ON SOLUTIONS TO WALCHER'S EXTENDED HOLOMORPHIC ANOMALY EQUATION

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Abstract. We give a generalization of Yamaguchi–Yau's result to Walcher's extended holomorphic anomaly equation.

1. Introduction

Let X be a nonsingular quintic hypersurface in $\mathbb{CP}^4$. The case of the X and its mirror is the most well-studied example of the mirror symmetry. After the construction of the mirror family of Calabi–Yau threefolds [10], the genus zero Gromov–Witten (GW) potential of X were computed via the Yukawa coupling of the mirror family [4]. The predicted mirror formula was proved first by Givental [7].

For higher genera, Bershadsky–Cecotti–Ooguri–Vafa (BCOV) [2] has predicted that the GW potential at genus g is obtained as a certain limit of the B-model closed topological string amplitude $\mathcal{F}^{(g)}$ of genus g [1]. They have also proposed a partial differential equation (PDE) for $\mathcal{F}^{(g)}$, called the BCOV holomorphic anomaly equation, which determines $\mathcal{F}^{(g)}$ up to a holomorphic function. The prediction of BCOV for the genus one GW potential was proved by Zinger [21].

Recently the open string analogue of the mirror symmetry has been developed by Walcher [18] for the pair (X, L) of the quintic 3-fold X defined over $\mathbb{R}$ (called a real quintic) and the set of real points $L = X(\mathbb{R})$ which is a Lagrangian submanifold of X. Open mirror symmetry gave the prediction for the generating function for the disc GW invariants of X with boundary in L and it was proved by Pandharipande–Solomon–Walcher [16]. Then, Walcher [19] further proposed the open string analogue of BCOV, the extended holomorphic anomaly equation, which is a PDE for the B-model topological string amplitude $\mathcal{F}^{(g,h)}$ for world-sheets with g handles and h boundaries [2].

At present there are two ways to solve the BCOV holomorphic anomaly equation. The one is to repeatedly use the identity called the special geometry relation, or equivalently to draw Feynman diagrams associated to the perturbative expansion of a certain path integral [2]. The other is to solve the system of PDE's due to Yamaguchi–Yau [20]. They showed that $\mathcal{F}^{(g)}$ multiplied by $(g-1)$-th powers of the Yukawa coupling, is a polynomial

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[1] For genus $g = 0$, the third covariant derivative of $\mathcal{F}^{(0)}$ is the Yukawa coupling, and for $g = 1$, it is recently proved that $\mathcal{F}^{(1)}$ is the Quillen's norm function [6]. For genus $g \geq 2$, the mathematical definition of $\mathcal{F}^{(g)}$ is yet to be known.

[2] There is also a proposal by Bonelli–Tanzini [3].

in finite number of generators and rewrite the holomorphic anomaly equation as PDE's with respect to these generators. This result were then reformulated into a more useful form by Hosono–Konishi [12, §3.4].

It is a natural problem to generalize these methods to Walcher's extended holomorphic anomaly equation. The generalization of the Feynman rule method can be obtained from the result of Cook–Ooguri–Yang [5]. The objective of this article is to generalize Yamaguchi–Yau's and Hosono–Konishi's results to the extended holomorphic anomaly equation. It gives more tractable method in computations than the one given by the Feynman rule.

The organization of the paper is as follows. In Section 2, we recall the special Kähler geometry of the B-model complex moduli space and Walcher's extended holomorphic anomaly equation. We also describe the Feynman rule. In Section 3, we rewrite the holomorphic anomaly equation as PDE's (Theorem 13). In Section 4, we compute several BPS numbers by fixing holomorphic ambiguities with certain assumptions. The assumptions in this section are experimental in a sense. In appendices we include the Feynman diagrams and the solution of the PDE's for $(g,h)=(0,4)$.

After we finished writing this paper, we were informed that Alim–Länge [1] also obtained a generalization of Yamaguchi–Yau's result.

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2. Walcher's extended holomorphic anomaly equation

2.1. **Special Kähler geometry.** Recall the mirror family of the quintic hypersurface $X \subset \mathbb{P}^4$ constructed in [10]. Let W_ψ be the hypersurface in $\mathbb{P}^4$ defined by

$$\sum_{i=0}^{4} x_i^5 - 5\psi \prod_{i=0}^{4} x_i = 0.$$

After taking the quotient by $(\mathbb{Z}/5\mathbb{Z})^3$ and a crepant resolution Y_ψ of $W_\psi/(\mathbb{Z}/5\mathbb{Z})^3$, we obtain a one-parameter family of Calabi–Yau threefolds $\pi : \mathcal{Y} \to \mathcal{M}_{cpl} := \mathbb{P}^1 \setminus \{0, \frac{1}{5^5}, \infty\}$, where a local coordinate z of $\mathcal{M}_{cpl}$ is given by $z = (5\psi)^{-5}$.

Consider the variation of Hodge structure of weight three on the middle cohomology groups $H^3(Y_z, \mathbb{C})$. Let $0 \subset F^3 \subset F^2 \subset F^1 \subset F^0 = R^3\pi_*\mathbb{C} \otimes \mathcal{O}_{\mathcal{M}_{cpl}}$ be the Hodge filtration and ∇ be the Gauss–Manin connection. The holomorphic line bundle $\mathcal{L} := F^3$ over $\mathcal{M}_{cpl}$ is called the vacuum line bundle (the fiber of $\mathcal{L}$ at z is $H^{3,0}(Y_z)$). Let $\Omega(z)$ be a local holomorphic section trivializing $\mathcal{L}$, i.e. a nowhere vanishing $(3,0)$-form on Y_z. The Yukawa coupling C_{zzz} is define by

$$C_{zzz} := \int_{X_z} \Omega(z) \wedge (\nabla_{\partial_z})^3 \Omega(z),$$

which is a holomorphic section of $\mathrm{Sym}^3(T^*_{\mathcal{M}_{cpl}}) \otimes (\mathcal{L}^*)^2$, where $T^*_{\mathcal{M}_{cpl}}$ denotes the holomorphic cotangent bundle of $\mathcal{M}_{cpl}$. A suitable choice of $\Omega(z)$ gives ([4])

$$C_{zzz} = \frac{5}{(1-5^5 z)z^3}.$$

It also gives the following Picard–Fuchs operator $\mathcal{D}$ which governs the periods of $\Omega(z)$:

$$\mathcal{D} = \theta_z^4 - 5z(\theta_z+1)(\theta_z+2)(\theta_z+3)(\theta_z+4),$$

where $\theta_z = z\frac{d}{dz}$.

Consider the pairing

$$(\phi, \psi) := \sqrt{-1} \int_{Y_z} \phi \wedge \psi, \quad \phi, \psi \in H^3(Y_z, \mathbb{C}).$$

Then $(\ ,\overline{\ \ })$ induces a Hermitian metric on $\mathcal{L}$. Let $K(z,\bar{z}) := -\log(\Omega(z), \overline{\Omega(z)})$. This defines a Kähler metric (the Weil-Peterson metric) $G_{z\bar{z}} := \partial_z \partial_{\bar{z}} K$ on $\mathcal{M}_{cpl}$. There is a unique holomorphic Hermitian connection D on $(T_{\mathcal{M}_{cpl}})^m \otimes \mathcal{L}^n$ whose $(1,0)$-part D_z is given by

$$D_z = \partial_z + m\Gamma^z_{zz} + n(-\partial_z K)\ ,$$

where $\Gamma^z_{zz} = G^{z\bar{z}} \partial_z G_{z\bar{z}}$. An important property of $G_{z\bar{z}}$ is the following identity called the special geometry relation [17]

$$\partial_{\bar{z}} \Gamma^z_{zz} = 2G_{z\bar{z}} - C_{zzz} C_{\bar{z}\bar{z}\bar{z}} e^{2K} G^{z\bar{z}} G^{z\bar{z}}, \tag{1}$$

where $C_{\bar{z}\bar{z}\bar{z}} := \overline{C_{zzz}}$.

