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EXPONENTS OF 2-MULTIARRANGEMENTS AND MULTIPLICITY LATTICES

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ABSTRACT. We introduce a concept of multiplicity lattices of 2-multiarrangements, determine the combinatorics and geometry of that lattice, and give a criterion and method to construct a basis for derivation modules effectively.

1. INTRODUCTION

Let \mathbb{K} be a field and V a two-dimensional vector space over \mathbb{K} . Fix a basis $\{x, y\}$ for V^* and define $S := \text{Sym}(V^*) \simeq \mathbb{K}[x, y]$. A *hyperplane arrangement* \mathcal{A} is a finite collection of affine hyperplanes in V . In this article, we assume that any $H \in \mathcal{A}$ contains the origin, in other words, all hyperplane arrangements are *central*. For each $H \in \mathcal{A}$, let us fix a linear form $\alpha_H \in V^*$ such that $\ker(\alpha_H) = H$. For a hyperplane arrangement \mathcal{A} , a map $\mu : \mathcal{A} \rightarrow \mathbb{N} = \mathbb{Z}_{\geq 0}$ is called a *multiplicity* and a pair (\mathcal{A}, μ) a *multiarrangement*. When we want to make it clear that all multiarrangements are considered in $V \simeq \mathbb{K}^2$, we use the term *2-multiarrangement*. (Ordinary, a 2-multiarrangement is defined as a pair (\mathcal{A}, m) of a central hyperplane arrangement \mathcal{A} and multiplicity function $m : \mathcal{A} \rightarrow \mathbb{Z}_{>0}$. From a 2-multiarrangement (\mathcal{A}, μ) in our definition, we can obtain a 2-multiarrangement (\mathcal{A}', m) in the original definition by assigning $\mathcal{A}' = \mu^{-1}(\mathbb{Z}_{>0})$ and $m = \mu|_{\mathcal{A}'}$. We identify ours with the original one in this manner.) To each multiarrangement (\mathcal{A}, μ) , we can associate the S -module $D(\mathcal{A}, \mu)$, called the *derivation module* by the following manner:

$$D(\mathcal{A}, \mu) := \left\{ \delta \in \text{Der}_{\mathbb{K}}(S) \mid \delta(\alpha_H) \in S \cdot \alpha_H^{\mu(H)} \ (\forall H \in \mathcal{A}) \right\},$$

where $\text{Der}_{\mathbb{K}}(S) := S \cdot \partial_x \oplus S \cdot \partial_y$ is the module of derivations. It is known that $D(\mathcal{A}, \mu)$ is a free graded S -module because we only consider 2-multiarrangements (see [8], [7] and [15]). If we choose a homogeneous basis $\{\theta, \theta'\}$ for $D(\mathcal{A}, \mu)$, then the *exponents* of (\mathcal{A}, μ) , denoted by $\text{exp}(\mathcal{A}, \mu)$, is a multiset defined by

$$\text{exp}(\mathcal{A}) := (\deg(\theta), \deg(\theta')),$$

where the degree is a polynomial degree.

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A multiarrangement is originally introduced by Ziegler in [15] and there are a lot of studies related to a multiarrangement and its derivation module. Especially, Yoshinaga characterized the freeness of hyperplane arrangements by using the freeness of multiarrangements ([13] and [14]). In particular, according to the results in [14], we can obtain the necessary and sufficient condition for a hyperplane arrangement in three-dimensional vector space to be free in terms of the combinatorics of hyperplane arrangements, and the explicit description of exponents of 2-multiarrangements. This is closely related to the Terao conjecture, which asserts that the freeness of hyperplane arrangements depends only on the combinatorics. However, instead of the simple description of the exponents of hyperplane arrangements, it is shown by Wakefield and Yuzvinsky in [11] that the general description of the exponents of 2-multiarrangements are very difficult. In fact, there are only few results related to them ([1], [2] and [10]). Recently, some theory to study the freeness of multiarrangements are developed by the first author, Terao and Wakefield in [5] and [6], and some results on the free multiplicities are appearing ([3]). In these papers, the importance of the exponents of 2-multiarrangements is emphasized too. Hence it is very important to establish some general theory for the exponents of 2-multiarrangements.

The aim of this article is to give some answers to this problem. Our idea is to introduce the concept of the *multiplicity lattice* of the fixed hyperplane arrangement. The aim of the study of this lattice is similar, but the method is contrary to the study in [11], for Wakefield and Yuzvinsky fixed one multiplicity and consider all hyperplane arrangements with it, but we fix one hyperplane arrangement and consider all multiplicities on it. Let us fix a central hyperplane arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ and the lattice $\Lambda = \mathbb{N}^{|\mathcal{A}|}$ of compositions of nonnegative integers whose lengths are $|\mathcal{A}|$. We identify $\mu \in \Lambda$ with the map $\mathcal{A} \rightarrow \mathbb{N}$ such that $\mu(H_i) = \mu_i$ for $H_i \in \mathcal{A}$. Define a map $\Delta : \Lambda \rightarrow \mathbb{Z}_{\geq 0}$ by

$$\Delta(\mu) := |\deg(\theta_\mu) - \deg(\theta'_\mu)|,$$

where $\{\theta_\mu, \theta'_\mu\}$ is a basis for $D(\mathcal{A}, \mu)$ such that $\deg(\theta_\mu) \leq \deg(\theta'_\mu)$. If we put $\Lambda' := \Lambda \setminus \Delta^{-1}(\{0\})$, then θ_μ is unique up to the scalar for each $\mu \in \Lambda'$, though θ'_μ is not. Hence θ_μ for $\mu \in \Lambda'$ is expected to have some good properties. Roughly speaking, our results about this problem are summarized as follows:

- (1) For a maximal connected component C of Λ' . Also, C has infinitely many elements if and only if C has unbalanced elements (Theorem 3.1).
- (2) A finite maximal connected component C of Λ' is a ball and $\Delta(\mu)$ is determined by the radius of C and the distance between

μ and the center of C for each $\mu \in C$. Equivalently, for each $\mu \in C$, $\exp(\mathcal{A}, \mu)$ is determined by them (Theorem 3.2).

- (3) If distinct maximal connected components C and C' of Λ' are next to each other, then θ_μ and $\theta_{\mu'}$ are S -linearly independent for $\mu \in C$, $\mu' \in C'$ (Theorem 3.4).

The results above and Saito's criterion (Theorem 4.1) imply that the set $\{\theta_\mu, \theta_{\mu'}\}$ forms a basis for $D(\mathcal{A}, \mu)$ and by using this basis we can describe any basis for $D(\mathcal{A}, \nu)$ such that $\mu \wedge \mu' \leq \nu \leq \mu \vee \mu'$, where $\mu \in C$ and $\mu' \in C'$ respectively attain the maximal values of Δ on distinct maximal connected components C and C' . (Theorem 3.9). Hence to determine the basis for 2-multiarrangements, it is effective to determine the shape, topology and combinatorics of Λ' .

