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# The characteristic quasi-polynomials of the arrangements of root systems

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## Abstract

For an irreducible root system  $R$ , consider a coefficient matrix  $S$  of the positive roots with respect to the associated simple roots. Then  $S$  defines an arrangement of “hyperplanes” modulo a positive integer  $q$ . The cardinality of the complement of this arrangement is a quasi-polynomial of  $q$ , which we call the characteristic quasi-polynomial of  $R$ . This paper gives the complete list of the characteristic quasi-polynomials of all irreducible root systems, and shows that the characteristic quasi-polynomial of an irreducible root system  $R$  is positive at  $q \in \mathbb{Z}_{>0}$  if and only if  $q$  is greater than or equal to the Coxeter number of  $R$ .

*Key words:* characteristic quasi-polynomial, elementary divisor, hyperplane arrangement, root system.

## 1 Introduction

Let  $S$  be an arbitrary  $m \times n$  integral matrix without zero columns. For each positive integer  $q \in \mathbb{Z}_{>0}$ , denote  $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$  and  $\mathbb{Z}_q^\times = \mathbb{Z}_q \setminus \{0\}$ . Consider the set

$$M_q(S) := \{\mathbf{z} = (z_1, \dots, z_m) \in \mathbb{Z}_q^m : \mathbf{z}S \in (\mathbb{Z}_q^\times)^n\},$$

and its cardinality  $|M_q(S)|$ . In our recent paper [3], we showed that there exists a monic **quasi-polynomial** (periodic polynomial)  $\chi_S(q)$  with integral coefficients of degree  $m$  such that

$$\chi_S(q) = |M_q(S)|, \quad q \in \mathbb{Z}_{>0}.$$

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Note that the set  $M_q(S)$  is the complement of an arrangement of hyperplanes in the following sense: Let  $S_1, S_2, \dots, S_n$  be the columns of  $S$ . Each set

$$H_{i,q} := \{\mathbf{z} = (z_1, \dots, z_m) \in \mathbb{Z}_q^m : \mathbf{z}S_i = 0\}, \quad 1 \leq i \leq n,$$

can be called a “hyperplane” in  $\mathbb{Z}_q^m$  by a slight abuse of terminology. Then

$$M_q(S) = \mathbb{Z}_q^m \setminus \bigcup_{i=1}^n H_{i,q}.$$

For a sufficiently large prime number  $q$ ,  $\chi_S(q)$  is known [1] to be equal to the **characteristic polynomial** [4, Def. 2.52] of the real arrangement consisting of the following hyperplanes (ignoring possible repetitions):

$$H_{i,\mathbb{R}} := \{\mathbf{z} = (z_1, \dots, z_m) \in \mathbb{R}^m : \mathbf{z}S_i = 0\}, \quad 1 \leq i \leq n.$$

It is thus natural to call the quasi-polynomial  $\chi_S(q)$  the **characteristic quasi-polynomial** of  $S$  as in [3].

In this paper, we define and determine the characteristic quasi-polynomial  $\chi_R(q)$  for **every irreducible root system**  $R$ . Let  $m$  be the rank of  $R$  and  $n = |R|/2$ . We assume that an  $m \times n$  integral matrix  $S = S(R) = [S_{ij}]$  satisfies

$$R_+ = \left\{ \sum_{i=1}^m S_{ij} \alpha_i : j = 1, \dots, n \right\},$$

where  $R_+$  is a set of positive roots and  $B(R) = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$  is the set of **simple roots** associated with  $R_+$ . In other words,  $S$  is a coefficient matrix of  $R_+$  with respect to the basis  $B(R)$ . Define the **characteristic quasi-polynomial**  $\chi_R(q) := \chi_S(q)$  for each irreducible root system  $R$ . Then  $\chi_R(q)$  depends only upon  $R$ .

For example, for the root system  $R = A_2 = \{\epsilon_i - \epsilon_j : 1 \leq i \leq 3, 1 \leq j \leq 3, i \neq j\}$ ,  $B(A_2) = \{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3\}$  and  $R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ , where  $\epsilon_1, \epsilon_2, \epsilon_3$  are orthonormal, one has

$$S = S(A_2) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

It is easy to see that  $\chi_{A_2}(q) = \chi_S(q) = (q-1)(q-2)$ , which is equal to the ordinary characteristic polynomial of type  $A_2$ . In other words, the minimum period of the quasi-polynomial  $\chi_{A_2}(q)$  is one. The minimum periods for all irreducible root systems are shown in the following table:

root system	minimum period	root system	minimum period
$A_m$	1	$E_6$	6
$B_m$	2	$E_7$	12
$C_m$	2	$E_8$	60
$D_m$	2	$F_4$	12
		$G_2$	6

The outline of this paper is as follows: In Section 2, we prove general results on  $\chi_S(q)$  which are used in Sections 5 and 6. In Section 3, we study the case of root system  $A_m$ , which is the easiest case. We investigate the root systems  $B_m, C_m$  and  $D_m$  using the coset method in Section 4. The characteristic quasi-polynomials of these three root systems are closely related to each other. The cases of  $G_2$  and  $F_4$  are studied in Section 5. In Section 6, we study the remaining root systems  $E_m$  ( $m = 6, 7, 8$ ) which require the hardest calculations in this paper. We are aided by the computer package PARI/GP [5] and the theoretical results from Section 2. Lastly in Section 7, we state two results obtained from our calculations and the classification of irreducible root systems. Throughout this paper we use the table of irreducible root systems in [2] as our standard reference.

## 2 Results on the characteristic quasi-polynomial of an integral matrix

Let  $\chi_S(t)$  be the characteristic quasi-polynomial of an  $m \times n$  integral matrix  $S$  without zero columns. Fix a nonempty  $J \subseteq [n] := \{1, 2, \dots, n\}$  and define an  $m \times |J|$  matrix  $S_J$  consisting of the columns of  $S$  corresponding to the set  $J$ . Let  $e_{J,1}, \dots, e_{J,\ell(J)} \in \mathbb{Z}_{>0}$  be the elementary divisors of  $S_J$  numbered so that  $e_{J,1} | e_{J,2} | \dots | e_{J,\ell(J)}$ , where  $\ell(J) := \text{rank } S_J$ . Write  $e(J) := e_{J,\ell(J)}$ , and define the **lcm period**  $\rho_0(S)$  of  $S$  by

$$\begin{aligned} \rho_0 = \rho_0(S) &:= \text{lcm}\{e(J) : J \subseteq [n], J \neq \emptyset\} \\ &= \text{lcm}\{e(J) : J \subseteq [n], 1 \leq |J| \leq \min\{m, n\}\}. \end{aligned}$$

Then it is known ([3, Theorem 2.4]) that the lcm period  $\rho_0$  is a period of  $\chi_S(t)$ .

It is further shown in [3] that the constituents of the quasi-polynomial  $\chi_S(t)$  are the same for all  $q$ 's with the same value of  $\gcd\{\rho_0, q\}$ . Let  $d$  be a positive integer which divides  $\rho_0$ , and define a monic polynomial  $P_d(t) = P_{S,d}(t)$  with integral coefficients of degree  $m$  by

$$\chi_S(q) = P_d(q) \quad \text{for all } q \in d + \rho_0 \mathbb{Z}_{\geq 0}.$$

Put

$$e(J, d) := \prod_{j=1}^{\ell(J)} \gcd\{e_{J,j}, d\}.$$

Then the following formula was essentially proved in our previous paper [3].

**Theorem 2.1.** *For each  $d \in \mathbb{Z}_{>0}$  with  $d | \rho_0$ , the polynomial  $P_d(t)$  is given by*

$$P_d(t) = \sum_{J \subseteq [n]} (-1)^{|J|} e(J, d) t^{m - \ell(J)},$$

where for  $J = \emptyset$ , we understand that  $\ell(\emptyset) = 0$  and that  $e(\emptyset, d) = 1$ .