Now we introduce the open disk amplitude with two insertions $\triangle_{zz}$, which is the open-sector analogue of the Yukawa coupling. Let $\mathcal{T}$ be a holomorphic section of $\mathcal{L}^*$ locally given by

$$\mathcal{T} = 60\ \tau(z), \qquad \tau(z) = \sum_{n=0}^{\infty} \frac{(\frac{7}{2})_{5n}}{((\frac{3}{2})_n)^5}\ z^{n+\frac{1}{2}}. \tag{2}$$

Here $(\alpha)_n$ is the Pochhammer symbol : $(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$ for $n > 0$ and $(\alpha)_0 := 1$. $\mathcal{T}$ is a solution to

$$\mathcal{D}\mathcal{T} = \frac{60}{2^4}\sqrt{z}. \tag{3}$$

Following [19], we define a C^∞–section $\triangle_{zz}$ of $\mathrm{Sym}^2(T^*_{\mathcal{M}_{cpl}}) \otimes \mathcal{L}^*$ by

$$\triangle_{zz} = D_z D_z \mathcal{T} - \frac{e^K C_{zzz}}{G_{z\bar{z}}} \overline{D}_{\bar{z}} \overline{\mathcal{T}}\ , \tag{4}$$

where $\overline{D}_{\bar{z}} = \partial_{\bar{z}} + \partial_{\bar{z}} K$ denotes the $(0,1)$-part of $\overline{D}$. By (1), it follows that $\triangle_{zz}$ satisfies the equation

$$\partial_{\bar{z}} \triangle_{zz} = -C_{zzz} e^K G^{z\bar{z}} \triangle_{\bar{z}\bar{z}}, \tag{5}$$

where $\triangle_{\bar{z}\bar{z}} := \overline{\triangle_{zz}}$.

Remark 1. In [19], it is argued that $\mathcal{T}$ and $\triangle_{zz}$ should be written as

$$\mathcal{T}(z) = \int_{Y_z} \Omega(z) \wedge \tilde{\nu}(z), \quad \triangle_{zz} = \int_{Y_z} \Omega(z) \wedge \nabla^2 \tilde{\nu}(z),$$

where $\tilde{\nu}$ is a C^∞–section of the Hodge bundle F^0 which is the 'real horizontal lift' of a certain Griffiths normal function ν associated to a family of homologically trivial 2-cycles [3]. The normal function ν should be determined from the Lagrangian submanifold $L \subset X$ under the mirror symmetry with D-branes.

2.2. **Extended holomorphic anomaly equation.** Let $\mathcal{F}^{(g,h)}$ be the B-model topological string amplitude of genus g with h boundaries, and let

$$\mathcal{F}_0^{(g,h)} := \mathcal{F}^{(g,h)}, \qquad \mathcal{F}_n^{(g,h)} := D_z \mathcal{F}_{n-1}^{(g,h)} \ (n \geq 1).$$

$\mathcal{F}_n^{(g,h)}$ is a C^∞–section of the line bundle $(T^*_{\mathcal{M}_{cpl}})^n \otimes \mathcal{L}^{2g-2+h}$. For $(g,h) = (0,0), (0,1)$,

$$\mathcal{F}_3^{(0,0)} = C_{zzz}, \quad \mathcal{F}_2^{(0,1)} = \triangle_{zz}. \tag{6}$$

For $(g,h) = (1,0), (0,2)$[4],

$$\begin{aligned} \mathcal{F}_1^{(1,0)} &= \frac{1}{2}\partial_z \log \left(e^{(4-\frac{\chi}{12})K} {G_{z\bar{z}}}^{-1} (1-5^5 z)^{-\frac{1}{6}} z^{-1-\frac{c_2 \cdot H}{12}} \right), \\ \mathcal{F}_1^{(0,2)} &= -\triangle_{zz}\triangle^z - \frac{1}{2} C_{zzz}\triangle^z\triangle^z + \frac{N}{2}\partial_z K + f^{(0,2)}, \quad f^{(0,2)} = \frac{75}{2(1-5^5 z)} \ , \end{aligned} \tag{7}$$

where $\chi = -200$, $c_2 \cdot H = 50$, $N = 1$ and $\triangle^z = -\frac{\triangle_{zz}}{C_{zzz}}$ (cf. §2.3).

As in [19], define

$$C_{\bar{z}}^{zz} = C_{\bar{z}\bar{z}\bar{z}} e^{2K} {G_{z\bar{z}}}^{-2} \ , \qquad \triangle_{\bar{z}}^{z} = \triangle_{\bar{z}\bar{z}} e^{K} {G_{z\bar{z}}}^{-1} \ .$$

Then Walcher's extended holomorphic anomaly equation for $(g,h) \neq (0,0), (1,0), (0,1), (0,2)$ is as follows.

$$\partial_{\bar{z}} \mathcal{F}^{(g,h)} = \frac{1}{2} C_{\bar{z}}^{zz} \Big(\sum_{g_1,g_2,h_1,h_2} \mathcal{F}_1^{(g_1,h_1)} \mathcal{F}_1^{(g_2,h_2)} + \mathcal{F}_2^{(g-1,h)} \Big) - \triangle_{\bar{z}}^{z} \mathcal{F}_1^{(g,h-1)} \ . \tag{8}$$

In the RHS, the summation is over $g_1, h_1, g_2, h_2 \geq 0$ satisfying $g_1 + g_2 = g$, $h_1 + h_2 = h$ and $(g_1,h_1), (g_2,h_2) \neq (0,0), (0,1)$. The second and the third terms in the RHS should be set to zero if $g = 0$ and $h = 0$, respectively.

[3] By definition, ν is a holomorphic and horizontal section of the intermediate Jacobian fibration $\mathcal{J}^3 \to \mathcal{M}_{cpl}$ of $\mathcal{Y} \to \mathcal{M}_{cpl}$. See, e.g., [9, 11].

[4] $\mathcal{F}_1^{(1,0)}$ and $\mathcal{F}_1^{(0,2)}$ are solutions to the following (extended) holomorphic anomaly equations [2][19].

$$\partial_{\bar{z}} \mathcal{F}_1^{(1,0)} = \frac{1}{2} C_{zzz} C_{\bar{z}}^{zz} - \Big(\frac{\chi}{24} - 1 \Big) G_{z\bar{z}}, \quad \partial_{\bar{z}} \mathcal{F}_1^{(0,2)} = -\triangle_{zz}\triangle_{\bar{z}}^{z} + \frac{N}{2} G_{z\bar{z}}.$$

2.3. **Propagators and Terminators.** We introduce the propagators S^{zz}, S^z, S and the terminators $\triangle^z, \triangle$ [2, 19]. By definition, they are solutions to

$$\begin{aligned}\partial_{\bar z} S^{zz} &= C^{zz}_{\bar z}, \quad \partial_{\bar z} S^z = S^{zz} G_{z\bar z}, \quad \partial_{\bar z} S = S^z G_{z\bar z}, \\ \partial_{\bar z} \triangle^z &= \triangle^z_{\bar z}, \qquad\qquad \partial_{\bar z} \triangle = \triangle^z G_{z\bar z}.\end{aligned} \tag{9}$$

These equation can be solved by using (1) and (5). The solutions of the propagators are [2, p.391].

$$\begin{aligned} S^{zz} &= \frac{1}{C_{zzz}}\big(2\partial_z \log(e^K|f|^2) - \partial_z \log(vG_{z\bar z})\big) \ , \\ S^z &= \frac{1}{C_{zzz}}\big((\partial_z \log(e^K|f|)^2 - v^{-1}\partial_z v \partial_z \log(e^K|f|^2)\big) \ , \\ S &= \big(S^z - \frac{1}{2}D_z S^{zz} - \frac{1}{2}(S^{zz})^2 C_{zzz}\big)\partial_z \log(e^K|f|^2) + \frac{1}{2}D_z S^z + \frac{1}{2}S^{zz}S^z C_{zzz} \ . \end{aligned} \tag{10}$$

Here f, v are holomorphic functions of z. We take $f = z^{-\frac{1}{5}}$ and $v = \frac{dz}{d\psi}$ $(z = \frac{1}{5^5\psi^5})$ so that S^{zz}, S^z, S do not diverge at $z = \infty$[5]. Solutions of the terminators are [19, (3.12)]

$$\triangle^z = -\frac{\triangle_{zz}}{C_{zzz}}, \quad \triangle = D_z \triangle^z. \tag{11}$$

2.4. **Feynman Rule.** We describe the Feynman rule which gives a solution to (8).

For non-negative integers g,h,m, and n, we define $\widetilde{C}^{(g,h)}_{n:m}$ recursively as follows.

$$\widetilde{C}^{(0,0)}_{0:m} = \widetilde{C}^{(0,0)}_{1:m} = \widetilde{C}^{(0,0)}_{2:m} = 0, \tag{12}$$

$$\widetilde{C}^{(0,1)}_{0:m} = \widetilde{C}^{(0,1)}_{1:m} = 0, \tag{13}$$

$$\widetilde{C}^{(0,2)}_{0:1} = -\frac{N}{2}, \tag{14}$$

$$\widetilde{C}^{(1,0)}_{0:0} = 0, \quad \widetilde{C}^{(1,0)}_{0:1} = \frac{\chi}{24} - 1, \tag{15}$$

$$C^{(g,h)}_{n} = \mathcal{F}^{(g,h)}_{n} \quad \text{if} \quad 2g - 2 + h + n \geq 1, \tag{16}$$

$$\widetilde{C}^{(g,h)}_{n:0} = C^{(g,h)}_{n}, \quad \text{if} \quad 2g - 2 + h + n \geq 1, \tag{17}$$

$$\widetilde{C}^{(g,h)}_{n:m+1} = (2g - 2 + h + n + m)\widetilde{C}^{(g,h)}_{n:m}. \tag{18}$$

Definition 2. A Feynman diagram G is a finite labeled graph

$$G = (V; E^{\text{in}}_0, E^{\text{in}}_1, E^{\text{in}}_2, E^{\text{out}}_0, E^{\text{out}}_1; j),$$

which consists of the following data.