Now the organization of this article is as follows. In Section 2, we introduce some notation. In Section 3, we state the main results. In Section 4, we recall elementary results about hyperplane arrangement theory and prove the main results. In Section 5, we show some applications of main results, especially determine some exponents of multiarrangements of the Coxeter type.

2. DEFINITION AND NOTATION

In this section, we introduce some basic terms and notation. Let \mathbb{K} be a field, V a two-dimensional vector space over \mathbb{K} , and S a symmetric algebra of V^* . By choosing a basis $\{x, y\}$ for V^* , S can be identified with a polynomial ring $\mathbb{K}[x, y]$. The algebra S can be graded by polynomial degree as $S = \bigoplus_{i \in \mathbb{N}} S_i$, where S_i is a vector space whose basis is $\{x^j y^{i-j} \mid j = 0, \dots, i\}$.

Let us fix a central hyperplane arrangement \mathcal{A} in V , i.e., a finite collection $\{H_1, \dots, H_n\}$ of linear hyperplanes in V . For $H \in \mathcal{A}$, fix $\alpha_H \in S_1$ such that $\ker(\alpha_H) = H$. The following new definition plays the key role in this article.

Definition 2.1. We define *the multiplicity lattice* Λ of \mathcal{A} by

$$\Lambda := (\mathbb{N})^{|\mathcal{A}|} = (\mathbb{N})^n.$$

Let us identify $\mu = (\mu_1, \dots, \mu_n) \in \Lambda$ with the multiplicity $\mu : \mathcal{A} \rightarrow \mathbb{N}$ defined by $\mu(H_i) := \mu_{H_i} = \mu_i$. Hence a pair (\mathcal{A}, μ) can be considered as a multiarrangement. The set Λ has a structure of a lattice as compositions of nonnegative integers, i.e.,

$$\mu \subset \nu \iff \mu_H \leq \nu_H \text{ for all } H.$$

For $\mu, \nu \in \Lambda$, the binary operations \wedge and \vee are defined by

$$\begin{aligned} \mu \wedge \nu &:= \inf \{ \mu, \nu \}, \\ \mu \vee \nu &:= \sup \{ \mu, \nu \}, \end{aligned}$$

i.e., $(\mu \wedge \nu)_H = \min \{ \mu_H, \nu_H \}$ and $(\mu \vee \nu)_H = \max \{ \mu_H, \nu_H \}$. For $\mu \in \Lambda$, we define the *size* $|\mu|$ of μ by $|\mu| := \sum_{H \in \mathcal{A}} \mu_H$. The composition

0, which is defined by $0_H = 0$ for all $H \in \mathcal{A}$, is the minimum element. The *covering relation* $\mu \dot{\subset} \nu$ is defined by $\mu \subset \nu$ and $|\mu| + 1 = |\nu|$. The graph whose set of edges is $\{(\mu, \nu) \in \Lambda^2 \mid \mu \dot{\subset} \nu\}$ and whose set of vertices is Λ is called the Hasse graph of Λ . We identify Λ and its subset with (the set of vertices of) the Hasse graph and its induced subgraph, respectively. For $\mu, \nu \in \Lambda$, we define the *distance* $d(\mu, \nu)$ by $d(\mu, \nu) := \sum_{H \in \mathcal{A}} |\mu_H - \nu_H|$. For $C, C' \subset \Lambda$, we define $d(C, C')$ by $d(C, C') := \min \{d(\mu, \mu') \mid \mu \in C, \mu' \in C'\}$. For $\mu \in \Lambda$ and $d \in \mathbb{N}$, we define the *ball* $B(\mu, d)$ with the radius d and center μ by $B(\mu, d) := \{\nu \in \Lambda \mid d(\mu, \nu) < d\}$.

Definition 2.2. We define a map $\Delta : \Lambda \rightarrow \mathbb{N}$ by

$$\Delta(\mu) := |d_1 - d_2|,$$

where (d_1, d_2) are the exponents of a free multiarrangement (\mathcal{A}, μ) .

Definition 2.3. Let Λ' denote the support $\Delta^{-1}(\mathbb{Z}_{>0})$. For $H \in \mathcal{A}$, let us define Λ_H to be the set

$$\left\{ \mu \in \Lambda \mid \mu_H > \frac{1}{2} |\mu| \right\}.$$

We define Λ_\emptyset and Λ'_\emptyset by

$$\Lambda_\emptyset := \Lambda \setminus \left(\bigcup_{H \in \mathcal{A}} \Lambda_H \right) = \left\{ \mu \in \Lambda \mid \mu_H \leq \frac{1}{2} |\mu| \ (\forall H \in \mathcal{A}) \right\},$$

$$\Lambda'_\emptyset := \Lambda_\emptyset \cap \Lambda'.$$

Roughly speaking, Λ_\emptyset consists of balanced elements while Λ_H consists of elements such that H monopolizes at least half of their multiplicities.

For each $\mu \in \Lambda'$, there exist θ_μ and θ'_μ such that $\deg(\theta_\mu) < \deg(\theta'_\mu)$ and $\{\theta_\mu, \theta'_\mu\}$ is a homogeneous basis for $D(\mathcal{A}, \mu)$. Since $\Delta(\mu) \neq 0$, θ_μ is unique up to nonzero scalar for each $\mu \in \Lambda'$. Hence we can define a map $\theta : \Lambda' \rightarrow D(\mathcal{A}, 0) = \text{Der}_{\mathbb{K}}(S)$ by $\theta(\mu) := \theta_\mu$ (up to scalar, or regard the image of θ as a one-dimensional vector space of $D(\mathcal{A}, 0)$).

Definition 2.4. Let us define $\text{cc}(\Lambda')$, $\text{cc}_0(\Lambda')$ and $\text{cc}_\infty(\Lambda')$ by

$$\text{cc}(\Lambda') := \{ \text{maximal connected components of } \Lambda' \},$$

$$\text{cc}_0(\Lambda') := \{ C \in \text{cc}(\Lambda') \mid |C| < \infty \},$$

$$\text{cc}_\infty(\Lambda') := \{ C \in \text{cc}(\Lambda') \mid |C| = \infty \},$$

where μ and ν are said to be *connected* if there exists a path from μ to ν in the induced subgraph Λ' of the Hasse graph. For $C \in \text{cc}(\Lambda')$, $\mu \in C$ and $H \in \mathcal{A}$, the set $C_{\mu, H}$ is defined by

$$C_{\mu, H} = \{ \nu \in C \mid \nu_{H'} = \mu_{H'} \text{ for each } H' \in \mathcal{A} \setminus \{H\} \}.$$

Definition 2.5. For $C \in \text{cc}_0(\Lambda')$, we define $P(C)$ by

$$P(C) := \{ \mu \in C \mid \Delta(\mu) = \max \{ \Delta(\nu) \mid \nu \in C \} \}$$

and $P(\Lambda')$ by

$$P(\Lambda') := \bigcup_{C \in \text{cc}_0(\Lambda')} P(C).$$

Definition 2.6. For a saturated chain ρ in Λ , i.e., a sequence $\rho = (\rho^{(0)}, \dots, \rho^{(k)})$ of elements in Λ satisfying $\rho^{(i)} \dot{\subset} \rho^{(i+1)}$, we define α_ρ^- by

$$\alpha_\rho^- = \prod_{i: \Delta(\rho^{(i)}) > \Delta(\rho^{(i+1)})} \alpha_i,$$

where $\alpha_i = \alpha_H$ such that $\rho_H^{(i)} + 1 = \rho_H^{(i+1)}$.