*Proof.* Obtained from [3, (10)] and the inclusion-exclusion principle. □

**Theorem 2.2** ([3] Theorem 2.5). *The polynomial*

$$P_1(t) = \sum_{J \subseteq [n]} (-1)^{|J|} t^{m-\ell(J)}$$

*is equal to the ordinary characteristic polynomial [4, Def. 2.52] of the real arrangement consisting of the hyperplanes (ignoring possible repetitions)  $H_{1,\mathbb{R}}, H_{2,\mathbb{R}}, \dots, H_{n,\mathbb{R}}$ .*

**Corollary 2.3.** *Suppose  $d, d' \in \mathbb{Z}_{>0}$  both divide  $\rho_0$ , and assume the following condition holds true for some positive integer  $s$ :  $\gcd\{e(J), d\} = \gcd\{e(J), d'\}$  for all  $J \subseteq [n]$  with  $|J| \leq s$ . Then*

$$\deg\{P_d(t) - P_{d'}(t)\} < m - s.$$

*In particular, we have  $\deg\{P_d(t) - P_1(t)\} < m - s$  if  $\gcd\{e(J), d\} = 1$  for all  $J \subseteq [n]$  with  $|J| \leq s$ .*

*Proof.* We apply Theorems 2.1 and 2.2. It is enough to show  $e(J, d) = e(J, d')$  for  $J \subseteq [n]$  with  $\ell(J) \leq s$ . We can choose a subset  $J' \subseteq J$  such that  $|J'| = \ell(J) \leq s$ . Then  $\gcd\{e(J'), d\} = \gcd\{e(J'), d'\}$ . Since  $e(J)|e(J')$  [3, Lemma 2.3],  $\gcd\{e(J), d\} = \gcd\{e(J), d'\}$ . This shows  $e(J, d) = \prod_{j=1}^{\ell(J)} \gcd\{e_{J,j}, d\} = \prod_{j=1}^{\ell(J)} \gcd\{e_{J,j}, e(J), d\} = \prod_{j=1}^{\ell(J)} \gcd\{e_{J,j}, e(J), d'\} = \prod_{j=1}^{\ell(J)} \gcd\{e_{J,j}, d'\} = e(J, d')$ .  $\square$

**Corollary 2.4.** *Suppose that  $d \in \mathbb{Z}_{>0}$  and  $d' \in \mathbb{Z}_{>0}$  both divide  $\rho_0$  and that  $\gcd\{d, d'\} = 1$ . In addition, we assume the following condition holds true for some positive integer  $s$ :*

$$(1) \quad \gcd\{e(J), d\} = 1 \quad \text{or} \quad \gcd\{e(J), d'\} = 1$$

*for all  $J \subseteq [n]$  with  $|J| \leq s$ . Then*

$$\deg\{P_1(t) + P_{dd'}(t) - P_d(t) - P_{d'}(t)\} < m - s.$$

*Proof.* Suppose  $J \subseteq [n]$  with  $\ell(J) \leq s$ . It is enough to show

$$1 + e(J, dd') - e(J, d) - e(J, d') = 0.$$

We can choose a subset  $J' \subseteq J$  such that  $|J'| = \ell(J) \leq s$ . Then either  $\gcd\{e(J'), d\} = 1$  or  $\gcd\{e(J'), d'\} = 1$  by (1). Since  $e(J)|e(J')$ ,

$$\gcd\{e(J), d\} = 1 \quad \text{or} \quad \gcd\{e(J), d'\} = 1.$$

This shows that either  $e(J, d) = 1$  or  $e(J, d') = 1$ . We finally have

$$\begin{aligned} 0 &= \{1 - e(J, d)\}\{1 - e(J, d')\} = 1 - e(J, d) - e(J, d') + e(J, d)e(J, d') \\ &= 1 - e(J, d) - e(J, d') + e(J, dd'). \end{aligned}$$

$\square$

**Corollary 2.5.** *Suppose that  $d \in \mathbb{Z}_{>0}$  and  $d' \in \mathbb{Z}_{>0}$  both divide  $\rho_0$  and that  $\gcd\{d, d'\} = 1$ . If  $e(J)$  are prime powers or one for all  $J$ , we have  $P_{dd'}(t) = P_d(t) + P_{d'}(t) - P_1(t)$ .*

*Proof.* Easily follows from Corollary 2.4.  $\square$

The results in Corollaries 2.3, 2.4 and 2.5 will be used to find characteristic quasi-polynomials of root systems.

### 3 Characteristic quasi-polynomial of $A_m$

We follow PLATE I in [2]. Let  $\{\epsilon_1, \dots, \epsilon_{m+1}\}$  be an orthonormal basis for an  $(m+1)$ -dimensional Euclidean space  $W$ , and define

$$V := \left\{ \sum_{i=1}^{m+1} c_i \epsilon_i \in W : \sum_{i=1}^{m+1} c_i = 0 \right\}.$$

Then

$$R := \{\pm(\epsilon_i - \epsilon_j) : 1 \leq i < j \leq m+1\} \subset V, \quad |R| = m(m+1),$$

is an irreducible root system in  $V$  of type  $A_m$ . Then we may choose a set of positive roots

$$R_+ := \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq m+1\}.$$

Define  $\alpha_i := \epsilon_i - \epsilon_{i+1}$ ,  $1 \leq i \leq m$ . Then  $B := \{\alpha_1, \dots, \alpha_m\}$  is the set of simple roots associated with  $R_+$ . We may express

$$R_+ = \left\{ \sum_{i \leq k \leq j} \alpha_k : 1 \leq i \leq j \leq m \right\}.$$

Let  $n := |R_+| = m(m+1)/2$ . Then the  $m \times n$  matrix  $S(A_m)$  consists of only 0's and 1's such that 1 appears consecutively in each column. For example

$$S(A_1) = [1], \quad S(A_2) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad S(A_3) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

The characteristic quasi-polynomial  $\chi_{A_m}(t)$  of the root system  $A_m$  is the characteristic quasi-polynomial of  $S(A_m)$  :  $\chi_{A_m}(t) := \chi_{S(A_m)}(t)$ . Let us enumerate the size of

$$M_q(S) = \{\mathbf{z} \in \mathbb{Z}_q^m : \mathbf{z}S \in (\mathbb{Z}_q^\times)^n\} = \{\mathbf{z} \in \mathbb{Z}_q^m : \sum_{i \leq k \leq j} z_k \neq 0 \quad (1 \leq i \leq j \leq m)\}.$$

First, there are  $(q-1)$  ways to choose  $z_1$ . Next, there are  $(q-2)$  ways to choose  $z_2$ , etc. Therefore we have

$$\chi_{A_m}(q) = |M_q(S)| = (q-1) \cdots (q-m).$$

Thus the characteristic quasi-polynomial  $\chi_{A_m}(q)$  of  $A_m$  is equal to the ordinary characteristic polynomial.

## 4 Characteristic quasi-polynomials of $B_m, C_m, D_m$

### 4.1 The coset method

Let  $P$  be a non-singular  $m \times m$  integral matrix. Consider a finite additive group  $G := \mathbb{Z}_q^m$ , and define a group homomorphism  $\pi : G \rightarrow G$  by  $\pi(\mathbf{z}) = \mathbf{z}P$ ,  $\mathbf{z} \in G$ . Consider the subgroup  $H := \text{im } \pi$  of  $G$ . Then it is not difficult to see that the index  $(G : H)$  is equal to  $b(q) := \gcd\{q, \det P\}$ . Note that every fiber  $\pi^{-1}(\mathbf{y})$  has the same cardinality  $b(q)$  for any  $\mathbf{y} \in H$ . Let us express the set  $G/H$  of cosets as

$$G/H = \{\mathbf{g}_i + H : 1 \leq i \leq b(q)\}$$

for some complete set of representatives  $\mathbf{g}_1, \dots, \mathbf{g}_{b(q)} \in G$ ,  $\mathbf{g}_1 \in H$ .

Let  $S$  be an  $m \times n$  integral matrix, and define an  $m \times n$  integral matrix  $T$  by  $T = PS$ . Then we can write  $f_S(q) := |\{\mathbf{y} \in G : \mathbf{y}S \in (\mathbb{Z}_q^\times)^n\}|$  as

$$\begin{aligned} f_S(q) &= |\{\mathbf{y} \in G : \mathbf{y}S \in (\mathbb{Z}_q^\times)^n\}| = \sum_{i=1}^{b(q)} |\{\mathbf{y} \in \mathbf{g}_i + H : \mathbf{y}S \in (\mathbb{Z}_q^\times)^n\}| \\ &= \frac{1}{b(q)} \sum_{i=1}^{b(q)} |\{\mathbf{z} \in G : (\mathbf{g}_i + \mathbf{z}P)S \in (\mathbb{Z}_q^\times)^n\}| \\ &= \frac{1}{b(q)} \sum_{i=1}^{b(q)} |\{\mathbf{z} \in G : \mathbf{z}T + \mathbf{g}_iS \in (\mathbb{Z}_q^\times)^n\}|. \end{aligned}$$

Define

$$(2) \quad f_i(q) := |\{\mathbf{z} \in G : \mathbf{z}T + \mathbf{g}_iS \in (\mathbb{Z}_q^\times)^n\}|$$

for  $i = 1, \dots, b(q)$ . Then we have:

**Theorem 4.1.** *For an  $m \times n$  integral matrix  $S$ , define  $f_S(q) = |\{\mathbf{y} \in G : \mathbf{y}S \in (\mathbb{Z}_q^\times)^n\}|$ . Then, for any non-singular  $m \times m$  integral matrix  $P$  and the  $m \times n$  integral matrix  $T$  defined by  $T = PS$ , we can write  $f_S(q)$  as*

$$(3) \quad f_S(q) = \frac{1}{b(q)} \sum_{i=1}^{b(q)} f_i(q),$$

where  $b(q) = \gcd\{q, \det P\}$ , and  $f_i(q)$ ,  $1 \leq i \leq b(q)$ , are defined in (2).

