;\tau\Big)+O(1).$$

By (6.10) and (6.3), this is equal to

$$\frac{(l-1)!}{(2\pi i)^{l-1}c^l}\mathrm{e}(\vec{m}\cdot(a\vec{x}+b\vec{y}))\sum_{k=0}^{l-1}\frac{(2\pi i d(r))^k}{k!d(Vr)^kX^{l-k}}\underline{F}^{(k)}\Big(-c\vec{x}-d\vec{y};-\frac{Y}{c}+az;\tau\Big)+O(1).$$

So we conclude that

$$L(X-\frac{d}{c}Y+z)=-\frac{(l-1)!}{(2\pi i)^{l-1}c^l}\mathrm{e}(\vec{m}\cdot(a\vec{x}+b\vec{y}))\times\sum_{k=0}^{l-1}\frac{(2\pi i)^k j(V;r)^{l-1-k}}{k!X^{l-k}}\underline{F}^{(k)}\Big(-c\vec{x}-d\vec{y};-\frac{Y}{c}+az;\tau\Big)+O(1). \tag{6.17}$$

On the other hand, we have

$$R(X-\frac{d}{c}Y+z)=-\frac{1}{c}\sum_{k=0}^{l-1}\binom{l-1}{k}j(V,r)^k \times\sum_{j',j(|c|)}\frac{k!}{(2\pi i)^k(cX)^{k+1}}\mathrm{e}(-\vec{m}\cdot\frac{\vec{j}+\vec{y}}{c})\underline{F}^{(l-1-k)}(-\frac{d\vec{j}+c\vec{x}+d\vec{y}}{c};-Y;\tau)+O(1).$$

Since $\vec{m}\cdot\vec{y}=a\vec{m}\cdot(c\vec{x}+d\vec{y})-c\vec{m}\cdot(a\vec{x}+b\vec{y})$, we get

$$R(X-\frac{d}{c}Y+z)=-\frac{\mathrm{e}(\vec{m}\cdot(a\vec{x}+b\vec{y}))}{c^l}\sum_{k=0}^{l-1}\binom{l-1}{k}j(V,r)^k \times\frac{k!}{(2\pi i)^k X^{k+1}}\underline{F}^{(l-1-k)}(-c\vec{x}-d\vec{y};-\frac{Y}{c}+az;\tau)+O(1).$$

This together with (6.17) implies that the function $LR(X)$ is holomorphic at $X\in -\frac{d}{c}Y+\frac{1}{c}(\tau\mathbb{Z}+\mathbb{Z})$, and it is an entire function.

We see from (6.13) that $LR(X)$ is a bounded function, and from Liouville's theorem that $LR(X)$ is a constant function. Since $\vec{y}\notin\mathbb{Z}^2$ by $\Xi\in\mathrm{SM}_2(r)$, we obtain $LR(X)=0$ by (6.13), which completes the proof because the set $\{Y|Y\notin\frac{1}{d(Vr)}(\tau\mathbb{Z}+\mathbb{Z})\}$ is dense in $\mathbb{C}$. $\square$

In order to obtain the transformation formula for elliptic Dedekind-Rademacher Sums $S^{\tau}_{1,l}(\Xi;r)$, we prepare a lemma which states that the functions $S^{\tau}_l(\Xi;{}^X_Y;r)$ and $R^{\tau}_l(\Xi;{}^X_Y;r;V)$ yield $S^{\tau}_{1,l}(\Xi;r)$ and $R^{\tau}_l(\Xi;r;V)$ respectively.

LEMMA 6.5. *For any meromorphic function $f(X,Y)$, we denote by $C_Y(f(X,Y))$ and $C_{Y,X}(f(X,Y))$ the coefficients of $Y^0(=1)$ in $f(X,Y)$ and X^0 in $C_Y(f(X,Y))$ respectively. If $V=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ is in $SL_2(\mathbb{Z})$ with $c\neq0$ and $j(V;r)>0$, the following three equations hold.*

$$C_{Y,X}(S^{\tau}_l(\Xi;{}^X_Y;r))=\frac{(2\pi i)^2}{l}\Big[-\frac{r^l}{l+1}B_{l+1}(\vec{y};\tau)+S^{\tau}_{1,l}(\Xi;r)\Big], \tag{6.18}$$