3. MAIN RESULTS

In this section we state the main results. First let us give three theorems which show the structure of Λ' .

Theorem 3.1. *We have the following:*

- For each $C \in \text{cc}_0(\Lambda')$, $C \subset \Lambda_\emptyset$. Moreover, $\bigcup_{C \in \text{cc}_0(\Lambda')} C = \Lambda'_\emptyset$.
- $\text{cc}_\infty(\Lambda') = \{ \Lambda_H \mid H \in \mathcal{A} \}$.
- Any maximal connected component of $\Lambda \setminus \Lambda' = \Delta^{-1}(\{0\})$ consists of one point, i.e., all of them are singletons.

Theorem 3.2. *Let $C \in \text{cc}_0(\Lambda')$ and $\mu \in P(C)$. Then*

$$C = B(\mu, \Delta(\mu)),$$

and, for $\nu \in C$,

$$\Delta(\nu) = \Delta(\mu) - d(\mu, \nu).$$

In particular, for $C \in \text{cc}_0(\Lambda')$, $P(C)$ consists of one point.

Corollary 3.3. *For $\mu \in P(C)$ and $\nu \in \Lambda$ satisfying $d(\mu, \nu) < \Delta(\mu) + 2$,*

$$\Delta(\nu) = |\Delta(\mu) - d(\mu, \nu)|.$$

Next result implies the independency of “low-degree” bases.

Theorem 3.4. *Let $C, C' \in \text{cc}_0(\Lambda')$ such that $d(C, C') = 2$. If $\mu \in C$ and $\mu' \in C'$, then $\{\theta_\mu, \theta_{\mu'}\}$ is S -linearly independent. Moreover, if $C \in \text{cc}(\Lambda')$ and $\mu, \mu' \in C$, then $\{\theta_\mu, \theta_{\mu'}\}$ is S -linearly dependent.*

The theorems above imply the following three corollaries, which enable us to construct the basis for $D(\mathcal{A}, \mu)$ effectively.

Corollary 3.5. *Let $N \subset \Lambda_\emptyset$ such that $\Lambda_\emptyset \setminus N$ does not have any connected component whose size is larger than 1, $\vartheta : N \rightarrow D(\mathcal{A}, 0)$ such that $\vartheta_\mu \in D(\mathcal{A}, \mu)$ and $\deg \vartheta_\mu < \frac{|\mu|}{2}$. Then the following are equivalent:*

- $\{\vartheta_\mu, \vartheta_\nu\}$ is S -linearly independent if

$$\min \left\{ d(\mu', \nu') \mid \begin{array}{l} \mu \text{ and } \mu' \text{ are in the same connected component in } N \\ \nu \text{ and } \nu' \text{ are in the same connected component in } N \end{array} \right\} = 2$$

- $N = \Lambda'_\emptyset$.

Corollary 3.6. *Let $N \subset \Lambda_\emptyset$ and $\vartheta : N \rightarrow D(\mathcal{A}, 0)$ such that $\vartheta_\mu \in D(\mathcal{A}, \mu)$, $\deg \vartheta_\mu < \frac{|\mu|}{2}$ and $\Delta'(\mu) = |\mu| - 2 \deg \vartheta_\mu > 0$. Assume that $B(\mu, \Delta'(\mu))$ and $B(\nu, \Delta'(\nu))$ are disjoint for $\mu \neq \nu \in N$, and that $\Lambda_\emptyset \setminus \bigcup_{\mu \in N} B(\mu, \Delta'(\mu))$ has no connected components whose size is larger than 1. Then the following are equivalent:*

- $\{\vartheta_\mu, \vartheta_\nu\}$ are S -linearly independent if $\Delta'(\mu) + \Delta'(\nu) = d(\mu, \nu)$.
- $N = P(\Lambda')$ and $\vartheta_\mu = \theta_\mu$ for each $\mu \in P(\Lambda')$.

Corollary 3.7. *Let $N = \{\mu \in \Lambda_\emptyset \mid |\mu| \text{ is odd.}\}$ and $\vartheta : N \rightarrow D(\mathcal{A}, 0)$ such that $\vartheta_\mu \in D(\mathcal{A}, \mu)$ and $\deg \vartheta_\mu < \frac{|\mu|}{2}$. Define the equivalence relation \sim generated by*

$$\mu \sim \nu \iff \{\vartheta_\mu, \vartheta_\nu\} \text{ is } S\text{-linearly dependent and } d(\mu, \nu) = 2.$$

Then the following are equivalent for $\mu, \nu \in N$:

- $\mu \sim \nu$.
- $\mu, \nu \in C \in \text{cc}_0(\Lambda')$.

Remark 3.8. In Corollaries 3.5, 3.6 and 3.7, we do not require the condition $\deg(\vartheta_\mu) = \deg(\theta_\mu)$.

Finally we state the theorems which describe the behavior of the basis near, or between the centers of connected balls.

Theorem 3.9. *Assume that $\mu, \nu \in \Lambda'$ belong to distinct connected components and satisfy $\Delta(\mu) + \Delta(\nu) = d(\mu, \nu)$. Let $\kappa \in \Lambda$ such that $\mu \wedge \nu \subset \kappa \subset \mu \vee \nu$, and*

$$\alpha_{\mu, \kappa} = \prod_{H \in \mathcal{A}} \alpha_H^{\max\{\kappa_H - \mu_H, 0\}},$$

$$\alpha_{\nu, \kappa} = \prod_{H \in \mathcal{A}} \alpha_H^{\max\{\kappa_H - \nu_H, 0\}}.$$

Then $\{\alpha_{\mu, \kappa} \theta_\mu, \alpha_{\nu, \kappa} \theta_\nu\}$ is a homogeneous basis for $D(\mathcal{A}, \kappa)$.

