Wavelets and modular inequalities in variable L^p spaces *

Mitsuo Izuki

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Abstract

The aim of this paper is to characterize variable L^p spaces $L^{p(\cdot)}(\mathbb{R}^n)$ using wavelets with proper smoothness and decay. We obtain conditions for the wavelet characterizations of $L^{p(\cdot)}(\mathbb{R}^n)$ with respect to the norm estimates and modular inequalities.

Keywords and Phrases. wavelet, unconditional basis, variable L^p space.

1 Introduction

Conditions for the boundedness of the Hardy-Littlewood maximal function M on variable L^p spaces $L^{p(\cdot)}(\Omega)$ have been studied in [4, 5, 15]. Cruz-Uribe, Fiorenza and Neugebauer [4] and Nekvinda [15] gave the sufficient conditions on the exponent function $p(\cdot)$ independently. Diening [5] studied the necessary and sufficient conditions in terms of the conjugate exponent function $p'(\cdot)$. In the case of $\Omega = \mathbb{R}^n$, he has proved that the boundedness of M on $L^{p(\cdot)}(\mathbb{R}^n)$ is equivalent to that on $L^{p'(\cdot)}(\mathbb{R}^n)$. Recently Cruz-Uribe, Fiorenza, Martell and Pérez [3] have showed that many important operators are bounded on $L^{p(\cdot)}(\Omega)$ when M is bounded on $L^{p(\cdot)}(\Omega)$. For example, their result ensures the boundedness of singular integral operators, commutators and fractional integral operators on $L^{p(\cdot)}(\Omega)$. In this paper we consider for the case of $\Omega = \mathbb{R}^n$, and give wavelet characterizations of $L^{p(\cdot)}(\mathbb{R}^n)$. Because we have to invoke the results in [3, 5] in order to obtain the characterizations, our results hold under the assumption that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. As an application, we also construct unconditional bases of $L^{p(\cdot)}(\mathbb{R}^n)$ in terms of wavelets.

We are also interested in modular inequalities on $L^{p(\cdot)}(\mathbb{R}^n)$. Lerner [13] considered modular inequalities for some classical operators. In particular he proved that the Hardy-Littlewood maximal function M satisfies the modular inequality

$$\int_{\mathbb{R}^n} Mf(x)^{p(x)}\,dx \le C \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx,$$

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precisely when the exponent function $p(\cdot)$ is some constant. In the present paper we study the modular inequalities for orthogonal projections associated with a multiresolution analysis and wavelet characterizations, and prove the results similar to those in [13].

2 Preliminaries

2.1 Variable L^p spaces and the Hardy-Littlewood maximal function

First of all, we define variable L^p spaces on $\mathbb{R}^n$, and introduce some known results on the boundedness of the Hardy-Littlewood maximal operator. We remark that some of the results are also true for variable L^p spaces on an open set of $\mathbb{R}^n$.

Throughout this paper, we consider a measurable function $p : \mathbb{R}^n \to [1,\infty)$ without notices. $p'(\cdot)$ means the conjugate exponent function of $p(\cdot)$, i.e., $p'(\cdot)$ satisfies $1/p(x) + 1/p'(x) = 1$ $(x \in \mathbb{R}^n)$. χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{R}^n$.

Definition 2.1. *The variable L^p space $L^{p(\cdot)}(\mathbb{R}^n)$ consists of all measurable functions f defined on $\mathbb{R}^n$ satisfying that there exists a constant $\lambda > 0$ such that $\rho_p(f/\lambda) < \infty$, where $\rho_p(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)}dx$.*

$L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space with the norm defined by

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\left\{\lambda > 0 : \rho_p(f/\lambda) \le 1\right\}.$$

Let $1 \le p_0 < \infty$ be a constant. If $p(\cdot) \equiv p_0$, then we have $L^{p(\cdot)}(\mathbb{R}^n) = L^{p_0}(\mathbb{R}^n)$. Now we define two classes of exponent functions. We use the following notation;

$$p_- := \operatorname{ess\,inf}\{p(x) : x \in \mathbb{R}^n\} \quad \text{and} \quad p_+ := \operatorname{ess\,sup}\{p(x) : x \in \mathbb{R}^n\}.$$

Definition 2.2. *The set $\mathcal{P}(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.*

Given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(y)|dy \quad (x \in \mathbb{R}^n),$$

where the supremum is taken over all cubes Q containing x.

Definition 2.3. *$\mathcal{B}(\mathbb{R}^n)$ is the set of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

Cruz-Uribe, Fiorenza and Neugebauer [4] and Nekvinda [15] showed the following sufficient conditions independently.

