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| Author(s) | GOTO, Shun'ichi; NAKAGAWA, Maki; OHTSUKA, Takeshi |
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UNIQUENESS AND EXISTENCE OF GENERALIZED MOTION FOR SPIRAL CRYSTAL GROWTH

SHUN'ICHI GOTO, MAKI NAKAGAWA, AND TAKESHI OHTSUKA

ABSTRACT. The uniqueness and existence of generalized solutions of ‘spiral curves’ for the mean curvature flow with driving force is studied by an adapted level set formulation. It is shown that the curves which are given by the level set formulation are unique with respect to initial spiral curves. For given spiral curves the method of a construction of an initial datum of a level set equation is also obtained by constructing a branch of the arguments from the centers of spiral curves, which has discontinuity at the given spiral curves.

1. INTRODUCTION

Various motions of spiral curves are observed on the growing surface of a crystal. This phenomena is come from spiral growth that occurs with aid of a screw dislocation intersecting the surface. In 1949, F. C. Frank pointed out important role of a screw dislocation to growth of a crystal in [5]. The theory of a growth of a crystal with aid of a screw dislocation is proposed by [1]. Recently this is called spiral growth of a crystal. According to the theory in [1], the spiral curves are described by steps on the surface of a crystal, and they move by the evolution law of the form

$$(1.1) \quad V = C - \kappa,$$

where V is the normal velocity, κ is the curvature of the curve, and C is a constant

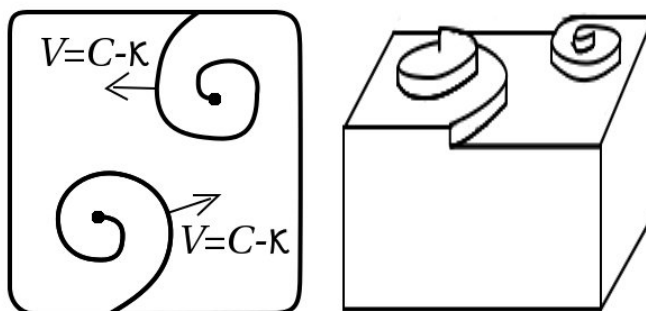


FIGURE 1. A crystal surface.

which denotes the driving force. This is an interface model of spiral crystal growth.

We have two types of mathematical models of this phenomena, phase-field models or interface models. A. Karma and M. Plapp ([12]) proposed a phase-field model

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for a situation such that there exist only one screw dislocation and one spiral curve. R. Kobayashi ([14]) proposed phase-field model for more complicated situations. P. Smereka ([20]) proposed a level set formulation of an interface model by using two auxiliary functions. The third author ([17]) proposed level set formulation by using a cross section of an auxiliary function and a Riemannian-like surface.

We consider a level set formulation for spirals due to the third author in [17]. In this formulation we describe a spiral curve by a cross section of an auxiliary surface and a helical surface. This idea is expressed by a zero-point set of $u - \theta$, where $u = u(t, x)$ is an auxiliary function, $\theta = \theta(x)$ is a function denoting a helical surface, which is of the form

$$\theta(x) := \sum_{j=1}^N m_j \arg(x - a_j),$$

a_1, \dots, a_N are screw dislocations on a surface of a crystal, $m_j \in \mathbb{Z} \setminus \{0\}$, and $\arg(x - a_j)$ denotes the argument of a vector $x - a_j$. The function θ is called a sheet structure function, which is introduced by [14] as a function illustrating the structure of the lattice of a crystal in his model. The quantity of $|m_j|$ denotes the mount of a slide at j -th screw dislocation, and the sign of m_j denotes the orientation of steps which come up to the surface by j -th screw dislocation.

By using the sheet structure function we express a spiral curve Γ_t at time t by

$$\Gamma_t := \{x \in \overline{W}; u(t, x) - \theta(x) \equiv 0 \pmod{2\pi m\mathbb{Z}}\},$$

where $W = \Omega \setminus \bigcup_{j=1}^N \overline{B_{\rho_j}(a_j)}$, Ω denotes a surface of a crystal, $B_{\rho_j}(a_j)$ denotes a screw dislocation, and m is the greatest common divisor of $|m_j|$ (see §2 for details of notations). Note that we need to treat the function θ as a multi-valued function to describe a spiral curve completely. However, we can define the normal vector field of Γ_t by $-\nabla(u - \theta)/|\nabla(u - \theta)|$ since $\nabla\theta$ is defined as a single-valued function. Therefore we derive the level set equation of (1.1). It is of the form

$$(1.2) \quad u_t - |\nabla(u - \theta)| \left\{ \operatorname{div} \frac{\nabla(u - \theta)}{|\nabla(u - \theta)|} + C \right\} = 0 \quad \text{in } (0, T) \times W,$$

where $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$, and $\operatorname{div} P = \partial P_1/\partial x_1 + \partial P_2/\partial x_2$ for $P = (P_1, P_2)$. We have obtained some mathematical results of (1.2) with Neumann boundary condition in [17], the comparison principle, and the existence of solutions in viscosity sense.

The aim of this paper is to discuss

- (i) the uniqueness of level sets with respect to the choice of initial data,
- (ii) how to construct an initial datum for the level set equation for a given initial curve.

The problem of the uniqueness of level sets arises since there is various choice of initial data for the level set equation for a given curve. In fact, initial data for the initial curve are not uniquely determined even if the initial curve is smooth. For example, consider the signed distance function of the initial curve. In usual level set formulation one can find many candidate of initial data. Indeed, there is the signed distance function ζ whose zero level set is a initial curve, and so is $c\zeta$ for arbitrary constant c . To prove the uniqueness of the generalized motion by the level set formulation, we create the framework to argue the uniqueness of level sets with respect to initial curves. The aim (i) is to prove the uniqueness of level sets in

the level set formulation for spirals. Fortunately, the uniqueness for the usual level set method has already shown by [2] or [3] if Γ_t is compact. For the case that Γ_t is unbounded [13] shows the uniqueness.

To prove the uniqueness for spirals we adapt methods as in [2] or [3]. The basic idea is to prove the comparison of super- or sub-level sets for solutions of the level set equation. For our problem it seems natural to consider $u - \theta$ to be a function w and apply the method to the super- and sub-level sets of w . However, it does not work since θ is a multi-valued function. The most crucial difference between an usual level set method and spirals is that spiral curves generally do not divide W into two domains. To overcome this difficulty, we introduce an idea of a covering space due to [17]. We can regard $u - \theta$ as a single-valued function on the covering space so that we can make sense super- and sub-level sets.

We also mention how to set up initial data for the initial spiral curves. In the usual level set method the signed distance function of an initial curve is one of candidates of an initial datum of the level set equation. However, this idea does not work because of a similar difficulty for the uniqueness. To overcome this difficulty we redefine the argument of vectors, whose branch-cut line is set to coincide with the initial curve. Once we obtain the new branch of the argument, initial data can be taken by adding a small constant to the new branch of the argument, and mollify it.