Note that when  $S$  has a zero column, (3) is trivially true because both sides are zero. Thus, we do not need the assumption that  $S$  has no zero column; when this assumption is satisfied,  $f_S(q)$  is the characteristic quasi-polynomial  $\chi_S(q)$ .

We also note the following:  $f_1(q) = |\{\mathbf{z} \in G : \mathbf{z}T \in (\mathbb{Z}_q^\times)^n\}| = f_T(q)$ . Hence, when  $b(q) = 1$  in particular (e.g., when  $P$  is unimodular), we have  $f_S(q) = f_1(q) = f_T(q)$ .

## 4.2 $B_m$

We follow PLATE II in [2]. Let  $\{\epsilon_1, \dots, \epsilon_m\}$  be an orthonormal basis for an  $m$ -dimensional Euclidean space  $V$ . Let  $m \geq 2$ . Then

$$R := \{ \pm\epsilon_i \ (1 \leq i \leq m), \ \pm(\epsilon_i - \epsilon_j) \ (1 \leq i < j \leq m), \ \\ \pm(\epsilon_i + \epsilon_j) \ (1 \leq i < j \leq m) \} \subset V, \quad |R| = 2m^2,$$

is an irreducible root system of type  $B_m$ . Then we may choose a set of positive roots

$$R_+ = \{ \epsilon_i \ (1 \leq i \leq m), \ \epsilon_i - \epsilon_j \ (1 \leq i < j \leq m), \ \epsilon_i + \epsilon_j \ (1 \leq i < j \leq m) \}.$$

Define  $\alpha_i := \epsilon_i - \epsilon_{i+1}$  ( $1 \leq i \leq m-1$ ),  $\alpha_m := \epsilon_m$ . Then  $B = \{\alpha_1, \dots, \alpha_m\}$  is the set of simple roots associated with  $R_+$ . We may express

$$R_+ = \left\{ \sum_{i \leq k \leq j} \alpha_k \ (1 \leq i \leq j \leq m), \ \sum_{i \leq k < j} \alpha_k + 2 \sum_{j \leq k \leq m} \alpha_k \ (1 \leq i < j \leq m) \right\}.$$

Let  $n := |R_+| = m^2$ . Then the  $m \times n$  matrix  $S := S(B_m)$  is the coefficient matrix of  $R_+$  with respect to the set of simple roots  $B$ . For example,

$$S(B_2) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}, \quad S(B_3) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 2 & 2 & 2 \end{bmatrix}.$$

We want to find the characteristic quasi-polynomial  $\chi_{B_m}(t) := \chi_{S(B_m)}(t)$  of  $B_m$ . Define an  $m \times m$  matrix

$$P := \begin{bmatrix} 1 & & & & & & & & & & \\ -1 & 1 & & & & & & & & & \\ & -1 & & & & & & & & & \\ & & \ddots & & & & & & & & \\ & & & \ddots & & & & & & & \\ & & & & 1 & & & & & & \\ & & & & -1 & 1 & & & & & \end{bmatrix}.$$

Then the  $m \times n$  matrix  $T = T(B_m) := PS$  is the coefficient matrix of  $R_+$  with respect to the orthonormal basis  $\epsilon_1, \dots, \epsilon_m$ . Since  $P$  is unimodular, we have

$$(4) \quad \chi_{B_m}(q) = \chi_{S(B_m)}(q) = \chi_{T(B_m)}(q).$$

## 4.3 $C_m$

We follow PLATE III in [2]. Let  $m \geq 3$ .

$$R := \{ \pm 2\epsilon_i \ (1 \leq i \leq m), \ \pm(\epsilon_i - \epsilon_j) \ (1 \leq i < j \leq m), \ \\ \pm(\epsilon_i + \epsilon_j) \ (1 \leq i < j \leq m) \} \subset V, \quad |R| = 2m^2,$$



is an irreducible root system in  $V$  of type  $C_m$ . Then we may choose a set of positive roots

$$R_+ = \{2\epsilon_i \ (1 \leq i \leq m), \ \epsilon_i - \epsilon_j \ (1 \leq i < j \leq m), \ \epsilon_i + \epsilon_j \ (1 \leq i < j \leq m)\}.$$

Define  $\alpha_i := \epsilon_i - \epsilon_{i+1}$  ( $1 \leq i \leq m-1$ ),  $\alpha_m := 2\epsilon_m$ . Then  $B = \{\alpha_1, \dots, \alpha_m\}$  is the set of simple roots associated with  $R_+$ . We may express

$$R_+ = \left\{ \sum_{i \leq k \leq j} \alpha_k \ (1 \leq i \leq j \leq m), \ \sum_{i \leq k < j} \alpha_k + 2 \sum_{j \leq k < m} \alpha_k + \alpha_m \ (1 \leq i \leq j < m) \right\}.$$

Let  $n := |R_+| = m^2$ . Then the  $m \times n$  matrix  $S = S(C_m)$  is the coefficient matrix of  $R_+$  with respect to the set of simple roots  $B$ . For example,

$$S(C_3) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

We want to find the characteristic quasi-polynomial  $\chi_{C_m}(t) := \chi_{S(C_m)}(t)$  of  $C_m$ . Define an  $m \times m$  matrix

$$P := \begin{bmatrix} 1 & & & & & & & & \\ -1 & 1 & & & & & & & \\ & -1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & & 1 & & & & \\ & & & & -1 & 2 & & & \end{bmatrix}.$$

Then the  $m \times n$  matrix  $T = T(C_m) := PS$  is the coefficient matrix of  $R_+$  with respect to  $\epsilon_1, \dots, \epsilon_m$ . Since  $\det P = 2$ , we have to consider two cases.

**Case 1: When  $q$  is odd.**

For odd  $q$ , we have  $b(q) := \gcd\{q, \det P\} = 1$  and thus

$$\chi_S(q) = \chi_{T(C_m)}(q).$$

**Case 2: When  $q$  is even.**

For  $\pi : G \rightarrow G$  defined by  $\pi(\mathbf{z}) = \mathbf{z}P$ ,  $\mathbf{z} \in G$ , we have

$$H := \text{im } \pi = \{(y_1, \dots, y_{m-1}, 2y_m) : y_1, \dots, y_m \in \mathbb{Z}_q\}.$$

Since  $b(q) = (G : H) = \gcd\{q, \det P\} = 2$ , we take  $\mathbf{g}_1 = \mathbf{0} \in H$  and  $\mathbf{g}_2 = (0, \dots, 0, 1) \in G \setminus H$ . By Theorem 4.1

$$\chi_S(q) = \frac{1}{2} \{f_1(q) + f_2(q)\},$$

where  $f_1(q) = \chi_{T(C_m)}(q)$  and

$$\begin{aligned}
f_2(q) &= |\{\mathbf{z} \in G : \mathbf{z}T + \mathbf{g}_2S \in (\mathbb{Z}_q^\times)^n\}| \\
&= |\{(z_1, \dots, z_m) \in \mathbb{Z}_q^m : 2z_i + 1 \neq 0 \ (1 \leq i \leq m), \\
&\quad z_i - z_j \neq 0 \ (1 \leq i < j \leq m), \\
&\quad z_i + z_j + 1 \neq 0 \ (1 \leq i < j \leq m)\}| \\
(5) \quad &= |\{(z_1, \dots, z_m) \in \mathbb{Z}_q^m : \\
&\quad z_i - z_j \neq 0 \ (1 \leq i < j \leq m), \\
&\quad z_i + z_j + 1 \neq 0 \ (1 \leq i < j \leq m)\}| \\
&= m! \times |\{(c_1, \dots, c_m) \in \mathbb{Z}^m : 0 \leq c_1 < \dots < c_m < q, \\
&\quad c_i + c_j \neq q - 1 \ (1 \leq i < j \leq m)\}| \\
&= m! \times |\{(c_1, \dots, c_m) \in \mathbb{Z}^m : 0 < c_1 < \dots < c_m < q + 1, \\
&\quad c_i + c_j \neq q + 1 \ (1 \leq i < j \leq m)\}| \\
&= \chi_{T(B_m)}(q + 1).
\end{aligned}$$

In the second equation of (5), we have used  $\{\sum_{i \leq k \leq j} \alpha_k : 1 \leq i \leq j < m\} = \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq m\}$ ,  $\{\sum_{i \leq k \leq m} \alpha_k \ (1 \leq i \leq m), \sum_{i \leq k < j} \alpha_k + 2 \sum_{j \leq k < m} \alpha_k + \alpha_m \ (1 \leq i \leq j < m)\} = \{2\epsilon_i \ (1 \leq i \leq m), \epsilon_i + \epsilon_j \ (1 \leq i < j \leq m)\}$  for  $R_+$ .

Therefore,

$$\chi_S(q) = \frac{1}{2} \{\chi_{T(C_m)}(q) + \chi_{T(B_m)}(q + 1)\}$$

for even  $q$ .