Corollary 3.10. *For each $\mu \in \Lambda_\emptyset$, we can construct a homogeneous basis for $D(\mathcal{A}, \mu)$ from the restricted map $\theta|_{P(\Lambda')}$.*

4. PROOFS OF MAIN RESULTS

In this section, we prove the main results. To prove them, first we recall a result about hyperplane arrangements and derivation modules. The following is the two-dimensional version of the famous Saito's criterion, which is very useful to find the basis for $D(\mathcal{A}, m)$. See Theorem 8 in [15] and Theorem 4.19 in [7] for the proof.

Theorem 4.1 (Saito's criterion). *Let (\mathcal{A}, μ) be a 2-multiarrangement and $\theta_1, \theta_2 \in D(\mathcal{A}, \mu)$. Then $\{\theta_1, \theta_2\}$ forms a basis for $D(\mathcal{A}, \mu)$ if and only if $\{\theta_1, \theta_2\}$ is independent and $\deg(\theta_1) + \deg(\theta_2) = |\mu|$.*

4.1. Proofs of Theorems 3.1 and 3.2.

Lemma 4.2. *If $\mu, \nu \in \Lambda$ and $\mu \dot{\subset} \nu$, then $|\Delta(\mu) - \Delta(\nu)| = 1$.*

Proof. It follows from the fact that $D(\mathcal{A}, \mu) \supset D(\mathcal{A}, \nu)$ and Saito's criterion. \square

Lemma 4.3. *Assume that $\mu, \nu \in \Lambda'$ and $\mu \dot{\subset} \nu$ with $\nu_H = \mu_H + 1$ for some $H \in \mathcal{A}$. Then*

$$\theta_\nu = \begin{cases} \alpha_H \theta_\mu & \text{if } \Delta(\mu) > \Delta(\nu), \\ \theta_\mu & \text{if } \Delta(\mu) < \Delta(\nu). \end{cases}$$

Proof. Fix a homogeneous basis $\{\theta_\mu, \theta'\}$ for $D(\mathcal{A}, \mu)$, where $\deg(\theta_\mu) < \deg(\theta')$. If $\Delta(\mu) > \Delta(\nu)$, then Saito's criterion implies $\theta_\mu \notin D(\mathcal{A}, \nu)$. Since $\alpha_H \theta_\mu \in D(\mathcal{A}, \nu)$, Lemma 4.2 implies $\alpha_H \theta_\mu$ is a part of a homogeneous basis for $D(\mathcal{A}, \nu)$. Hence we may assume that $\{\alpha_H \theta_\mu, \theta''\}$ is a basis for $D(\mathcal{A}, \nu)$. If $\Delta(\mu) < \Delta(\nu)$, then $\theta_\mu \in D(\mathcal{A}, \nu)$, which completes the proof. \square

Corollary 4.4. *Let $\mu, \nu \in C \in \text{cc}(\Lambda')$ with $\mu \subset \nu$, and ρ be a saturated chain from μ to ν . Then $\theta_\nu = \alpha_\rho^- \theta_\mu$.*

Proof. Apply Lemma 4.3 repeatedly. \square

Lemma 4.5. *For $C \in \text{cc}_0(\Lambda')$, $\mu \in C$ and $H \in \mathcal{A}$, $\Delta|_{C_{\mu,H}}$ is unimodal, i.e., there exists a unique element $\kappa \in C_{\mu,H}$ such that*

$$\Delta(\nu') \leq \Delta(\nu) \text{ for } \nu' \subset \nu \subset \kappa \text{ or } \kappa \subset \nu \subset \nu'.$$

Proof. Let $\nu, \nu', \nu'' \in C_{\mu,H}$ satisfy $\nu \dot{\subset} \nu' \dot{\subset} \nu''$. Assume that $\Delta(\nu) > \Delta(\nu') < \Delta(\nu'')$. By Lemma 4.3, we may choose a basis $\{\alpha_H \theta_\nu, \theta'\}$ for $D(\mathcal{A}, \nu')$ such that $\{\alpha_H \theta_\nu, \alpha_H \theta'\}$ is a basis for $D(\mathcal{A}, \nu'')$. Hence $\alpha_H \theta_\nu(\alpha_H) \in S \cdot \alpha_H^{\nu''} = S \cdot \alpha_H^{\nu_H+2}$ and $\theta_\nu(\alpha_H) \in S \cdot \alpha_H^{\nu_H+1}$. Then $\theta_\nu \in D(\mathcal{A}, \nu')$, which is a contradiction. \square

Definition 4.6. For $H \in \mathcal{A}$, $C \in \text{cc}_0(\Lambda')$ and $\mu \in C$, we may choose, by Lemma 4.5, the unique element $\kappa \in C_{\mu,H}$ such that $\Delta(\kappa) \geq \Delta(\mu')$ for any $\mu' \in C_{\mu,H}$. We call this κ the *peak element with respect to $C_{\mu,H}$* .

Corollary 4.7. *Let $C \in \text{cc}_0(\Lambda')$, $\mu \in C$ and $H \in \mathcal{A}$. Let $\kappa \in C$ be the peak element with respect to $C_{\mu,H}$. Then, for $\mu' \in C_{\mu,H}$,*

$$\theta_{\mu'} = \begin{cases} \theta_\kappa & (\mu' \subset \kappa), \\ \alpha_H^{|\mu'| - |\kappa|} \theta_\kappa & (\kappa \subset \mu'). \end{cases}$$

Proof. Apply Lemmas 4.3 and 4.5. \square

Lemma 4.8. *Let $C \in \text{cc}(\Lambda')$, and $\kappa, \mu, \mu', \nu \in \Lambda$. Assume that $\kappa \dot{\subset} \mu \dot{\subset} \nu$, $\kappa \dot{\subset} \mu' \dot{\subset} \nu$, and $\mu \neq \mu'$.*

(1) Assume that $\kappa, \mu, \mu' \in C$. Then

$$\begin{aligned} \Delta(\kappa) > \Delta(\mu), \Delta(\kappa) > \Delta(\mu') &\Rightarrow \Delta(\mu) > \Delta(\nu), \Delta(\mu') > \Delta(\nu), \\ \Delta(\kappa) < \Delta(\mu), \Delta(\kappa) < \Delta(\mu') &\Rightarrow \Delta(\mu) < \Delta(\nu), \Delta(\mu') < \Delta(\nu), \\ \Delta(\kappa) < \Delta(\mu), \Delta(\kappa) > \Delta(\mu') &\Rightarrow \Delta(\mu) > \Delta(\nu), \Delta(\mu') < \Delta(\nu). \end{aligned}$$