Proposition 2.4. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Suppose*

$$|p(x) - p(y)| \le \frac{C}{-\log(|x-y|)} \quad if \quad |x-y| \le 1/2 \tag{1}$$

and

$$|p(x) - p(y)| \le \frac{C}{\log(e+|x|)} \quad if \quad |y| \ge |x|, \tag{2}$$

where $C > 0$ is a constant independent of x and y. Then we have $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.

For example, $p(x) = a + b(1 - e^{-c|x|})$ $(a > 1,\ b > 1-a,\ c \ge e^{-1})$ satisfies the conditions (1) and (2).

For any $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, we see that

$$|p'(x) - p'(y)| \le \frac{|p(x) - p(y)|}{(p_- - 1)^2}.$$

Thus, if $p(\cdot)$ satisfies (1) and (2), then so does $p'(\cdot)$. Diening proved the following equivalence (cf. Theorem 8.1 in [5]).

Proposition 2.5. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then the following four conditions are equivalent:*

1. *$p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.*
2. *$p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.*
3. *There exists a constant $q \in (1, p_-)$ such that $p(\cdot)/q \in \mathcal{B}(\mathbb{R}^n)$.*
4. *There exists a constant $q \in (1, p_-)$ such that $(p(\cdot)/q)' \in \mathcal{B}(\mathbb{R}^n)$.*

2.2 Wavelets and MRAs

First let us recall the definition of wavelet [9, 14, 17]. Let $\{\psi^l : l = 1, 2, \ldots, 2^n - 1\}$ be a set of functions belonging to $L^2(\mathbb{R}^n)$. Define

$$\begin{aligned} \psi^l_{j,k}(x) &:= 2^{jn/2}\psi^l(2^j x - k) \\ &= 2^{jn/2}\psi^l(2^j x_1 - k_1, \ldots, 2^j x_n - k_n) \quad (x = (x_1, \ldots, x_n) \in \mathbb{R}^n) \end{aligned}$$

for each $l = 1, 2, \ldots, 2^n - 1$, $j \in \mathbb{Z}$ and $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$. The sequence $\{\psi^l : l = 1, 2, \ldots, 2^n - 1\}$ is a wavelet set if $\{\psi^l_{j,k} : l = 1, 2, \ldots, 2^n - 1,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$ forms an orthonormal basis in $L^2(\mathbb{R}^n)$. Then $\{\psi^l_{j,k} : l = 1, 2, \ldots, 2^n - 1,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$ is a wavelet basis in $L^2(\mathbb{R}^n)$ and each ψ^e is a wavelet.

We generally need suitable smoothness or decay on wavelets in order to obtain wavelet characterizations of function spaces. Now we define the class of functions $\mathcal{R}^0(\mathbb{R}^n)$ [9]. The set $\mathcal{R}^0(\mathbb{R}^n)$ consists of all functions f satisfying the following three conditions:

1. $\int_{\mathbb{R}^n} f(x)\,dx = 0$.

2. There exist two constants $C_0, \gamma > 0$ such that
$$|f(x)| \leq C_0(1+|x|)^{-(2+\gamma)n}. \tag{3}$$

3. $f \in C^1(\mathbb{R}^n)$, and there exist constants $\varepsilon > 0$ and $C_j > 0$ for each $j = 1, 2, \ldots, n$ such that
$$\left|\frac{\partial}{\partial x_j} f(x)\right| \leq C_j(1+|x|)^{-(1+\varepsilon)n}. \tag{4}$$

In this paper we use a wavelet set which consists of wavelets in $\mathcal{R}^0(\mathbb{R}^n)$. We can construct it by means of an MRA (multiresolution analysis) [9, 14, 17].

Definition 2.6. *An MRA (multiresolution analysis) is a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^n)$ such that the following six conditions hold:*

1. *$V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$.*

2. *$\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^n)$.*

3. *$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.*

4. *$f \in V_j$ holds if and only if $f(2^{-j}x) \in V_0$ for all $j \in \mathbb{Z}$.*

5. *$f \in V_0$ holds if and only if $f(x-k) \in V_0$ for every $k \in \mathbb{Z}^n$.*

6. *There exists a function $\varphi \in V_0$ such that the system $\{\varphi(x-k)\}_{k\in\mathbb{Z}^n}$ is an orthonormal basis in V_0. φ is called a scaling function of $\{V_j\}_{j\in\mathbb{Z}}$.*

Given an MRA $\{V_j\}_{j\in\mathbb{Z}}$ with a scaling function φ, define
$$P_j f := \sum_{k\in\mathbb{Z}^n} \langle f, \varphi_{j,k}\rangle \varphi_{j,k} \quad (f \in L^2(\mathbb{R}^n)) \tag{5}$$
for every $j \in \mathbb{Z}$, where $\langle\cdot,\cdot\rangle$ means the L^2-inner product. Then each operator $P_j : L^2(\mathbb{R}^n) \to V_j$ is an orthogonal projection.