(i) Each vertex $v \in V$ is labeled by a pair of non-negative integers (g_v, h_v).

(ii) There are three kinds of inner edges $E^{\text{in}} = E^{\text{in}}_0 \sqcup E^{\text{in}}_1 \sqcup E^{\text{in}}_2$ and two kinds of outer edges $E^{\text{out}} = E^{\text{out}}_0 \sqcup E^{\text{out}}_1$. The end points of the edges are specified by the collection of maps $j = (j^{\text{in}}_0, j^{\text{in}}_1, j^{\text{in}}_2, j^{\text{out}}_0, j^{\text{out}}_1)$:

$$j^{\text{in}}_0 : E^{\text{in}}_0 \to (V \times V)/\sigma, \quad j^{\text{in}}_1 : E^{\text{in}}_1 \to V \times V, \quad j^{\text{in}}_2 : E^{\text{in}}_2 \to (V \times V)/\sigma,$$
$$j^{\text{out}}_0 : E^{\text{out}}_0 \to V, \quad j^{\text{out}}_1 : E^{\text{out}}_0 \to V,$$

where $\sigma : V \times V \to V \times V$ is the involution interchanging the first and the second factors.

[5] If rewritten in the ψ-coordinate, (10) are the same as those used in [19, 3.11][20, (2.21)].

In a more plain language, an edge of type E_i^{in} has both endpoints attached to vertices, and an edge of type E_i^{out} has only one endpoint attached to a vertex. We represent edges of types E_0^{in} and E_0^{out} by solid lines, edges of types E_2^{in} and E_1^{out} by dashed lines and an edge of type E_1^{in} by a half-solid, half-dashed line. See Fig. 1.

For a vertex $v \in V$, we set

$$L_{i,v} = \{e \in E_i^{\mathrm{in}} \mid j_i^{\mathrm{in}}(e) = \{v, v\}\}, \quad L_i = \textstyle\bigsqcup_{v\in V} L_{i,v}, \ \ (i = 0, 2),$$
$$L_{1,v} = \{e \in E_1^{\mathrm{in}} \mid j_1^{\mathrm{in}}(e) = (v, v)\}.$$

In other words, $L_{i,v}$ is the number of self-loops attached to the vertex v whose edges are of the type E_i^{in}. Define non-negative integers n_v^{in}, n_v^{out}, m_v^{in} and m_v^{out} by

$$n_v^{\mathrm{in}} = \#\{e \in E_2^{\mathrm{in}} \mid v \in j_2^{\mathrm{in}}(e)\} + \#\{e \in E_1^{\mathrm{in}} \mid j_1^{\mathrm{in}}(e) = (v, *)\} + \#L_{2,v} + \#L_{1,v},$$
$$m_v^{\mathrm{in}} = \#\{e \in E_0^{\mathrm{in}} \mid v \in j_0^{\mathrm{in}}(e)\} + \#\{e \in E_1^{\mathrm{in}} \mid j_1^{\mathrm{in}}(e) = (*, v)\} + \#L_{0,v} + \#L_{1,v},$$
$$n_v^{\mathrm{out}} = \#\{e \in E_1^{\mathrm{out}} \mid v \in j_1^{\mathrm{out}}(e)\}, \quad m_v^{\mathrm{out}} = \#\{e \in E_0^{\mathrm{out}} \mid v \in j_0^{\mathrm{out}}(e)\}.$$

The valence $\mathrm{val}(v)$ of $v \in V$ is given by $\mathrm{val}(v) = n_v + m_v$, where $n_v := n_v^{\mathrm{in}} + n_v^{\mathrm{out}}$ (the number of solid lines attached to v), $m_v := m_v^{\mathrm{in}} + m_v^{\mathrm{out}}$ (the number of dashed lines attached to v). See Fig. 2.

Definition 3. (i) For a Feynman diagram G, define

$$F_G = \prod_{v\in V} \widetilde{C}^{(g_v,h_v)}_{n_v:m_v} \cdot \prod_{e\in E_0^{\mathrm{in}}} (-2S) \cdot \prod_{e\in E_1^{\mathrm{in}}} (-S^z) \cdot \prod_{e\in E_2^{\mathrm{in}}} (-S^{zz}) \cdot \prod_{e\in E_0^{\mathrm{out}}} \Delta \cdot \prod_{e\in E_1^{\mathrm{out}}} \Delta^z. \tag{19}$$

(ii) Let $\mathrm{Aut}(G)$ be the automorphism group of G. Define the group A_G by

$$A_G = \prod_{e\in L_0 \sqcup L_2} \mathbb{Z}/2\mathbb{Z} \ \ltimes \mathrm{Aut}(G),$$

i.e. A_G fits into the following exact sequence:

$$1 \to (\mathbb{Z}/2\mathbb{Z})^{\#L_0+\#L_2} \to A_G \to \mathrm{Aut}(G) \to 1.$$

This means that each self-loop of type E_0^{in} and E_2^{in} contributes the factor 2 to $\#A_G$.

Definition 4. Let $\mathbb{G}(g, h)$ be the set of (isomorphism classes of) Feynman diagrams G which satisfy the following conditions.
(i) G is connected.
(ii) For any $v \in V$, $\widetilde{C}^{(g_v,h_v)}_{n_v:m_v} \neq 0$.
(iii) G satisfies $\sum_{v\in V} g_v + \#E^{\mathrm{in}} - \#V + 1 = g$ and $\sum_{v\in V} h_v + \#E^{\mathrm{out}} = h$.
(iv) For any $v \in V$, $\mathrm{val}(v) > 0$.

Note that the set $\mathbb{G}(g, h)$ is a finite set. Note also that the graph whose amplitude is $\mathcal{F}^{(g,h)}$, i.e. the graph with only one vertex with label (g, h) and without edges is not a member of $\mathbb{G}(g, h)$ by (iv).

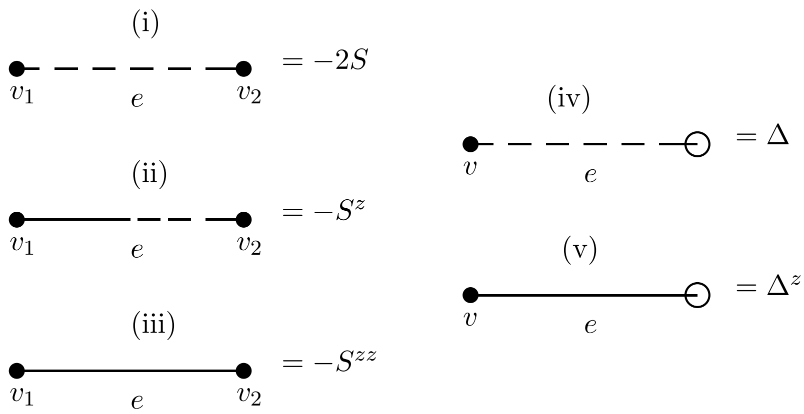

FIGURE 1. Three types of inner edges and propagators: (i) $e \in E_0^{\mathrm{in}}$, $j_0^{\mathrm{in}}(e) = \{v_1, v_2\}$, (ii) $e \in E_1^{\mathrm{in}}$, $j_1^{\mathrm{in}}(e) = (v_1, v_2)$, (iii) $e \in E_2^{\mathrm{in}}$, $j_2^{\mathrm{in}}(e) = \{v_1, v_2\}$. Two types of outer edges and terminators: (iv) $e \in E_0^{\mathrm{out}}$, $j_0^{\mathrm{out}}(e) = v$, (v) $e \in E_1^{\mathrm{out}}$, $j_1^{\mathrm{out}}(e) = v$.

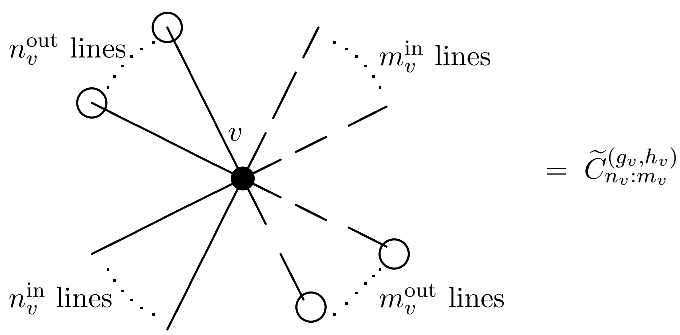

FIGURE 2. A vertex v labeled by (g_v, h_v) to which $n_v = n_v^{\mathrm{in}} + n_v^{\mathrm{out}}$ solid lines and $m_v = m_v^{\mathrm{in}} + m_v^{\mathrm{out}}$ dashed lines are attached and its value.