(2) Assume that $\mu, \mu', \nu \in C$. Then

$$\begin{aligned} \Delta(\mu) > \Delta(\nu), \Delta(\mu') > \Delta(\nu) &\Rightarrow \Delta(\kappa) > \Delta(\mu), \Delta(\kappa) > \Delta(\mu'), \\ \Delta(\mu) < \Delta(\nu), \Delta(\mu') < \Delta(\nu) &\Rightarrow \Delta(\kappa) < \Delta(\mu), \Delta(\kappa) < \Delta(\mu'), \\ \Delta(\mu) < \Delta(\nu), \Delta(\mu') > \Delta(\nu) &\Rightarrow \Delta(\kappa) > \Delta(\mu), \Delta(\kappa) < \Delta(\mu'). \end{aligned}$$

(3) Assume that $\kappa, \mu, \nu \in C$. Then

$$\begin{aligned} \Delta(\kappa) > \Delta(\mu) > \Delta(\nu) &\Rightarrow \Delta(\kappa) > \Delta(\mu') > \Delta(\nu), \\ \Delta(\kappa) < \Delta(\mu) < \Delta(\nu) &\Rightarrow \Delta(\kappa) < \Delta(\mu') < \Delta(\nu), \\ \Delta(\kappa) < \Delta(\mu) > \Delta(\nu) &\Rightarrow \Delta(\kappa) > \Delta(\mu') < \Delta(\nu), \\ \Delta(\kappa) > \Delta(\mu) < \Delta(\nu) &\Rightarrow \Delta(\kappa) < \Delta(\mu') > \Delta(\nu). \end{aligned}$$

Proof. (1) Assume that $\kappa_H + 1 = \mu_H$ and $\kappa_{H'} + 1 = \mu'_{H'}$ for some $H \neq H' \in \mathcal{A}$. Since $\nu = \mu \vee \mu'$, $\mu_{H'} + 1 = \nu_{H'}$ and $\mu'_{H'} + 1 = \nu_{H'}$. First we consider the case when $\Delta(\kappa) > \Delta(\mu)$ and $\Delta(\kappa) > \Delta(\mu')$. Then $\Delta(\mu) = \Delta(\mu')$. It follows from Lemma 4.3 that $\theta_\mu = \alpha_H \theta_\kappa$, $\theta_{\mu'} = \alpha_{H'} \theta_\kappa$. If $\Delta(\mu) = \Delta(\mu') < \Delta(\nu)$, then $\Delta(\nu) > 0$, i.e., $\nu \in \Lambda'$. Then Lemma 4.3 implies that

$$\alpha_{H'} \theta_\kappa = \theta_{\mu'} = \theta_\nu = \theta_\mu = \alpha_H \theta_\kappa,$$

which is a contradiction.

Next we consider the case when $\Delta(\kappa) < \Delta(\mu)$ and $\Delta(\kappa) < \Delta(\mu')$. Then $\Delta(\mu) = \Delta(\mu')$ and $\Delta(\kappa) \leq \Delta(\nu)$. Hence $\nu \in C$. It follows from Lemma 4.3 that $\theta_\mu = \theta_\kappa$, $\theta_{\mu'} = \theta_\kappa$. If $\Delta(\mu) = \Delta(\mu') > \Delta(\mu)$, then Lemma 4.3 implies that

$$\alpha_H \theta_\kappa = \alpha_H \theta_{\mu'} = \theta_\nu = \alpha_{H'} \theta_\mu = \alpha_{H'} \theta_\kappa,$$

which is a contradiction.

Finally we consider the case when $\Delta(\kappa) < \Delta(\mu)$ and $\Delta(\kappa) > \Delta(\mu')$. Then $\Delta(\mu) - 1 = \Delta(\mu') + 1 = \Delta(\kappa)$. Hence $\Delta(\nu) = \Delta(\mu) - 1 = \Delta(\mu') + 1 = \Delta(\kappa)$.

The same argument is valid for (2) and (3), which completes the proof. \square

Remark 4.9. In case (1), (2) and (3) in Lemma 4.8, $\Delta(\nu) = 0$, $\Delta(\kappa) = 0$ and $\Delta(\mu') = 0$ may happen, respectively.

Lemma 4.10. *Let $\mu, \mu' \in \Lambda$ such that $|\mu| = |\mu'|$ and $\mu \neq \mu'$. Then the following are equivalent:*

- *At least three of $\{\mu \wedge \mu', \mu, \mu', \mu \vee \mu'\}$ are in the same connected component $C \in \text{cc}(\Lambda')$.*

- $(\Delta(\mu \vee \mu') - \Delta(\mu'))(\Delta(\mu) - \Delta(\mu \wedge \mu')) > 0$.
- $(\Delta(\mu \vee \mu') - \Delta(\mu))(\Delta(\mu') - \Delta(\mu \wedge \mu')) > 0$.

Proof. It directly follows from Lemma 4.8. \square

Lemma 4.11. For $\mu \in C \in \text{cc}(\Lambda')$, define X_μ by $X_\mu := \bigcup_{H \in \mathcal{A}} C_{\mu, H}$. If μ satisfies $\Delta(\mu) = \max \{ \Delta(\nu) \mid \nu \in X_\mu \}$, then

$$\Delta(\kappa) = \Delta(\mu) - d(\kappa, \mu)$$

for $\kappa \in \Lambda$ with $d(\kappa, \mu) \leq \Delta(\mu)$. In particular, C is the ball $B(\mu, \Delta(\mu))$.

Proof. Apply Lemma 4.10 repeatedly. \square

Proof of Theorem 3.2. Let $C \in \text{cc}_0(\Lambda')$ and $\mu \in P(C)$. Then it follows from Lemma 4.5 that $\Delta|_{C_{\mu, H}}$ is unimodal for all $H \in \mathcal{A}$. Hence Lemma 4.11 completes the proof. \square

Lemma 4.12. Let $H \in \mathcal{A}$ and $\mu, \nu \in C \in \text{cc}_\infty(\Lambda')$ satisfy $\mu \dot{<} \mu'$ with $\mu_H + 1 = \mu'_H$. If $|C_{\mu, H'}| < \infty$ for some $H' \in \mathcal{A} \setminus \{H\}$, then $|C_{\mu', H'}| < \infty$. Moreover, for $H' \in \mathcal{A} \setminus \{H\}$, μ is the peak element with respect to $C_{\mu, H'}$ if and only if μ' is the peak element with respect to $C_{\mu', H'}$.

Proof. Let $|C_{\mu, H'}| < \infty$. By Lemmas 4.10 and 4.5, $|C_{\mu', H'}| = |C_{\mu, H'}| \pm 2$. Hence $|C_{\mu', H'}| < \infty$.

Let μ be the peak element with respect to $C_{\mu, H'}$. It follows from Lemma 4.10 that μ' is the peak element with respect to $C_{\mu', H'}$, and vice versa. \square

Lemma 4.13. Let $C \in \text{cc}(\Lambda')$. If there exists $\mu \in C$ satisfying $|C_{\mu, H}| < \infty$ for any $H \in \mathcal{A}$, then $C \in \text{cc}_0(\Lambda')$. Hence, for $\mu \in C \in \text{cc}_\infty(\Lambda')$, there exists $H \in \mathcal{A}$ such that $|C_{\mu, H}| = \infty$.