In summary,

$$(6) \quad \chi_{C_m}(q) = \chi_{S(C_m)}(q) = \begin{cases} \chi_{T(C_m)}(q) & \text{if } q \text{ is odd,} \\ \frac{1}{2} \{\chi_{T(C_m)}(q) + \chi_{T(B_m)}(q + 1)\} & \text{if } q \text{ is even.} \end{cases}$$

#### 4.4 $D_m$

We follow PLATE IV in [2]. Let  $m \geq 4$ .

$$R := \{\pm(\epsilon_i - \epsilon_j) \ (1 \leq i < j \leq m), \pm(\epsilon_i + \epsilon_j) \ (1 \leq i < j \leq m)\} \subset V, \quad |R| = 2m(m - 1),$$

is an irreducible root system in  $V$  of type  $D_m$ . Then we may choose a set of positive roots

$$R_+ = \{\epsilon_i - \epsilon_j \ (1 \leq i < j \leq m), \epsilon_i + \epsilon_j \ (1 \leq i < j \leq m)\}.$$

Define  $\alpha_i := \epsilon_i - \epsilon_{i+1}$  ( $1 \leq i \leq m - 1$ ),  $\alpha_m := \epsilon_{m-1} + \epsilon_m$ . Then  $B = \{\alpha_1, \dots, \alpha_m\}$  is the set of simple roots associated with  $R_+$ . We may express

$$R_+ = \left\{ \sum_{i \leq k \leq j} \alpha_k \ (1 \leq i \leq j < m), \right.$$

$$\left. \begin{aligned} & \sum_{i \leq k < j} \alpha_k + 2 \sum_{j \leq k < m-1} \alpha_k + \alpha_{m-1} + \alpha_m \quad (1 \leq i < j < m), \\ & \sum_{i \leq k < m-1} \alpha_k + \alpha_m \quad (1 \leq i < m) \end{aligned} \right\}.$$

Let  $n := |R_+| = m(m-1)$ . Then the  $m \times n$  matrix  $S = S(D_m)$  is the coefficient matrix of  $R_+$  with respect to the set of simple roots  $B$ . For example,

$$S(D_4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

We want to find the characteristic quasi-polynomial  $\chi_{D_m}(t) := \chi_{S(D_m)}(t)$  of  $D_m$ . Define an  $m \times m$  matrix

$$P := \begin{bmatrix} 1 & & & & & & & & & & & \\ -1 & 1 & & & & & & & & & & \\ & & -1 & & & & & & & & & \\ & & & \ddots & & & & & & & & \\ & & & & & & & & 1 & 1 & & \\ & & & & & & & & -1 & 1 & & \end{bmatrix}.$$

Then the  $m \times n$  matrix  $T = T(D_m) := PS$  is the coefficient matrix of  $R_+$  with respect to  $\epsilon_1, \dots, \epsilon_m$ . Since  $\det P = 2$ , we have to consider two cases.

**Case 1: When  $q$  is odd.**

For odd  $q$ , we have  $b(q) := \gcd\{q, \det P\} = 1$  and thus

$$\chi_S(q) = \chi_{T(D_m)}(q).$$

**Case 2: When  $q$  is even.**

We have

$$H = \text{im } \pi = \{(y_1, \dots, y_{m-1}, y_{m-1} + 2y_m) : y_1, \dots, y_m \in \mathbb{Z}_q\}.$$

Since  $b(q) = (G : H) = \gcd\{q, \det P\} = 2$ , we take  $\mathbf{g}_1 = \mathbf{0} \in H$  and  $\mathbf{g}_2 = (0, \dots, 0, 1) \in G \setminus H$ . By Theorem 4.1

$$\chi_S(q) = \frac{1}{2} \{f_1(q) + f_2(q)\},$$

where  $f_1(q) = \chi_{T(D_m)}(q)$  and

$$f_2(q) = |\{\mathbf{z} \in G : \mathbf{z}T + \mathbf{g}_2S \in (\mathbb{Z}_q^\times)^n\}|$$

$$\begin{aligned}
(7) \quad &= |\{(z_1, \dots, z_m) \in \mathbb{Z}_q^m : z_i - z_j \neq 0 \ (1 \leq i < j \leq m), \\
&\quad z_i + z_j + 1 \neq 0 \ (1 \leq i < j \leq m)\}| \\
&= \chi_{T(B_m)}(q+1)
\end{aligned}$$

by (5). In the second equation of (7), we have used  $\{\sum_{i \leq k \leq j} \alpha_k : 1 \leq i \leq j < m\} = \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq m\}$ ,  $\{\sum_{i \leq k < j} \alpha_k + 2 \sum_{j \leq k < m-1} \alpha_k + \alpha_{m-1} + \alpha_m \ (1 \leq i < j < m)\}$ ,  $\{\sum_{i \leq k < m-1} \alpha_k + \alpha_m \ (1 \leq i < m)\} = \{\epsilon_i + \epsilon_j : 1 \leq i < j \leq m\}$  for  $R_+$ .

Therefore,

$$\chi_S(q) = \frac{1}{2} \{ \chi_{T(D_m)}(q) + \chi_{T(B_m)}(q+1) \}$$

for even  $q$ .

In summary,

$$(8) \quad \chi_{D_m}(q) = \chi_{S(D_m)}(q) = \begin{cases} \chi_{T(D_m)}(q) & \text{if } q \text{ is odd,} \\ \frac{1}{2} \{ \chi_{T(D_m)}(q) + \chi_{T(B_m)}(q+1) \} & \text{if } q \text{ is even.} \end{cases}$$

## 4.5 Orthonormal basis

### 4.5.1 $\chi_{T(B_m)}(q)$ and $\chi_{T(D_m)}(q)$

We first prove the following lemma.

**Lemma 4.2.** *Assume that a matrix  $A$  satisfies the following three conditions:*

- (1) *each entry lies in  $\{0, \pm 1, \pm 2\}$ ,*
- (2) *each column contains at most two nonzero entries, and*
- (3) *each column contains at most one entry from  $\{\pm 2\}$ .*

*Then the elementary divisors of  $A$  lie in  $\{1, 2\}$ .*

*Proof.* Let us temporarily say that a matrix is of type (T) if it satisfies these three conditions. Denote the set of elementary divisors of  $A$  by  $ED(A)$ . Argue by an induction on the number of columns. When a matrix has only one column, the statement is obviously true. Suppose that a matrix  $A$  has more than one column.

Case 1. When  $A = O$ ,  $ED(A) = \emptyset$ .

Case 2. When  $A \neq O$  and each entry of  $A$  lies in  $\{0, \pm 2\}$ , then  $ED(A) = \{2\}$ .

Case 3. If  $A$  has a column with only one nonzero entry  $a \in \{\pm 1\}$ , then  $A$  is equivalent to

$$\left[ \begin{array}{c|ccc} 1 & * & * & * \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & & \end{array} \right] \begin{array}{c} \\ \\ B \\ \end{array}$$

with  $B$  of type (T). Since  $ED(B) \subseteq \{1, 2\}$  by the induction assumption,  $ED(A) \subseteq \{1, 2\}$ .

Case 4. If  $A$  has a column with exactly two nonzero entries, then  $A$  is equivalent to

$$A_1 = \left[ \begin{array}{c|ccc} 1 & * & * & * \\ 1 & * & * & * \\ 0 & * & * & * \\ \vdots & * & * & * \\ 0 & * & * & * \end{array} \right].$$

By clearing the first row using the first column of  $A_1$  we see that  $A$  is equivalent to

$$\left[ \begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ 1 & & & \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{array} \right].$$

$C$

Since

$$\begin{aligned} (1, 1, 0, \dots, 0)^t - (1, -1, 0, \dots, 0)^t &= (0, 2, 0, \dots, 0)^t, \\ (1, 1, 0, \dots, 0)^t - (1, 0, -1, \dots, 0)^t &= (0, 1, 1, \dots, 0)^t, \\ (1, 1, 0, \dots, 0)^t - (1, 0, 1, \dots, 0)^t &= (0, 1, -1, \dots, 0)^t, \end{aligned}$$

and so on,  $C$  is of type (T). Since  $ED(C) \subseteq \{1, 2\}$  by the induction assumption,  $ED(A) \subseteq \{1, 2\}$ .  $\square$

In the cases of  $T(B_m)$  and  $T(D_m)$ , we have by Lemma 4.2 that  $\gcd\{\rho_0, q\} = 1$  for odd  $q \in \mathbb{Z}_{>0}$ . Therefore,  $\chi_{T(B_m)}(q)$  and  $\chi_{T(D_m)}(q)$  for odd  $q$  are equal to the values of the characteristic polynomials of the real arrangements determined by the columns of  $T(B_m)$  and  $T(D_m)$ , respectively ([3, Theorem 2.5]). Hence we have the following proposition.