We are now in position to mention somewhat recent results. The third author, Y.-H. R. Tsai and Y. Giga ([18]) showed numerical computations of (1.2). They also gave a numerical point of view to the construction of an initial datum. Y. Giga, N. Ishimura and Y. Kohsaka ([7]) obtained the uniqueness and the existence of a ‘spiral solution’ for the geometric equation (1.1), which contains an anisotropy of the motion. They also proved the stability of solutions in the sense of Liapunov. T. Ogiwara and K.-I. Nakamura ([15], [16]) showed the existence, the uniqueness and the stability in the sense of Liapunov of ‘spiral traveling wave solution’ for the equation by [12] or [14]. N. Ishimura ([11]) showed the existence of a family of expanding spiral-like self-similar solutions for the curvature flow, and its asymptotic behavior. He also showed the nonexistence of a family of shrinking spiral-like self-similar solutions. B. Fiedler, J.-S. Guo and J.-C. Tsai ([4]) propose a center manifold approach to analyse the motion of rotating spirals. J.-S. Guo, K.-I. Nakamura, T. Ogiwara and J.-C. Tsai ([9]) gave some classification of rotating spirals. H. Imai, N. Ishimura and T. Ushijima ([10]) showed the existence of solutions for an ordinary differential equation which describes the motion of spirals by crystalline curvature, and showed numerical examples.

2. MAIN RESULTS

2.1. Preliminaries. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a C^2 boundary, which denotes a surface of a crystal. Let $a_1, a_2, \dots, a_N \in \Omega$, which denotes edges of spirals. Here we assume that the edges of spirals are fixed. Let $\rho_1, \rho_2, \dots, \rho_N$ be positive constants satisfying

$$(2.1) \quad \overline{B_{\rho_i}(a_i)} \cap \overline{B_{\rho_j}(a_j)} = \emptyset, \quad \overline{B_{\rho_j}(a_j)} \cap \Omega^c = \emptyset \quad \text{for } i, j = 1, \dots, N, \quad i \neq j,$$

where $B_r(a) = \{x \in \mathbb{R}^2; |x - a| < r\}$. Let $W = \Omega \setminus (\bigcup_{j=1}^N \overline{B_{\rho_j}(a_j)})$. We observe that ∂W is smooth.

We define two kinds of spiral curves and a bunch of spirals.

Definition 2.1. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^2 boundary.

- (i) Let $a \in \Omega$ and $\rho > 0$. Set $W = \Omega \setminus \overline{B_\rho(a)}$. Assume that ρ satisfies (2.1). We say $\Gamma = \Gamma^a$ is a principal spiral on \overline{W} if $\Gamma = \{P(s) \in \overline{W}; 0 \leq s \leq \ell\}$ satisfies
- ($\Gamma 1$) Γ is a C^1 curve,
 - ($\Gamma 2$) $P(0) \in \partial B_\rho(a)$, $P(\ell) \in \partial\Omega$, and $P(s) \notin \partial W$ for $s \in (0, \ell)$,
 - ($\Gamma 3$) $P(s_1) \neq P(s_2)$ if $s_1 \neq s_2$,
 - ($\Gamma 4$) there exists $\delta > 0$ such that, for any $x \in \overline{W}$ satisfying $\text{dist}(x, \Gamma) < \delta$, there exists a unique $y \in \Gamma$ satisfying $\text{dist}(x, \Gamma) = |x - y|$.
- (ii) Let $a_1, a_2 \in \Omega$ and $\rho_1, \rho_2 > 0$. Set $W = \Omega \setminus (\overline{B_{\rho_1}(a_1)} \cup \overline{B_{\rho_2}(a_2)})$. Assume that ρ_1 and ρ_2 satisfy (2.1). We say $\Gamma = \Gamma^{a_1 a_2}$ is a connecting spiral on \overline{W} between a_1 and a_2 if $\Gamma = \{P(s) \in \overline{W}; 0 \leq s \leq \ell\}$ satisfies ($\Gamma 1$), ($\Gamma 3$), ($\Gamma 4$) and
- ($\Gamma 2'$) $P(0) \in \partial B_{\rho_1}(a_1)$, $P(\ell) \in \partial B_{\rho_2}(a_2)$, and $P(s) \notin \partial W$ for $s \in (0, \ell)$.
- (iii) Let $a_1, \dots, a_N \in \Omega$ and $\rho_1, \dots, \rho_N > 0$. Set $W = \Omega \setminus (\bigcup_{j=1}^N \overline{B_{\rho_j}(a_j)})$. Assume that all of ρ_j satisfy (2.1). We say Γ is a bunch of spirals if Γ is given by $\Gamma = \bigcup_{j=1}^M \Gamma_j$, where $\Gamma_1, \dots, \Gamma_M$ are principal or connecting spirals such that $\Gamma_i \cap \Gamma_j = \emptyset$ if $i \neq j$.

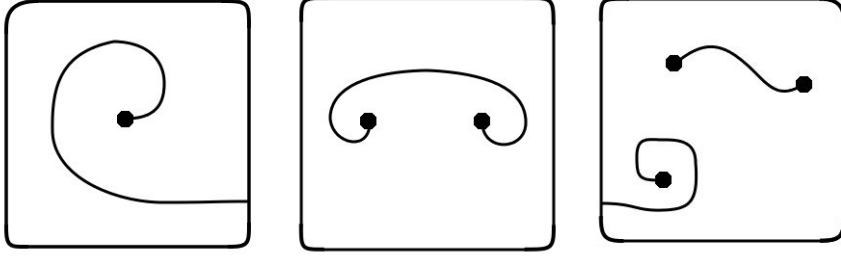


FIGURE 2. Figures of a principal spiral, a connecting spiral, and a bunch of spirals(from left to right).

Remark 2.2. (i) One can find $|m_j|/m$ pieces of principal spirals in a neighborhood of $B_{\rho_j}(a_j)$. However, it holds that $M \leq \sum_{j=1}^N |m_j|/m$ because of connecting spirals, in general.

- (ii) For a connecting spiral $\Gamma = \Gamma^{a_1 a_2}$ between a_1 and a_2 , there exist principal spirals Γ^{a_1} and Γ^{a_2} , which is respectively started from a_1 and a_2 , satisfying

$$\Gamma^{a_1 a_2} = \Gamma^{a_1} \setminus (\Gamma^{a_2} \cup B_{\rho_2}(a_2)).$$

2.2. Level set formulation. We consider a family of set $\{\Gamma_t\}_{t \geq 0}$ which moves by the evolution law of the form

$$(2.2) \quad V = C - \kappa \quad \text{on } \Gamma_t,$$

$$(2.3) \quad \Gamma_t \perp \partial W$$

with an initial spiral curve Γ_0 , where V is the velocity of the curve in the direction of the normal vector field \mathbf{n} of Γ_t , κ is the curvature of the curve, and C is a

constant which denotes a driving force. We recall a level set formulation by [17]. We define

$$(2.4) \quad \Gamma_t := \{x \in \overline{W}; u(t, x) - \theta(x) \equiv 0 \pmod{2\pi m\mathbb{Z}}\}, \quad \vec{\mathbf{n}} = -\frac{\nabla(u - \theta)}{|\nabla(u - \theta)|},$$

where $u(t, x)$ is an auxiliary function, $\theta(x)$ is the sheet structure function, which is introduced by [14], defined by

$$\theta(x) = \sum_{j=1}^N m_j \arg(x - a_j), \quad m_j \in \mathbb{Z} \setminus \{0\},$$

and m is the greatest common divisor of $|m_j|$. Note that the function $\arg x$ is treated as a multi-valued function. By this formulation (2.4) we derive the level set equation of the form

$$(2.5) \quad u_t - |\nabla(u - \theta)| \left\{ \operatorname{div} \frac{\nabla(u - \theta)}{|\nabla(u - \theta)|} + C \right\} = 0 \quad \text{in } (0, T) \times W,$$

$$(2.6) \quad \langle \nu, \nabla(u - \theta) \rangle = 0 \quad \text{on } (0, T) \times \partial W,$$

$$(2.7) \quad u(0, \cdot) = u_0(\cdot) \quad \text{on } \overline{W},$$

where ν is the unit outer normal vector field of ∂W , and $\langle \cdot, \cdot \rangle$ denotes the usual inner product on \mathbb{R}^2 . Note that $\nabla\theta$ is determined as a single-valued function so that the equations (2.5)–(2.6) is well-defined.