It is well-known that there exists an MRA $\{V_j\}_{j\in\mathbb{Z}}$ with a scaling function φ which has the decay and the smoothness (3) and (4). Then we can construct a wavelet set $\{\psi^l : l = 1, 2, \ldots, 2^n - 1\}$ associated with $\{V_j\}_{j\in\mathbb{Z}}$ satisfying that each wavelet ψ^l is in $\mathcal{R}^0(\mathbb{R}^n)$ and $\{\psi^l_{j,k} : l = 1, 2, \ldots, 2^n - 1,\ k \in \mathbb{Z}^n\}$ forms an orthonormal basis in W_j for every $j \in \mathbb{Z}$. Here W_j is the orthogonal complement of V_j in V_{j+1}.

2.3 A_{p_0} weights and A_∞ weights

First we define the class of weights A_{p_0} with $1 < p_0 < \infty$. We say that a non-negative and locally integrable function defined on $\mathbb{R}^n$ is a weight on $\mathbb{R}^n$.

Definition 2.7. *Let $1 < p_0 < \infty$ be a constant, and w be a weight on $\mathbb{R}^n$ such that $w^{-1/(p_0-1)}$ is locally integrable. w is said to be an A_{p_0} weight if w satisfies*

$$A_{p_0,w} := \sup_{Q:\text{cube}} \frac{1}{|Q|} \int_Q w(x)\, dx \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p_0-1)}\, dx \right)^{p_0-1} < \infty,$$

where $|Q|$ means the Lebesgue measure of Q. A_{p_0} denotes the set of all A_{p_0} weights.

Next we define A_∞ weights.

Definition 2.8.

1. *A weight w is said to be an A_∞ weight if for all $0 < \alpha < 1$, there exists a constant $0 < \beta < 1$ such that for every cube Q and measurable set $E \subset Q$ satisfying $|E| \geq \alpha |Q|$,*

$$\beta \int_Q w(x)\, dx \leq \int_E w(x)\, dx.$$

   A_∞ denotes the set of all A_∞ weights.

2. *A family of weights $\{w_t\}_{t>0}$ is said to be a family of A_∞ weights uniformly in t if each w_t is an A_∞ weight and the constant β is independent of t.*

There are several conditions which are equivalent to the definition of A_∞ above [8]. For example, it is well-known that $A_\infty = \bigcup_{1<p_0<\infty} A_{p_0}$. Lerner gave the following characterization of families of A_∞ weights (Lemma 2.1 in [13]).

Lemma 2.9. *Let $p(\,\cdot\,)$ be a non-negative and measurable function on $\mathbb{R}^n$. The family of weights $\left\{t^{p(\,\cdot\,)}\right\}_{t>0}$ is a family of A_∞ weights uniformly in t if and only if $p(\,\cdot\,) \equiv p_0$ for some constant p_0.*

2.4 The extrapolation theorem

In this subsection we describe the extrapolation theorem given by Cruz-Uribe, Fiorenza, Martell and Pérez (Corollary 1.11 in [3]). Let us recall the definition of weighted L^{p_0} space.

Definition 2.10. *Let $1 \leq p_0 < \infty$ be a constant, and w be a weight on $\mathbb{R}^n$. The weighted L^{p_0} space $L^{p_0}_w(\mathbb{R}^n)$ is the space of all measurable functions f with*

$$\|f\|_{L^{p_0}_w(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^{p_0} w(x)\, dx \right)^{1/p_0} < \infty.$$

$L^{p_0}_w(\mathbb{R}^n)$ is a Banach space with the norm $\|\cdot\|_{L^{p_0}_w(\mathbb{R}^n)}$.

Cruz-Uribe, Fiorenza, Martell and Pérez have proved the boundedness of many important operators on variable L^p spaces by applying the following result, provided that M is bounded.