$$C_{Y,X}(S^{\tau}_l(V\Xi;V\left({}^X_Y\right);Vr))=\frac{(2\pi i)^2}{l}\Big[\frac{(-1)^{l-1}}{(l+1)c^l j(V;r)^l}B_{l+1}(c\vec{x}+d\vec{y};\tau) +\frac{lc}{(l+1)d(r)d(Vr)^l}B_{l+1}(n(r)\vec{y}-d(r)\vec{x};\tau)+S^{\tau}_{1,l}(V\Xi;Vr)\Big], \tag{6.19}$$

$$C_{Y,X}(R_l^\tau(\Xi;\, {}^X_Y\,;r;V)) = \frac{(2\pi i)^2}{l}\Big[\frac{-r^l}{l+1}B_{l+1}(\vec{y};\tau) + \frac{(-1)^l}{(l+1)c^l j(V;r)}B_{l+1}(c\vec{x}+d\vec{y};\tau) + R_l^\tau(\Xi;r;V)\Big]. \tag{6.20}$$

Proof. We find from (6.16) and Lemma 6.3 that

$$C_Y(S_l^\tau(\Xi;\, {}^X_Y\,;r)) = \frac{2\pi i}{l}\Big[\frac{1}{d(r)}\sum_{j',j(d(r))}\underline{F}\Big(\frac{\vec{j}+\vec{y}}{d(r)};-d(r)X;\tau\Big) \times B_l\Big(n(r)\frac{\vec{j}+\vec{y}}{d(r)}-\vec{x};;\tau\Big) - r^l\underline{F}^{(l)}(\vec{y};-X;\tau)\Big].$$

By using (6.16) again, one gets (6.18). We can also obtain (6.19) and (6.20) in a similar way, so we omit the proof. □

We are in a position to prove Theorem 6.1 now.

Proof of Theorem 6.1. The case that c is zero is trivial, so suppose that c is not zero. If $j(V;r)$ is positive, (6.1) follows from Proposition 6.4 and Lemma 6.5. It is seen that $j(-V;r) = -j(V;r)$, $S_{1,l}^\tau(-V\Xi;(-V)r) = (-1)^{l-1}S_{1,l}^\tau(V\Xi;Vr)$, and $R_l^\tau(\Xi;r;-V) = R_l^\tau(\Xi;r;V)$, thus (6.1) with $j(V;r)<0$ is reduced to the case that $j(V;r)>0$. □

As a corollary of Theorem 6.1, we shall reproduce a part of Halbritter's result, i.e., the transformation formula for the sums $S_{1,n}(\,{}^x_y\,;r)$. In analogy to the function $R_l^\tau(\Xi;z;V)$, we define a sum of generalized Dedekind-Rademacher sums as follows:

$$R_l\begin{pmatrix}x\\y\end{pmatrix};z;V\Big) := \begin{cases}\dfrac{(-1)^l}{l+1}\displaystyle\sum_{k=-1}^{l}\binom{l+1}{k+1}\big(-j(V,z)\big)^k S_{k+1,l-k}\begin{pmatrix}-x\\y\end{pmatrix};\dfrac{d}{c}\Big) & (c>0),\\ 0 & (c=0).\end{cases} \tag{6.21}$$

When c is negative, $R_l(\,{}^x_y\,;z;V)$ also denotes $R_l(\,{}^x_y\,;z;-V)$. By (2.12), we have

$$\operatorname{Re}\lim_{\tau\to i\infty}R_l^\tau(\Xi;z;V) = \begin{cases}R_l(\,{}^x_y\,;z;V)+\dfrac{1}{4}\mathfrak{c}\big(\,{}^{-x'}_{y'}\,;d/c\big) & (l=1,x,y\in\mathbb{Z}),\\ R_l(\,{}^x_y\,;z;V) & (\textit{otherwise}),\end{cases} \tag{6.22}$$

where $\mathfrak{c}\big(\,{}^{-x'}_{y'}\,;(\pm1)/0\big)$ means zero.