**Proposition 4.3.** *For odd integers  $q \in \mathbb{Z}_{>0}$ , we have*

$$\begin{aligned} (9) \quad \chi_{T(B_m)}(q) &= (q-1)(q-3) \cdots (q-2m+1), \\ (10) \quad \chi_{T(D_m)}(q) &= (q-1)(q-3) \cdots (q-2m+3)(q-m+1). \end{aligned}$$

Next, let us find  $\chi_{T(B_m)}(q)$  and  $\chi_{T(D_m)}(q)$  for even  $q \in \mathbb{Z}_{>0}$ .

**Lemma 4.4.** *We have the following equalities:*

$$\begin{aligned} (11) \quad \chi_{T(B_m)}(q) &= \chi_{T(D_m)}(q-1) \quad \text{for even } q \in \mathbb{Z}_{>0}, \\ (12) \quad \chi_{T(D_m)}(q) &= \chi_{T(B_m)}(q) + m\chi_{T(B_{m-1})}(q) \quad \text{for all } q \in \mathbb{Z}_{>0}. \end{aligned}$$

*Proof.* For even  $q \in \mathbb{Z}_{>0}$ ,

$$\chi_{T(B_m)}(q) = |\{(z_1, \dots, z_m) \in \mathbb{Z}_q^m : z_i \neq 0 \ (1 \leq i \leq m)\}|,$$

$$\begin{aligned}
& |z_i \neq \pm z_j \ (1 \leq i < j \leq m)| \\
= & m! \times |\{(c_1, \dots, c_m) \in \mathbb{Z}^m : 0 < c_1 < \dots < c_m < q, \\
& c_i + c_j \neq q \ (1 \leq i < j \leq m)\}| \\
= & m! \times |\{(c_1, \dots, c_m) \in \mathbb{Z}^m : 1 - \frac{q}{2} \leq c_1 < \dots < c_m < \frac{q}{2}, \\
& c_i + c_j \neq 0 \ (1 \leq i < j \leq m)\}| \\
= & |\{(z_1, \dots, z_m) \in \mathbb{Z}_{q-1}^m : z_i \neq \pm z_j \ (1 \leq i < j \leq m)\}| \\
= & \chi_{T(D_m)}(q-1),
\end{aligned}$$

where the third equality is seen by considering  $c_i - (q/2)$  instead of  $c_i$ ,  $1 \leq i \leq m$ .

On the other hand, for all integers  $q \in \mathbb{Z}_{>0}$ , we have

$$\begin{aligned}
\chi_{T(D_m)}(q) &= |\{(z_1, \dots, z_m) \in \mathbb{Z}_q^m : z_i \neq \pm z_j \ (1 \leq i < j \leq m)\}| \\
&= m! \times |\{(c_1, \dots, c_m) \in \mathbb{Z}^m : 0 \leq c_1 < \dots < c_m < q, \\
& c_i + c_j \neq q \ (1 \leq i < j \leq m)\}| \\
&= m! \times |\{(c_1, \dots, c_m) \in \mathbb{Z}^m : 0 < c_1 < \dots < c_m < q, \\
& c_i + c_j \neq q \ (1 \leq i < j \leq m)\}| \\
&+ m! \times |\{(0, c_2, \dots, c_m) \in \mathbb{Z}^m : 0 < c_2 < \dots < c_m < q, \\
& c_i + c_j \neq q \ (2 \leq i < j \leq m)\}| \\
&= \chi_{T(B_m)}(q) + m\chi_{T(B_{m-1})}(q).
\end{aligned}$$

□

**Theorem 4.5.** *For even integers  $q \in \mathbb{Z}_{>0}$ , we have*

$$\begin{aligned}
(13) \quad \chi_{T(B_m)}(q) &= (q-2)(q-4) \cdots (q-2m+2)(q-m), \\
(14) \quad \chi_{T(D_m)}(q) &= (q-2)(q-4) \cdots (q-2m+4) \\
&\quad \times \{q^2 - 2(m-1)q + m(m-1)\}.
\end{aligned}$$

*Proof.* Equation (13) follows from (11) and (10); then (14) follows from (12) and (13). □

Proposition 4.3 and Theorem 4.5 imply in particular that each of the characteristic quasi-polynomials  $\chi_{T(B_m)}(t)$  and  $\chi_{T(D_m)}(t)$  has the minimum period two.

#### 4.5.2 $\chi_{T(C_m)}(q)$

Lemma 4.2 implies that  $\gcd\{\rho_0, q\} = 1$  for odd  $q \in \mathbb{Z}_{>0}$  also in the case of  $T(C_m)$ . Since  $T(B_m)$  and  $T(C_m)$  define the same real arrangement, we have  $\chi_{T(C_m)}(q) = \chi_{T(B_m)}(q)$  for odd  $q \in \mathbb{Z}_{>0}$ .

**Proposition 4.6.** *For odd integers  $q \in \mathbb{Z}_{>0}$ , we have*

$$\chi_{T(C_m)}(q) = (q-1)(q-3) \cdots (q-2m+1).$$

For even  $q \in \mathbb{Z}_{>0}$ , we can derive the following result:

**Theorem 4.7.** *For even integers  $q \in \mathbb{Z}_{>0}$ , we have*

$$\begin{aligned}\chi_{T(C_m)}(q) &= \chi_{T(C_m)}(q-1) \\ &= (q-2)(q-4)\cdots(q-2m).\end{aligned}$$

*Proof.* For even  $q \in \mathbb{Z}_{>0}$ ,

$$\begin{aligned}\chi_{T(C_m)}(q) &= |\{(z_1, \dots, z_m) \in \mathbb{Z}_q^m : 2z_i \neq 0 \ (1 \leq i \leq m), \\ &\quad z_i \neq \pm z_j \ (1 \leq i < j \leq m)\}| \\ &= m! \times |\{(c_1, \dots, c_m) \in \mathbb{Z}^m : 0 < c_1 < \cdots < c_m < q, \\ &\quad c_i \neq \frac{q}{2} \ (1 \leq i \leq m), \ c_i + c_j \neq q \ (1 \leq i < j \leq m)\}| \\ &= m! \times |\{(c_1, \dots, c_m) \in \mathbb{Z}^m : 0 < c_1 < \cdots < c_m < q-1, \\ &\quad c_i + c_j \neq q-1 \ (1 \leq i < j \leq m)\}| \\ &= \chi_{T(B_m)}(q-1) = \chi_{T(C_m)}(q-1),\end{aligned}$$

where the third equality is confirmed by transforming  $c_i \mapsto c_i - 1$  for those  $c_i$ 's with  $c_i > q/2$ . Thus we obtain the theorem by Proposition 4.6.  $\square$

We see that  $\chi_{T(C_m)}(t)$  has also the minimum period two.

## 4.6 Conclusion on $B_m, C_m$ and $D_m$

By equations (4), (6), (8) and the results in Section 4.5, we can obtain the characteristic quasi-polynomials of  $B_m, C_m$  and  $D_m$ :

**Theorem 4.8.** *The characteristic quasi-polynomials of  $B_m, C_m$  and  $D_m$  are*

$$\begin{aligned}\chi_{B_m}(q) &= \chi_{C_m}(q) = \begin{cases} (q-1)(q-3)\cdots(q-2m+1) & \text{if } q \text{ is odd,} \\ (q-2)(q-4)\cdots(q-2m+2)(q-m) & \text{if } q \text{ is even,} \end{cases} \\ \chi_{D_m}(q) &= \begin{cases} (q-1)(q-3)\cdots(q-2m+3)(q-m+1) & \text{if } q \text{ is odd,} \\ (q-2)(q-4)\cdots(q-2m+4)\left\{q^2 - 2(m-1)q + \frac{m(m-1)}{2}\right\} & \text{if } q \text{ is even.} \end{cases}\end{aligned}$$

Thus the minimum periods for  $B_m, C_m$  and  $D_m$  are equal to two.

## 5 Characteristic quasi-polynomials of $G_2, F_4$

In the rest of this paper we use the notation

$$\mathcal{E}_s := \{e(J) : J \subseteq [n], |J| \leq s\}$$

for the  $m \times n$  matrix  $S = S(R)$  for a root system  $R$  and  $s \in \mathbb{Z}_{>0}$ .