Proof. For $H \in \mathcal{A}$, $C \in \text{cc}(\Lambda')$ and $\mu \in C$, define $m_{\mu, H}$ and \mathcal{B}_μ by $m_{\mu, H} := \max \{ \Delta(\mu') \mid \mu' \in C_{\mu, H} \}$ and $\mathcal{B}_\mu := \{ H \in \mathcal{A} \mid \Delta(\mu) = m_{\mu, H} \}$. Assume that $|C_{\mu, H}| < \infty$ for all $H \in \mathcal{A}$. Let us construct ν as follows:

- (1) Let ν be μ .
- (2) Repeat the following until $\mathcal{A} = \mathcal{B}_\nu$:
 - (a) Choose $H_0 \in \mathcal{A} \setminus \mathcal{B}_\nu$ and the peak element ν' with respect to C_{ν, H_0} .
 - (b) Let ν be ν' .

By the assumption and Lemma 4.12, $|C_{\nu', H}| < \infty$ for all $H \in \mathcal{A}$ and $\Delta(\nu') = m_{\nu', H}$ for all $H \in \mathcal{B}_\nu$. Hence, by Lemma 4.12, $\mathcal{B}_{\nu'} = \mathcal{B}_\nu \amalg \{H_0\}$. Since $|\mathcal{A}| < \infty$, we can always find $\nu \in C$ such that $\Delta(\nu) = m_{\nu, H}$ for all $H \in \mathcal{A}$. Hence Lemma 4.11 implies that $C \in \text{cc}_0(\Lambda')$. \square

Lemma 4.14. $\text{cc}_\infty(\Lambda') = \{ \Lambda'_H \mid H \in \mathcal{A} \}$.

Proof. Lemma 4.13 implies that, for $\mu \in C \in \text{cc}_\infty(\Lambda')$, there exists H such that $|C_{\mu,H}| = \infty$. Hence if $\nu \in \Lambda$ satisfies

$$\nu(H') = \begin{cases} \mu(H) + |\mu| & (H = H'), \\ \mu(H') & (H \neq H'), \end{cases}$$

then $\nu \in C_{\mu,H}$. By definition, $\nu \in \Lambda'_H$. Since μ and ν belong to the same component C , μ is also in Λ'_H . On the other hand, $\Lambda'_H \in \text{cc}_\infty(\Lambda')$. Since Λ'_H is connected, $C = \Lambda'_H$. \square

Proof of Theorem 3.1. Apply Lemma 4.2 and 4.14. \square

4.2. Proof of Theorem 3.4. In this subsection we prove Theorem 3.4. Roughly speaking, the proof is based on the observation of θ_μ for μ in some finite balls in Theorem 3.2.

Lemma 4.15. *Let $C \in \text{cc}_0(\Lambda')$, $\kappa \in C$ and $\mu \in P(C)$. Then we can construct θ_μ from θ_κ , and vice versa.*

Proof. By Theorem 3.2, $\mu \wedge \kappa \in C$. It follows from Lemma 4.4 that

$$\begin{aligned} \theta_\mu &= \alpha_\rho^- \theta_{\mu \wedge \kappa}, \\ \theta_\kappa &= \alpha_{\rho'}^- \theta_{\mu \wedge \kappa} \end{aligned}$$

for some saturated chains ρ and ρ' . Hence we have

$$\begin{aligned} \theta_\mu &= \frac{\alpha_\rho^-}{\alpha_{\rho'}^-} \theta_\kappa, \\ \theta_\kappa &= \frac{\alpha_{\rho'}^-}{\alpha_\rho^-} \theta_\mu. \end{aligned}$$

\square

Lemma 4.16. *Let $C \in \text{cc}_0(\Lambda')$ and $\mu, \nu \in C$. Then $\{\theta_\mu, \theta_\nu\}$ is S -linearly dependent.*

Proof. The lemma follows from Lemma 4.15. \square

Lemma 4.17. *Let $\mu, \nu \in \Lambda'$ satisfy $d(\mu, \nu) = 2$. If $\Delta(\kappa) = 0$ for all $\kappa \in \Lambda$ such that $d(\mu, \kappa) = d(\nu, \kappa) = 1$, then $\{\theta_\mu, \theta_\nu\}$ is S -linearly independent.*

Proof. First assume that $\mu_H + 2 = \nu_H$ for some $H \in \mathcal{A}$ and $\mu_{H''} = \nu_{H''}$ for $H'' \in \mathcal{A} \setminus \{H\}$. Let $\kappa \in \Lambda$ be the element such that $\mu \dot{<} \kappa \dot{<} \nu$. Since $\Delta(\kappa) = 0$, $\theta_\mu \notin D(\mathcal{A}, \kappa)$. Hence $\alpha_H \theta_\mu \in D(\mathcal{A}, \kappa)$ and is a part of basis. Let $\{\alpha_H \theta_\mu, \theta'\}$ be a basis for the S -module $D(\mathcal{A}, \kappa)$. Since $D(\mathcal{A}, \nu) \subset D(\mathcal{A}, \kappa)$, $\theta_\nu = a\alpha_H \theta_\mu + b\theta'$ for some $a, b \in \mathbb{K}$. If $\{\theta_\mu, \theta_\nu\}$ is S -linearly dependent, then $b = 0$, i.e., $\theta_\nu = a\alpha_H \theta_\mu$. Since $\alpha_H \theta_\mu \in D(\mathcal{A}, \nu)$, $\alpha_H \theta_\mu(\alpha_H) \in S \cdot \alpha_H^{\nu_H} = S \cdot \alpha_H^{\kappa_H + 1}$. Hence $\theta_\mu(\alpha_H) \in S \cdot \alpha_H^{\kappa_H}$ and $\theta_\mu \in D(\mathcal{A}, \kappa)$, which is a contradiction.