Define

$$\mathcal{F}_{\mathrm{FD}}^{(g,h)} := - \sum_{G \in \mathbb{G}(g,h)} \frac{1}{\# A_G} F_G. \tag{20}$$

The next result follows from [5].

Proposition 5. $\partial_{\bar{z}} \mathcal{F}_{\mathrm{FD}}^{(g,h)} =$ *the RHS of* (8).

Therefore, the general solution $\mathcal{F}^{(g,h)}$ of Walcher's holomorphic anomaly equation is of the form

$$\mathcal{F}^{(g,h)} = \mathcal{F}_{\mathrm{FD}}^{(g,h)} + f^{(g,h)}, \tag{21}$$

where $f^{(g,h)}$ is the holomorphic ambiguity which can not be determined from the equation (8).

2.5. **Holomorphic ambiguity.** Recall that the holomorphic ambiguity $f^{(g,0)}$ $(g \geq 2)$ is of the form [2][20, (2.30)]

$$f^{(g,0)} = \frac{a_0 + a_1 z + \cdots + a_{2g-1} z^{2g-1}}{(1-5^5 z)^{2g-2}} + \sum_{i=0}^{\lfloor \frac{2g-2}{5} \rfloor} z^j$$

for the closed sector $h = 0$. Huang–Klemm–Quackenbush [13] determined the holomorphic ambiguity up to $g \leq 51$ by using the vanishing of the BPS numbers n_d^g (cf. footnote 4), the gap condition at the conifold point $z = \frac{1}{5^5}$ and the regularity condition at the orbifold point $z = \infty$.

For $h > 0$, we assume that $\mathcal{F}^{(g,h)}$ has poles of order at most $2g - 2 + h$ at $z = \frac{1}{5^5}$ and also that the asymptotic behaviour at $z = \infty$ is $F^{(g,h)} \sim z^{\frac{2g-2+h}{2}}$ [19, §3.3]. Therefore we put the following ansatz for $f^{(g,h)}$:

$$\begin{aligned} f^{(g,h)} &= \frac{a_0 + a_1 z + \cdots + a_{3g-3+\frac{3h}{2}} z^{3g-3+\frac{3h}{2}}}{(1-5^5 z)^{2g-2+h}} \quad (h \text{ even}), \\ f^{(g,h)} &= \frac{\sqrt{z}(a_0 + a_1 z + \cdots + a_{3g-3+\frac{3h-1}{2}} z^{3g-3+\frac{3h-1}{2}})}{(1-5^5 z)^{2g-2+h}} \quad (h \text{ odd}). \end{aligned} \tag{22}$$

3. Polynomiality and PDE's for $\mathcal{F}^{(g,h)}$

In this section, we consider extending Yamaguchi–Yau's and Hosono–Konishi's results [20, 12] to $\mathcal{F}^{(g,h)}$.

3.1. The generators of polynomial ring.

3.1. The generators of polynomial ring. Let $\theta_z = z\frac{\partial}{\partial z}$.

We define

$$\begin{aligned} A_p &= \frac{\theta_z G_{z\bar{z}}}{G_{z\bar{z}}}\ , \qquad B_p = \frac{\theta_z e^{-K}}{e^{-K}} \qquad (p = 1, 2, \dots)\ , \\ Q_p &= z^{\frac{1}{2}} \theta_z \mathcal{T} \qquad\qquad\qquad (p = 0, 1, 2, \dots)\ , \\ R_1 &= z^{\frac{5}{2}} \frac{e^K C_{zzz}}{G_{z\bar{z}}} \overline{D}_{\bar{z}} \overline{\mathcal{T}}\ , \qquad R_2 = z^{\frac{7}{2}} e^K C_{zzz} \mathcal{T}. \end{aligned} \tag{23}$$

The generators A_p's and B_p's were defined in [20]. The new ingredients are Q_p's, R_1 and R_2 which are necessary for incorporating $\triangle_{zz}$.

Consider the polynomial ring

$$I = \mathbb{C}(z)[A_1, B_1, B_2, B_3, Q_0, Q_1, Q_2, Q_3, R_1, R_2] \tag{24}$$

with coefficients in the field of rational functions $\mathbb{C}(z)$.

Lemma 6. *1.* $A_p \in I$ $(p \geq 2)$, $B_p \in I$ $(p \geq 4)$, $Q_p \in I$ $(p \geq 4)$.
2. $\theta_z I \subseteq I$

Proof. First, notice that the logarithmic derivation θ_z acts as follows:

$$\begin{aligned}
&\theta_z A_p = A_{p+1} - A_p A_1 \ , \qquad \theta_z B_p = B_{p+1} - B_p B_1 \ , \\
&\theta_z Q_p = \frac{1}{2} Q_p + Q_{p+1} \ , \\
&\theta_z R_1 = \Big(\frac{5}{2} - A_1 - B_1 + \frac{\theta_z C_{zzz}}{C_{zzz}} \Big) R_1 + R_2, \\
&\theta_z R_2 = \Big(\frac{7}{2} - B_1 + \frac{\theta_z C_{zzz}}{C_{zzz}} \Big) R_2 \ .
\end{aligned} \tag{25}$$

Next we show $A_2, B_4, Q_4 \in I$. By the special geometry relation (1), we have

$$A_2 = -2A_1B_1 + 2B_1^2 + 2B_1 - 4B_2 + \frac{\theta_z(zC_{zzz})}{zC_{zzz}}(1 + A_1 + 2B_1) + h(z) \ . \tag{26}$$

Here $h(z)$ is determined by comparing the behaviour of the RHS and the LHS at $z = 0$:

$$h(z) = \frac{1 - 3 \cdot 5^4 z}{1 - 5^5 z} \ .$$

Let us write the Picard–Fuchs operator as $\mathcal{D} = \sum_{p=0}^{4} H_p(z) {\theta_z}^p$ where $H_p(z) \in \mathbb{C}[z]$. Since $\mathcal{D} e^{-K} = 0$, B_4 satisfies

$$B_4 = -\sum_{p=1}^{3} \frac{H_p(z)}{H_4(z)} B_p - \frac{H_0(z)}{H_4(z)} = 0 \ . \tag{27}$$

Moreover, since $\mathcal{T}$ satisfies (3),

$$Q_4 = -\sum_{p=0}^{3} \frac{H_p(z)}{H_4(z)} Q_p + \frac{60}{2^4} z. \tag{28}$$

These together with (25) implies that I is closed with respect to the logarithmic derivation θ_z. Moreover, by applying θ_z recursively, we can show that $A_p \in I$ $(p \geq 3)$, $B_p \in I$ $(p \geq 5)$, $Q_p \in I$ $(p \geq 5)$. $\square$

3.2. **Polynomiality.** For simplicity, we will use the notation

$$V_1 = A_1 + 2B_1 + 1, \qquad V_2 = B_2 - B_1 V_1 \tag{29}$$

from here on.

Since D_z acts on $(T_{\mathcal{M}_{cpl}})^m \otimes \mathcal{L}^n$ as

$$D_z = \frac{1}{z}(\theta_z + mA_1 + nB_1) \ ,$$

we have the following

Lemma 7. *Let f be a section of $(T_{\mathcal{M}_{cpl}})^m \otimes \mathcal{L}^n$. Then $D_z f \in I$ if $f \in I$ and $D_z f \in \sqrt{z} I$ if $f \in \sqrt{z} I$.*

Lemma 8. $\mathcal{F}_n^{(g,h)} \in z^{\frac{h}{2}} \ I$.

Proof. We prove the lemma by induction on (g,h).