## 5.1 Characteristic quasi-polynomial of $G_2$

We follow PLATE IX in [2]. Let  $\{\epsilon_1, \epsilon_2, \epsilon_3\}$  be an orthonormal basis for a 3-dimensional Euclidean space  $W$ , and define

$$V := \left\{ \sum_{i=1}^3 c_i \epsilon_i \in W : \sum_{i=1}^3 c_i = 0 \right\}.$$

Then

$$R := \{\pm(\epsilon_i - \epsilon_j) (1 \leq i < j \leq 3), \pm(2\epsilon_1 - \epsilon_2 - \epsilon_3), \pm(2\epsilon_2 - \epsilon_1 - \epsilon_3), \pm(2\epsilon_3 - \epsilon_1 - \epsilon_2)\} \subset V, \\ |R| = 12,$$

is an irreducible root system in  $V$  of type  $G_2$ . Then we may choose a set of positive roots

$$R_+ := \{\epsilon_1 - \epsilon_2, -2\epsilon_1 + \epsilon_2 + \epsilon_3, \epsilon_3 - \epsilon_1, \epsilon_3 - \epsilon_2, -2\epsilon_2 + \epsilon_1 + \epsilon_3, 2\epsilon_3 - \epsilon_1 - \epsilon_2\}.$$

Define  $\alpha_1 := \epsilon_1 - \epsilon_2, \alpha_2 := -2\epsilon_1 + \epsilon_2 + \epsilon_3$ . Then  $B := \{\alpha_1, \alpha_2\}$  is the set of simple roots associated with  $R_+$ . We may express

$$R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.$$

Thus the  $6 \times 3$  matrix  $S(G_2)$  is given by

$$S(G_2) = \begin{bmatrix} 1 & 0 & 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}.$$

We find  $\chi_{G_2}(q)$  as follows: First, the exponents of  $G_2$  are 1, 5. So we have

$$P_1(q) = q^2 - 6q + 5 = (q - 1)(q - 5),$$

which is the ordinary characteristic polynomial of type  $G_2$ . Next we compute:

$$\mathcal{E}_1 = \{1\}, \quad \mathcal{E}_2 = \{1, 2, 3\}.$$

Thus  $\rho_0 = \text{lcm } \mathcal{E}_2 = \text{lcm}\{1, 2, 3\} = 6$ . By Corollary 2.3, we have  $P_d(q) = q^2 - 6q + \dots$  for any  $d|6$ . Since

$$P_6 = P_2 + P_3 - P_1$$

by Corollary 2.5, it is enough to find  $P_2$  and  $P_3$ . Therefore the special values

$$P_2(2) = |M_2(S)| = 0, \quad P_3(3) = |M_3(S)| = 0$$

are enough for us to obtain:

$$\chi_{G_2}(q) = \begin{cases} q^2 - 6q + 5 = (q - 1)(q - 5), & \gcd\{6, q\} = 1, \\ q^2 - 6q + 8 = (q - 2)(q - 4), & \gcd\{6, q\} = 2, \\ q^2 - 6q + 9 = (q - 3)^2, & \gcd\{6, q\} = 3, \\ q^2 - 6q + 12, & \gcd\{6, q\} = 6. \end{cases}$$

Thus the minimum period for  $G_2$  is 6.



## 5.2 Characteristic quasi-polynomial of $F_4$

We follow PLATE VIII in [2]. Let  $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$  be an orthonormal basis for a 4-dimensional Euclidean space  $V$ .

$$R := \{\pm\epsilon_i \ (1 \leq i \leq 4), \ \pm(\epsilon_i - \epsilon_j) \ (1 \leq i < j \leq 4), \ \pm(\epsilon_i + \epsilon_j) \ (1 \leq i < j \leq 4), \\ \frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) \ (16 \text{ of them})\} \subset V,$$

$|R| = 48$ , is an irreducible root system in  $V$  of type  $F_4$ . Then we may choose a set of positive roots

$$R_+ = \{\epsilon_i \ (1 \leq i \leq 4), \ \epsilon_i - \epsilon_j \ (1 \leq i < j \leq 4), \ \epsilon_i + \epsilon_j \ (1 \leq i < j \leq 4), \\ \frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) \ (8 \text{ of them})\}.$$

Define  $\alpha_1 := \epsilon_2 - \epsilon_3$ ,  $\alpha_2 := \epsilon_3 - \epsilon_4$ ,  $\alpha_3 := \epsilon_4$ , and  $\alpha_4 := \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)$ . Then  $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$  is the set of simple roots associated with  $R_+$ . We may express

$$R_+ = \left\{ \sum_{i \leq k \leq j} \alpha_k \ (1 \leq i \leq j \leq 4), \alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + \alpha_4, \right. \\ \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \\ \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4, \\ \left. \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \right\}.$$

Thus the  $4 \times 24$  matrix  $S = S(F_4)$ , which is the coefficient matrix of  $R_+$  with respect to the set of simple roots  $B$ , is:

$$S(F_4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 2 \end{bmatrix}.$$

We find  $\chi_{F_4}(q)$  as follows: First, the exponents of  $F_4$  are 1, 5, 7, 11. So we have

$$P_1(q) = q^4 - 24q^3 + 190q^2 - 552q + 385 = (q-1)(q-5)(q-7)(q-11),$$

which is the ordinary characteristic polynomial of type  $F_4$ . Next we compute:

$$\mathcal{E}_1 = \{1\}, \quad \mathcal{E}_2 = \{1, 2\}, \quad \mathcal{E}_3 = \{1, 2, 4\}, \quad \mathcal{E}_4 = \{1, 2, 3, 4\}.$$

Thus  $\rho_0 = \text{lcm } \mathcal{E}_4 = \text{lcm}\{1, 2, 3, 4\} = 12$ . By Corollary 2.3, we have  $P_d(q) = q^4 - 24q^3 + \dots$  for any  $d|12$ . Since

$$P_6 = P_2 + P_3 - P_1, \quad P_{12} = P_3 + P_4 - P_1$$

by Corollary 2.5, it is enough to find  $P_2, P_3$  and  $P_4$ . Also

$$\deg(P_2 - P_4) < 2, \quad \deg(P_3 - P_1) < 1$$

by Corollary 2.3. Therefore the following special values

$$\begin{aligned} P_2(2) = |M_2(S)| = 0, \quad P_3(3) = |M_3(S)| = 0, \quad P_4(4) = |M_4(S)| = 0, \\ P_4(8) = |M_8(S)| = 0, \quad P_2(10) = |M_{10}(S)| = 0, \quad P_2(14) = |M_{14}(S)| = 3456 \end{aligned}$$

are enough for us to obtain:

$$\chi_{F_4}(q) = \begin{cases} q^4 - 24q^3 + 190q^2 - 552q + 385 \\ \quad = (q-1)(q-5)(q-7)(q-11), \quad \gcd\{12, q\} = 1, \\ q^4 - 24q^3 + 208q^2 - 768q + 880 \\ \quad = (q-2)(q-10)(q^2 - 12q + 44), \quad \gcd\{12, q\} = 2, \\ q^4 - 24q^3 + 190q^2 - 552q + 513 \\ \quad = (q-3)(q-9)(q^2 - 12q + 19), \quad \gcd\{12, q\} = 3, \\ q^4 - 24q^3 + 208q^2 - 768q + 1024 \\ \quad = (q-4)^2(q-8)^2, \quad \gcd\{12, q\} = 4, \\ q^4 - 24q^3 + 208q^2 - 768q + 1008 \\ \quad = (q-6)^2(q^2 - 12q + 28), \quad \gcd\{12, q\} = 6, \\ q^4 - 24q^3 + 208q^2 - 768q + 1152, \quad \gcd\{12, q\} = 12. \end{cases}$$

Thus the minimum period for  $F_4$  is 12.

## 6 Characteristic quasi-polynomials of $E_6, E_7, E_8$

For each of the root systems  $E_6, E_7$  and  $E_8$ , we can find the characteristic quasi-polynomial by a similar method to the method in the previous section. First we compute  $\mathcal{E}_s$  for each  $s$ . Then we have the lcm period  $\rho_0$ . For each constituent  $P_d(t)$ ,  $d|\rho_0$ , apply Corollaries 2.3, 2.4 and 2.5 to get as much information as possible. Finally we actually count  $|M_q(S)|$  for a large enough number of  $q$ 's with  $\gcd\{\rho_0, q\} = d$  and interpolate a polynomial. In this way, we obtain the characteristic quasi-polynomials for  $E_6, E_7$  and  $E_8$ . For the evaluations of  $\rho_0$ 's, we used PARI/GP [5].

