FRACTIONAL PROCESSES WITH LONG-RANGE DEPENDENCE

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Abstract. We introduce a class of Gaussian processes with stationary increments which exhibit long-range dependence. The class includes fractional Brownian motion with Hurst parameter $H > 1/2$ as a typical example. We establish infinite and finite past prediction formulas for the processes in which the predictor coefficients are given explicitly in terms of the MA(∞) and AR(∞) coefficients. We apply the formulas to prove an analogue of Baxter's inequality, which concerns the L^1-estimate of the difference between the finite and infinite past predictor coefficients.

1. Introduction

Let $(X(t) : t \in \mathbf{R})$ be a centered Gaussian process with stationary increments, defined on a probability space $(\Omega, \mathcal{F}, P)$, that admits the *moving-average* representation

$$X(t) = \int_{-\infty}^{\infty} \{g(t-s) - g(-s)\}\, dW(s), \qquad t \in \mathbf{R}, \tag{1.1}$$

where $(W(t) : t \in \mathbf{R})$ is a Brownian motion, and $g(t)$ is a function of the form

$$g(t) = \int_0^t c(s)ds, \qquad t \in \mathbf{R}, \tag{1.2}$$

$$c(t) := I_{(0,\infty)}(t) \int_0^\infty e^{-ts}\nu(ds), \qquad t \in \mathbf{R}, \tag{1.3}$$

with some Borel measure ν on $(0, \infty)$ satisfying

$$\int_0^\infty \frac{1}{1+s}\nu(ds) < \infty. \tag{1.4}$$

We will also assume some extra conditions such as

$$\lim_{t\to 0+} c(t) = \infty, \tag{1.5}$$

$$g(t) \sim t^{H-(1/2)}\ell(t) \cdot \frac{1}{\Gamma(\frac{1}{2} + H)}, \qquad t \to \infty, \tag{1.6}$$

where $\ell(t)$ is a slowly varying function at infinity and H is a constant such that

$$1/2 < H < 1. \tag{1.7}$$

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In (1.6), and throughout the paper, $a(t) \sim b(t)$ as $t \to \infty$ means $\lim_{t\to\infty} a(t)/b(t) = 1$. We call $c(t)$ (rather than $g(t)$) the *MA*(∞) *coefficient* of $(X(t))$.

A typical example of ν is

$$\nu(ds) = \frac{\sin\{\pi(H - \frac{1}{2})\}}{\pi} s^{1/2-H} ds \quad \text{on } (0, \infty) \tag{1.8}$$

with (1.7). For this ν, $g(t)$ becomes

$$g(t) = I_{(0,\infty)}(t) t^{H-1/2} \frac{1}{\Gamma(\frac{1}{2} + H)}, \qquad t \in \mathbf{R}, \tag{1.9}$$

and $(X(t))$ reduces to *fractional Brownian motion* $(B_H(t))$ with *Hurst parameter* H (see Example 2.3 below). Fractional Brownian motion, abbreviated fBm, was introduced by Kolmogorov [20]. For $1/2 < H < 1$, fBm has both *self-similarity* and *long-range dependence* (Samorodnitsky and Taqqu [27]), and plays an important role in various fields such as network traffic (see, e.g., Mikosch et al. [22]) and finance (see, e.g., Hu et al. [10]); see also Taqqu [28] and other papers in the same volume. Because of its importance, stochastic calculus for fBm has been developed by many authors; see, e.g., Decreusefond and Üstünel [8], and Nualart [24]. Other important examples of $(X(t))$ are the processes with long-range dependence which, unlike fBm, have two different indices H_0 and H describing the local properties (path properties) and long-time behavior of $(X(t))$, respectively (see Example 2.4 below).

Let t_0, t_1 and T be real constants such that

$$-\infty < -t_0 \le 0 \le t_1 < T < \infty, \qquad -t_0 < t_1. \tag{1.10}$$

For $I = (-\infty, t_1]$ or $[-t_0, t_1]$, we write $P_I X(T)$ for the predictor of the future value $X(T)$ based on the observable $(X(s) : s \in I)$ (see Section 3 below). One of the fundamental prediction problems for $(X(t))$ is to express $P_I X(T)$ using the segment $(X(s) : s \in I)$ and some deterministic quantities. Another is to express the variance of the prediction error $P_I^\perp X(T) := X(T) - P_I X(T)$. Results of this type become important tools in the analysis of non-Markovian processes and systems modulated by them (see, e.g., Norros et al. [23], Anh et al. [3], Inoue et al. [19] and Inoue and Nakano [18]). One of our main purposes here is to derive such results for $(X(t))$.

We establish the following infinite and finite past prediction formulas for $(X(t))$ (see Theorems 3.8 and 4.12 below):

$$P_{(-\infty,t_1]} X(T) = X(t_1) + \int_{-\infty}^{t_1} \left\{ \int_0^{T-t_1} b(t_1 - s, \tau) d\tau \right\} dX(s), \tag{1.11}$$

$$P_{[-t_0,t_1]} X(T) = X(t_1) + \int_{-t_0}^{t_1} \left\{ \int_0^{T-t_1} h(s + t_0, u) du \right\} dX(s). \tag{1.12}$$

The significance of (1.11) and (1.12) is that the predictor coefficients $b(t, s)$ and $h(t, s)$ are given explicitly in terms of the MA(∞) coefficient $c(t)$ and *AR*(∞) *coefficient* $a(t)$ of $(X(t))$. We will find that $a(t)$ has a nice integral representation similar to (1.3) (see (3.3) below). It turns out that the existence of such a nice AR(∞) coefficient, in addition to the nice MA(∞) coefficient, is a key to the solution to the prediction problems above.

We apply the results above to the proof of *Baxter's inequality* for $(X(t))$, which concerns the L^1-estimate of the difference between the predictor coefficients $b(t, s)$

and $h(t,s)$. The original inequality of Baxter [4] is an assertion for stationary time series $(Y_n : n \in \mathbf{N})$ with short memory. It takes the form

$$\sum_{j=1}^{n} |\phi_{n,j} - \phi_j| \le K \sum_{k=n+1}^{\infty} |\phi_k|, \qquad \forall n \ge 1, \tag{1.13}$$

where K is a positive constant, and ϕ_j and $\phi_{n,j}$ are the infinite and finite past predictor coefficients in

$$P_{(-\infty,-1]}Y_0 = \sum_{j=1}^{\infty} \phi_j Y_{-j}, \qquad P_{[-n,-1]}Y_0 = \sum_{j=1}^{n} \phi_{n,j} Y_{-j},$$

respectively, with $P_{(-\infty,-1]}Y_0$ and $P_{[-n,-1]}Y_0$ being defined similarly. See Berk [5], Cheng and Pourahmadi [7], and Inoue and Kasahara [17] for related work; for a textbook account, see Pourahmadi [26, Section 7.6.2]. Using the explicit representations of $b(t,s)$ and $h(t,s)$, we can prove an analogue of (1.13) for $(X(t))$ which are continuous-time stationary-increment processes with long-range dependence.

For fBm with $1/2 < H < 1$, the predictor coefficients $b(t,s)$ and $h(t,s)$ are given in Gripenberg and Norros [9] (see (3.13) and (5.3) below). See [23] and [25] for different proofs. Fractional Brownian motion has a variety of nice properties, and the methods of proof of [9, 23, 25] naturally rely on such special properties of fBm, hence are not applicable to $(X(t))$. The method of this paper is based on the *alternating projections to the past and future* (see Section 4.1 below). As for fBm with $0 < H < 1/2$, its infinite and finite past prediction formulas also exist, and are due to Yaglom [29] and Nuzman and Poor [25], respectively (see also Anh and Inoue [2]); see Inoue and Anh [15] for an extension to these results, which have different forms from (1.11) and (1.12) since no stochastic integrals appear there.