For $(g,h)=(0,0),(1,0),(0,1),(0,2)$, the lemma is true since

$$\begin{aligned}
\mathcal{F}_3^{(0,0)} &= C_{zzz} \in I, \\
\mathcal{F}_1^{(1,0)} &= \frac{1}{2z}\Big[-A_1 - \frac{62}{3}B_1 - \frac{31}{6} - \frac{1}{6}\frac{\theta_z(1-5^5z)}{(1-5^5z)}\Big] \ \in I, \\
\mathcal{F}_2^{(0,1)} &= \triangle_{zz} = z^{-\frac{5}{2}}[Q_2 - V_1Q_1 - V_2Q_0 - R_1] \ \in \sqrt{z}\ I, \\
\mathcal{F}_1^{(0,2)} &= \frac{1\triangle_{zz}}{2C_{zzz}} - \frac{B_1}{2z} + f^{(0,2)} \ \in I.
\end{aligned} \tag{30}$$

For $(g,h)\neq(0,0),(1,0),(0,1),(0,2)$, assume that $\mathcal{F}_n^{(g',h')}\in z^{\frac{h}{2}}I$ holds for every $(g',h')\neq(g,h)$ such that $g'\le g$ and $h'\le h$. Consider the contribution F_G from a Feynman diagram $G\in\mathbb{G}(g,h)$ to $\mathcal{F}_{\mathrm{FD}}^{(g,h)}$ (19). The assumption of the induction implies that a vertex factor satisfies $\widetilde{C}^{(g_v,h_v)}_{n_v;m_v}\in z^{\frac{h_v}{2}}I$. As for edge factors, the followings hold. From (10),

$$S^{zz} = \frac{1}{zC_{zzz}}\Big(-A_1 - 2B_1 - \frac{8}{5}\Big)\in I, \quad S^{z} = \frac{1}{z^2C_{zzz}}\Big(B_2 + 3B_1 + \frac{2}{25}\Big)\in I. \tag{31}$$

By Lemma 7, S also satisfies $S\in I$. Similarly by (30) the terminators (11) satisfy

$$\triangle^z, \triangle \ \in \sqrt{z}I.$$

Therefore, by the condition (iii) in Definition 4, we have $F_G\in z^{\frac{h}{2}}I$ and thus $\mathcal{F}_{\mathrm{FD}}^{(g,h)}\in z^{\frac{h}{2}}I$. As to the holomorphic ambiguity $f^{(g,h)}$, it satisfies $f^{(g,h)}\in z^{\frac{h}{2}}\mathbb{C}(z)\subset z^{\frac{h}{2}}\ I$ by assumption (22). Therefore $\mathcal{F}^{(g,h)}\in I$. For $n\geq 1$, $\mathcal{F}_n^{(g,h)}\in I$ by Lemma 7. □

Definition 9. Let $g,h,n\geq 0$ be integers satisfying $2g-2+h+n>0$. We define

$$P_n^{(g,h)} = (z^3C_{zzz})^{g+h-1}z^{\frac{h}{2}}\mathcal{F}_n^{(g,h)}\ , \quad P^{(g,h)} := P_0^{(g,n)}. \tag{32}$$

For other values of (g,h,n), we set $P_n^{(g,h)}=0$.

Lemma 8 implies that

$$P_n^{(g,h)} \ \in \ I.$$

Remark 10. Let $x=z^3C_{zzz}=\frac{5}{1-5^5z}$. Consider the graded ring

$$\mathbb{C}[x, A_1, B_1, B_2, B_3, Q_0, \dots, Q_3, R_1, R_2],$$

where the grading is given by deg $x=1$, deg $A_1=1$, deg $B_p=p$ $(p=1,2,3)$, deg $Q_p=p$ $(p=0,1,2,3)$, deg $R_1=2$ and deg $R_2=3$. Then $P^{(g,h)}$ belongs to this ring and its degree is at most $3(g+h-1)$.

3.3. **Rewriting the extended holomorphic anomaly equation** (8)**.** There are relations among the $\partial_{\bar{z}}$-derivatives of the generators (23).

Lemma 11.

$$
\begin{aligned}
\partial_{\bar z} B_2 &= V_1 \partial_{\bar z} B_1\ , \\
\partial_{\bar z} B_3 &= (A_2 + 2A_1 + 3B_1 + 3B_2 + 3A_1 B_1 + 1)\partial_{\bar z} B_1 \\
&= \Big(- V_2 + \frac{\theta_z(z^3 C_{zzz})}{z^3 C_{zzz}} V_1 + h(z) - 1 \Big) \partial_{\bar z} B_1 \qquad (33) \\
\partial_{\bar z} Q_p &= 0 \qquad (p = 0, 1, 2, \ldots)\ , \\
\partial_{\bar z} R_2 &= - R_1 \partial_{\bar z} B_1\ .
\end{aligned}
$$

Proof. The first and the second equations were obtained from (1) in [20]. The third is trivial since Q_p's do not depend on $\bar z$. The calculation of $\partial_{\bar z} R_2$ is as follows.

$$
\partial_{\bar z} R_2 = z^{\alpha+1} C_{zzz} (\partial_{\bar z} \bar T + \partial_{\bar z} K \cdot \bar T) = z G_{z\bar z} R_1 = - R_1 \partial_{\bar z} B_1
$$

where we have used the identity $G_{z\bar z} = \partial_z \partial_{\bar z} K(z, \bar z) = - \partial_{\bar z} B_1 / z$. $\square$

If one assumes that $\partial_{\bar z} A_1$, $\partial_{\bar z} B_1$, $\partial_{\bar z} R_1$ are independent, the Walcher's extended holomorphic equation (8) is rewritten as follows.

Lemma 12. *The equation* (8) *is equivalent to the system of PDE's:*

$$
\begin{aligned}
&\Big[- R_1 \frac{\partial}{\partial R_2} - 2 \frac{\partial}{\partial A_1} + \frac{\partial}{\partial B_1} + V_1 \frac{\partial}{\partial B_2} \qquad (34) \\
&+ \Big(- V_2 + \frac{\theta_z(z^3 C_{zzz})}{z^3 C_{zzz}} V_1 + h(z) - 1 \Big) \frac{\partial}{\partial B_3} \Big] P^{(g,h)} = 0\ ,
\end{aligned}
$$

$$
\frac{\partial P^{(g,h)}}{\partial A_1} = - \frac{1}{2} \Big(\sum_{\substack{g_1+g_2=g,\\ h_1+h_2=h}} P_1^{(g_1,h_1)} P_1^{(g_2,h_2)} + P_2^{(g-1,h)} \Big) + (B_1 Q_0 - Q_1) P_1^{(g,h-1)}\ , \tag{35}
$$

$$
\frac{\partial P^{(g,h)}}{\partial R_1} = - P_1^{(g,h-1)}\ . \tag{36}
$$

Here the summation in (35) *runs over* $(g_1, h_1), (g_2, h_2)$ *such that* $(g_i, h_i) \neq (0,0), (0,1)$.

Proof. By (8), we have

$$
\begin{aligned}
\partial_{\bar z} P^{(g,h)} = &\frac{1}{2} \partial_{\bar z} (z C_{zzz} S^{zz}) \cdot \Big(\sum_{\substack{g_1+g_2=g,\\ h_1+h_2=h}} P_1^{(g_1,h_1)} P_1^{(g_2,h_2)} + P_2^{(g-1,h)} \Big) \\
&- \partial_{\bar z} (z^{\frac{5}{2}} C_{zzz} \triangle^z) \cdot P_1^{(g,h-1)}\ .
\end{aligned}
$$

Note that, by (31)(33),

$$
\begin{aligned}
&\partial_{\bar z} (z C_{zzz} S^{zz}) = - (\partial_{\bar z} A_1 + 2 \partial_{\bar z} B_1), \\
&\partial_{\bar z} (z^{\frac{5}{2}} C_{zzz} \triangle^z) = - (\partial_{\bar z} A_1 + 2 \partial_{\bar z} B_1)(-Q_1 + B_1 Q_0) + \partial_{\bar z} R_1\ .
\end{aligned}
$$

On the other hand, by (33), $\partial_{\bar z}$ in the LHS is as follows :

$$
\begin{aligned}
\partial_{\bar z} = &\partial_{\bar z} R_1 \frac{\partial}{\partial R_1} + \partial_{\bar z} A_1 \frac{\partial}{\partial A_1} + \partial_{\bar z} B_1 \Big[- R_1 \frac{\partial}{\partial R_2} + \frac{\partial}{\partial B_1} + V_1 \frac{\partial}{\partial B_2} \\
&+ \Big(- V_2 + \frac{\theta_z(z^3 C_{zzz})}{z^3 C_{zzz}} V_1 + h(z) - 1 \Big) \frac{\partial}{\partial B_3} \Big].
\end{aligned}
$$

Inserting these and comparing the coefficients of $\partial_{\bar{z}}A_1, \partial_{\bar{z}}B_1, \partial_{\bar{z}}R_1$, one obtains Lemma 12.