Next assume that $\mu_H + 1 = \nu_H$ and $\mu_{H'} + 1 = \nu_{H'}$ for some $H, H' \in \mathcal{A}$ and $\mu_{H''} = \nu_{H''}$ for $H'' \in \mathcal{A} \setminus \{H, H'\}$. Let $\kappa \in \Lambda$ be the element such that $\kappa_H = \mu_H + 1$ and $\kappa_{H''} = \nu_{H''}$ for $H'' \in \mathcal{A} \setminus \{H\}$, and $\kappa' \in \Lambda$ such that $\kappa_{H'} = \mu_{H'} + 1$ and $\kappa_{H''} = \nu_{H''}$ for $H'' \in \mathcal{A} \setminus \{H'\}$. By the assumption, $\Delta(\kappa) = \Delta(\kappa') = 0$. Hence $\theta_\mu \notin D(\mathcal{A}, \kappa)$ and $\theta_\mu \notin D(\mathcal{A}, \kappa')$. Let $\{\alpha_H \theta_\mu, \theta'\}$ be a basis for the S -module $D(\mathcal{A}, \kappa)$. Since $\theta_\nu \in D(\mathcal{A}, \nu) \subset D(\mathcal{A}, \kappa)$, $\theta_\nu = a\alpha_H \theta_\mu + b\theta'$ for some $a, b \in \mathbb{K}$. If $\{\theta_\mu, \theta_\nu\}$ is S -linearly dependent, then $\theta_\nu = a\alpha_H \theta_\mu$. Since $\theta_\nu(\alpha_{H'}) = a\alpha_H \theta_\mu(\alpha_{H'}) \in S \cdot \alpha_{H'}^{\nu_{H'}} = S \cdot \alpha_{H'}^{\kappa'_{H'}}$, $\theta_\mu(\alpha_{H'}) \in S \cdot \alpha_{H'}^{\kappa'_{H'}}$. Hence $\theta_\mu \in D(\mathcal{A}, \kappa')$, which is contradiction.

Finally assume that $\mu_H + 1 = \nu_H$, $\mu_{H'} = \nu_{H'} + 1$ for some $H, H' \in \mathcal{A}$ and $\mu_{H''} = \nu_{H''}$ for $H'' \in \mathcal{A} \setminus \{H, H'\}$. Let $\kappa = \mu \wedge \nu$ and $\kappa' = \mu \vee \nu$. By the assumption, $\Delta(\kappa') = \Delta(\kappa) = 0$. Hence $\theta_\mu, \theta_\nu \notin D(\mathcal{A}, \kappa')$. We may choose a basis $\{\theta_\mu, \theta'\}$ for $D(\mathcal{A}, \kappa)$ such that $\{\theta_\mu, \alpha_H \theta'\}$ is a basis for $D(\mathcal{A}, \mu)$. Since $D(\mathcal{A}, \nu) \subset D(\mathcal{A}, \kappa)$, $\theta_\nu = a\theta_\mu + b\theta'$ for some $a, b \in \mathbb{K}$. If $\{\theta_\mu, \theta_\nu\}$ is S -linearly dependent, then $\theta_\nu = a\theta_\mu$. Since $\theta_\nu = a\theta_\mu \in D(\mathcal{A}, \mu) \cap D(\mathcal{A}, \nu)$, $\theta_\nu \in D(\mathcal{A}, \kappa')$ which is a contradiction. \square

Lemma 4.18. *If $\mu, \nu \in P(\Lambda')$ satisfy $d(\mu, \nu) = \Delta(\mu) + \Delta(\nu)$, then $\{\theta_\mu, \theta_\nu\}$ is S -linearly independent.*

Proof. By the assumption, there exist some $\mu', \nu' \in \Lambda'$ such that

- $d(\mu', \nu') = 2$,
- $\Delta(\kappa) = 0$ for all $\kappa \in \Lambda$ such that $d(\mu', \kappa) = d(\nu', \kappa) = 1$,
- $\mu, \mu' \in C \in \text{cc}_0(\Lambda')$, and
- $\nu, \nu' \in C' \in \text{cc}_0(\Lambda')$.

By Lemma 4.17 $\{\theta_{\mu'}, \theta_{\nu'}\}$ is S -linearly independent. Hence Lemma 4.15 completes proof. \square

Proof of Theorem 3.4. Apply Lemma 4.16 and 4.18. \square

4.3. Proof of Theorem 3.9.

Lemma 4.19. *Assume that $\mu, \nu \in \Lambda'$ satisfy $\Delta(\mu) + \Delta(\nu) = d(\mu, \nu)$, and that $\{\theta_\mu, \theta_\nu\}$ is S -linearly independent. Then $\{\theta_\mu, \theta_\nu\}$ is a basis for $D(\mathcal{A}, \mu \wedge \nu)$.*

Proof. Since $(\mu \wedge \nu)_H = \min\{\mu_H, \nu_H\}$ for $H \in \mathcal{A}$,

$$|(\mu \wedge \nu)| = \sum_{H \in \mathcal{A}} \min\{\mu_H, \nu_H\}.$$

On the other hand,

$$\begin{aligned}
\deg(\theta_\mu) + \deg(\theta_\nu) &= \frac{|\mu| - \Delta(\mu)}{2} + \frac{|\nu| - \Delta(\nu)}{2} \\
&= \frac{|\mu| + |\nu| - \Delta(\mu) - \Delta(\nu)}{2} \\
&= \frac{|\mu| + |\nu| - d(\mu, \nu)}{2} \\
&= \sum_{H \in \mathcal{A}} \frac{\mu_H + \nu_H - |\mu_H - \nu_H|}{2} \\
&= \sum_{H \in \mathcal{A}} \min\{\mu_H, \nu_H\} = |\mu \wedge \nu|.
\end{aligned}$$

Since $\mu \wedge \nu \subset \mu, \nu$, it follows that $\{\theta_\mu, \theta_\nu\}$ is a basis for $D(\mathcal{A}, \mu \wedge \nu)$. \square

Lemma 4.20. *Assume that $\mu, \nu \in \Lambda'$ satisfy $\Delta(\mu) + \Delta(\nu) = d(\mu, \nu)$ and that $\{\theta_\mu, \theta_\nu\}$ is S -linearly independent. For $\kappa \in \Lambda$ such that $\mu \wedge \nu \subset \kappa \subset \mu \vee \nu$, let us define*

$$\begin{aligned}
\alpha_{\mu, \kappa} &= \prod_{H \in \mathcal{A}} \alpha_H^{\max\{\kappa_H - \mu_H, 0\}}, \\
\alpha_{\nu, \kappa} &= \prod_{H \in \mathcal{A}} \alpha_H^{\max\{\kappa_H - \nu_H, 0\}}.
\end{aligned}$$

Then $\{\alpha_{\mu, \kappa}\theta_\mu, \alpha_{\nu, \kappa}\theta_\nu\}$ is a basis for $D(\mathcal{A}, \kappa)$.

Proof. Note that $\deg(\alpha_{\mu, \kappa}) + \deg(\alpha_{\nu, \kappa}) = d(\kappa, \mu \wedge \nu)$ and that $\alpha_{\mu, \kappa}\theta_\mu, \alpha_{\nu, \kappa}\theta_\nu \in D(\mathcal{A}, \kappa)$. Thus Saito's criterion completes the proof. \square

Proof of Theorem 3.9. By Theorem 3.4, $\{\theta_\mu, \theta_\nu\}$ is S -linearly independent. Hence Lemma 4.20 completes the proof. \square

5. APPLICATION

In this section, we consider the case when a group acts on V . Let W be a group acting on V from the left. Canonically, this action induces actions on S and $\text{Der}_{\mathbb{K}}(S)$, i.e., W acts on S and $\text{Der}_{\mathbb{K}}(S)$ by $(\sigma f)(v) = f(\sigma^{-1}v)$ and $(\sigma\delta)(f) = \sigma(\delta(\sigma^{-1}f))$ for $\sigma \in W$, $f \in S$, $\delta \in \text{Der}_{\mathbb{K}}(S)$ and $v \in V$. For each $\sigma \in W$, we assume $\mathcal{A} = \sigma\mathcal{A}$. In this case, W also acts on \mathcal{A} as a subgroup of the symmetric group of \mathcal{A} . Hence W also acts on Λ by $(\sigma\mu)_H = \mu_{\sigma^{-1}H}$.