We provide the basic properties and examples of $(X(t))$ in Section 2. We consider the infinite and finite past prediction problems for $(X(t))$ in Sections 3 and 4, respectively. In Section 5, we prove an analogue of Baxter's inequality for $(X(t))$, using the results in Sections 3 and 4.

2. Basic properties and examples

In this section, we assume (1.2)–(1.4) and

$$\int_1^{\infty} c(t)^2 dt < \infty. \tag{2.1}$$

Then, as in [15, Lemma 2.1], we have $\int_{-\infty}^{\infty} |g(t-s) - g(-s)|^2 ds < \infty$ for $t \in \mathbf{R}$. Therefore, for a one-dimensional standard Brownian motion $(W(t) : t \in \mathbf{R})$ with $W(0) = 0$, we may define the centered stationary-increment Gaussian process $(X(t) : t \in \mathbf{R})$ by (1.1).

For $s > 0$ and $t \in \mathbf{R}$, we put $\Delta_s X(t) := X(t+s) - X(t)$. Then, by definition, $(\Delta_s X(t) : t \in \mathbf{R})$ is a stationary process.

Lemma 2.1. *Let $s \in (0,\infty)$. We assume* (1.6) *and* (1.7). *Then*

$$E\left[\Delta_s X(t) \cdot \Delta_s X(0)\right] \sim t^{2H-2} \ell(t)^2 \cdot \frac{s^2 \Gamma(2-2H) \sin\{(H-\frac{1}{2})\pi\}}{\pi}, \qquad t \to \infty.$$

Since $-1 < 2H - 2 < 0$ in Lemma 2.1, we see from this lemma that $(\Delta_s X(t))$, whence $(X(t))$, has long-range dependence.

We put $\sigma(t) := E[|X(t+s) - X(s)|^2]^{1/2}$ for $t \ge 0$ and $s \in \mathbf{R}$.

Lemma 2.2. *Let $H_0 \in (1/2, 1)$ and $\ell_0(\cdot)$ a slowly varying function at infinity. We assume*

$$g(t) \sim t^{H_0-(1/2)}\ell_0(1/t) \cdot \frac{1}{\Gamma(\frac{1}{2}+H_0)}, \qquad t \to 0+. \tag{2.2}$$

Then

$$\sigma(t) \sim t^{H_0}\ell(1/t)\sqrt{v(H_0)}, \qquad t \to 0+,$$

where $v(H_0) := \Gamma(2-2H_0)\cos(\pi H_0)/\{\pi H_0(1-2H_0)\}$. In particular, we have

$$H_0 = \sup\{\beta : \sigma(t) = o(t^\beta), \quad t \to 0+\} = \inf\{\beta : t^\beta = o(\sigma(t)), \quad t \to 0+\}.$$

From Lemma 2.2, we see that the index H_0 describes the path properties of $(X(t))$ (see Adler [1, Section 8.4]).

By the monotone density theorem (cf. Bingham et al. [6, Theorem 1.7.5]), (1.6) with (1.7) implies

$$c(t) \sim t^{H-(3/2)}\ell(t) \cdot \frac{1}{\Gamma(H-\frac{1}{2})}, \qquad t \to \infty. \tag{2.3}$$

Similarly, (2.2) implies

$$c(t) \sim t^{H_0-(3/2)}\ell_0(1/t) \cdot \frac{1}{\Gamma(H_0-\frac{1}{2})}. \qquad t \to 0+. \tag{2.4}$$

Lemmas 2.1 and 2.2 follow from (2.3) and (2.4), respectively, by standard arguments. However, since we do not use these results in the arguments below, we omit the details.

Example 2.3. For $H \in (1/2, 1)$, let ν be as in (1.8). Then we have (1.9); and so all the conditions above are satisfied. The resulting process $(X(t))$ is fBm $(B_H(t))$:

$$B_H(t) = \frac{1}{\Gamma(\frac{1}{2}+H)}\int_{-\infty}^{\infty}\left\{((t-s)_+)^{H-(1/2)} - ((-s)_+)^{H-(1/2)}\right\} dW(s), \tag{2.5}$$

where $(x)_+ := \max(0, x)$ for $x \in \mathbf{R}$. The representation (2.5) of fBm is due to the pioneering work of Mandelbrot and Van Ness [21].

Example 2.4. Let $f(\cdot)$ be a nonnegative, locally integrable function on $(0, \infty)$. For $H_0, H \in (1/2, 1)$ and slowly varying functions $\ell_0(\cdot)$ and $\ell(\cdot)$ at infinity, we assume

$$f(s) \sim \frac{\sin\{\pi(H_0-\frac{1}{2})\}}{\pi} s^{(1/2)-H}\ell(1/s), \qquad s \to 0+,$$
$$f(s) \sim \frac{\sin\{\pi(H_0-\frac{1}{2})\}}{\pi} s^{(1/2)-H_0}\ell_0(s), \qquad s \to \infty.$$

Let $\nu(ds) = f(s)ds$. Then, by Abelian theorems for Laplace transforms (cf. [6, Section 1.7]), we have (2.3), whence (1.6). Similarly, we have (2.4), whence (2.2). Thus all the conditions above are satisfied. As we have seen above, the indices H_0 and H describe the path properties and long-time behavior of $(X(t))$, respectively.

3. Infinite past prediction problems

In this section, we assume (1.1)–(1.5), (2.1) and

$$\lim_{t\to\infty} g(t) = \infty. \tag{3.1}$$

Notice that, for the processes $(X(t))$ in Examples 2.3 and 2.4, all these conditions are satisfied. We also assume (1.10).

We write $M(X)$ for the real Hilbert space spanned by $(X(t) : t \in \mathbf{R})$ in $L^2(\Omega, \mathcal{F}, P)$, and $\|\cdot\|$ for its norm. Let I be a closed interval of $\mathbf{R}$ such as $[-t_0, t_1]$, $(-\infty, t_1]$, and $[-t_0, \infty)$. Let $M_I(X)$ be the closed subspace of $M(X)$ spanned by $(X(t) : t \in I)$. We write P_I for the orthogonal projection operator from $M(X)$ to $M_I(X)$, and $P_I^\perp$ for its orthogonal complement: $P_I^\perp Z = Z - P_I Z$ for $Z \in M(X)$. Note that, since $(X(t))$ is a Gaussian process, we have

$$P_I Z = E\left[Z|\ \sigma(X(s) : s \in I)\right], \qquad Z \in M(X).$$

3.1. MA and AR coefficients.

The conditions (1.5) and (3.1) imply $\nu(0,\infty) = \infty$ and $\int_0^\infty s^{-1}\nu(ds) = \infty$, respectively. Therefore, by [15, Theorem 3.2], there exists a unique Borel measure μ on $(0,\infty)$ satisfying

$$\int_0^\infty \frac{1}{1+s}\mu(ds) < \infty, \quad \mu(0,\infty) = \infty, \quad \int_0^\infty \frac{1}{s}\mu(ds) = \infty$$

and

$$-iz\left\{\int_0^\infty e^{izt}c(t)dt\right\}\left\{\int_0^\infty e^{izt}\alpha(t)dt\right\} = 1, \qquad \Im z > 0, \tag{3.2}$$

with

$$\alpha(t) := \int_0^\infty e^{-st}\mu(ds), \qquad t > 0.$$

We define the $AR(\infty)$ *coefficient* $a(t)$ of $(X(t))$ by

$$a(t) := -\frac{d\alpha}{dt}(t) = \int_0^\infty e^{-st}s\mu(ds), \qquad t > 0. \tag{3.3}$$