$\square$

To write the equations in a more useful form, we change the generators. We define

$$
\begin{aligned}
&u = B_1, \qquad v_1 = V_1 + \frac{3}{5}, \qquad v_2 = V_2 + \frac{2}{25},\\
&v_3 = B_3 - B_1\Big(- V_2 + \frac{\theta_z(z^3C_{zzz})}{z^3C_{zzz}} V_1 + h(z) - 1\Big) + s(z),\\
&m_1 = \frac{2}{25}Q_0 + \frac{3}{5}Q_1 + Q_2 - R_1,\\
&m_2 = Q_0\Big(s(z) - \frac{2}{25}\frac{\theta_z(z^3C_{zzz})}{z^3C_{zzz}}\Big) + Q_1\Big(\frac{23}{25} - h(z)\Big) - Q_2\frac{\theta_z(z^3C_{zzz})}{z^3C_{zzz}}\\
&\quad + Q_3 - R_2 - B_1R_1,
\end{aligned}
\tag{37}
$$

where

$$
s(z) = \frac{12}{25} - \frac{1}{5}h(z) + \frac{3}{25}\frac{\theta_z(z^3C_{zzz})}{z^3C_{zzz}}. \tag{38}
$$

Define the ring

$$
J := \mathbb{C}(z)[u, v_1, v_2, v_3, Q_0, Q_1, Q_2, Q_3, m_1, m_2].
$$

It is isomorphic to I since (37) is invertible. Notice that $\theta_z : J \to J$ increases the degree in u at most by 1.

Now we regard $P^{(g,h)} \in J$. Then (34) implies $P^{(g,h)}$ is independent of u. In turn, $P_n^{(g,h)} \in J$ has degree at most n in u. Following [12, (3-4.c)], let us define u-independent polynomials $Y_0, Y_1, W_0, W_1, W_2 \in J$ by

$$
\begin{aligned}
&Y_0 + u\ Y_1 = P_1^{(g,h-1)},\\
&W_0 + uW_1 + u^2W_2 = (\text{the RHS of (35)}).
\end{aligned}
\tag{39}
$$

Then applying the change of generators (37) to the equations (34)(35)(36), we obtain

Theorem 13. *The equation* (8) *is equivalent to the following system of PDE's for $P^{(g,h)} \in J$:*

$$
\begin{aligned}
&\frac{\partial}{\partial u}P^{(g,h)} = 0,\\
&\frac{\partial}{\partial m_1}P^{(g,h)} = Y_0, \quad \frac{\partial}{\partial m_2}P^{(g,h)} = Y_1,\\
&\frac{\partial}{\partial v_1}P^{(g,h)} = W_0, \quad \frac{\partial}{\partial v_2}P^{(g,h)} = -W_1 + \frac{\theta_z(z^3C_{zzz})}{z^3C_{zzz}}W_2, \quad \frac{\partial}{\partial v_3}P^{(g,h)} = -W_2.
\end{aligned}
\tag{40}
$$

Let us comment on the constant of integration. Decompose $P^{(g,h)}$ as

$$
P^{(g,h)} = \hat{P}^{(g,h)} + P^{(g,h)}|_{v_1,v_2,v_3,m_1,m_2=0}
$$

where $\hat{P}^{(g,h)}$ consists of terms of degree ≥ 1 with respect to at least one of v_1, v_2, v_3, m_1, m_2. The equations (40) can determine $\hat{P}^{(g,h)}$, but not the second term. The latter is a priori a polynomial in Q_0, Q_1, Q_2, Q_3 with $\mathbb{C}(z)$ coefficients. However, the choice of the new generators (37) is "good" (cf. [12, (3-4.d)]) so that we have the following

Proposition 14.

$$P^{(g,h)}|_{v_1,v_2,v_3,m_1,m_2=0} = (z^3C_{zzz})^{g+h-1}z^{\frac{h}{2}}f^{(g,h)}.$$

Proof. We have

$$\begin{aligned}
S^{zz} &= -\frac{v_1}{zC_{zzz}}, \quad S^z = \frac{uv_1+v_2}{z^2C_{zzz}},\\
S &= \frac{1}{z^3C}\Big[-\frac{1}{2}u^2v_1 - \Big(u+\frac{5^5z}{2(1-5^5z)}\Big)v_2 + \frac{v_3}{2}\Big],\\
\triangle^z &= \frac{1}{z^{\frac{5}{2}}C_{zzz}}(-m_1+Q_1v_1+Q_0v_2),\\
\triangle &= \frac{1}{z^{\frac{7}{2}}C_{zzz}}\Big[um_1 - m_2 - uQ_0v_1 - v_2\Big(uQ_0 + \frac{5^5z}{1-5^5z}Q_0 + Q_1\Big) + Q_0v_3\Big].
\end{aligned}$$

Notice that every monomial term in the propagators S^{zz}, S^z, S and the terminators $\triangle^z, \triangle$ contains at least one of v_1, v_2, v_3, m_1, m_2. Therefore the Feynman diagram part $\mathcal{F}^{(g,h)}_{FD}$ of $\mathcal{F}^{(g,h)}$ has degree at least one with respect to one of v_1, v_2, v_3, m_1, m_2 by (19)(20). This implies that the first term in the RHS of

$$P^{(g,h)} = (z^3C_{zzz})^{g+h-1}z^{\frac{h}{2}}\mathcal{F}^{(g,h)}_{FD} + (z^3C_{zzz})^{g+h-1}z^{\frac{h}{2}}f^{(g,h)}$$

vanishes as v_1, v_2, v_3, m_1, m_2 go to zero. This proves the proposition. □

4. Fixing holomorphic ambiguity and $n_d^{(g,h)}$

Let $\omega_0(z)$,$\omega_1(z)$, $\omega_2(z)$, $\omega_3(z)$ be the following solutions to the Picard–Fuchs equation $\mathcal{D}\omega = 0$ about $z = 0$.

$$\omega_i(z) = \partial_\rho^i\Big(\sum_{n\geq 0}\frac{(5\rho+1)_{5n}}{{(\rho+1)_n}^5}z^{n+\rho}\Big)\Big|_{\rho=0}.$$

Let $t = \omega_1(z)/\omega_0(z)$ be the mirror map and consider the inverse $z = z(q)$ where $q = e^t$. Explicitly, these are

$$\begin{aligned}
\omega_0(z) &= 1 + 120z + 113400z^2 + \cdots,\\
\omega_1(z) &= \omega_0(z)\log z + 770z + 810225z^2 + \cdots,\\
t &= 770z + 717825z^2 + \frac{3225308000}{3}z^3 + \cdots,\\
z &= q - 770q^2 + 171525q^3 + \cdots.
\end{aligned}$$

Let

$$F_A^{(g,h)} = \lim_{\bar{z}\to 0}\mathcal{F}^{(g,h)}\omega_0(z)^{2g+h-2}, \tag{41}$$

for (g,h) satisfying $2g+h-2>0$ [6]. The limit $\bar{z}\to 0$ in the RHS means

$$G_{z\bar{z}} \to \frac{dt}{dz}, \quad e^K \to \omega_0(z), \quad \triangle_{zz} \to D_z D_z \mathcal{T}.$$

Define $n_d^{(g,h)}$ for $h>0$ [7] by the formula [15] [14] [19, (3.22)]:

$$\begin{aligned} &\text{the terms in positive powers in } q \text{ of } \sum_{g=0}^{\infty} {g_s}^{2g+h-2} F_A^{(g,h)} \\ &= \sum_{g=0}^{\infty}\sum_{d}\sum_{k} n_d^{(g,h)} \frac{1}{k}\Big(2\sin\frac{kg_s}{2}\Big)^{2g+h-2} q^{\frac{kd}{2}}. \end{aligned} \tag{42}$$

Here the summation of k is over positive odd integers and that of d is over positive even (resp. odd) integers when h is even (resp. odd).

Remark 15. It is expected that $F_A^{(g,h)}$ is the A-model topological string amplitude of genus g with h boundaries for the real quintic 3-fold (X,L), and that $n_d^{(g,h)}$ be the BPS invariants in the class $d\in H_2(X,L;Z)$. See [7, 21] for $(g,h)=(0,0),(1,0)$ and [18, 16] for $(g,h)=(0,1)$.

In order to fix the holomorphic ambiguity, we put the following assumptions.

(i) If h is even, the q-constant term in $F_A^{(g,h)}$ vanishes except for $(g,h)=(0,2)$.

(ii) $n_d^{(g,h)}=0$ for $d\le d_0$ where d_0 is the smallest number necessary to completely determine unknown parameters in $f^{(g,h)}$. For example, $d_0=3$ for $(g,h)=(0,3),(1,1)$, $d_0=6$ for $(g,h)=(1,2),(0,4)$ and $d_0=9$ for $(g,h)=(1,3),(0,5)$.

The numbers $n_d^{(g,h)}$ obtained under these assumptions are listed in Tables 1 and 2.

Remark 16. The boundary conditions proposed in [19] are the condition (i) and the condition that

$$n_d^{(g,h)}=0 \text{ if } n_d^{2g+h-1}=0. \tag{43}$$

These do not give enough equations to fix the unknown parameters of $f^{(g,h)}$, unless $(g,h)=(0,1),(0,2),(0,3),(1,1)$. For this reason we assumed (ii) instead of (43).

Remark 17. For the cases listed in Tables 1 and 2, $n_d^{(g,h)}$ turn out to be integers. However, for $(g,h)=(0,7),(1,5),(2,1)$, the holomorphic ambiguities determined by our assumptions do not give integral $n_d^{(g,h)}$'s.