Lemma 2.11. *Let $p(\,\cdot\,) \in \mathcal{P}(\mathbb{R}^n)$, and $\mathcal{F}$ be a family of ordered pairs of non-negative and measurable functions (f, g). Suppose that*

$$(p(\,\cdot\,)/p_1)' \in \mathcal{B}(\mathbb{R}^n) \text{ for some constant } p_1 \in (1, p_-), \tag{6}$$

and that there exists a constant $p_0 \in (1, \infty)$ such that for all $w \in A_{p_0}$ and all $(f, g) \in \mathcal{F}$ with $f \in L^{p_0}_w(\mathbb{R}^n)$,

$$\|f\|_{L^{p_0}_w(\mathbb{R}^n)} \le C_0 \|g\|_{L^{p_0}_w(\mathbb{R}^n)},$$

where C_0 is a constant depending only on n, p_0 and $A_{p_0,w}$. Then it follows that for all $(f, g) \in \mathcal{F}$ such that $f \in L^{p(\,\cdot\,)}(\mathbb{R}^n)$,

$$\|f\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)} \le C_1 \|g\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)},$$

where C_1 is a constant independent of (f, g).

We note that Lemma 2.11 also holds for variable L^p spaces on an open set of $\mathbb{R}^n$, however we state only the case of $L^{p(\,\cdot\,)}(\mathbb{R}^n)$ in the present paper.

As is mentioned in Proposition 2.5, there are some equivalent conditions to the assumption (6). In particular, $p(\,\cdot\,)$ satisfies (6) if and only if $p(\,\cdot\,) \in \mathcal{B}(\mathbb{R}^n)$.

2.5 Banach function spaces

First we describe the definition of Banach function space and absolutely continuous norm.

Definition 2.12. *Let $\mathcal{M}(\mathbb{R}^n)$ be the set of all measurable functions on $\mathbb{R}^n$.*

1. *A linear space $X \subset \mathcal{M}(\mathbb{R}^n)$ is said to be a Banach function space if there exists a functional $\|\cdot\| : \mathcal{M}(\mathbb{R}^n) \to \mathbb{R}$ which has the norm property and satisfies the following conditions:*

   (a) $f \in X$ if and only if $\|f\| < \infty$.

   (b) $\|f\| = \|\,|f|\,\|$ for all $f \in \mathcal{M}(\mathbb{R}^n)$.

   (c) If $0 \le f_1 \le f_2 \le \dots$ and $\lim_{n\to\infty} f_n = f$, then $\|f_1\| \le \|f_2\| \le \dots$ and $\lim_{n\to\infty} \|f_n\| = \|f\|$.

   (d) For all $E \subset \mathbb{R}^n$ with $|E| < \infty$, it follows that $\|\chi_E\| < \infty$.

(e) *For all* $E \subset \mathbb{R}^n$ *with* $|E| < \infty$*, there exists a constant* $c_E > 0$ *such that* $\int_E |f(x)|\,dx \le c_E \|f\|$ *for all* $f \in X$*.*

2. *Let* X *be a Banach function space with the norm* $\|\cdot\|$*. The norm* $\|\cdot\|$ *is said to be an absolutely continuous norm if* $\lim_{j\to\infty} \left\|f\chi_{E_j}\right\| = 0$ *for all* $f \in X$ *and all sequences of measurable sets* $\{E_j\}_{j=1}^{\infty}$ *such that* $\lim_{j\to\infty} E_j = \emptyset$*.*

We will need the next lemma in order to obtain unconditional bases in terms of wavelets later. The lemma is the dominated convergence theorem for Banach function spaces with absolutely continuous norm (Chapter 1, Proposition 3.6 in [2]).

Lemma 2.13. *Let* $(X, \|\cdot\|)$ *be a Banach function space with absolutely continuous norm,* $f \in X$ *and* $\{f_j\}_{j=1}^{\infty} \subset X$*. Suppose that* $\lim_{j\to\infty} f_j = f$ *a.e. and there exists a positive function* $g \in X$ *such that* $|f_j| \le g$ *a.e. for all* $j \in \mathbb{N}$*. Then we have* $\lim_{j\to\infty} \|f_j - f\| = 0$*.*

The following two lemmas imply that $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space with the absolutely continuous norm $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ (cf. [6]). Hence we can apply Lemma 2.13 to $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 2.14 ([2])**.** *Let* $(X, \|\cdot\|)$ *be a Banach function space. Then* X *is separable if and only if the norm* $\|\cdot\|$ *is absolutely continuous on* X*.*

Lemma 2.15 ([11])**.** *Suppose that* $p_+ < \infty$*. Then* $L^{p(\cdot)}(\mathbb{R}^n)$ *is separable.*

2.6 Weakly positive kernels

Aimar, Bernardis and Martín-Reyes defined weakly positive kernels, and studied the boundedness of operators having such kernels on $L^{p_0}_w(\mathbb{R}^n)$ with $w \in A_{p_0}$ [1].