We define the positive kernel $b(t,s)$ by

$$b(t,s) := \int_0^s c(u)a(t+s-u)du, \qquad t, s > 0.$$

Then, by [15, Lemma 3.4], the following equalities hold:

$$\int_0^\infty b(t,s)dt = 1, \qquad s > 0, \tag{3.4}$$

$$c(t+s) = \int_0^t c(t-u)b(u,s)du, \qquad t, s > 0. \tag{3.5}$$

3.2. Stochastic integrals.

Let I be a closed interval of $\mathbf{R}$. We define

$$\mathcal{H}_I(X) := \left\{ f\ :\ \begin{array}{l} f \text{ is a real-valued measurable function on } I \text{ such} \\ \text{that } \int_{-\infty}^\infty \left\{\int_I |f(u)|c(u-s)du\right\}^2 ds < \infty. \end{array}\right\}.$$

This is the class of functions f for which we can define the stochastic integral $\int_I f(s)dX(s)$. We define a subclass $\mathcal{H}_I^0$ of $\mathcal{H}_I(X)$ by

$$\mathcal{H}_I^0 := \left\{\sum\nolimits_{k=1}^m a_k I_{(t_{k-1},t_k]}(s) : \begin{array}{l} m \in \mathbf{N},\ -\infty < t_0 < t_1 < \cdots < t_m < \infty \\ \text{with } (t_0, t_m] \subset I,\ a_k \in \mathbf{R}\ (k = 1, \ldots, m) \end{array}\right\}.$$

Each member of $f \in \mathcal{H}_I^0$ a *simple function* on I.

Definition 3.1. For $f = \sum_{k=1}^m a_k I_{(t_{k-1},t_k]} \in \mathcal{H}_I^0$, we define

$$\int_I f(s)dX(s) := \sum_{k=1}^m a_k \{X(t_k) - X(t_{k-1}\}.$$

We see that $\int_I f(s)dX(s) \in M_I(X)$ for $f \in \mathcal{H}_I^0$.

Proposition 3.2. *For $f \in \mathcal{H}_I^0$, we have*

$$\int_I f(s)dX(s) = \int_{-\infty}^{\infty} \left\{ \int_I f(u)c(u-s)du \right\} dW(s). \tag{3.6}$$

Proof. For $-\infty < a < b < \infty$ with $(a,b] \subset I$, we have

$$X(b) - X(a) = \int_{-\infty}^{\infty} \left\{ \int_I I_{(a,b]}(u)c(u-s)du \right\} dW(s),$$

which implies (3.6) for $f = I_{(a,b]}$. The general case follows easily from this. □

Proposition 3.3. *Let $f \in \mathcal{H}_I(X)$ such that $f \geq 0$, and let f_n $(n = 1, 2, \dots)$ be a sequence of simple functions on I such that $0 \leq f_n \uparrow f$ a.e. Then, in $M(X)$,*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(s)dX(s) = \int_{-\infty}^{\infty} \left\{ \int_I f(u)c(u-s)du \right\} dW(s).$$

Proof. By Proposition 3.2 and the monotone convergence theorem, we have

$$\begin{aligned} &\left\| \int_I f_n(s)dX(s) - \int_{-\infty}^{\infty} \left\{ \int_I f(u)c(u-s)du \right\} dW(s) \right\|^2 \\ &\leq \int_{-\infty}^{\infty} \left\{ \int_I (f(u) - f_n(u))c(u-s)du \right\}^2 ds \ \downarrow \ 0, \qquad n \to \infty. \end{aligned}$$

Thus the proposition follows. □

For a real-valued function f on I, we write $f(x) = f^+(x) - f^-(x)$, where

$$f^+(x) := \max(f(x), 0), \quad f^-(x) := \max(-f(x), 0), \qquad x \in I.$$

Definition 3.4. For $f \in \mathcal{H}_I(X)$, we define

$$\int_I f(s)dX(s) := \lim_{n\to\infty} \int_I f_n^+(s)dX(s) - \lim_{n\to\infty} \int_I f_n^-(s)dX(s) \quad \text{in } M(X),$$

where $\{f_n^+\}$ and $\{f_n^-\}$ are arbitrary sequences of non-negative simple functions on I such that $f_n^+ \uparrow f^+$, $f_n^- \uparrow f^-$, as $n \to \infty$, a.e.

From the definition above, we see that $\int_I f(s)dX(s) \in M_I(X)$ for $f \in \mathcal{H}_I(X)$. The next proposition follows immediately from Proposition 3.3.

Proposition 3.5. *The equality* (3.6) *also holds for $f \in \mathcal{H}_I(X)$.*

3.3. **Infinite past prediction formulas.** We denote by $\mathcal{D}(\mathbf{R})$ the space of all $\phi \in C^\infty(\mathbf{R})$ with compact support, endowed with the usual topology. For a random distribution Y (cf. [11, Section 2] and [3, Section 2]), we write DY for its derivative. For $t \in \mathbf{R}$, we write $M_{(-\infty,t]}(Y)$ for the closed linear hull of $\{Y(\phi) : \phi \in \mathcal{D}(\mathbf{R}),\ \mathrm{supp}\ \phi \subset (-\infty, t]\}$ in $L^2(\Omega, \mathcal{F}, P)$. Notice that $M_I(X)$ here coincides with that defined above.

As in [15, Proposition 2.4], we have the next proposition.

Proposition 3.6. *The derivative DX of $(X(t))$ is a purely nondeterministic stationary random distribution, and $(W(t) : t \in \mathbf{R})$ is a canonical Brownian motion of DX in the sense that $M_{(-\infty,t]}(DX) = M_{(-\infty,t]}(DW)$ for every $t \in \mathbf{R}$.*

Here is the infinite past prediction formula for $\int_t^\infty f(s)dX(s)$.

Theorem 3.7. *For $t \in [0, \infty)$ and $f \in \mathcal{H}_{[t,\infty)}(X)$, the following assertions hold:*

(a) $\int_0^\infty b(t - \cdot, \tau)f(t + \tau)d\tau \in \mathcal{H}_{(-\infty,t]}(X)$.

(b) $P_{(-\infty,t]} \int_t^\infty f(s)dX(s) = \int_{-\infty}^t \left\{\int_0^\infty b(t - s, \tau)f(t + \tau)d\tau\right\} dX(s)$.