THEOREM 6.6 (cf. [Ha] Theorem 2)**.** *Let $r = n(r)/d(r)$ be a rational number, x,y real numbers, and V a matrix in $SL_2(\mathbb{Z})$ with $j(V;r)\neq 0$.*
(i) *If l is an integer more than one, then we have*

$$S_{1,l}(\,{}^x_y\,;r) - j(V;r)^{l-1}S_{1,l}(V(\,{}^x_y\,);Vr) = R_l(\,{}^x_y\,;r;V) + \frac{lc\tilde{B}_{l+1}(n(r)y-d(r)x)}{(l+1)d(r)^{l+1}j(V;r)}. \tag{6.23}$$

(ii) *Put* $p = n(r)$, $q = d(r)$ *and* $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *If* c *is not zero, then we have*

$$(6.24)\quad S_{1,1}(\tbinom{x}{y}; r) - S_{1,1}(V\tbinom{x}{y}; Vr) + S_{1,1}(\tbinom{-x}{y}; d/c) = \frac{1}{2}\Big(\frac{c}{q(cp+dq)}\tilde{B}_2(py-qx)$$

$$+ \frac{cp+dq}{cq}\tilde{B}_2(y) + \frac{q}{(cp+dq)c}\tilde{B}_2(cx+dy)\Big) + \begin{cases} -\dfrac{\text{sign } c(cp+dq)}{4} & (x, y \in \mathbb{Z}), \\ 0 & (otherwise). \end{cases}$$

Proof. Suppose that $x, y \in \mathbb{Z}$. By (2.12), (6.1) and (6.22), one has

$$S_{1,1}(\tbinom{x}{y}; r) - S_{1,1}(V\tbinom{x}{y}; Vr) + S_{1,1}(\tbinom{-x}{y}; d/c) = \frac{1}{2}\Big(\frac{c\tilde{B}_2(n(r)y - d(r)x)}{d(r)^2 j(V; r)}$$

$$+ \frac{j(V;r)}{c}\tilde{B}_2(y) + \frac{1}{cj(V;r)}\tilde{B}_2(cx+dy)\Big) + \frac{1}{4}\Big(\mathfrak{c}(\tbinom{x'}{y'}; r) - \mathfrak{c}(V\tbinom{x'}{y'}; Vr) + \mathfrak{c}(\tbinom{-x'}{y'}; d/c)\Big).$$

Since a calculation shows that

$$\lim_{y' \to i\infty}\lim_{x' \to i\infty}\Big(\mathfrak{c}(\tbinom{x'}{y'}; r) - \mathfrak{c}(V\tbinom{x'}{y'}; Vr) + \mathfrak{c}(\tbinom{-x'}{y'}; d/c)\Big) = -\text{sign } c(cp+dq),$$

we obtain (6.24). The other cases immediately follow from (2.12), (6.1) and (6.22). □

REMARK 6.7. If p and q are relatively prime integers with $q \neq 0$,

$$(6.25)\qquad S_{1,l}(\tbinom{x}{y}; p/q) = \text{sign } q \sum_{j(|q|)} \tilde{B}_1\Big(\frac{j+y}{q}\Big)\tilde{B}_l\Big(p\frac{j+y}{q} - x\Big).$$

Thus (6.23) and (6.24) imply a part of Halbritter's result [Ha, Theorem 2]. In particular, (6.24) with $x = y = 0$ yields (1.2), and (6.24) with $x = y = a = d = 0$ and $-b = c = 1$ gives the reciprocity formula for Apostol-Dedekind sums.

7. Proofs of Proposition 2.1 and 3.2

We prove Proposition 2.1 and 3.2. Throughout this section, let $\Xi = \left(\begin{smallmatrix} x' & x \\ y' & y \end{smallmatrix}\right)$ be in $\mathrm{M}_2(\mathbb{R})$, and $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{Z})$. We denote the vectors (x', x) and (y', y) by $\vec{x}$ and $\vec{y}$ respectively.

For a proof of Proposition 2.1, suppose that $r = n(r)/d(r)$ is a rational number and Ξ is in $\mathrm{SM}_2(r)$.