Definition 2.16 ([1])**.** *Let* $K_j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *be a measurable function for each* $j \in \mathbb{Z}$*. The family* $\{K_j\}_{j\in\mathbb{Z}}$ *is said to be a family of weakly positive kernels if there exist a constant* $\mathcal{C} > 0$ *and a sequence* $\{l_j\}_{j\in\mathbb{Z}}$ *such that the following three conditions hold:*

1. $0 < l_{j+1} < l_j < \infty$ *for all* $j \in \mathbb{Z}$*.*

2. $\lim_{j\to\infty} l_j = 0$ *and* $\lim_{j\to-\infty} l_j = \infty$*.*

3. *For every* $j \in \mathbb{Z}$*,* $K_j(x, y) > \mathcal{C}(l_{j+1})^{-n}$ *whenever* $|x - y| < l_j$*.*

We are interested in kernels of orthogonal projections associated with an MRA which has a scaling function with proper decay.

Definition 2.17.

1. *A function η is said to be radial decreasing, if η itself is radial and*
$$[0,\infty) \ni t \longmapsto \eta(t,0,0,\dots,0)$$
is decreasing.

2. *The set $\mathcal{RB}(\mathbb{R}^n)$ consists of all functions ϕ such that there exists a non-negative, bounded and radial decreasing function $\eta \in L^1(\mathbb{R}^n)$ such that $|\phi| \le \eta$ a.e. $\mathbb{R}^n$.*

It is clear that any function satisfying (3) and (4) belongs to $\mathcal{RB}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. Let φ be a scaling function of an MRA. If $\varphi \in \mathcal{RB}(\mathbb{R}^n)$, we can write

$$P_j f(x) = \int_{\mathbb{R}^n} K_j(x,y) f(y)\, dy \quad (f \in L^2(\mathbb{R}^n)),$$

where the kernel $K_j(x,y)$ is defined by

$$K_j(x,y) := \sum_{k\in\mathbb{Z}^n} \varphi_{j,k}(x)\overline{\varphi_{j,k}(y)} = 2^{jn} \sum_{k\in\mathbb{Z}^n} \varphi(2^j x - k)\overline{\varphi(2^j y - k)}$$

for each $j \in \mathbb{Z}$. Under the assumption $\varphi \in \mathcal{RB}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, $\{K_j\}_{j\in\mathbb{Z}}$ is a family of weakly positive kernels with $l_j = c\,2^{-j}$ for some constant $c > 0$ [1].

3 Main results

3.1 Norm estimates on $L^{p(\cdot)}(\mathbb{R}^n)$

Our first result is the boundedness of orthogonal projections associated with an MRA.

Theorem 3.1. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\varphi \in \mathcal{RB}(\mathbb{R}^n) \cap L^{p'(\cdot)}(\mathbb{R}^n)$ and $\tilde{\varphi} \in \mathcal{RB}(\mathbb{R}^n)$. Define*
$$\tilde{P}_j f := \sum_{k\in\mathbb{Z}^n} \langle f, \varphi_{j,k}\rangle \tilde{\varphi}_{j,k}.$$
Then there exists a constant $C > 0$ such that for all $j \in \mathbb{Z}$ and $f \in L^{p(\cdot)}(\mathbb{R}^n)$,
$$\left\| \tilde{P}_j f \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \left\| f \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Remark 3.2.

1. If φ is a scaling function of an MRA and $\varphi = \tilde{\varphi}$, then we see that $\tilde{P}_j f = P_j f$, where P_j is the orthogonal projection defined in (5) for each $j \in \mathbb{Z}$.

2. By virtue of the generalized Hölder inequality (Theorem 2.1 in [11]), we have

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \le r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, where $r_p := 1 + 1/p_- - 1/p_+$. Hence the L^2-inner product $\langle f, \varphi_{j,k}\rangle$ above makes sense.

We can immediately obtain Theorem 3.1 using the two estimates below.

Lemma 3.3 (Lemma 2.8 in [10]). *For any $\phi, \tilde{\phi} \in \mathcal{RB}(\mathbb{R}^n)$, there exists a non-negative, bounded and radial decreasing function $H \in L^1(\mathbb{R}^n)$ such that*

$$\sum_{k\in\mathbb{Z}^n} \left|\tilde{\phi}(x-k)\phi(y-k)\right| \le H(x-y).$$

Lemma 3.4 (p.63 in [16]). *Let $H \in L^1(\mathbb{R}^n)$ be a positive, bounded and radial decreasing function. Then there exists a constant $C_n > 0$ depending only on n such that for all $j \in \mathbb{Z}$,*

$$\int_{\mathbb{R}^n} 2^{jn} H(2^j(x-y))|f(y)|\, dy \le C_n \|H\|_{L^1(\mathbb{R}^n)} Mf(x).$$