Proof. Since $f \in \mathcal{H}_{[t,\infty)}(X)$ iff $|f| \in \mathcal{H}_{[t,\infty)}(X)$, we may assume $f \geq 0$. Since

$$c(u) = 0, \qquad t \leq 0, \tag{3.7}$$

it follows from (3.5) and the Fubini–Tonelli theorem that, for $s < t$,

$$\begin{aligned}\int_t^\infty f(u)c(u - s)du &= \int_0^\infty d\tau f(t + \tau) \int_0^{t-s} c(t - s - u)b(u, \tau)du \\ &= \int_{-\infty}^t duc(u - s) \int_0^\infty b(t - u, \tau)f(t + \tau)d\tau.\end{aligned} \tag{3.8}$$

Thus we obtain (a). By Proposition 3.6 and [3, Proposition 2.3 (2)], we have

$$M_{(-\infty,t]}(X) = M_{(-\infty,t]}(DW). \tag{3.9}$$

This and Proposition 3.5 yield

$$P_{(-\infty,t]} \int_t^\infty f(s)dX(s) = \int_{-\infty}^t \left\{\int_t^\infty f(u)c(u - s)du\right\} dW(s).$$

By (3.7), (3.8) and Proposition 3.5, the integral on the right-hand side is

$$\begin{aligned}&\int_{-\infty}^t \left\{\int_{-\infty}^t duc(u - s) \int_0^\infty b(t - u, \tau)f(t + \tau)d\tau\right\} dW(s) \\ &= \int_{-\infty}^t \left\{\int_0^\infty b(t - s, \tau)f(t + \tau)d\tau\right\} dX(s).\end{aligned}$$

Thus (b) follows. □

By putting $f(s) = I_{(t_1,T]}(s)$ in Theorem 3.7 (b), we immediately obtain the next infinite past prediction formula for $(X(t))$.

Theorem 3.8. *Let $0 \leq t_1 < T < \infty$. Then $\int_0^{T-t_1} b(t_1 - \cdot, \tau)d\tau \in \mathcal{H}_{(-\infty,t_1]}(X)$ and the infinite past prediction formula* (1.11) *holds.*

Using the Hilbert space isomorphism $\theta : M(X) \to M(X)$ characterized by $\theta(X(t)) = X(-t)$ for $t \in \mathbf{R}$, we obtain the next theorem from Theorem 3.7 (see the proof of [3, Theorem 3.6]).

Theorem 3.9. *For $t \in [0, \infty)$ and $f \in \mathcal{H}_{[t,\infty)}(X)$, the following assertions hold:*

(a) $\int_0^\infty b(t+\cdot, \tau) f(t+\tau) d\tau \in \mathcal{H}_{[-t,\infty)}(X)$.

(b) $P_{[-t,\infty)} \int_{-\infty}^{-t} f(-s) dX(s) = \int_{-t}^{\infty} \left\{ \int_0^\infty b(t+s,\tau) f(t+\tau) d\tau \right\} dX(s)$.

As in [3, Definition 2.2], we define another Brownian motion $(W^*(t) : t \in \mathbf{R})$ by

$$W^*(t) := \theta(W(-t)), \qquad t \in \mathbf{R}. \tag{3.10}$$

Proposition 3.10. *Let I be a closed interval of $\mathbf{R}$ and let $f \in \mathcal{H}_I(X)$. Then*

$$\int_I f(s) dX(s) = \int_{-\infty}^{\infty} \left\{ \int_I f(u) c(s-u) du \right\} dW^*(s).$$

The proof of Proposition 3.10 is the same as that of [3, Proposition 3.5], whence we omit it. We need Theorem 3.9 and Proposition 3.10 in the next section.

Example 3.11. As in Example 2.3, we consider fBm $(B_H(t))$ with $1/2 < H < 1$. Then the MA(∞) coefficient $c(t)$ is given by

$$c(t) = t^{H-3/2} \frac{1}{\Gamma(H-\frac{1}{2})}, \qquad t > 0, \tag{3.11}$$

so that $\int_0^\infty e^{izt} c(t) dt = (-iz)^{1/2-H}$ for $\Im z > 0$. From (3.2), we have

$$\int_0^\infty e^{izt} \alpha(t) dt = (-iz)^{H-3/2}.$$

Hence, $\alpha(t) = t^{\frac{1}{2}-H}/\Gamma(\frac{3}{2}-H)$, so that the AR($\infty$) coefficient $a(t)$ is given by

$$a(t) = t^{-(H+\frac{1}{2})} \frac{H-\frac{1}{2}}{\Gamma(\frac{3}{2}-H)}, \qquad t > 0. \tag{3.12}$$

By the change of variable $u = sv$, $\int_0^s (s-u)^{H-(3/2)}(t+u)^{-H-(1/2)} du$ becomes

$$s^{H-\frac{1}{2}} t^{-H-\frac{1}{2}} \int_0^1 (1-v)^{H-\frac{3}{2}} \{1+(s/t)v\}^{-H-\frac{1}{2}} dv = \frac{1}{(H-\frac{1}{2})} \left(\frac{s}{t}\right)^{H-\frac{1}{2}} \frac{1}{t+s},$$

where we have used the equality

$$\int_0^1 (1-v)^{p-1}(1+xv)^{-p-1} dv = \frac{1}{p(x+1)}, \qquad p > 0,\ x > -1.$$

Thus

$$b(t,s) = \frac{\sin\{\pi(H-\frac{1}{2})\}}{\pi} \left(\frac{s}{t}\right)^{H-\frac{1}{2}} \frac{1}{t+s}, \qquad t > 0,\ s > 0; \tag{3.13}$$

and so, from Theorem 3.8, we see that, for $0 \le t < T$,

$$\begin{aligned} &E\left[B_H(T) \middle| \sigma(B_H(s) : -\infty < s \le t)\right] \\ &= B_H(t) + \frac{\sin\{\pi(H-\frac{1}{2})\}}{\pi} \int_{-\infty}^{t} \left\{ \int_0^{T-t} \left(\frac{\tau}{t-s}\right)^{H-\frac{1}{2}} \frac{1}{t-s+\tau} d\tau \right\} dB_H(s). \end{aligned}$$

This prediction formula was obtained in [9, Theorem 3.1] by a different method.

4. Finite past prediction problems

In this section, we assume (1.1)–(1.7) and (1.10). Notice that (1.6) with (1.7) implies (3.1) as well as (2.3), whence (2.1). For t_0, t_1, and T in (1.10), we put

$$t_2 := t_0 + t_1, \quad t_3 := T - t_1.$$

4.1. **Alternating projections to the past and future.** For $n \in \mathbf{N}$, we define the orthogonal projection operator P_n by

$$P_n := \begin{cases} P_{(-\infty,t_1]}, & n = 1, 3, 5, \dots, \\ P_{[-t_0,\infty)}, & n = 2, 4, 6, \dots. \end{cases}$$

It should be noted that $\{P_n\}_{n=1}^{\infty}$ is merely an alternating sequence of projection operators, first to $M_{(-\infty,t_1]}(X)$, then to $M_{[-t_0,\infty)}(X)$, and so on. This sequence plays a key role in the proof of the finite past prediction formula for $(X(t))$.

For $t, s \in (0, \infty)$ and $n \in \mathbf{N}$, we define $b_n(t, s) = b_n(t, s; t_2)$ iteratively by

$$\begin{cases} b_1(t, s) := b(t, s), \\ b_n(t, s) := \int_0^{\infty} b(t, u) b_{n-1}(t_2 + u, s) du, \quad n = 2, 3, \dots. \end{cases} \tag{4.1}$$

Proposition 4.1. *For $f \in \mathcal{H}_{[t_1,\infty)}(X)$, the following assertions hold:*

(a) $\int_0^{\infty} b_n(t_1 - \cdot, \tau) f(t_1 + \tau) d\tau \in \mathcal{H}_{(-\infty,t_1]}(X)$ *for* $n = 1, 3, 5, \dots$.

(b) $\int_0^{\infty} b_n(t_0 + \cdot, \tau) f(t_1 + \tau) d\tau \in \mathcal{H}_{[-t_0,\infty)}(X)$ *for* $n = 2, 4, 6, \dots$.