It is convenient for the theory of viscosity solutions to introduce other expression of the equation (2.5)–(2.6) of the form;

$$(2.8) \quad u_t + F(\nabla(u - \theta), \nabla^2(u - \theta)) = 0 \quad \text{in } (0, T) \times W,$$

$$(2.9) \quad B(x, \nabla(u - \theta)) = 0 \quad \text{on } (0, T) \times \partial W,$$

where $F: (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{S}^2 \rightarrow \mathbb{R}$ and $B: \partial W \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are defined by

$$(2.10) \quad F(p, X) = -\operatorname{trace} \left[\left(I - \frac{p \otimes p}{|p|^2} \right) X \right] - C|p|,$$

$$(2.11) \quad B(x, p) = \langle \nu(x), p \rangle,$$

and \mathbb{S}^2 is the space of 2×2 symmetric matrices. In other words, F stands for the operator of the mean curvature flow with driving force, and B denotes the Neumann boundary condition. The usual properties of viscosity solutions for (2.5)–(2.6), for examples, the comparison principle, stability, rescaling invariance and the existence of global viscosity solutions for the continuous initial data, are shown by [17].

The generalized motion of spirals by (2.2)–(2.3) is defined as

Definition 2.3. We say that the family of the curves $\{\Gamma_t\}_{t \geq 0}$ is a generalized motion of spirals by (2.2)–(2.3) started from Γ_0 if Γ_t is expressed by (2.4) with a viscosity solution $u(t, x)$ of (2.5)–(2.6) with an initial datum $u(0, x) = u_0(x)$, which satisfies

$$\Gamma_0 = \{x \in \overline{W}; u_0(x) - \theta(x) \equiv 0 \pmod{2\pi m\mathbb{Z}}\}.$$

2.3. Uniqueness and existence. We now state the theorem of the uniqueness of a generalized motion for spirals.

Theorem 2.4 (Uniqueness). *Let Γ_0 be a closed subset in \overline{W} and $u_0, v_0 \in C(\overline{W})$ satisfy*

$$\begin{aligned}\Gamma_0 &= \{x \in \overline{W}; u_0(x) - \theta(x) \equiv 0 \pmod{2\pi m\mathbb{Z}}\} \\ &= \{x \in \overline{W}; v_0(x) - \theta(x) \equiv 0 \pmod{2\pi m\mathbb{Z}}\}.\end{aligned}$$

Let $T > 0$, and let u and v be a viscosity solution of (2.5)–(2.6) on $(0, T) \times \overline{W}$ with $u(0, \cdot) = u_0$ and $v(0, \cdot) = v_0$, respectively. Then

$$\begin{aligned}\{x \in \overline{W}; u(t, x) - \theta(x) \equiv 0 \pmod{2\pi m\mathbb{Z}}\} \\ = \{x \in \overline{W}; v(t, x) - \theta(x) \equiv 0 \pmod{2\pi m\mathbb{Z}}\}\end{aligned}$$

for $t \in [0, T)$.

The proof will be given in §3. The basic idea is to prove the comparison of sub- and super-level sets (see [2] or [3] for the usual level set method). However, we have to clarify the sense of sub- and super level sets since θ is multi-valued. For this purpose we use an idea of a covering space due to [17]. This idea gives us an another point of view such that $u - \theta$ is regarded as a single-valued function.

We also state the existence of generalized motion for spirals;

Theorem 2.5. *Let Γ_0 be spiral curves on \overline{W} . Then there exists a unique generalized motion of spirals $\{\Gamma_t\}_{t \geq 0}$.*

The proof is divided into two steps; prove the existence of an initial datum for initial spiral curves, and prove the existence of a viscosity solution of (2.5)–(2.6). The second step is established by the argument of [17]. Therefore it suffices to overcome the first step; construct an initial datum $u_0 \in C(\overline{W})$ satisfying

$$\Gamma_0 = \{x \in \overline{W}; u_0(x) - \theta(x) \equiv 0 \pmod{2\pi m\mathbb{Z}}\}$$

for given spiral curves Γ_0 . It is not clear because the signed distance function of initial curves Γ_0 does not work well. To overcome this difficulty, we redefine the sheet structure function θ as a smooth, single-valued function in $\overline{W} \setminus \Gamma_0$. And also, we add a small constant to the sheet structure function, and mollify it. The construction will be demonstrated in §4.

3. UNIQUENESS OF THE MOTION OF SPIRALS

In this section we discuss about a uniqueness of a motion of spirals. We first see an adapted rescaling invariance, and we next see a comparison of super- or sub-level sets. A usual method of super- or sub-level sets is not useful for the formulation (2.4). So we consider the super- or sub-level sets on a covering space as in [17].

3.1. Rescaling invariance. We recall the property of geometric for the initial and boundary value problem

$$(3.1) \quad \begin{cases} u_t + \tilde{F}(x, \nabla u, \nabla^2 u) = 0 & \text{in } (0, T) \times \Omega, \\ \tilde{B}(x, \nabla u) = 0 & \text{on } (0, T) \times \partial\Omega, \\ u|_{t=0} = u_0 & \text{on } \overline{\Omega} \end{cases}$$

for some $\tilde{F}: \Omega \times (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{S}^2 \rightarrow \mathbb{R}$ and $\tilde{B}: \partial\Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$. The problem (3.1) is called geometric if \tilde{F} satisfies

$$(3.2) \quad \begin{aligned} \tilde{F}(x, \lambda p, \lambda X + \mu p \otimes p) &= \lambda \tilde{F}(x, p, X) \\ \text{for } \lambda > 0, \mu \in \mathbb{R}, \text{ and } (x, p, X) &\in \Omega \times (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{S}^2, \end{aligned}$$

and \tilde{B} satisfies

$$(3.3) \quad \tilde{B}(x, \lambda p) = \lambda \tilde{B}(x, p) \quad \text{for } \lambda > 0 \text{ and } (x, p) \in \Omega \times \mathbb{R}^2.$$

For a geometric problem we have the following.