Proof of Proposition 2.1. Set $F(x, X; \tau) := \dfrac{\theta'(0;\tau)\theta(x+X;\tau)}{\theta(x;\tau)\theta(X;\tau)}$. This function has the modular property (see the theorem in [Za, Section 3]):

$$F(x, X; \tau) = \frac{1}{c\tau+d}\mathrm{e}(-\frac{cxX}{c\tau+d})F(\frac{x}{c\tau+d}, \frac{X}{c\tau+d}; V\tau).$$

We note that $\theta(x;\tau)$ is equal to $i\theta_\tau(2\pi i x)$ in [Za]. Since the function $\underline{F}(\vec{x}; X; \tau)$ equals $\mathrm{e}(xX)F(-x' + x\tau, X; \tau)$,

$$(7.1)\qquad \underline{F}(\vec{x}; X; \tau) = \frac{1}{j(V;\tau)}\underline{F}(\vec{x}\,{}^tV; \frac{X}{j(V;\tau)}; V\tau)$$

from which we deduce $B_m(\vec{x};\tau) = \dfrac{1}{j(V;\tau)^m} B_m(\vec{x}\ {}^tV; V\tau)$. Thus we have

$$S^{\tau}_{m,n}(\Xi;r) \\ = \frac{1}{d(r)j(V;\tau)^{m+n}} \sum_{j',j(d(r))} B_m\Big(\frac{\vec{j}+\vec{y}}{d(r)}\ {}^tV; V\tau\Big) B_n\Big(\big(n(r)\frac{\vec{j}+\vec{y}}{d(r)} - \vec{x}\big)\ {}^tV; V\tau\Big).$$

From the fact that the determinant of V is one, it follows that $\vec{j}\ {}^tV$ runs over all elements in $(\mathbb{Z}/d(r)\mathbb{Z})^2$ when $\vec{j}$ does too. This together with the above equation implies (2.11). (2.12) is a special case of [Ma, Eq. (25)]. □

In order to give a proof of Proposition 3.2, let α be an irrational real algebraic number, l an integer with $l \geq 3$, and $\vec{y} \notin \mathbb{Z}^2$.

Proof of Proposition 3.2. Let $I^{\tau} = \sum'_{m',m} \dfrac{\mathrm{e}(\vec{m}\cdot\vec{x})}{(\tau m'+m)^l} \underline{F}(-\vec{y}; \alpha(\tau m'+m);\tau)$, and $\vec{x},\vec{y}$ be fixed. For any vector $\vec{m} = (m',m) \in \mathbb{Z}^2$, let $\vec{n} = (n',n)$ be the vector with $\vec{m} = \vec{n}\ V$. We see from (7.1) that

$$I^{\tau} = \frac{1}{j(V;\tau)^{l+1}} \sum_{m',m}{}' \frac{\mathrm{e}(\vec{n}\cdot(\vec{x}\ {}^tV))}{(V\tau n'+n)^l} \underline{F}(-\vec{y}\ {}^tV; \alpha(V\tau n'+n); V\tau).$$

Since the determinant of V is one, (n',n) runs over all elements in $\mathbb{Z}^2$ except $(0,0)$ when (m',m) does too. Thus we obtain (3.5).

In order to prove (3.6), we introduce the Fourier expansion for the function $F(\xi,X;\tau)$ (see [We, p. 70], [Za, p. 456]). If $|\mathrm{Im}\ \xi|, |\mathrm{Im}\ X| < |\mathrm{Im}\ \tau|$, then

$$F(\xi,X;\tau) = \pi\big(\cot\pi\xi + \cot\pi X\big) - 2\pi i \sum_{i,j=1}^{\infty} (\mathrm{e}(i\xi+jX) - \mathrm{e}(-i\xi-jX))\mathrm{e}(ij\tau).$$

Let ξ',ξ,X' and X be a real number with $|X'| \leq 1/2$ and $|\xi| < 1$. We find from this that

$$\underline{F}(\xi',\xi;X'\tau+X;\tau) = \pi\mathrm{e}(\xi(X'\tau+X))\Big(\cot\pi(-\xi'+\xi\tau) + \cot\pi(X+X'\tau)\Big) \\ -2\pi i\mathrm{e}(\xi X)\sum_{i,j=1}^{\infty}\Big(\mathrm{e}(-i\xi'+jX)\mathrm{e}((i+X')(j+\xi)\tau) - \mathrm{e}(i\xi'-jX)\mathrm{e}((i-X')(j-\xi)\tau)\Big). \tag{7.2}$$