Proof. We may assume that $f \geq 0$. By Theorem 3.7, (a) holds for $n = 1$. By the Fubini–Tonelli theorem, we have, for $s > -t_0$,

$$\int_0^{\infty} du b(t_0 + s, u) \int_0^{\infty} b_1(t_2 + u, \tau) f(t_1 + \tau) d\tau = \int_0^{\infty} b_2(t_0 + s, \tau) f(t_1 + \tau) d\tau.$$

Hence, by Theorem 3.9, we have (b) for $n = 2$. Repeating this procedure, we obtain the proposition. □

Let $f \in \mathcal{H}_{[t_1,\infty)}(X)$. By Proposition 4.1, we may define the random variables $G_n(f)$ by

$$G_n(f) := \begin{cases} \int_{-t_0}^{t_1} \left\{ \int_0^{\infty} b_n(t_1 - s, \tau) f(t_1 + \tau) d\tau \right\} dX(s), & n = 1, 3, \dots, \\ \int_{-t_0}^{t_1} \left\{ \int_0^{\infty} b_n(t_0 + s, \tau) f(t_1 + \tau) d\tau \right\} dX(s), & n = 2, 4, \dots. \end{cases}$$

We may also define the random variables $\epsilon_n(f)$ by $\epsilon_0(f) := \int_{t_1}^{\infty} f(s) dX(s)$ and

$$\epsilon_n(f) := \begin{cases} \int_{-\infty}^{-t_0} \left\{ \int_0^{\infty} b_n(t_1 - s, \tau) f(t_1 + \tau) d\tau \right\} dX(s), & n = 1, 3, \dots, \\ \int_{t_1}^{\infty} \left\{ \int_0^{\infty} b_n(t_0 + s, \tau) f(t_1 + \tau) d\tau \right\} dX(s), & n = 2, 4, \dots. \end{cases}$$

Proposition 4.2. *Let $f \in \mathcal{H}_{[t_1,\infty)}(X)$ and $n \in \mathbf{N}$. Then*

$$P_n P_{n-1} \cdots P_1 \int_{t_1}^{\infty} f(s) dX(s) = \epsilon_n(f) + \sum_{k=1}^{n} G_k(f). \tag{4.2}$$

We can prove (4.2) using Proposition 4.1 and the facts

$$M_{[-t_0,t_1]}(X) \subset M_{(-\infty,t_1]}(X) \cap M_{[-t_0,\infty)}(X), \tag{4.3}$$

$$G_k \in M_{[-t_0,t_1]}(X), \qquad k = 1, 2, \dots. \tag{4.4}$$

Since the proof is similar to that of [3, Proposition 4.4], we omit the details.

We are about to investigate the limit of (4.2) as $n \to \infty$ (see Lemma 4.9 below).

For $f \in \mathcal{H}_{[t_1,\infty)}(X)$ and $s > 0$, we define $D_n(s, f) = D_n(s, f; t_1, t_2)$ by

$$D_n(s, f) := \begin{cases} \int_0^{\infty} c(u) f(t_1 + s + u) du, & n = 0, \\ \int_0^{\infty} du c(u) \int_0^{\infty} b_n(t_2 + u + s, \tau) f(t_1 + \tau) d\tau, & n = 1, 2, \dots. \end{cases}$$

From the proof of the next proposition, we see that these integrals converge absolutely. Recall $(W^*(t))$ from (3.10).

Proposition 4.3. *Let $f \in \mathcal{H}_{[t_1,\infty)}(X)$. Then*

$$P_{n+1}^{\perp}\epsilon_n(f) = \begin{cases} \int_{t_1}^{\infty} D_n(s-t_1,f)dW(s), & n=0,2,4,\dots, \\ \int_{-\infty}^{-t_0} D_n(-t_0-s,f)dW^*(s), & n=1,3,5,\dots. \end{cases}$$

Proof. By (3.9) and Proposition 3.5,

$$P_1^{\perp}\epsilon_0(f) = \int_{t_1}^{\infty}\left\{\int_s^{\infty} f(u)c(u-s)du\right\}dW(s) = \int_{t_1}^{\infty} D_0(s-t_1,f)dW(s).$$

Thus the assertion holds for $n=0$. Let $n=1,3,\dots$. Then, by Proposition 3.10,

$$\epsilon_n(f) = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{-t_0} duc(s-u)\int_0^{\infty} b_n(t_1-u,\tau)f(t_1+\tau)d\tau\right\}dW^*(s).$$

Hence, using [3, Proposition 2.3 (7)] and (3.7),

$$\begin{aligned} P_{n+1}^{\perp}\epsilon_n(f) &= \int_{-\infty}^{-t_0}\left\{\int_{-\infty}^{s} duc(s-u)\int_0^{\infty} b_n(t_1-u,\tau)f(t_1+\tau)d\tau\right\}dW^*(s) \\ &= \int_{-\infty}^{-t_0}\left\{\int_0^{\infty} duc(u)\int_0^{\infty} b_n(t_2+u-t_0-s,\tau)f(t_1+\tau)d\tau\right\}dW^*(s) \\ &= \int_{-\infty}^{-t_0} D_n(-t_0-s,f)dW^*(s). \end{aligned}$$

Thus we obtain the assertion for $n=1,3,\dots$. The proof for $n=2,4,\dots$ is similar; and so we omit it. □

From Propositions 4.2 and 4.3, we immediately obtain the next proposition (cf. the proof of [3, Proposition 4.9]).

Proposition 4.4. *Let $f \in \mathcal{H}_{[t_1,\infty)}(X)$. Then the following assertions hold:*

(a) $\|P_1^{\perp}\int_{t_1}^{\infty} f(s)dX(s)\|^2 = \int_0^{\infty} D_0(s,f)^2 ds$.
(b) $\|P_{n+1}^{\perp}P_nP_{n-1}\cdots P_1\int_{t_1}^{\infty} f(s)dY(s)\|^2 = \int_0^{\infty} D_n(s,f)^2 ds$ *for* $n=1,2,\dots$.

We write Q for the orthogonal projection operator from $M(X)$ onto the intersection $M_{(-\infty,t_1]}(X)\cap M_{[-t_0,\infty)}(X)$. Then, by von Neumann's alternating projection theorem (see, e.g., [26, Theorem 9.20]), we have $Q = \operatorname*{s-lim}_{n\to\infty} P_nP_{n-1}\cdots P_1$. Using this, (4.3) and Proposition 4.4, we immediately obtain the next proposition (cf. the proof of [3, Proposition 4.9 (3)]).

Proposition 4.5. *Let $f \in \mathcal{H}_{[t_1,\infty)}(X)$. Then $\lim_{n\to\infty}\int_0^{\infty} D_n(s,f)^2 ds = 0$.*

We need the next proposition.

Proposition 4.6. *Let $f \in \mathcal{H}_{[t_1,\infty)}(X)$. Then, for $t>0$ and $n=0,1,\dots$, we have*

$$\int_0^{\infty} b_{n+1}(t,\tau)f(t_1+\tau)d\tau = \int_0^{\infty} a(t+u)D_n(u,f)du.$$

Proof. We may assume $f \geq 0$. By the Fubini–Tonelli theorem, we have, for $t > 0$,

$$\int_0^\infty b_1(t,\tau) f(t_1+\tau) d\tau = \int_0^\infty \left\{ \int_0^\tau c(\tau-u) a(t+u) du \right\} f(t_1+\tau) d\tau$$
$$= \int_0^\infty a(t+u) \left\{ \int_0^\infty c(\tau) f(t_1+u+\tau) d\tau \right\} du = \int_0^\infty a(t+u) D_0(u,f) du.$$

Thus the assertion holds for $n = 0$. Now we assume that $n \geq 1$. Since we have

$$b_{n+1}(t,\tau) = \int_0^\infty a(t+v) \left\{ \int_0^\infty c(u) b_n(t_2+u+v,\tau) du \right\} dv, \quad t,\tau > 0,$$

we obtain the assertion, again using the Fubini–Tonelli theorem. □

For $t, s > 0$, we define $k(t,s) = k(t,s;t_2)$ by

$$k(t,s) := \int_0^\infty c(t+u) a(t_2+u+s) du.$$

Notice that $k(t,s) < \infty$ for $t, s > 0$ since $k(t,s) \leq c(t) \int_{t_2+s}^\infty a(u) du$.