Rescaling invariance. *Let $G: \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing, uniformly continuous function. If v is a viscosity supersolution of a geometric problem (3.1), then $G(v)$ is also a viscosity supersolution of (3.1).*

See [2, §5] or [19, §4] for the details. Apparently, our equation (2.8)–(2.9) is a geometric problem for $u - \theta$, so we shall show an adapted version of the rescaling invariance in this subsection.

We first recall a stability for (2.5)–(2.6).

Lemma 3.1 (Stability). *Let $\{u_n\}$ be a family of viscosity supersolutions (resp. subsolutions) of (2.5)–(2.6). Assume that there exists u satisfying $u_n \rightarrow u$ locally uniformly as $n \rightarrow \infty$. Then u is a viscosity supersolution (resp. subsolution) of (2.5)–(2.6).*

Proof. Apply [19, Proposition 3.3] with $F_k(x, p, X) = F(p - \nabla\theta(x), X - \nabla^2\theta(x))$. It is also easy to check the boundary condition $B(x, p - \nabla\theta(x)) = 0$. \square

To investigate (2.5)–(2.6) and (2.8)–(2.9), $G(u)$ in the statement of rescaling invariance will be replaced by $G(u - \theta)$. For this revision we need to make sure the sense of $G(u - \theta)$. We now consider

$$\mathcal{L}_j := \{x \in \overline{W}; (x - a_j)/|x - a_j| = (-1, 0)\}, \quad \mathcal{L} := \bigcup_{j=1}^N \mathcal{L}_j$$

and define the function $\Theta_j: \overline{W} \setminus \mathcal{L}_j \rightarrow (-\pi, \pi)$ by

$$\Theta_j(x) := \text{Arg}(x - a_j),$$

where $\text{Arg}x \in (-\pi, \pi]$ is the principal value of the argument of $x \in \mathbb{R}^2 \setminus \{0\}$. We now clarify the sense of a rescaling for our problem.

Definition 3.2. Let $G: \mathbb{R} \rightarrow \mathbb{R}$ be an upper (resp. a lower) semicontinuous, monotone nondecreasing function satisfying

$$(3.4) \quad G(s + 2\pi|m_j|) = G(s) + 2\pi|m_j| \quad \text{for } j = 1, 2, \dots, N \text{ and } s \geq s_0$$

for some $s_0 \in \mathbb{R}$. For a function $f: \overline{W} \rightarrow \mathbb{R}$ we define $g: \overline{W} \rightarrow \mathbb{R}$ by

$$g(x) := \begin{cases} G(f(x) - \Theta(x)) + \Theta(x) & \text{for } x \in \overline{W} \setminus \mathcal{L}, \\ \overline{\lim}_{y \rightarrow x} [G(f(x) - \Theta(y)) + \Theta(y)] & \text{for } x \in \mathcal{L}, \end{cases}$$

(resp. $g(x) := \underline{\lim}_{y \rightarrow x} [G(f(x) - \Theta(y)) + \Theta(y)]$ for $x \in \mathcal{L}$),

where $\Theta(x) := \sum_{j=1}^N \Theta_j(x) - 2\pi m \ell_0$, $\ell_0 \in \mathbb{N}$ satisfies

$$\inf_{\overline{W}} (f - \Theta_*) - 2\pi \sum_{j=1}^N |m_j| \geq s_0,$$

and $\Theta_* : \overline{W} \rightarrow \mathbb{R}$ is a lower semicontinuous envelope of Θ .

Note that g is well-defined. It suffices to consider only the case $x \in \mathcal{L}$. Moreover, it suffices to show it only the case G is upper semicontinuous, since the proof is similar. Let $\{y_n\}$ and $\{z_n\}$ be sequences which converge to $x \in \mathcal{L}$. We may assume that $\Theta_j(y_n) < 0$ and $\Theta_j(z_n) > 0$ for all $n \in \mathbb{N}$ and $j \in \Lambda_x$, where $\Lambda_x := \{j; x \in \mathcal{L}_j\}$. Then we observe that

$$\begin{aligned} \lim_{n \rightarrow \infty} (\Theta_j(y_n) + 2\pi) &= \lim_{n \rightarrow \infty} \Theta_j(z_n) \quad \text{for } j \in \Lambda_x, \\ \lim_{n \rightarrow \infty} \Theta_j(y_n) &= \lim_{n \rightarrow \infty} \Theta_j(z_n) \quad \text{otherwise.} \end{aligned}$$

By straightforward calculation we obtain

$$\begin{aligned} g(y_n) &= G \left(f(x) - \left[\sum_{j \notin \Lambda_x} m_j \Theta_j(y_n) + \sum_{j \in \Lambda_x} m_j \Theta_j(y_n) \right] \right) \\ &\quad + \sum_{j \notin \Lambda_x} m_j \Theta_j(y_n) + \sum_{j \in \Lambda_x} m_j \Theta_j(y_n) \\ &= G \left(f(x) - \left[\sum_{j \notin \Lambda_x} m_j \Theta_j(y_n) + \sum_{j \in \Lambda_x} m_j (\Theta_j(y_n) + 2\pi) \right] \right) \\ &\quad + \sum_{j \notin \Lambda_x} m_j \Theta_j(y_n) + \sum_{j \in \Lambda_x} m_j (\Theta_j(y_n) + 2\pi), \end{aligned}$$

since $f(x) - \Theta(x) - 2\pi \sum_{j \in \Lambda_x} m_j \geq s_0$. This implies

$$\overline{\lim}_{n \rightarrow \infty} g(y_n) = \overline{\lim}_{n \rightarrow \infty} g(z_n).$$

Therefore we conclude that $g(x)$ for $x \in \mathcal{L}$ is well-defined.

Here and hereafter we use a symbolic notation $g = G(f - \theta) + \theta$. We next verify regularities of $g = G(\varphi - \theta) + \theta$ for the smooth function φ .

Lemma 3.3. *Let $G: \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing smooth function satisfying (3.4). Let $\varphi \in C^{1,2}((0, T) \times \overline{W})$. Then $g(t, x) := [G(\varphi - \theta) + \theta](t, x) \in C^{1,2}((0, T) \times \overline{W})$. Moreover we obtain*

$$\begin{aligned} g_t(t, x) &= G'(\varphi(t, x) - \Theta_*(x)) \varphi_t(t, x), \\ \nabla g(t, x) &= G'(\varphi(t, x) - \Theta_*(x)) (\nabla \varphi(t, x) - \nabla \theta(x)) + \nabla \theta(x), \\ \nabla^2 g(t, x) &= G''(\varphi(t, x) - \Theta_*(x)) (\nabla \varphi(t, x) - \nabla \theta(x)) \otimes (\nabla \varphi(t, x) - \nabla \theta(x)) \\ &\quad + G'(\varphi(t, x) - \Theta_*(x)) (\nabla^2 \varphi(t, x) - \nabla^2 \theta(x)) + \nabla^2 \theta(x). \end{aligned}$$

Proof. It is easy to see the assertion when $x \notin \mathcal{L}$. If $x \in \mathcal{L}$, we consider $y \in B_r(x)$ for some $0 < r < \min\{\rho_1, \dots, \rho_N\}$. Then, there exist smooth functions $\hat{\Theta}_k: B_r(x) \rightarrow (0, 2\pi)$ for $k \in \Lambda_x$ such that $\hat{\Theta}_k(x) \equiv \arg(x - a_k) \pmod{2\pi\mathbb{Z}}$, where

$\Lambda_x = \{j; x \in \mathcal{L}_j\}$. These functions yield that

$$\begin{aligned} g(t, y) = & G(\varphi(t, y) - \sum_{j \notin \Lambda_x} m_j(\Theta_j)_*(y) - \sum_{j \in \Lambda_x} m_j(\hat{\Theta}_j)_*(y) + 2\pi) \\ & + \sum_{j \notin \Lambda_x} m_j(\Theta_j)_*(y) + \sum_{j \in \Lambda_x} m_j(\hat{\Theta}_j)_*(y) \quad \text{for } y \in B_r(x). \end{aligned}$$

This implies that g is smooth in $B_r(x)$. By straightforward calculation we obtain the formulas of g_t , ∇g and $\nabla^2 g$. \square

We are now in position to prove an adapted rescaling invariance.