### 6.1 Characteristic quasi-polynomial of $E_6$

We use PLATE V in [2] to get the  $6 \times 36$  matrix  $S = S(E_6)$ :

$$S(E_6) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

We find  $\chi_{E_6}(q)$  as follows: First, the exponents of  $E_6$  are 1, 4, 5, 7, 8, 11. So we have

$$\begin{aligned} P_1(q) &= q^6 - 36q^5 + 510q^4 - 3600q^3 + 13089q^2 - 22284q + 12320 \\ &= (q-1)(q-4)(q-5)(q-7)(q-8)(q-11), \end{aligned}$$

which is the ordinary characteristic polynomial of type  $E_6$ . Next we compute:

$$\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_3 = \{1\}, \quad \mathcal{E}_4 = \mathcal{E}_5 = \{1, 2\}, \quad \mathcal{E}_6 = \{1, 2, 3\}.$$

Thus  $\rho_0 = \text{lcm } \mathcal{E}_6 = \text{lcm}\{1, 2, 3\} = 6$ . By Corollary 2.3,

$$P_d(q) = q^6 - 36q^5 + 510q^4 - 3600q^3 + \dots$$

for any  $d|6$ . Since

$$P_6 = P_2 + P_3 - P_1$$

by Corollary 2.5, it is enough to find  $P_2$  and  $P_3$ . Also

$$\deg(P_3 - P_1) < 1$$

by Corollary 2.3. Therefore the following special values

$$P_2(2) = |M_2(S)| = 0, \quad P_3(3) = |M_3(S)| = 0, \quad P_2(4) = |M_4(S)| = 0, \quad P_2(8) = |M_8(S)| = 0$$

are enough for us to obtain:

$$\chi_{E_6}(q) = \begin{cases} q^6 - 36q^5 + 510q^4 - 3600q^3 + 13089q^2 - 22284q + 12320 \\ \quad = (q-1)(q-4)(q-5)(q-7)(q-8)(q-11), \\ \quad \quad \text{gcd}\{6, q\} = 1, \\ q^6 - 36q^5 + 510q^4 - 3600q^3 + 13224q^2 - 23904q + 16640 \\ \quad = (q-2)(q-4)(q-8)(q-10)(q^2 - 12q + 26), \\ \quad \quad \text{gcd}\{6, q\} = 2, \\ q^6 - 36q^5 + 510q^4 - 3600q^3 + 13089q^2 - 22284q + 12960 \\ \quad = (q-3)(q-9)(q^4 - 24q^3 + 195q^2 - 612q + 480), \\ \quad \quad \text{gcd}\{6, q\} = 3, \\ q^6 - 36q^5 + 510q^4 - 3600q^3 + 13224q^2 - 23904q + 17280 \\ \quad = (q-6)^2(q^4 - 24q^3 + 186q^2 - 504q + 480), \\ \quad \quad \text{gcd}\{6, q\} = 6. \end{cases}$$

Thus the minimum period for  $E_6$  is 6.

## 6.2 Characteristic quasi-polynomial of $E_7$

We use PLATE VI in [2] to get the  $7 \times 63$  matrix  $S = S(E_7)$ :

$$S(E_7) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 \\ 2 & 2 & 3 & 3 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 4 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 2 & 3 & 3 & 3 & 3 \\ 2 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

We find  $\chi_{E_7}(q)$  as follows: First, the exponents of  $E_7$  are 1, 5, 7, 9, 11, 13, 17. So we have

$$\begin{aligned} P_1(q) &= q^7 - 63q^6 + 1617q^5 - 21735q^4 + 162939q^3 - 663957q^2 + 1286963q - 765765 \\ &= (q-1)(q-5)(q-7)(q-9)(q-11)(q-13)(q-17), \end{aligned}$$

which is the ordinary characteristic polynomial of type  $E_7$ . Next we compute:

$$\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_3 = \{1\}, \quad \mathcal{E}_4 = \mathcal{E}_5 = \{1, 2\}, \quad \mathcal{E}_6 = \{1, 2, 3\}, \quad \mathcal{E}_7 = \{1, 2, 3, 4\}.$$

Thus  $\rho_0 = \text{lcm } \mathcal{E}_7 = \text{lcm}\{1, 2, 3, 4\} = 12$ . By Corollary 2.3,

$$P_d(q) = q^7 - 63q^6 + 1617q^5 - 21735q^4 + \dots$$

for any  $d|12$ . Since

$$P_6 = P_2 + P_3 - P_1, \quad P_{12} = P_4 + P_3 - P_1$$

by Corollary 2.5, it is enough to find  $P_2, P_3$  and  $P_4$ . Also

$$\deg(P_3 - P_1) < 2, \quad \deg(P_2 - P_4) < 1$$

by Corollary 2.3. Therefore the following special values

$$P_2(2) = |M_2(S)| = 0, P_3(3) = |M_3(S)| = 0, P_4(4) = |M_4(S)| = 0, P_3(9) = |M_9(S)| = 0, \\ P_2(10) = |M_{10}(S)| = 0, P_2(14) = |M_{14}(S)| = 0, P_2(22) = |M_{22}(S)| = 36288000$$

are enough for us to obtain:

$$\chi_{E_7}(q) = \begin{cases} q^7 - 63q^6 + 1617q^5 - 21735q^4 + 162939q^3 - 663957q^2 + 1286963q - 765765 \\ = (q-1)(q-5)(q-7)(q-9)(q-11)(q-13)(q-17), \\ \gcd\{12, q\} = 1, \\ q^7 - 63q^6 + 1617q^5 - 21735q^4 + 163884q^3 - 689472q^2 + 1495808q - 1244880 \\ = (q-2)(q-10)(q-13)(q-14)(q^3 - 24q^2 + 155q - 342), \\ \gcd\{12, q\} = 2, \\ q^7 - 63q^6 + 1617q^5 - 21735q^4 + 162939q^3 - 663957q^2 + 1304883q - 927045 \\ = (q-3)(q-9)(q-15)(q^4 - 36q^3 + 438q^2 - 2052q + 2289), \\ \gcd\{12, q\} = 3, \\ q^7 - 63q^6 + 1617q^5 - 21735q^4 + 163884q^3 - 689472q^2 + 1495808q - 1290240 \\ = (q-4)(q-5)(q-8)(q-16)(q^3 - 30q^2 + 263q - 504), \\ \gcd\{12, q\} = 4, \\ q^7 - 63q^6 + 1617q^5 - 21735q^4 + 163884q^3 - 689472q^2 + 1513728q - 1406160 \\ = (q-6)(q^6 - 57q^5 + 1275q^4 - 14085q^3 + 79374q^2 - 213228q + 234360), \\ \gcd\{12, q\} = 6, \\ q^7 - 63q^6 + 1617q^5 - 21735q^4 + 163884q^3 - 689472q^2 + 1513728q - 1451520 \\ = (q-12)(q^6 - 51q^5 + 1005q^4 - 9675q^3 + 47784q^2 - 116064q + 120960), \\ \gcd\{12, q\} = 12. \end{cases}$$

Thus the minimum period for  $E_7$  is 12.

### 6.3 Characteristic quasi-polynomial of $E_8$

We use PLATE VII in [2] to get the  $8 \times 120$  matrix  $S = S(E_8)$ :

$$S(E_8) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

0 0 1 1 0 0 0 0 0 1 1 0 0 1 0 0 1 1 1 0 0 1 0 1 1 1  
0 0 1 0 1 1 1 0 0 1 1 1 1 0 1 0 1 1 1 1 0 1 1 1 1 1  
0 0 1 1 1 1 0 1 0 1 1 1 1 1 0 1 1 1 2 1 1 1 1 1 2 1  
1 0 1 1 1 2 1 1 1 2 1 2 1 1 1 1 1 2 2 2 2 1 1 2 2 2  
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 2 1 1  
1 1 0 1 1 0 1 1 1 0 1 1 1 1 1 1 1 1 0 1 1 1 1 1 1 1  
1 1 0 0 0 0 1 1 1 0 0 0 1 1 1 1 1 0 0 0 1 1 1 0 0 1  
0 1 0 0 0 0 0 0 1 0 0 0 0 0 1 1 0 0 0 0 0 1 1 0 0 0  
0 1 0 1 1 1 0 0 1 1 1 1 0 1 1 1 1 1 1 1 0 1 1 1 1 1  
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 2 1 1 1  
1 1 1 2 2 1 1 1 1 2 2 1 1 2 1 2 2 2 2 1 1 2 2 2 2 1  
2 1 2 2 2 2 2 2 3 2 2 2 2 2 3 2 3 2 2 2 3 3 3 2 2  
2 1 1 2 1 2 2 2 1 2 2 2 2 1 2 2 2 2 2 2 2 2 2 2 2  
1 1 1 1 1 1 2 1 1 1 1 2 2 1 1 1 2 1 1 2 2 2 1 1 2 2  
1 1 1 0 1 1 1 1 1 0 1 1 1 1 1 0 1 1 1 1 2 1 1 1 1 2  
0 1 1 0 0 0 0 1 1 0 0 0 1 1 1 0 0 0 1 1 1 0 0 1 1 1  
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 2 1 1  
1 2 2 1 1 2 2 1 1 2 2 2 1 2 2 2 1 2 2 2 2 2 2 2 2  
2 2 2 2 2 2 2 2 2 2 2 2 3 2 2 2 3 3 2 2 3 3 2  
3 3 3 3 2 3 3 3 3 4 3 3 3 4 4 3 3 4 4 4 3 4 4 4  
3 2 2 2 2 3 2 3 2 3 3 2 3 3 3 3 3 3 3 3 3 3 3 3  
2 2 1 2 2 2 2 2 2 2 2 2 2 2 2 2 3 2 2 2 3 2 2 3  
1 1 1 1 2 1 1 1 2 1 1 2 2 1 1 2 2 1 1 2 2 1 2 2  
0 0 1 1 1 0 1 1 1 0 1 1 1 0 1 1 1 0 1 1 1 1 1 1  
2 1 1 2 1 1 2 1 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2  
2 2 2 2 2 2 2 3 2 3 2 3 3 3 3 3 3 3 3 3 3 3 3 3  
3 3 2 3 3 3 3 3 3 3 3 4 4 4 4 4 4 4 4 4 4 4 4 4  
4 4 4 4 4 5 4 5 5 5 5 5 6 6 6 6 6 6 6 6 6 6 6 6  
3 3 4 3 4 4 4 4 4 4 4 4 4 4 4 5 5 5 5 5 5 5 5 5  
2 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 4 4 4 4 4 4 4 4  
2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 3 3 3 3 3 3 3  
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