Proposition 4.7. *Let $f \in \mathcal{H}_{[t_1,\infty)}(X)$. Then*

$$P_{n+1}\epsilon_n(f) = \begin{cases} \int_{-\infty}^{t_1} \left\{ \int_0^\infty k(t_1-s,u) D_{n-1}(u,f) du \right\} dW(s), & n = 2,4,\ldots, \\ \int_{-t_0}^\infty \left\{ \int_0^\infty k(t_0+s,u) D_{n-1}(u,f) du \right\} dW^*(s), & n = 1,3,\ldots. \end{cases}$$

Proof. We assume $n = 2,4,\ldots$. Then, by Propositions 3.5 and 4.6, we have

$$P_{n+1}\epsilon_n(f) = \int_{-\infty}^{t_1} \left\{ \int_{t_1}^\infty du c(u-s) \int_0^\infty b_n(t_0+u,\tau) f(t_1+\tau) d\tau \right\} dW(s)$$
$$= \int_{-\infty}^{t_1} \left\{ \int_0^\infty dv c(t_1-s+v) \int_0^\infty a(t_2+v+u) D_{n-1}(u,f) du \right\} dW(s)$$
$$= \int_{-\infty}^{t_1} \left\{ \int_0^\infty k(t_1-s,u) D_{n-1}(u,f) du \right\} dW(s).$$

The proof of the case $n = 1,3,\ldots$ is similar. □

We need the next L^2-boundedness theorem.

Theorem 4.8. *Let $p \in (0,1/2)$ and let $\ell(\cdot)$ be a slowly varying function at infinity. Let $C(\cdot)$ and $A(\cdot)$ be nonnegative and decreasing functions on $(0,\infty)$. We assume $C(\cdot) \in L^1_{\mathrm{loc}}[0,\infty)$ and $A(0+) < \infty$. We also assume*

$$A(t) \sim t^{-(1+p)} \ell(t) p, \qquad t \to \infty,$$
$$C(t) \sim \frac{t^{-(1-p)}}{\ell(t)} \cdot \frac{\sin(p\pi)}{\pi}, \qquad t \to \infty.$$

Then

$$\sup_{0<x<\infty} \int_0^\infty K(x,y) (x/y)^{1/2} dy < \infty,$$
$$\sup_{0<y<\infty} \int_0^\infty K(x,y) (y/x)^{1/2} dx < \infty,$$

where $K(x,y) := \int_0^\infty C(x+u) A(u+y) du$ for $x, y > 0$. In particular, the integral operator K defined by $(Kf)(x) := \int_0^\infty K(x,y) f(y) dy$ for $x > 0$ is a bounded operator on $L^2((0,\infty), dy)$.

The proof of Theorem 4.8 is similar to that of [15, Theorem 5.1], whence we omit it.

By putting $z = iy$ in (3.2), we get

$$y\left\{\int_0^\infty e^{-yt}c(t)dt\right\}\left\{\int_0^\infty e^{-yt}\alpha(t)dt\right\} = 1, \qquad y > 0.$$

By Karamata's Tauberian theorem (cf. [6, Theorem 1.7.6]) applied to this, (2.3) implies

$$\alpha(t) \sim \frac{t^{-(H-\frac{1}{2})}}{\ell(t)} \cdot \frac{1}{\Gamma(\frac{3}{2}-H)}, \qquad t \to \infty.$$

This and the monotone density theorem give

$$a(t) \sim \frac{t^{-(H+\frac{1}{2})}}{\ell(t)} \cdot \frac{(H-\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}, \qquad t \to \infty. \tag{4.5}$$

The next lemma is a key to our arguments.

Lemma 4.9. *Let $f \in \mathcal{H}_{[t_1,\infty)}(X)$. Then $\|\epsilon_n(f)\| \to 0$ as $n \to \infty$.*

Proof. It follows from (2.3), (4.5) and Theorem 4.8 below that the integral operator K defined by $Kf(t) := \int_0^\infty k(t,s)f(s)ds$ is a bounded operator on $L^2((0,\infty), ds)$. Hence, by Propositions 4.3, 4.5 and 4.7, we have

$$\begin{aligned}\|\epsilon_n(f)\|^2 &= \int_0^\infty D_n(s,f)^2 ds + \int_0^\infty \left\{\int_0^\infty k(s,u)D_{n-1}(u,f)du\right\}^2 ds \\ &\le \int_0^\infty D_n(s,f)^2 ds + \|K\|^2 \int_0^\infty D_{n-1}(s,f)^2 ds \to 0, \qquad n \to \infty.\end{aligned}$$

Thus the lemma follows. □

We can now state the conclusions of the arguments above.

Theorem 4.10. *The following assertions hold:*

(a) $M_{[-t_0,t_1]}(X) = M_{(-\infty,t_1]}(X) \cap M_{[-t_0,\infty)}(X)$.

(b) $P_{[-t_0,t_1]} = \operatorname*{s-lim}_{n\to\infty} P_n P_{n-1} \cdots P_1$.

(c) $\|P^{\perp}_{[-t_0,t_1]}Z\|^2 = \left\|P_1^{\perp}Z\right\|^2 + \sum_{n=1}^\infty \left\|(P_{n+1})^{\perp}P_n \cdots P_1 Z\right\|^2$ *for* $Z \in M(X)$.

We can prove Theorem 4.10 using Proposition 4.2 and Lemma 4.9. Since the proof is similar to that of [3, Theorem 4.6], we omit the details.

4.2. **Finite past prediction formulas.** We define $h(s,u) = h(s,u;t_2)$ by

$$h(s,u) := \sum_{k=1}^\infty \left\{b_{2k-1}(t_2 - s, u) + b_{2k}(s,u)\right\}, \qquad 0 < s < t_2,\ u > 0. \tag{4.6}$$

Here is the finite past prediction formula for $\int_{t_1}^\infty f(s)dX(s)$.

Theorem 4.11. *Let $f \in \mathcal{H}_{[t_1,\infty)}(X)$. Then the following assertions hold:*

(a) $\int_0^\infty h(t_0 + \cdot, u)f(t_1+u)du \in \mathcal{H}_{[-t_0,t_1]}(X)$.

(b) $P_{[-t_0,t_1]}\int_{t_1}^\infty f(s)dX(s) = \int_{-t_0}^{t_1}\left\{\int_0^\infty h(t_0+s,u)f(t_1+u)du\right\}dX(s)$.

(c) $\|P^{\perp}_{[-t_0,t_1]}\int_{t_1}^\infty f(s)dX(s)\|^2 = \sum_{n=0}^\infty \int_0^\infty D_n(s,f)^2 ds$.