Lemma 3.4 (Rescaling invariance). *Let G be an upper (resp. a lower) semi-continuous, monotone nondecreasing function satisfying (3.4). Assume that $G \not\equiv +\infty$ (resp. $G \not\equiv -\infty$). Let $T > 0$ and let $v: [0, T) \times \overline{W}$ be a viscosity supersolution (resp. subsolution) of (2.5)–(2.6) on $(0, T) \times \overline{W}$ with $v(0, \cdot) = v_0$. Let $w: [0, T) \times \overline{W} \rightarrow \mathbb{R}$ be a function defined by $w(t, x) := [G(v - \theta) + \theta](t, x)$. Then w is a viscosity supersolution (resp. subsolution) of (2.5)–(2.6) on $[0, T) \times \overline{W}$ with $w(0, \cdot) = w_0(\cdot) := [G(v - \theta) + \theta](0, \cdot)$.*

Proof. We prove only a viscosity supersolution case, since the proof of a subsolution version is parallel to that. Let $(\hat{t}, \hat{x}) \in (0, T) \times \overline{W}$ and $\varphi \in C^{1,2}((0, T) \times \overline{W})$ satisfy

$$(3.5) \quad w(t, x) - \varphi(t, x) \geq w(\hat{t}, \hat{x}) - \varphi(\hat{t}, \hat{x}) = 0 \quad \text{for } (t, x) \in (0, T) \times \overline{W}.$$

We may assume that $\varphi(t, x) - \Theta_*(x) > s_0$ without loss of generality.

Case 1. Assume that G is smooth and $G' > 0$. Since $G' > 0$, there exists a smooth function $H = G^{-1}$. We observe that H satisfies $H' > 0$ and

$$H(s + 2\pi|m_j|) = H(s) + 2\pi|m_j|$$

for $j = 1, 2, \dots, N$ and $s \geq G(s_0)$. In fact, we have that $H'(s) = G'(H(s))^{-1} > 0$ and $H(s) \geq H(G(s_0)) = s_0$. Therefore we see that

$$G(H(s) + 2\pi|m_j|) = G(H(s)) + 2\pi|m_j| = s + 2\pi|m_j| = G(H(s + 2\pi|m_j|)).$$

This implies $H(s + 2\pi|m_j|) = H(s) + 2\pi|m_j|$ for $s \geq G(s_0)$.

Here we define

$$\psi(t, x) = [H(\varphi - \theta) + \theta](t, x).$$

By Lemma 3.3 we observe that $\psi \in C^{1,2}((0, T) \times \overline{W})$ and satisfies

$$(3.6) \quad v(t, x) - \psi(t, x) \geq v(\hat{t}, \hat{x}) - \psi(\hat{t}, \hat{x}) = 0.$$

In fact, we get

$$\psi(t, x) \leq H(w(t, x) - \Theta(x)) + \Theta(x) = H(G(v(t, x) - \Theta(x))) + \Theta(x) = v(t, x).$$

and $\psi(\hat{t}, \hat{x}) = v(\hat{t}, \hat{x})$ by (3.5). By Lemma 3.3 we have

$$(3.7) \quad \begin{aligned} \psi_t &= H'(h)\varphi_t, \\ \nabla\psi &= H'(h)(\nabla\varphi - \nabla\theta) + \nabla\theta, \\ \nabla^2\psi &= H''(h)(\nabla\varphi - \nabla\theta) \otimes (\nabla\varphi - \nabla\theta) + H'(h)(\nabla^2\varphi - \nabla^2\theta) + \nabla^2\theta, \end{aligned}$$

where $h = \varphi(t, x) - \Theta_*(x)$.

Case 1.1. If $\hat{x} \in W$, we have

$$(3.8) \quad \psi_t + F^*(\nabla(\psi - \theta), \nabla^2(\psi - \theta)) \geq 0 \quad \text{at } (\hat{t}, \hat{x}),$$

since v is a viscosity supersolution of (2.5)–(2.6) and $v - \psi$ satisfies (3.6). By (3.7) and (3.2) we also have

$$\psi_t + F^*(\nabla(\psi - \theta), \nabla^2(\psi - \theta)) = H'(h)(\varphi_t + F^*(\nabla(\varphi - \theta), \nabla^2(\varphi - \theta))).$$

Therefore we have

$$(3.9) \quad \varphi_t + F(\nabla\varphi - \nabla\theta, \nabla^2\varphi - \nabla^2\theta) \geq 0,$$

since $H' > 0$.

Case 1.2. If $\hat{x} \in \partial W$, v satisfies (3.8) or

$$(3.10) \quad \langle \nu, \nabla(\psi - \theta) \rangle \geq 0 \quad \text{at } (\hat{t}, \hat{x}).$$

If (3.8) holds, then we have (3.9) at $(\hat{t}, \hat{x}) \in (0, T) \times \partial W$ by similar way in Case 1.1. If (3.10) holds, then we have

$$\langle \nu, \nabla\psi - \nabla\theta \rangle = H'(h)\langle \nu, \nabla\varphi - \nabla\theta \rangle \geq 0$$

by (3.7). Therefore we have

$$(3.11) \quad \langle \nu, \nabla\varphi - \nabla\theta \rangle \geq 0 \quad \text{at } (\hat{t}, \hat{x}).$$

These and Case 1.1 yields that w is a viscosity supersolution of (2.5)–(2.6) with initial data $w(0, \cdot) = w_0$.

Case 2. Assume that $G \in C(\mathbb{R})$. Let $\mu \in C_0^\infty(\mathbb{R})$ such that $\text{supp}\mu \subset [-1, 1]$, $\mu \geq 0$ and $\int_{\mathbb{R}} \mu = 1$. Set $\mu_k(s) = k\mu(ks)$. We define

$$G_k(s) = \left(1 - \frac{1}{k}\right) \int_{\mathbb{R}} G(s - \sigma)\mu_k(\sigma)d\sigma + \frac{s}{k}.$$

Thus, G_k has the following properties;

- (i) $G_k \in C^\infty(\mathbb{R})$,
- (ii) $G_k \rightarrow G$ locally uniformly as $k \rightarrow \infty$,
- (iii) $G'_k > 0$,
- (iv) $G_k(s + 2\pi|m_j|) = G_k(s) + 2\pi|m_j|$ for $j = 1, 2, \dots, N$ and $s \geq s_0 + 1/k$, where s_0 is as in (3.4).