Our second result is a wavelet characterization of $L^{p(\cdot)}(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. We define a dyadic cube

$$Q_{j,k} := \prod_{i=1}^{n} \left[2^{-j}k_i, 2^{-j}(k_i+1)\right)$$

and denote $\chi_{j,k} := 2^{jn/2}\chi_{Q_{j,k}}$ for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$. Given a wavelet set $\{\psi^l : l = 1, 2, \dots, 2^n - 1\}$, we consider the following two square functions in order to obtain wavelet characterizations:

$$Vf := \left(\sum_{l=1}^{2^n-1} \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^n} \left|\langle f, \psi^l_{j,k}\rangle \psi^l_{j,k}\right|^2\right)^{1/2},$$

$$Wf := \left(\sum_{l=1}^{2^n-1} \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^n} \left|\langle f, \psi^l_{j,k}\rangle \chi_{j,k}\right|^2\right)^{1/2}.$$

The next wavelet characterizations of weighted L^{p_0} spaces have been proved.

Lemma 3.5 ([7, 12]). *Let $p_0 \in (1, \infty)$ be a constant, $w \in A_{p_0}$, and $\{\psi^l : l = 1, 2, \dots, 2^n - 1\}$ be a wavelet set such that each wavelet ψ^l is in $\mathcal{R}^0(\mathbb{R}^n)$. Then there exist constants $c, C, c', C' > 0$ depending only on n, p_0, $\{\psi^l\}$ and $A_{p_0,w}$ such that for all $f \in L^{p_0}_w(\mathbb{R}^n)$,*

$$c\|f\|_{L^{p_0}_w(\mathbb{R}^n)} \le \|Vf\|_{L^{p_0}_w(\mathbb{R}^n)} \le C\|f\|_{L^{p_0}_w(\mathbb{R}^n)}$$

and

$$c'\|f\|_{L^{p_0}_w(\mathbb{R}^n)} \le \|Wf\|_{L^{p_0}_w(\mathbb{R}^n)} \le C'\|f\|_{L^{p_0}_w(\mathbb{R}^n)}.$$

We will apply Lemma 2.11 in order to obtain the wavelet characterizations of $L^{p(\cdot)}(\mathbb{R}^n)$. We also need the density. $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ denotes the set of all infinitely differentiable functions with compact support. Kováčik and Rákosník showed the following density (Theorem 2.11 in [11]).

Lemma 3.6. *Suppose that $p_+ < \infty$. Then $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$.*

Using these facts, we have the next result.

Theorem 3.7. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, and $\{\psi^l : l = 1, 2, \ldots, 2^n - 1\}$ be a wavelet set such that each wavelet ψ^l is in $\mathcal{R}^0(\mathbb{R}^n)$. Then there exist constants $c, C, c', C' > 0$ such that for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$,*

$$c \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \|Vf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{7}$$

and

$$c' \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \|Wf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C' \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{8}$$

Proof of Theorem 3.7. By using Proposition 2.5, we see that $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ is equivalent to condition (6). Thus Lemmas 3.5 and 2.11 imply that the estimates (7) and (8) hold for all $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$. By virtue of Lemma 3.6, these inequalities are also valid for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$. □

3.2 Unconditional bases in $L^{p(\cdot)}(\mathbb{R}^n)$

Applying the wavelet characterizations (Theorem 3.7) and the dominated convergence theorem (Lemma 2.13), we can construct unconditional bases of $L^{p(\cdot)}(\mathbb{R}^n)$ in terms of wavelets.

Theorem 3.8. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, and $\{\psi^l : l = 1, 2, \ldots, 2^n - 1\}$ be a wavelet set such that each wavelet ψ^l is in $\mathcal{R}^0(\mathbb{R}^n)$. Then the wavelet basis $\{\psi^l_{j,k} : l = 1, 2, \ldots, 2^n - 1, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ forms an unconditional basis on $L^{p(\cdot)}(\mathbb{R}^n)$.*

Proof of Theorem 3.8. For convenience, we denote $\Lambda := \{1, 2, \ldots, 2^n - 1\} \times \mathbb{Z} \times \mathbb{Z}^n$, and $S_A f := \displaystyle\sum_{(l,j,k)\in A} \langle f, \psi^l_{j,k}\rangle \psi^l_{j,k}$ for $A \subset \Lambda$. It suffices to check the following two conditions:

1. There exists a constant $C > 0$ such that $\|S_A f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ for all $A \subset \Lambda$ and all $f \in L^{p(\cdot)}(\mathbb{R}^n)$.
2. $\operatorname{span}\left\{\psi^l_{j,k} : (l, j, k) \in \Lambda\right\}$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$.