Lemma 5.1. *For $\mu \in \Lambda$ and $\sigma \in W$, $\Delta(\mu) = \Delta(\sigma\mu)$.*

Proof. If $\{\theta, \theta'\}$ be a homogeneous basis for $D(\mathcal{A}, \mu)$, then $\{\sigma\theta, \sigma\theta'\}$ is a homogeneous basis for $D(\mathcal{A}, \sigma\mu)$. \square

Next we assume that $\mathcal{A}^W = \emptyset$, i.e., for each $H \in \mathcal{A}$, there exists $\sigma_H \in W$ such that $\sigma_H H \neq H$.

Lemma 5.2. *Let $\mu \in \Lambda$ satisfy $\sigma\mu = \mu$ for all $\sigma \in W$. Let $\mu, \nu \in \Lambda$ satisfy that $\mu \subset \nu$, $\mu' \subset \nu$ for all $\mu \dot{\subset} \mu'$, and $\Delta(\mu) - \Delta(\nu) > d(\mu, \nu) - 4$. Let $\kappa \in \Lambda$ satisfy $\kappa \subset \mu'$ for all $\mu' \dot{\subset} \mu$ and $\Delta(\kappa) - \Delta(\nu) > d(\kappa, \nu) - 4$. Then $\mu \in P(\Lambda')$.*

Proof. It suffices to show that $\Delta(\mu') < \Delta(\mu)$ if $\mu \dot{\subset} \mu'$ or $\mu' \dot{\subset} \mu$. First let us assume $\mu \dot{\subset} \mu'$, $\Delta(\mu') > \Delta(\mu)$ and $\mu'_H \neq \mu_H$. Since $\mathcal{A}^W = \emptyset$, $H \neq \sigma H$ for some $\sigma \in W$. For such σ , we have $\sigma\mu' \neq \mu'$. By Lemma 4.10,

$$\begin{aligned} \Delta(\mu' \vee \sigma\mu') &= \Delta(\mu') + 1 \\ &= \Delta(\mu) + 2. \end{aligned}$$

By Lemma 4.2, $\Delta(\mu' \vee \sigma\mu') = \Delta(\mu) + 2 \leq d(\nu, \mu' \vee \sigma\mu') + \Delta(\nu)$. Since $d(\nu, \mu' \vee \sigma\mu') + \Delta(\nu) = d(\nu, \mu) - 2 + \Delta(\nu)$, $\Delta(\mu) - \Delta(\nu) \leq d(\nu, \mu) - 4$, which is a contradiction.

The same argument is valid for the case where $\mu' \dot{\subset} \mu$, $\Delta(\mu') > \Delta(\mu)$ and $\mu'_H \neq \mu_H$. Hence we have the lemma. \square

As an application of the results above, we consider the exponents of Coxeter arrangements of type B_2 and G_2 defined as follows:

Definition 5.3. We call a hyperplane arrangement

$$\mathcal{A} = \{ \ker(x), \ker(y), \ker(x+y), \ker(x-y) \}$$

a Coxeter arrangement of type B_2 and

$$\mathcal{A} = \left\{ \begin{array}{l} \ker(x), \ker(x+\sqrt{3}y), \ker(x-\sqrt{3}y), \\ \ker(y), \ker(\sqrt{3}x+y), \ker(\sqrt{3}x-y) \end{array} \right\}$$

a Coxeter arrangement of type G_2 .

It is shown by Teramo in [9] that the constant multiplicity is free and the exponents are also determined. We give the meaning of Teramo's result from our point of view, i.e., the role of constant multiplicity in the multiplicity lattice.

Proposition 5.4. *Let \mathcal{A} be a Coxeter arrangement of type B_2 or G_2 . Then $(2k+1, \dots, 2k+1) \in P(\Lambda')$.*

Proof. First we consider the case when the type of \mathcal{A} is G_2 . In this case, we can take the Weyl group of type G_2 as W . Let $\nu = (2k+2, \dots, 2k+2)$ and $\kappa = (2k, \dots, 2k)$. Then $d(\mu, \nu) = d(\mu, \kappa) = 6$. Since $\Delta(\mu) = 4$ and $\Delta(\nu) = \Delta(\kappa) = 0$ by [9], it follows from Lemma 5.2 that $\mu \in P(\Lambda)$.

Next we consider the case when the type of \mathcal{A} is B_2 and W is the Weyl group of type B_2 . Then $d(\mu, \nu) = d(\mu, \kappa) = 4$, where μ, ν, κ are the same as the above. Since $\Delta(\mu) = 2$ and $\Delta(\nu) = \Delta(\kappa) = 0$ by [9], $\mu \in P(\Lambda)$. \square

Now we can determine the basis and exponents of multiplicities on Coxeter arrangements when they are near the constant one, which is based on the primitive derivation methods in [9] and [12].

Corollary 5.5. *Let \mathcal{A} be a Coxeter arrangement of type B_2 , $\mu = (2k + 1, \dots, 2k + 1) \in \Lambda$ and $i \in \mathbb{Z}^{|\mathcal{A}|}$ such that $\sum_H |i_H| < |\mathcal{A}|$. If $\nu \in \Lambda$ is defined by $\nu_H = \mu_H + i_H$, then $\exp(\mathcal{A}, \nu) = (4k + 1 + \sum_H |i_H|, 4k + 3)$.*

Corollary 5.6. *Let \mathcal{A} be a Coxeter arrangement of type G_2 , $\mu = (2k + 1, \dots, 2k + 1) \in \Lambda$ and $i \in \mathbb{Z}^{|\mathcal{A}|}$ such that $\sum_H |i_H| < |\mathcal{A}|$. If $\nu \in \Lambda$ is defined by $\nu_H = \mu_H + i_H$, then $\exp(\mathcal{A}, \nu) = (6k + 1 + \sum_H |i_H|, 6k + 5)$.*

The proofs of above corollaries are completed by applying Corollary 3.3 and Proposition 5.4.

Remark 5.7. In the forthcoming paper [4] the argument in this section is generalized to determine the centers of the balls of any 2-multiarrangements which admit some cyclic group actions.

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