We find  $\chi_{E_8}(q)$  as follows: First, the exponents of  $E_8$  are 1, 7, 11, 13, 17, 19, 23, 29. So we have

$$\begin{aligned}
P_1(q) &= q^8 - 120q^7 + 6020q^6 - 163800q^5 + 2616558q^4 \\
&\quad - 24693480q^3 + 130085780q^2 - 323507400q + 215656441 \\
&= (q - 1)(q - 7)(q - 11)(q - 13)(q - 17)(q - 19)(q - 23)(q - 29)
\end{aligned}$$

which is the ordinary characteristic polynomial of type  $E_8$ . Next we compute:

$$\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_3 = \{1\}, \quad \mathcal{E}_4 = \mathcal{E}_5 = \{1, 2\},$$

$$\mathcal{E}_6 = \{1, 2, 3\}, \quad \mathcal{E}_7 = \{1, 2, 3, 4\}, \quad \mathcal{E}_8 = \{1, 2, 3, 4, 5, 6\}$$

aided by PARI/GP [5]. Thus  $\rho_0 = \text{lcm } \mathcal{E}_8 = \text{lcm}\{1, 2, 3, 4, 5, 6\} = 60$ . By Corollary 2.3,

$$P_d(q) = q^8 - 120q^7 + 6020q^6 - 163800q^5 + \dots$$

for any  $d|60$ . Since

$$\begin{aligned} P_{10} &= P_2 + P_5 - P_1, & P_{15} &= P_3 + P_5 - P_1, & P_{20} &= P_4 + P_5 - P_1, \\ P_{30} &= P_6 + P_5 - P_1, & P_{60} &= P_{12} + P_5 - P_1 \end{aligned}$$

by Corollary 2.4, it is enough to find  $P_2, P_3, P_4, P_5, P_6$  and  $P_{12}$ . Also

$$\deg(P_1 + P_6 - P_2 - P_3) < 1, \quad \deg(P_1 + P_{12} - P_3 - P_4) < 1$$

by Corollary 2.4 again. Moreover,

$$\deg(P_3 - P_1) < 3, \quad \deg(P_2 - P_4) < 2, \quad \deg(P_5 - P_1) < 1$$

by Corollary 2.3. Therefore the following special values

$$\begin{aligned} P_2(2) &= |M_2(S)| = 0, & P_3(3) &= |M_3(S)| = 0, & P_4(4) &= |M_4(S)| = 0, & P_5(5) &= |M_5(S)| = 0, \\ P_6(6) &= |M_6(S)| = 0, & P_4(8) &= |M_8(S)| = 0, & P_3(9) &= |M_9(S)| = 0, & P_{12}(12) &= |M_{12}(S)| = 0, \\ P_2(14) &= |M_{14}(S)| = 0, & P_3(21) &= |M_{21}(S)| = 0, & P_2(22) &= |M_{22}(S)| = 0, \\ P_2(26) &= |M_{26}(S)| = 0, & P_2(34) &= |M_{34}(S)| = 6967296000 \end{aligned}$$

are enough for us to obtain:

$$\begin{aligned} P_1(q) &= q^8 - 120q^7 + 6020q^6 - 163800q^5 + 2616558q^4 \\ &\quad - 24693480q^3 + 130085780q^2 - 323507400q + 215656441 \\ &= (q-1)(q-7)(q-11)(q-13)(q-17)(q-19)(q-23)(q-29), \\ P_2(q) &= q^8 - 120q^7 + 6020q^6 - 163800q^5 + 2626008q^4 \\ &\quad - 25260480q^3 + 141860480q^2 - 418876800q + 435250816 \\ &= (q-2)(q-14)(q-22)(q-26)(q^4 - 56q^3 + 1068q^2 - 8344q + 27176), \\ P_3(q) &= q^8 - 120q^7 + 6020q^6 - 163800q^5 + 2616558q^4 \\ &\quad - 24693480q^3 + 130802580q^2 - 345011400q + 348264441 \\ &= (q-3)(q-9)(q-21)(q-27)(q^4 - 60q^3 + 1250q^2 - 10500q + 22749), \\ P_4(q) &= q^8 - 120q^7 + 6020q^6 - 163800q^5 + 2626008q^4 \\ &\quad - 25260480q^3 + 141860480q^2 - 424320000q + 516898816 \\ &= (q-4)(q-8)(q-16)(q-28)(q^4 - 64q^3 + 1428q^2 - 12536q + 36056), \\ P_5(q) &= q^8 - 120q^7 + 6020q^6 - 163800q^5 + 2616558q^4 \\ &\quad - 24693480q^3 + 130085780q^2 - 323507400q + 243525625 \end{aligned}$$

$$\begin{aligned}
&= (q-5)(q-25)(q^6 - 90q^5 + 3195q^4 - 56700q^3 + 516183q^2 - 2120490q + 1948205), \\
P_6(q) &= q^8 - 120q^7 + 6020q^6 - 163800q^5 + 2626008q^4 \\
&\quad - 25260480q^3 + 142577280q^2 - 440380800q + 587212416 \\
&= (q-6)(q-18)(q^6 - 96q^5 + 3608q^4 - 66840q^3 + 632184q^2 - 2869344q + 5437152), \\
P_{10}(q) &= q^8 - 120q^7 + 6020q^6 - 163800q^5 + 2626008q^4 \\
&\quad - 25260480q^3 + 141860480q^2 - 418876800q + 463120000 \\
&= (q-10)(q^7 - 110q^6 + 4920q^5 - 114600q^4 \\
&\quad + 1480008q^3 - 10460400q^2 + 37256480q - 46312000), \\
P_{12}(q) &= q^8 - 120q^7 + 6020q^6 - 163800q^5 + 2626008q^4 \\
&\quad - 25260480q^3 + 142577280q^2 - 445824000q + 668860416 \\
&= (q-12)(q-24)(q^6 - 84q^5 + 2708q^4 - 42120q^3 + 329784q^2 - 1257696q + 2322432), \\
P_{15}(q) &= q^8 - 120q^7 + 6020q^6 - 163800q^5 + 2616558q^4 \\
&\quad - 24693480q^3 + 130802580q^2 - 345011400q + 376133625 \\
&= (q-15)^2(q^6 - 90q^5 + 3095q^4 - 50700q^3 + 399183q^2 - 1310490q + 1671705), \\
P_{20}(q) &= q^8 - 120q^7 + 6020q^6 - 163800q^5 + 2626008q^4 \\
&\quad - 25260480q^3 + 141860480q^2 - 424320000q + 544768000 \\
&= (q-20)(q^7 - 100q^6 + 4020q^5 - 83400q^4 \\
&\quad + 958008q^3 - 6100320q^2 + 19854080q - 27238400), \\
P_{30}(q) &= q^8 - 120q^7 + 6020q^6 - 163800q^5 + 2626008q^4 \\
&\quad - 25260480q^3 + 142577280q^2 - 440380800q + 615081600, \\
P_{60}(q) &= q^8 - 120q^7 + 6020q^6 - 163800q^5 + 2626008q^4 \\
&\quad - 25260480q^3 + 142577280q^2 - 445824000q + 696729600.
\end{aligned}$$

Thus the minimum period for  $E_8$  is 60.

## 7 Two results

The following two results are obtained from our calculations and the classification of irreducible root systems.

**Theorem 7.1.** *For an irreducible root system  $R$ , the minimum period of the quasi-polynomial  $\chi_R(q)$  is equal to the lcm period  $\rho_0$ .*

**Theorem 7.2.** *Let  $q$  be a positive integer. For an irreducible root system  $R$  with its Coxeter number  $h$ ,  $\chi_R(q) > 0$  if and only if  $q \geq h$ .*

**Remark 7.3.** *It is easy to see the point  $(1, 1, \dots, 1) \in \mathbb{Z}_q^m$  lies in  $M_q(S)$  if  $q \geq h$  because the sum of the elements of each column of  $S$  does not exceed the largest exponent. This shows  $\chi_R(q) > 0$  if  $q \geq h$ . However, our proof of the “only if” part of Theorem 7.2 still requires the classification.*



## References

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