By usual theory of mollifier we have (i) and (ii). We only verify (iii) and (iv). To see (iii), we obtain

$$G_k(s + \beta) - G_k(s) = \left(1 - \frac{1}{k}\right) \int_{\mathbb{R}} (G(s + \beta - \sigma) - G(s - \sigma))\mu_k(\sigma)d\sigma + \frac{\beta}{k} \geq \frac{\beta}{k},$$

since G is nondecreasing. Hence we have $G'(s) \geq 1/k > 0$. We next see (iv). For $s \geq s_0 + 1/k$ we have

$$\begin{aligned} G_k(s + 2\pi|m_j|) &= \left(1 - \frac{1}{k}\right) \int_{[-k^{-1}, k^{-1}]} G(s + 2\pi|m_j| - \sigma)\mu_k(\sigma)d\sigma + \frac{s + 2\pi|m_j|}{k} \\ &= \left(1 - \frac{1}{k}\right) \int_{[-k^{-1}, k^{-1}]} (G(s - \sigma) + 2\pi|m_j|)\mu_k(\sigma)d\sigma + \frac{s + 2\pi|m_j|}{k} \\ &= G_k(s) + 2\pi|m_j| \left(1 - \frac{1}{k}\right) + \frac{2\pi|m_j|}{k} = G_k(s) + 2\pi|m_j|, \end{aligned}$$

since $\text{supp}\mu_k \subset [-k^{-1}, k^{-1}]$.

Define $w_k(t, x) := [G_k(v - \theta) + \theta](t, x)$. Then we obtain w_k is a supersolution of (2.5)–(2.6) for $k \in \mathbb{N}$, and $w_k \rightarrow w(t, x) := [G(v - \theta) + \theta](t, x)$ locally uniformly as $k \rightarrow \infty$. It implies that w is still a viscosity supersolution of (2.5)–(2.6).

Case 3. Assume that G is upper semicontinuous. We approximate G by a sequence of continuous functions G_k as in [8, Lemma 4.1]. We fix $k \in \mathbb{N}$. For $j \in \mathbb{Z}$ we set

$$G_k \left(\frac{j}{2^k} \right) = G \left(\frac{j+1}{2^k} \right) =: \alpha_j.$$

We now define

$$G_k(s) = \alpha_j + \frac{\alpha_{j+1} - \alpha_j}{2^{-k}}(s - 2^{-k}j) \quad \text{if } s \in [2^{-k}j, 2^{-k}(j+1)].$$

We have a sequence $\{G_k\}_{k \in \mathbb{N}}$ and it converges to G locally uniformly as $k \rightarrow \infty$, since G is upper semicontinuous and then right continuous. We now complete the proof of Lemma 3.4. \square

Remark 3.5. (i) By symmetric arguments we also obtain the rescaling invariance for a rescaling function \tilde{G} satisfying

$$(3.12) \quad \tilde{G}(s + 2\pi|m_j|) = \tilde{G}(s) + 2\pi|m_j| \quad \text{for } j = 1, 2, \dots, N \text{ and } s \leq s_0$$

for some $s_0 \in \mathbb{R}$. The function $G(\varphi - \theta) + \theta$ can be defined similarly as Definition 3.2.

(ii) Lemma 3.4 make clear the argument as in [17, §4]. We change [17, Proposition 4.8] to Lemma 3.4 of this paper. The rescaling functions σ_+ and σ_- satisfy the assumptions of (3.12) or (3.4), respectively. The rescaled functions $f_{\varepsilon, y}$ and $g_{\varepsilon, y}$ are regarded as functions defined in the sense of Definition 3.2. By these revisions the proofs in [17] are still valid.

3.2. Construction of transformation. In this section we construct a transformation function to prove Theorem 2.4. It is convenient to introduce a covering space of \overline{W} of the form

$$\mathfrak{X} = \{(x, \xi) \in \overline{W} \times \mathbb{R}^N; \xi = (\xi_1, \xi_2, \dots, \xi_N), (\cos \xi_j, \sin \xi_j) = (x - a_j)/|x - a_j|\}.$$

We may assume that

- (A1) $u_0(x) = v_0(x)$ on Γ_0 ,
- (A2) $\{(x, \xi) \in \mathfrak{X}; u_0(x) - \sum_{j=1}^N m_j \xi_j > 0\} = \{(x, \xi) \in \mathfrak{X}; v_0(x) - \sum_{j=1}^N m_j \xi_j > 0\}$

without loss of generality. We first give an upper semicontinuous function \tilde{G} to rescale v_0 so that $u_0 - \theta \leq \tilde{G}(v_0 - \theta)$ in some sense.

Lemma 3.6. *Let $u_0, v_0 \in C(\overline{W})$, and assume that*

$$(3.13) \quad \{(x, \xi) \in \mathfrak{X}; u_0(x) - \sum_{j=1}^N m_j \xi_j > 0\} \subset \{(x, \xi) \in \mathfrak{X}; v_0(x) - \sum_{j=1}^N m_j \xi_j > 0\}.$$

Then there exists $\tilde{G}: \mathbb{R} \rightarrow \mathbb{R}$ satisfying

- (i) \tilde{G} is nondecreasing,
- (ii) $\tilde{G}(s) = 0$ for $s \leq 0$,
- (iii) $u_0(x) - \sum_{j=1}^N m_j \xi_j \leq \tilde{G}(v_0(x) - \sum_{j=1}^N m_j \xi_j)$ for $(x, \xi) \in \mathfrak{X}$,

- (iv) $\tilde{G}(s + 2\pi|m_j|) = \tilde{G}(s) + 2\pi|m_j|$ for $j = 1, 2, \dots, N$ and $s \in \tilde{G}^{-1}(0, \infty)$.
 Moreover, we have $\tilde{G}(s) = \max\{\tilde{G}(s + 2\pi|m_j|) - 2\pi|m_j|, 0\}$ for $s \in \mathbb{R}$ and $j = 1, 2, \dots, N$,
 (v) \tilde{G} is upper semicontinuous in \mathbb{R} and continuous at $s = 0$.

The property (iii) means $u_0 - \theta \leq \tilde{G}(v_0 - \theta)$. This inequality makes sense by (iv) and the inequality $u_0(x) \leq [\tilde{G}(v_0 - \theta) + \theta](x)$. In [2] or [3] we have a way to construct \tilde{G} satisfying (i)–(iii) and (v) (see also [6]). However, we need (iv) for our problem.

Proof. Set $\tilde{u}_0(x, \xi) = u_0(x) - \sum_{j=1}^N m_j \xi_j$ and $\tilde{v}_0(x, \xi) = v_0(x) - \sum_{j=1}^N m_j \xi_j$. If $\{(x, \xi) \in \mathfrak{X}; \tilde{u}_0(x, \xi) > 0\} = \emptyset$, we set

$$\tilde{G}(s) = \begin{cases} s & \text{if } s \geq 0, \\ 0 & \text{if } s < 0, \end{cases}$$

in other words, $\tilde{G}(\tilde{v}_0) = (\tilde{v}_0)_+$, where $(u)_+ = \max(u, 0)$. It is easy to check (i)–(v) of Lemma 3.6.