Let S be the last term in the right hand side of (7.2). If $c = \min\{1\pm\xi, 1/2\}$, then

$$|S| \leq 4\pi \sum_{i,j=1}^{\infty} \mathrm{e}(c^2 ij\tau). \tag{7.3}$$

For any non-zero integer m', let $y_{m'}$ be the real number with $y_{m'} \equiv -y \pmod 1$, $|y_{m'}| < 1$ and $y_{m'}\langle\langle\alpha m'\rangle\rangle \geq 0$, and let y_0 be $\{-y\}$. Since it follows from (2.4) and (2.5) that

$$\underline{F}(-\vec{y};\alpha(\tau m'+m);\tau) = \mathrm{e}(-[[\alpha m']]y')\underline{F}(-y', y_{m'}; \langle\langle\alpha m'\rangle\rangle\tau + \alpha m;\tau),$$

we find from (7.2) and (7.3) that

$$\lim_{\tau\to i\infty} I^\tau = \lim_{\tau\to i\infty} \pi \sum_{m',m}{}' \frac{\mathrm{e}(\vec{m}\cdot\vec{x})}{(\tau m'+m)^l}\mathrm{e}(-[[\alpha m']]y') \times \mathrm{e}(y_{m'}(\langle\langle \alpha m'\rangle\rangle\tau+\alpha m))\Big(\cot\pi(y'+y_{m'}\tau)+\cot\pi(\alpha m+\langle\langle\alpha m'\rangle\rangle\tau)\Big). \tag{7.4}$$

Let $S_{m',m}$'s be the summands of the summation in right hand side of (7.4). If

$$\lim_{\tau\to i\infty} \sum_{\substack{m',m \\ (m'\neq 0)}}{}' S_{m',m} = 0, \tag{7.5}$$

then we have

$$\begin{aligned} \lim_{\tau\to i\infty} I^\tau &= \pi \sum_m{}' \frac{\mathrm{e}(mx)}{m^l} \lim_{\tau\to i\infty} \mathrm{e}(\{-y\}\alpha m)\Big(\cot\pi(y'+\{-y\}\tau)+\cot\pi\alpha m\Big) \\ &= \begin{cases} \pi\Big(\xi_l(x;\alpha)+\cot\pi y' \sum_m{}' \dfrac{\mathrm{e}(mx)}{m^l}\Big) & (y\in\mathbb{Z}), \\ \pi\Big(\xi_l(x+\{-y\}\alpha;\alpha) - i\sum_m{}' \dfrac{\mathrm{e}(m(x+\{-y\}\alpha))}{m^l}\Big) & (y\notin\mathbb{Z}) \end{cases} \end{aligned}$$

from which we deduce (3.6). So we may prove (7.5). Let c be a positive real number such that if $|\mathrm{Re}\ z|, |\mathrm{Im}\ z| \le 1/2$, then $|z\cot\pi z| \le c$. Since $\cot\pi z$ converges at $\pm i$ when z tends to $\mp i\infty$, there is a positive real number c' such that if $|\mathrm{Re}\ z|\le 1/2$ and z is not zero, then

$$|\cot\pi z| \le \frac{c}{|z|} + c'.$$

Set $C=\max\{c,c'\}$. Since $y_{m'}\langle\langle\alpha m'\rangle\rangle \ge 0$, one obtains

$$\sum_{\substack{m',m \\ (m'\neq 0)}}{}' |S_{m',m}| \le \sum_{\substack{m',m \\ (m'\neq 0)}}{}' \frac{C}{|\tau m'+m|^l}\Big(\frac{1}{|\langle\langle y'\rangle\rangle + y_{m'}\tau|} + \frac{1}{|\langle\langle \alpha m\rangle\rangle + \langle\langle\alpha m'\rangle\rangle\tau|} + 2\Big).$$

This together with (3.2) implies (7.5), which completes the proof. □

Acknowledgments

The author expresses his gratitude to Professor SHIBUKAWA Youichi and Professor YAMASHITA Hiroshi for their encouragements. He also thanks Professor ASAI Tetsuya for sending his lecture note about (1.2) at Shizuoka university.

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Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido, 060-0810, Japan

E-mail address: machide@math.sci.hokudai.ac.jp