First we check the condition 1. By Theorem 3.7 and the orthonormality, it follows that for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\|S_A f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C_0 \|W(S_A f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C_0 \|Wf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C_0 C_1 \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \quad (9)$$

where $C_0, C_1 > 0$ are constants independent of f. This completes 1.

Next we check the condition 2. It suffices to show $\lim_{A\to\Lambda} \|f - S_A f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = 0$. We see that $W(f - S_A f) \le Wf$ and $\|Wf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C_1 \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ by (9). By Lemmas 2.14 and 2.15, $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space with the absolutely continuous norm $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. Thus Lemma 2.13 gives $\lim_{A\to\Lambda} \|W(f - S_A f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} = 0$. On the other hand, (9) implies $\|f - S_A f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C_0 \|W(f - S_A f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. Namely we obtain $\lim_{A\to\Lambda} \|f - S_A f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = 0$. □

3.3 Modular inequalities in $L^{p(\cdot)}(\mathbb{R}^n)$

Finally we study the relation between wavelets and modular inequalities in $L^{p(\cdot)}(\mathbb{R}^n)$. First we consider the modular inequality for operators with weakly positive kernels. $L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ denotes the set of all bounded and compactly supported functions.

Theorem 3.9. *Let $\{K_j\}_{j\in\mathbb{Z}}$ be a family of weakly positive kernels and define*

$$T_j f(x) := \int_{\mathbb{R}^n} K_j(x,y) f(y)\, dy \quad (f \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)).$$

Suppose $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If there exists a constant $C_0 > 0$ such that for all $j \in \mathbb{Z}$ and $f \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |T_j f(x)|^{p(x)}\, dx \le C_0 \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx,$$

then $p(\cdot) \equiv p_0$ for some constant $1 < p_0 < \infty$.

Proof. Let $\{l_j\}_{j\in\mathbb{Z}}$ and $\mathcal{C}$ be the corresponding sequence and constant appearing in Definition 2.16 respectively. Take a cube Q arbitrarily. We denote the diameter of Q by $d(Q) := \sup\{|x-y| \,:\, x, y \in Q\}$ and the side length of Q by $\ell(Q)$. Since $|Q| = \ell(Q)^n$ and $d(Q) = \sqrt{n}\,\ell(Q)$, we have $|Q| = n^{-n/2} d(Q)^n$. On the other hand, there exists $j_Q \in \mathbb{Z}$ uniquely such that $l_{j_Q+1} \le d(Q) < l_{j_Q}$. Now let us take $0 < \alpha < 1$ arbitrarily, and consider a measurable set $E \subset Q$ such that $|E| \ge \alpha |Q|$. Fix $t > 0$. It follows that for each $x \in Q$,

$$\begin{aligned} \left|T_{j_Q}(t\,\chi_E)(x)\right| &= t\left|\int_E K_{j_Q}(x,y)\,dy\right| \ge t\,\mathcal{C}\,(l_{j_Q+1})^{-n} \cdot |E| \ge t\,\mathcal{C}\,d(Q)^{-n} \cdot \alpha\,|Q| \\ &= t\,\mathcal{C}\,n^{-n/2}\,|Q|^{-1} \cdot \alpha\,|Q| = t\,\mathcal{C}\,n^{-n/2}\,\alpha. \end{aligned}$$

Thus we get

$$\int_Q \left|T_{jQ}(t\,\chi_E)(x)\right|^{p(x)}\,dx \geq \int_Q \left(t\,\mathcal{C}\,n^{-n/2}\,\alpha\right)^{p(x)}\,dx \geq C_1 \int_Q t^{p(x)}\,dx,$$

where $C_1 > 0$ is a constant depending only on $\mathcal{C}$, n, α, p_+ and p_-. On the other hand, we obtain

$$\begin{aligned}\int_Q \left|T_{jQ}(t\,\chi_E)(x)\right|^{p(x)}\,dx &\leq \int_{\mathbb{R}^n} \left|T_{jQ}(t\,\chi_E)(x)\right|^{p(x)}\,dx \\ &\leq C_0 \int_{\mathbb{R}^n} |t\,\chi_E(x)|^{p(x)}\,dx \\ &= C_0 \int_E t^{p(x)}\,dx.\end{aligned}$$

Therefore we have

$$\int_Q t^{p(x)}\,dx \leq C_0\,C_1^{-1} \int_E t^{p(x)}\,dx.$$

Hence $\left\{t^{p(\,\cdot\,)}\right\}_{t>0}$ is a family of A_∞ weights uniformly in t. By virtue of Lemma 2.9, we have $p(\,\cdot\,) \equiv p_0$ for some constant $1 < p_0 < \infty$. $\square$

As is mentioned, if $\varphi \in \mathcal{RB}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is a scaling function of an MRA, then the family of kernels $\left\{\sum_{k\in\mathbb{Z}^n} \varphi_{j,k}(x)\overline{\varphi_{j,k}(y)}\right\}_{j\in\mathbb{Z}}$ is a family of weakly positive kernels in words of [1]. Thus we have the following.