Assume that $\{(x, \xi) \in \mathfrak{X}; \tilde{u}_0(x, \xi) > 0\} \neq \emptyset$. We define

$$(3.14) \quad \tilde{G}(s) := \sup\{(\tilde{u}_0(y, \eta))_+; (y, \eta) \in \mathfrak{X}, \tilde{v}_0(y, \eta) \leq s\}.$$

Note that \tilde{G} is well-defined. Indeed, since u_0 and v_0 are continuous, there exists $M_0 > 0$ satisfying $\max\{\|u_0\|_\infty, \|v_0\|_\infty\} \leq M_0$. If $\tilde{v}_0(y, \eta) \leq s$, then we obtain

$$-\sum_{j=1}^N m_j \xi_j \leq M_0 + s,$$

which implies

$$0 \leq (\tilde{u}_0(y, \eta))_+ \leq M_0 + M_0 + s = 2M_0 + s < +\infty,$$

so that \tilde{G} is well-defined.

We shall verify (i)–(iii). The monotonicity of \tilde{G} is obtained by its definition. The property (ii) is established by (3.13). Indeed, we have $(\tilde{u}_0(y, \eta))_+ = 0$ for $(y, \eta) \in \{(x, \xi) \in \mathfrak{X}; \tilde{v}_0(x, \xi) \leq 0\}$, which implies $\tilde{G}(s) \leq 0$ for $s \leq 0$. Since $\tilde{G}(s) \geq 0$ for $s \in \mathbb{R}$, we obtain (ii). Moreover, we have

$$\tilde{u}_0(x, \xi) \leq (\tilde{u}_0(x, \xi))_+ \leq \tilde{G}(\tilde{v}_0(x, \xi))$$

since $(x, \xi) \in \{(y, \eta) \in \mathfrak{X}; \tilde{v}_0(y, \eta) \leq \tilde{v}_0(x, \xi)\}$, which is (iii).

We verify (iv). Fix $j = 1, 2, \dots, N$ arbitrarily. We first assume that $m_j > 0$. Let $(y, \eta) \in \mathfrak{X}$ satisfy $\tilde{v}_0(y, \eta) \leq s + 2\pi m_j$. Then we observe that $(y, \eta + 2\pi e_j) \in \mathfrak{X}$, and $\tilde{v}_0(y, \eta + 2\pi e_j) \leq s$. These yield

$$\begin{aligned} (\tilde{u}_0(y, \eta))_+ &= (\tilde{u}_0(y, \eta + 2\pi e_j) + 2\pi m_j)_+ \\ &\leq (\tilde{u}_0(y, \eta + 2\pi e_j))_+ + 2\pi m_j \leq \tilde{G}(s) + 2\pi m_j. \end{aligned}$$

Hence, $\tilde{G}(s + 2\pi m_j) - 2\pi m_j \leq \tilde{G}(s)$. Since $\tilde{G} \geq 0$, we obtain

$$\tilde{G}(s) \geq \max\{\tilde{G}(s + 2\pi m_j) - 2\pi m_j, 0\}.$$

It remains to show that $\tilde{G}(s) \leq \max\{\tilde{G}(s + 2\pi m_j) - 2\pi m_j, 0\}$. Let $\varepsilon > 0$. If $s > 0$ satisfies $\tilde{G}(s) > 0$, there exists $(y_\varepsilon, \eta_\varepsilon) \in \{(x, \xi); \tilde{v}_0(x, \xi) \leq s\}$ satisfying $\tilde{u}_0(y_\varepsilon, \eta_\varepsilon) > 0$ and

$$\tilde{G}(s) - \varepsilon \leq \tilde{u}_0(y_\varepsilon, \eta_\varepsilon).$$

Therefore we obtain

$$\tilde{G}(s) + 2\pi m_j - \varepsilon \leq \tilde{u}_0(y_\varepsilon, \eta_\varepsilon) + 2\pi m_j = \tilde{u}_0(y_\varepsilon, \eta_\varepsilon - 2\pi e_j) \leq \tilde{G}(s + 2\pi m_j).$$

We have used $\tilde{v}_0(y_\varepsilon, \eta_\varepsilon - 2\pi e_j) \leq s + 2\pi m_j$ in the last inequality. This implies

$$\tilde{G}(s) + 2\pi m_j \leq \tilde{G}(s + 2\pi m_j)$$

and we obtain (iv) if $s \in \{s; \tilde{G}(s) > 0\}$. Assume that $\tilde{G}(s) = 0$. It suffices to see $\tilde{G}(s + 2\pi m_j) \leq 2\pi m_j$. Indeed, we observe that

$$\tilde{u}_0(x, \xi) \leq 0 \quad \text{for } (x, \xi) \in \{(y, \eta) \in \mathfrak{X}; \tilde{v}_0(y, \eta) \leq s\}.$$

If $(y, \eta) \in \mathfrak{X}$ satisfies $\tilde{v}_0(y, \eta) \leq s + 2\pi m_j$, then we observe that $(y, \eta + 2\pi e_j) \in \mathfrak{X}$ and $\tilde{v}_0(y, \eta + 2\pi e_j) \leq s$. This implies $\tilde{u}_0(y, \eta + 2\pi e_j) \leq 0$ and so $\tilde{u}_0(y, \eta) \leq 2\pi m_j$. Therefore we obtain $\tilde{G}(s + 2\pi m_j) \leq 2\pi m_j$, which implies

$$\tilde{G}(s) = 0 = \max\{\tilde{G}(s + 2\pi m_j) - 2\pi m_j, 0\}.$$

For the case $m_j < 0$ we consider $-m_j$ and $-e_j$ instead of m_j and e_j on above, respectively.

Before verifying (v), we give some representation of \tilde{G} . First we demonstrate

$$\tilde{G}(s) = \hat{G}(s) := \sup\{(\tilde{u}_0(x, \xi))_+; \tilde{v}_0(x, \xi) \leq s \text{ and } -M_0 - s \leq \sum_{j=1}^N m_j \xi_j \leq M_0\}.$$

By definitions we have $\tilde{G}(s) \geq \hat{G}(s)$. If we assume that $\tilde{G}(s) > \hat{G}(s)$, then there exists $(x, \xi) \in \mathfrak{X}$ satisfying that $\tilde{v}_0(x, \xi) \leq s$, $\sum_{j=1}^N m_j \xi_j \notin [-M_0 - s, M_0]$, and

$$\tilde{u}_0(x, \xi) > \tilde{G}(s) \geq 0.$$

However we observe that $\sum_{j=1}^N m_j \xi_j > M_0$ since $\tilde{v}_0(x, \xi) \leq s$. Indeed, this yields

$$\sum_{j=1}^N m_j \xi_j \geq v_0(x) - s \geq -M_0 - s.$$

This implies $\tilde{u}_0(x, \xi) < 0$, which is a contradiction.