[6] For $(g,h)=(0,0),(1,0),(0,1),(0,2)$, one should consider

$$\partial_t^n F_A^{(g,h)} = \Big(\frac{dz}{dt}\Big)^n \lim_{\bar{z}\to 0} \mathcal{F}_n^{(g,h)} \omega_0^{2g+h-2}$$

where $n=3,1,2,1$, respectively.

[7] For $h=0$, the BPS number n_d^g is defined by [8]

$$\sum_{g=0}^{\infty} {g_s}^{2g-2} F_A^{(g,0)} = \sum_{g=0}^{\infty}\sum_{d>0}\sum_{k>0} n_d^g \frac{1}{k}\Big(2\sin\frac{kg_s}{2}\Big)^{2g-2} q^{kd} + \text{ polynomial in } \log q.$$

d	$n_d^{(0,4)}$	d	$n_d^{(0,5)}$
2	0	1	0
4	0	3	0
6	0	5	0
8	−307669500	7	0
10	−1290543544800	9	0
12	−4192442370526500	11	−101052180000
14	−11974312128284645400	13	−6448499064000
16	−31709386561589633978460	15	2809704427965432000
18	−79870219101822591783739800	17	19034205058652662269000
20	−194146223749422074623095454800	19	85987169904148441092385200

d	$n_d^{(0,6)}$
2	0
4	0
6	0
8	0
10	0
12	0
14	10969992383850000
16	88807052603386080000
18	453871851092663617206000
20	1856308715086126538509560000

Table 1. $n_d^{(g,h)}$ for $(g,h) = (0,4), (0,5), (0,6)$

Remark 18. As a final remark, let us comment on the expansion about the conifold point $z = \frac{1}{5^5}$. By expanding $\mathcal{F}^{(0,4)}$ about $z = \frac{1}{5^5}$, we see that there is no gap condition such as the one found in [13, (1.2)]. On the other hand, if one imposes the gap condition to $\mathcal{F}^{(0,4)}$ instead of $n_6^{(0,4)} = 0$, then the integrality of $n_d^{(0,4)}$'s does not hold.

Appendix A. Examples of Feynman diagrams

Feynman diagrams for $\mathcal{F}_{\mathrm{FD}}^{(0,3)}$ and $\mathcal{F}_{\mathrm{FD}}^{(1,1)}$ have been given in eqs. (2.109) and (2.108) of [19] respectively ($\#\mathbb{G}(0,3) = \#\mathbb{G}(1,1) = 4$). For $(g,h) = (0,4)$, we have $\#\mathbb{G}(0,4) = 19$. See Fig. 3. It is clear that the number of Feynman diagrams grows rapidly as g and h increase. For example, one can check that $\#\mathbb{G}(0,5) = 83$, $\#\mathbb{G}(1,2) = 29$, $\#\mathbb{G}(2,1) = 97$.

d	$n_d^{(1,1)}$	d	$n_d^{(1,2)}$
1	0	2	0
3	0	4	0
5	−222535	6	0
7	−472460880	8	−1798092240
9	−970639017980	10	−3910898328975
11	−1925950714205525	12	−3254492224834500
13	−3771152449472734885	14	1174928171611889000
15	−7341083828377813532445	16	7585803372459666836250
17	−14254813486499789264497980	18	284100639663878543462155290
19	−27655486644196368361422400900	20	88156839926773091360811758000

d	$n_d^{(1,3)}$	d	$n_d^{(1,4)}$
1	0	2	0
3	0	4	0
5	0	6	0
7	0	8	0
9	0	10	0
11	59476704611850	12	0
13	376498723243912410	14	−510835096894879500
15	1597793312432171312570	16	−4625213168889849497100
17	5622302692504776557418000	18	−26075494174267321098602160
19	17697465511801448466779111250	20	−116382815077174964736448167150

TABLE 2. $n_d^{(g,h)}$ for $(g,h) = (1,1), (1,2), (1,3), (1,4)$

APPENDIX B. $f^{(0,4)}$ AND $P^{(0,4)}$

$$f^{(0,4)} = \frac{2 - 20125\,z + 70618750\,z^2 - 86493078125\,z^3}{10000\,(1-3125\,z)^2}.$$

$$\begin{aligned}
P^{(0,4)} = {} & \frac{z^2\,(-2 + 20125\,z - 70618750\,z^2 + 86493078125\,z^3)}{80\,(-1+3125\,z)^5} - \frac{z\,(2 - 9500\,z + 16015625\,z^2)\,{m_1}^2}{20\,(-1+3125\,z)^3} \\
& + \frac{(-9 + 12500\,z)\,{m_1}^4}{120\,(-1+3125\,z)} + \frac{75\,z^2\,(-1+3145\,z)\,m_2}{4\,(-1+3125\,z)^3} + \frac{{m_1}^3\,m_2}{6} + \frac{5\,z\,{m_2}^2}{4\,(-1+3125\,z)} \\
& + m_1\,(\frac{375\,z^3\,(-3+3125\,z)}{2\,(-1+3125\,z)^4} + \frac{375\,z^2\,m_2}{2\,(-1+3125\,z)^2}) - \frac{{Q_1}^4\,{v_1}^5}{8} + (\frac{(-3+25000\,z)\,{Q_0}^4}{40\,(-1+3125\,z)} - \frac{{Q_0}^3\,Q_1}{6})\,{v_2}^4 \\
& + {v_1}^4\,(\frac{m_1\,{Q_1}^3}{2} + \frac{(-9+12500\,z)\,{Q_1}^4}{120\,(-1+3125\,z)} - \frac{Q_0\,{Q_1}^3\,v_2}{2}) + (\frac{25\,z^2}{8\,(-1+3125\,z)^2} - \frac{75\,z^2\,(-1+3145\,z)\,Q_0}{4\,(-1+3125\,z)^3} \\
& - \frac{375\,z^2\,m_1\,Q_0}{2\,(-1+3125\,z)^2} - \frac{{m_1}^3\,Q_0}{6} - \frac{5\,z\,m_2\,Q_0}{2\,(-1+3125\,z)})v_3 + \frac{5\,z\,{Q_0}^2\,{v_3}^2}{4\,(-1+3125\,z)} \\
& + {v_2}^2\,(\frac{z\,(-1+4750\,z+119921875\,z^2)\,{Q_0}^2}{10\,(-1+3125\,z)^3} + m_1\,(\frac{-5\,z\,Q_0}{2\,(-1+3125\,z)} + \frac{m_2\,{Q_0}^2}{2}) \\
& - \frac{8000\,z^2\,Q_0\,Q_1}{(-1+3125\,z)^2} + \frac{5\,z\,{Q_1}^2}{4\,(-1+3125\,z)} + {m_1}^2\,(\frac{(-9+43750\,z)\,{Q_0}^2}{20\,(-1+3125\,z)} - \frac{Q_0\,Q_1}{2}) - \frac{m_1\,{Q_0}^3\,v_3}{2})
\end{aligned}$$