Corollary 3.10. *Suppose $p(\,\cdot\,) \in \mathcal{P}(\mathbb{R}^n)$, and $\varphi \in \mathcal{RB}(\mathbb{R}^n) \cap C(\mathbb{R}^n) \cap L^{p'(\,\cdot\,)}(\mathbb{R}^n)$ be a scaling function of an MRA. Then the following two conditions are equivalent:*

1. *There exists a constant $C > 0$ such that for all $j \in \mathbb{Z}$ and $f \in L^{p(\,\cdot\,)}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |P_j f(x)|^{p(x)}\,dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx.$$

2. *$p(\,\cdot\,) \equiv p_0$ for some constant $1 < p_0 < \infty$.*

Lerner [13] gave the equivalence above for the Hardy-Littlewood maximal function. If φ satisfies the same assumption as Corollary 3.10, then Lemmas 3.3 and 3.4 lead the estimate

$$|P_j f(x)| \leq CMf(x)$$

for all $j \in \mathbb{Z}$ and $f \in L^{p(\,\cdot\,)}(\mathbb{R}^n)$. Thus we can regard Corollary 3.10 as a generalization of Lerner's result.

Our final result is the modular inequality for the square function Vf on $L^{p(\,\cdot\,)}(\mathbb{R}^n)$.

Theorem 3.11. *Let $\{\psi^l : l = 1, 2, \ldots, 2^n - 1\}$ be a wavelet set satisfying the following two conditions:*

- *Each wavelet ψ^l is in $\mathcal{R}^0(\mathbb{R}^n)$.*
- *$\{\psi^l : l = 1, 2, \ldots, 2^n - 1\}$ is associated with an MRA which has a scaling function φ satisfying (3) and (4).*

Suppose $p(\,\cdot\,) \in \mathcal{P}(\mathbb{R}^n)$. Then the following two conditions 1. and 2. are equivalent:

1. *There exist two constants $C_0, C_1 > 0$ such that for all $f \in L^{p(\,\cdot\,)}(\mathbb{R}^n)$,*

$$C_0 \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx \le \int_{\mathbb{R}^n} |Vf(x)|^{p(x)}\, dx \le C_1 \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx.$$

2. *$p(\,\cdot\,) \equiv p_0$ for some constant $1 < p_0 < \infty$.*

Proof. The fact *2.⇒1.* is well-known [9, 14, 17]. We show *1.⇒2.*. Let us assume *1.*. Fix $m \in \mathbb{Z}$ and take $f \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ arbitrarily. Now we write

$$S_m f := \sum_{l=1}^{2^n-1} \sum_{j=-\infty}^{m-1} \sum_{k \in \mathbb{Z}^n} \langle f, \psi^l_{j,k} \rangle \psi^l_{j,k}.$$

From the fundamental wavelet theory, it follows that $P_m f(x) = S_m f(x)$ a.e. $x \in \mathbb{R}^n$. Thus we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} |P_m f(x)|^{p(x)}\, dx &= \int_{\mathbb{R}^n} |S_m f(x)|^{p(x)}\, dx \\
&\le C_0^{-1} \int_{\mathbb{R}^n} |V(S_m f)(x)|^{p(x)}\, dx \\
&= C_0^{-1} \int_{\mathbb{R}^n} \left(\sum_{l=1}^{2^n-1} \sum_{j=-\infty}^{m-1} \sum_{k \in \mathbb{Z}^n} \left| \langle f, \psi^l_{j,k} \rangle \psi^l_{j,k}(x) \right|^2 \right)^{p(x)/2} dx \\
&\le C_0^{-1} \int_{\mathbb{R}^n} |Vf(x)|^{p(x)}\, dx \\
&\le C_0^{-1} C_1 \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx.
\end{aligned}$$

In view of Theorem 3.9, we have $p(\,\cdot\,) \equiv p_0$ for some constant $1 < p_0 < \infty$. □

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Author's address:
Department of Mathematics
Faculty of Science
Hokkaido University
Kita 10 Nishi 8, Kita-ku, Sapporo, Hokkaido 060-0810
JAPAN
E-mail: **mitsuo@math.sci.hokudai.ac.jp**