Proof. We may assume that $f \geq 0$. By Theorem 4.10 (b), Proposition 4.2 and Lemma 4.9, we have, in $M(X)$,

$$\begin{aligned} P_{[-t_0,t_1]} \int_{t_1}^{\infty} f(s)dX(s) &= \lim_{n\to\infty} P_n P_{n-1} \cdots P_1 \int_{t_1}^{\infty} f(s)dX(s) \\ &= \lim_{n\to\infty} \int_{-t_0}^{t_1} \left\{ \int_0^{\infty} h_n(t_0+u, v) f(t_1+v)dv \right\} dX(s), \end{aligned}$$

where, for $0 < s < t_2$ and $u > 0$, we define $h_n(s,u) = h_n(s,u;t_2)$ by

$$h_n(s,u) = \begin{cases} b_1(t_2-s,u) + b_2(s,u) + \cdots + b_n(t_2-s,u), & n = 1,3,5,\ldots, \\ b_1(t_2-s,u) + b_2(s,u) + \cdots + b_n(s,u), & n = 2,4,6,\ldots. \end{cases}$$

Since $h_n(s,u) \uparrow h(s,u)$ as $n \to \infty$, we obtain (a) and (b) using the monotone convergence theorem. Finally, (c) follows immediately from Theorem 4.11 (c) and Proposition 4.4. $\square$

For $s, u > 0$, we define $D_n(s) = D_n(s;t_2,t_3)$ by

$$D_n(s) := \int_0^{\infty} du c(u) \int_0^{t_3} b_n(t_2+u+s,\tau)d\tau, \qquad n = 1,2,\ldots.$$

Here are the solutions to the finite past prediction problems for $(X(t))$.

Theorem 4.12. *The finite past prediction formula* (1.12) *and the following equality for the mean-square prediction error hold:*

$$\left\| P_{[-t_0,t_1]}^{\perp} X(T) \right\|^2 = \int_0^{T-t_1} g(s)^2 ds + \sum_{n=1}^{\infty} \int_0^{\infty} D_n(s)^2 ds.$$

Proof. We put $f(s) = I_{(t_1,T]}(s)$. Then $\int_{t_1}^{\infty} f(s)dX(s) = X(T) - X(t_1)$ and

$$\int_0^{\infty} h(t_0+s,u)f(t_1+u)du = \int_0^{t_3} h(t_0+s,u)du, \qquad -t_0 < s < t_1.$$

We also have $D_n(s,f) = D_n(s)$ for $n = 1,2,\ldots$ and $D_0(s,f) = g(t_3-s)$. Thus the theorem follows from Theorem 4.11. $\square$

5. Baxter's inequality

In this section, we assume (1.1)–(1.7) and (1.10). Let $t_2 := t_0 + t_1$ as before. By (4.6), the infinite and finite past predictor coefficients $b(t,s)$ and $h(s,u) = h(s,u;t_2)$ satisfy, for $s \in (-t_0, t_1)$ and $u > 0$,

$$h(s+t_0,u) - b(t_1-s,u) = \sum_{k=1}^{\infty} \{b_{2k}(s+t_0,u) + b_{2k+1}(t_1-s,u)\} > 0, \tag{5.1}$$

where we recall that $b_n(t,s) = b_n(t,s;t_2)$ from (4.1).

The aim here is to prove Baxter's inequality for $(X(t))$.

Theorem 5.1. *There exists a positive constant K such that, for all $t_0 \geq 1$,*

$$\int_{-t_0}^{t_1} ds \int_0^{T-t_1} \{h(s+t_0,u) - b(t_1-s,u)\}du \leq K \int_{-\infty}^{-t_0} ds \int_0^{T-t_1} b(t_1-s,u)du.$$

5.1. **Representation in terms of β.** We define a positive function $\beta(t)$ by

$$\beta(t) := \int_0^\infty c(v)a(t+v)dv, \qquad t>0.$$

We next derive the representation of the finite past prediction coefficient $h(s,u) = h(s,u;t_2)$ in terms of $\beta(t)$ (and $c(t)$ and $a(t)$). We need this result in Section 5.2. See [16, 17, 14] for the usefulness of such expressions in terms of $\beta(t)$ in the discrete-time setting.

For $t,u,v>0$, we define $\delta_1(u,v;t) := \beta(t+v+u)$,

$$\delta_2(u,v;t) := \int_0^\infty dw_1 \beta(t+v+w_1)\beta(t+w_1+u),$$

and, for $k=3,4,\ldots$,

$$\begin{aligned}\delta_k(u,v;t) := \int_0^\infty dw_{k-1}\cdots\int_0^\infty dw_1 \beta(t+v+w_{k-1})&\\ \times\left\{\prod\nolimits_{l=1}^{k-2}\beta(t+w_{l+1}+w_l)\right\}\beta(t+w_1+u).&\end{aligned}$$

For $t,s>0$, we define $B_1(t,s;t_2) := b(t,s)$, and, for $k\ge 2$,

$$B_k(t,s;t_2) := \int_0^s dvc(s-v)\int_0^\infty a(t+u)\delta_{k-1}(u,v;t_2)du$$

The next proposition gives the desired representation of $h(s,u)$.

Proposition 5.2. *For $t,s>0$ and $k\ge 1$, $b_k(t,s;t_2) = B_k(t,s;t_2)$, that is,*

$$b_k(t,s;t_2) = \int_0^s dvc(s-v)\int_0^\infty a(t+u)\delta_{k-1}(u,v;t_2)du, \quad k=2,3,\ldots.$$

Proof. It is enough to show that, for $t,s>0$ and $k=1,2,\ldots$,

$$B_{k+1}(t,s;t_2) = \int_0^\infty b(t,\tau)B_k(t_2+\tau,s;t_2)d\tau.$$

However, from the Fubini–Tonelli theorem, we see that

$$\begin{aligned}
&\int_0^\infty b(t,\tau)B_k(t_2+\tau,s;t_2)d\tau\\
&\quad=\int_0^\infty\left\{\int_0^\tau c(\tau-z)a(t+z)dz\right\}\\
&\qquad\qquad\times\left\{\int_0^s dvc(s-v)\int_0^\infty a(t_2+\tau+u)\delta_{k-1}(u,v;t_2)du\right\}d\tau\\
&\quad=\int_0^s dvc(s-v)\int_0^\infty dza(t+z)\\
&\qquad\qquad\int_0^\infty du\left\{\int_z^\infty dvc(\tau-z)a(t_2+\tau+u)\right\}\delta_{k-1}(u,v;t_2)\\
&\quad=\int_0^s dvc(s-v)\int_0^\infty dza(t+z)\int_0^\infty \beta(t_2+z_u)\delta_{k-1}(u,v;t_2)du\\
&\quad=\int_0^s dvc(s-v)\int_0^\infty dza(t+z)\delta_k(z,v;t_2)\\
&\quad=B_{k+1}(t,s;t_2).
\end{aligned}$$

Thus the proposition follows. $\square$

5.2. **Proof of Baxter's inequality.** For simplicity, we write $d := H - \frac{1}{2}$. Then $0 < d < 1/2$. From (2.3), (4.5) and [11, Proposition 4.3], we have

$$\beta(t) \sim t^{-1} \cdot \frac{\sin(\pi d)}{\pi}, \qquad t \to \infty. \tag{5.2}$$

As in [12, Section 6], [13, Section 3] and [17, Section 3]], we put, for $u \geq 0$,

$$f_1(u) := \frac{1}{\pi(1+u)}, \quad f_2(u) := \frac{1}{\pi^2} \int_0^\infty \frac{ds_1}{(s_1+1)(s_1+1+u)},$$

and, for $k = 3, 4, \ldots$,

$$f_k(u) := \frac{1}{\pi^k} \int_0^\infty ds_{k-1} \cdots \int_0^\infty ds_1 \frac{1}{(1+s_{k-1})} \times \left\{ \prod\nolimits_{l=1}^{k-2} \frac{1}{(1+s_{l+1}+s_l)} \right\} \times \frac{1}{(1+s_1+u)}.$$