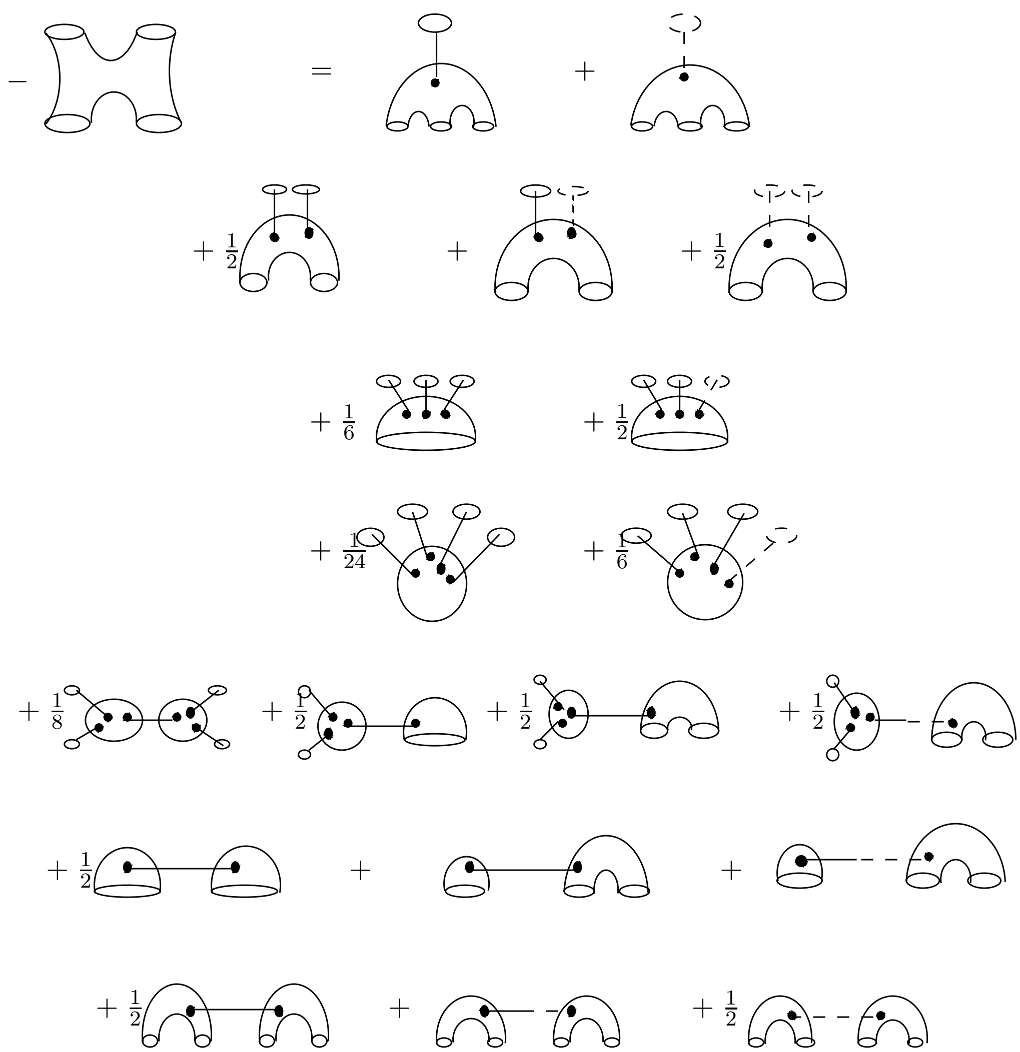

FIGURE 3. The elements G in $\mathbb{G}(0,4)$ and the orders of A_G. The vertices are expressed as bordered Riemann surfaces to visualize the labeling.

$$
\begin{aligned}
&+ {v_2}^3 \left(\frac{5\,z\,{Q_0}^2}{4\,(-1+3125\,z)} - \frac{m_2\,{Q_0}^3}{6} + m_1 \left(\frac{-((-9+59375\,z)\,{Q_0}^3)}{30\,(-1+3125\,z)} + \frac{{Q_0}^2\,Q_1}{2}\right) + \frac{{Q_0}^4\,v_3}{6}\right) \\
&+ {v_1}^3 \left(\frac{-375\,z^2\,{Q_1}^2}{4\,(-1+3125\,z)^2} - \frac{3\,{m_1}^2\,{Q_1}^2}{4} - \frac{(-9+12500\,z)\,m_1\,{Q_1}^3}{30\,(-1+3125\,z)} - \frac{m_2\,{Q_1}^3}{6}\right. \\
&+ \left(\frac{3\,m_1\,Q_0\,{Q_1}^2}{2} + \frac{3\,Q_0\,{Q_1}^3}{10} - \frac{{Q_1}^4}{6}\right) v_2 - \frac{3\,{Q_0}^2\,{Q_1}^2\,{v_2}^2}{4} + \left.\frac{Q_0\,{Q_1}^3\,v_3}{6}\right) \\
&+ v_2 \left(\frac{81875\,z^3}{8\,(-1+3125\,z)^3} - \frac{236625\,z^3\,Q_0}{4\,(-1+3125\,z)^3} + {m_1}^2 \left(\frac{5\,z}{4\,(-1+3125\,z)} - \frac{m_2\,Q_0}{2}\right)\right. \\
&+ {m_1}^3 \left(\frac{-3\,Q_0}{10} + \frac{Q_1}{6}\right) + \frac{75\,z^2\,(-1+3145\,z)\,Q_1}{4\,(-1+3125\,z)^3} + m_1 \left(\frac{z\,(-1+1625\,z)\,Q_0}{5\,(-1+3125\,z)^2} + \frac{375\,z^2\,Q_1}{2\,(-1+3125\,z)^2}\right) \\
&+ m_2 \left(\frac{-8000\,z^2\,Q_0}{(-1+3125\,z)^2} + \frac{5\,z\,Q_1}{2\,(-1+3125\,z)}\right) + \left.\left(\frac{8000\,z^2\,{Q_0}^2}{(-1+3125\,z)^2} + \frac{{m_1}^2\,{Q_0}^2}{2} - \frac{5\,z\,Q_0\,Q_1}{2\,(-1+3125\,z)}\right) v_3\right) \\
&+ v_1 \left(\frac{-140625\,z^4}{8\,(-1+3125\,z)^4} - \frac{{m_1}^4}{8} - \frac{375\,z^3\,(-3+3125\,z)\,Q_1}{2\,(-1+3125\,z)^4} + \frac{z\,(2-9500\,z+16015625\,z^2)\,m_1\,Q_1}{10\,(-1+3125\,z)^3}\right. \\
&- \frac{(-9+12500\,z)\,{m_1}^3\,Q_1}{30\,(-1+3125\,z)} - \frac{375\,z^2\,m_2\,Q_1}{2\,(-1+3125\,z)^2} + {m_1}^2 \left(\frac{-375\,z^2}{4\,(-1+3125\,z)^2} - \frac{m_2\,Q_1}{2}\right) \\
&+ \left(\frac{m_1\,{Q_0}^3}{2} + \frac{(-9+59375\,z)\,{Q_0}^3\,Q_1}{30\,(-1+3125\,z)} - \frac{{Q_0}^2\,{Q_1}^2}{2}\right) {v_2}^3 - \frac{{Q_0}^4\,{v_2}^4}{8} \\
&+ \left(\frac{375\,z^2\,Q_0\,Q_1}{2\,(-1+3125\,z)^2} + \frac{{m_1}^2\,Q_0\,Q_1}{2}\right) v_3 + v_2 \left(\frac{{m_1}^3\,Q_0}{2} - \frac{z\,(-1+1625\,z)\,Q_0\,Q_1}{5\,(-1+3125\,z)^2} - \frac{375\,z^2\,{Q_1}^2}{2\,(-1+3125\,z)^2}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ m_1 \left(\frac{375\, z^2\, Q_0}{2\,(-1 + 3125\, z)^2} - \frac{5\, z\, Q_1}{2\,(-1 + 3125\, z)} + m_2\, Q_0\, Q_1\right) + {m_1}^2 \left(\frac{9\, Q_0\, Q_1}{10} - \frac{{Q_1}^2}{2}\right) - m_1\, {Q_0}^2\, Q_1\, v_3)\\
&+ {v_2}^2 \left(\frac{-375\, z^2\, {Q_0}^2}{4\,(-1 + 3125\, z)^2} - \frac{3\, {m_1}^2\, {Q_0}^2}{4} + \frac{5\, z\, Q_0\, Q_1}{2\,(-1 + 3125\, z)} - \frac{m_2\, {Q_0}^2\, Q_1}{2}\right.\\
&+ m_1 \left(\frac{-((-9 + 43750\, z)\, {Q_0}^2\, Q_1)}{10\,(-1 + 3125\, z)} + Q_0\, {Q_1}^2\right) + \frac{{Q_0}^3\, Q_1\, v_3}{2}))\\
&+ {v_1}^2 \left(\frac{{m_1}^3\, Q_1}{2} - \frac{z\,(2 - 9500\, z + 16015625\, z^2)\, {Q_1}^2}{20\,(-1 + 3125\, z)^3} + \frac{(-9 + 12500\, z)\, {m_1}^2\, {Q_1}^2}{20\,(-1 + 3125\, z)}\right.\\
&+ m_1 \left(\frac{375\, z^2\, Q_1}{2\,(-1 + 3125\, z)^2} + \frac{m_2\, {Q_1}^2}{2}\right) + \left(\frac{3\, m_1\, {Q_0}^2\, Q_1}{2} + \frac{(-9 + 43750\, z)\, {Q_0}^2\, {Q_1}^2}{20\,(-1 + 3125\, z)} - \frac{Q_0\, {Q_1}^3}{2}\right) {v_2}^2\\
&- \frac{{Q_0}^3\, Q_1\, {v_2}^3}{2} - \frac{m_1\, Q_0\, {Q_1}^2\, v_3}{2} + v_2 \left(\frac{-375\, z^2\, Q_0\, Q_1}{2\,(-1 + 3125\, z)^2} - \frac{3\, {m_1}^2\, Q_0\, Q_1}{2} + \frac{5\, z\, {Q_1}^2}{4\,(-1 + 3125\, z)}\right.\\
&- \frac{m_2\, Q_0\, {Q_1}^2}{2} + m_1 \left(\frac{-9\, Q_0\, {Q_1}^2}{10} + \frac{{Q_1}^3}{2}\right) + \frac{{Q_0}^2\, {Q_1}^2\, v_3}{2})).
\end{aligned}
$$

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