Proposition 5.3. *The following assertions hold:*

(a) *For $r \in (1, \infty)$, there exists $N > 0$ such that*

$$0 < \delta_k(u, v; t) \leq \frac{f_k(0)\{r \sin(\pi d)\}^k}{t}, \quad u, v > 0,\ k \in \mathbf{N},\ t \geq N.$$

(b) *For $k \in \mathbf{N}$ and $u > 0$, $\delta_k(tu, v; t) \sim t^{-1} f_k(u) \sin^k(\pi d)$ as $t \to \infty$.*

For example, we see from (5.2) that, formally,

$$\begin{aligned} t\delta_k(tu, v; t) &= \int_0^\infty ds_{k-1} \ldots \int_0^\infty ds_1 t\beta(t(1 + (v/t) + s_{k-1})) \\ &\qquad \times \left\{ \prod\nolimits_{l=1}^{k-2} t\beta(t(1+s_{l+1}+s_l)) \right\} t\beta(t(1+s_1+u)) \\ &\to f_k(u) \sin^k(\pi d), \qquad t \to \infty, \end{aligned}$$

which is (b) of the proposition above. Since we can prove the two assertions rigorously as in the proof of [17, Proposition 3.2], we omit the details.

Theorem 5.1 follows immediately from the next more precise result.

Lemma 5.4. *For $0 \leq t_1 < T$, we have, as $t_0 \to \infty$,*

$$\begin{aligned} &\int_{-t_0}^{t_1} ds \int_0^{T-t_1} \{h(s+t_0, u; t_2) - b(t_1 - s, u)\} du \\ &\sim t_2 a(t_2) \cdot \left\{ \int_0^{T-t_1} ds \int_0^s c(v) dv \right\} \cdot \left\{ \int_0^1 s^{-d-1}[(1-s)^{-d} - 1] ds \right\} \\ &\sim \int_{-\infty}^{-t_0} ds \int_0^{T-t_1} b(t_1 - s, u) du \cdot d \int_0^1 s^{-d-1}[(1-s)^{-d} - 1] ds. \end{aligned}$$

Proof. By (5.1) and Proposition 5.2, we have

$$\begin{aligned}
&\int_{-t_0}^{t_1} ds \int_0^{T-t_1} \{h(s+t_0, u; t_2) - b(t_1 - s, u)\} du \\
&\quad = \sum_{k=2}^{\infty} \int_0^{T-t_1} ds \int_0^{t_2} d\tau b_k(\tau, s; t_2) \\
&\quad = \sum_{k=2}^{\infty} \int_{-t_0}^{t_1} ds \int_0^{s} dv c(s-v) \int_0^{t_2} d\tau \int_0^{\infty} a(\tau+u) \delta_{k-1}(u, v; t_2) du \\
&\quad = (t_2)^2 \sum_{k=2}^{\infty} \int_{-t_0}^{t_1} ds \int_0^{s} dv c(s-v) \int_0^{t_2} d\tau \int_0^{\infty} a(t_2(\tau+u)) \delta_{k-1}(t_2 u, v; t_2) du.
\end{aligned}$$

Therefore, by (4.5), Proposition 5.3 and standard arguments involving Potter's theorem (cf. [6, Theorem 1.5.6]) and the dominated convergence theorem,

$$\begin{aligned}
&\frac{1}{t_2 a(t_2)} \int_{-t_0}^{t_1} ds \int_0^{T-t_1} \{h(s+t_0, u; t_2) - b(t_1 - s, u)\} du \\
&= \sum_{k=2}^{\infty} \int_{-t_0}^{t_1} ds \int_0^{s} dv c(s-v) \int_0^{t_2} d\tau \int_0^{\infty} \frac{a(t_2(\tau+u))}{a(t_2)} t_2 \delta_{k-1}(t_2 u, v; t_2) du \\
&\to \quad \left\{ \int_0^{T-t_1} ds \int_0^{s} c(v) dv \right\} \sum_{m=1}^{\infty} \left\{ \int_0^{\infty} du f_m(u) \int_0^{1} \frac{1}{(\tau+u)^{d+1}} d\tau \right\} \sin^m(\pi d)
\end{aligned}$$

as $t_0 \to \infty$. On the other hand,

$$\begin{aligned}
&\frac{1}{t_2 a(t_2)} \int_{-\infty}^{-t_0} ds \int_0^{T-t_1} b(t_1 - s, u) du \\
&= \int_0^{T-t_1} du \int_0^{u} dv c(u-v) \int_0^{\infty} \frac{a(t_2(1+s+(v/t_2)))}{a(t_2)} ds \\
&\to \quad \int_0^{T-t_1} du \int_0^{u} du c(u-v) \int_0^{\infty} \frac{1}{(1+s)^{d+1}} ds = \frac{1}{d} \int_0^{T-t_1} du \int_0^{u} dv c(u-v)
\end{aligned}$$

as $t_0 \to \infty$. Therefore, by Lemma 5.5 below, we obtain the lemma. $\square$

Lemma 5.5. *For $0 < d < 1/2$, it holds that*

$$\sum_{m=1}^{\infty} \left\{ \int_0^{\infty} du f_m(u) \int_0^{1} \frac{1}{(\tau+u)^{d+1}} d\tau \right\} \sin^m(\pi d) = \int_0^1 s^{-d-1} [(1-s)^{-d} - 1] ds.$$

Proof. Though the lemma is a general result, we give a proof based on the results for fBm. Thus we take fBm $(B_H(t))$ as $(X(t))$. Then we have (3.11) and (3.12). Also, by [9], we have (3.13) and

$$h(s, u; t_2) = \frac{\sin\left\{\pi(H - \frac{1}{2})\right\}}{\pi} (t_2 - s)^{-(H-\frac{1}{2})} s^{-(H-\frac{1}{2})} \frac{\{u(u+t_2)\}^{H-\frac{1}{2}}}{u + t_2 - s}, \tag{5.3}$$

whence, as $t_0 \to \infty$,

$$
\begin{aligned}
&\frac{1}{t_2 a(t_2)} \int_{-t_0}^{t_1} ds \int_0^{T-t_1} \{h(s+t_0, u; t_2) - b(t_1 - s, u)\} du \\
&= \frac{1}{t_2 a(t_2)} \int_0^{t_2} ds \int_0^{T-t_1} \{h(t_2 - s, u; t_2) - b(s, u)\} du \\
&= \frac{1}{\Gamma(d+1)} \int_0^{T-t_1} du u^d \int_0^1 ds \frac{s^{-d} t_2}{u + t_2 s} \left\{ \left(\frac{u}{t_2} + 1 \right)^d (1-s)^{-d} - 1 \right\} \\
&\quad \to \quad \frac{(T-t_1)^{d+1}}{\Gamma(d+2)} \int_0^1 s^{-d-1} [(1-s)^{-d} - 1] ds.
\end{aligned}
$$

However, by the proof of Theorem 5.1, this limit must be equal to

$$
\left\{ \int_0^{T-t_1} ds \int_0^s c(v) dv \right\} \sum_{m=1}^{\infty} \left\{ \int_0^{\infty} du f_m(u) \int_0^1 \frac{1}{(\tau + u)^{d+1}} d\tau \right\} \sin^m(\pi d).
$$

Thus the lemma follows. $\square$

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