Next we remark that, for any $s \in \mathbb{R}$, there exists a positive constant R_s satisfying (3.15)

$$\tilde{G}(s) = \sup\{(\tilde{u}_0(x, \xi))_+; \tilde{v}_0(x, \xi) \leq s \text{ and } \xi_j \in [-R_s, R_s] \text{ for } j = 1, 2, \dots, N\}.$$

In fact, for each $j = 1, 2, \dots, N-1$, there exists $k_j \in \mathbb{Z}$ and $\xi'_j \in [0, 2\pi|m_N|)$ satisfying

$$\xi_j = \xi'_j + 2\pi m_N k_j.$$

Then we obtain

$$\tilde{u}_0(x, \xi) = u_0(x) - \sum_{j=1}^{N-1} m_j \xi'_j - m_N(\xi_N + 2\pi \sum_{j=1}^{N-1} m_j k_j).$$

Set $\xi'_N = \xi_N + 2\pi \sum_{j=1}^{N-1} m_j k_j$ and $\xi' = (\xi'_1, \dots, \xi'_N)$. Then we obtain $(x, \xi') \in \mathfrak{X}$ and

$$\tilde{u}_0(x, \xi) = \tilde{u}_0(x, \xi').$$

Since $-M_0 - s \leq \sum_{j=1}^N m_j \xi_j \leq M_0$ and $\sum_{j=1}^N m_j \xi_j = \sum_{j=1}^N m_j \xi'_j$, we obtain

$$-M_0 - s - \sum_{j=1}^{N-1} m_j \xi'_j \leq m_N \xi'_N \leq M_0 - \sum_{j=1}^{N-1} m_j \xi'_j.$$

We thus have $-M_0 - s - S \leq m_N \xi'_N \leq M_0 + S$, where $S = 2\pi |m_N| \sum_{j=1}^{N-1} |m_j|$. Set $R_s = \max\{2\pi |m_N|, (M_0 + S + |s|)/m_N\}$. Then we observe that, for each $(x, \xi) \in \mathfrak{X}$, there exists $(x, \xi') \in \overline{W} \times [-R_s, R_s]^N$ satisfying

$$\tilde{u}_0(x, \xi) = \tilde{u}_0(x, \xi').$$

Therefore we obtain (3.15).

We are now in position to verify (v). Since \tilde{G} is monotone nondecreasing, we have $\overline{\lim}_{s \rightarrow s_0-0} \tilde{G}(s) \leq \tilde{G}(s_0)$. Therefore it suffices to see that $\overline{\lim}_{s \rightarrow s_0+0} \tilde{G}(s) \leq \tilde{G}(s_0)$. We first assume that $s_0 \geq \hat{s} := \sup\{\sigma; \tilde{G}(\sigma) = 0\}$. Since $\tilde{G}(s_0 + k^{-1}) > 0$ for $k \in \mathbb{N}$, there exists (y_k, η_k) satisfying

$$\begin{aligned} \tilde{G}(s_0 + k^{-1}) - k^{-1} &\leq \tilde{u}_0(y_k, \eta_k), \\ \tilde{v}_0(y_k, \eta_k) &\leq s_0 + k^{-1}. \end{aligned}$$

Note that $s_0 \geq 0$ by its definition, and $R_s \leq R_{s'}$ if $s' \geq s \geq 0$. Then we have $(y_k, \eta_k) \in \overline{W} \times [-R_{s_0+1}, R_{s_0+1}]^N$. Therefore there exists (y_0, η_0) satisfying

$$\begin{aligned} (y_k, \eta_k) &\rightarrow (y_0, \eta_0) \text{ as } k \rightarrow \infty, \\ \tilde{v}_0(y_0, \eta_0) &\leq s_0. \end{aligned}$$

For $\varepsilon > 0$ we take $k \in \mathbb{N}$ satisfying $k^{-1} < \varepsilon/2$ and $\tilde{u}_0(y_k, \eta_k) \leq \tilde{u}_0(y_0, \eta_0) + \varepsilon/2$. Moreover we take $r_0 > 0$ satisfying $r_0 \leq k^{-1}$. Then we obtain for $r < r_0$

$$\tilde{G}(s_0 + r) \leq \tilde{G}(s_0 + k^{-1}) \leq \tilde{u}_0(y_k, \eta_k) + k^{-1} \leq \tilde{u}_0(y_0, \eta_0) + \varepsilon \leq \tilde{G}(s_0) + \varepsilon,$$

which implies that $\overline{\lim}_{s \rightarrow s_0+0} \tilde{G}(s) \leq \tilde{G}(s_0)$ at $s_0 \in [\hat{s}, \infty)$. We next consider the case $s_0 \in (-\infty, \hat{s})$. By the monotonicity of \tilde{G} we have $\tilde{G}(s) = 0$ for $s \in (-\infty, \hat{s})$. Therefore \tilde{G} is continuous in $(-\infty, \hat{s})$, in particular, \tilde{G} is upper semicontinuous.

Finally we verify the continuity of \tilde{G} at $s = 0$. By (3.13) and (3.14) we have $\hat{s} \geq 0$. Since $\tilde{G} = 0$ in $(-\infty, \hat{s})$, the continuity at $s = 0$ holds if $\hat{s} > 0$. If $\hat{s} = 0$, it suffices to see $\lim_{s \rightarrow +0} \tilde{G}(s) = 0$. By the proof of upper semicontinuity of \tilde{G} , we have

$$\begin{aligned} \overline{\lim}_{s \rightarrow +0} \tilde{G}(s) &\leq \tilde{u}_0(y_0, \eta_0), \\ \tilde{v}_0(y_0, \eta_0) &\leq 0. \end{aligned}$$

By (3.13) we have $\tilde{u}_0(y_0, \eta_0) \leq 0$. Therefore we obtain $\overline{\lim}_{s \rightarrow +0} \tilde{G}(s) \leq 0$, which implies $\overline{\lim}_{s \rightarrow +0} \tilde{G}(s) = 0$. We thus obtain $\lim_{s \rightarrow 0} \tilde{G}(s) = 0$. \square

To prove Theorem 2.4 we need a uniform continuous function $G: \mathbb{R} \rightarrow \mathbb{R}$ satisfying (i)–(iv) in Lemma 3.6.

Lemma 3.7. *There exists a function $G: \mathbb{R} \rightarrow \mathbb{R}$ satisfying*

- (i) G is nondecreasing,
- (ii) $G(s) = 0$ for $s \leq 0$,
- (iii) $u_0(x) - \sum_{j=1}^N m_j \xi_j \leq G(v_0(x) - \sum_{j=1}^N m_j \xi_j)$ for $(x, \xi) \in \mathfrak{X}$,

- (iv) $G(s) = G(s + 2\pi m_j) - 2\pi m_j$ for $j = 1, 2, \dots, N$ and $s \in G^{-1}(0, \infty)$.
 Moreover, $G(s) = \max\{G(s + 2\pi m_j) - 2\pi m_j, 0\}$ for $s \in \mathbb{R}$ and $j = 1, 2, \dots, N$,
 (v) G is uniform continuous on \mathbb{R} .

Proof. Let m is the greatest common divisor of $\{|m_1|, |m_2|, \dots, |m_N|\}$. Set

$$\beta_k = \begin{cases} 2\pi m & \text{if } k = 0, \\ (7 - k)m & \text{if } k = 1, \dots, 6, \\ 2^{-(k-6)}m & \text{if } k \geq 7. \end{cases}$$

For $\ell \in \mathbb{N} \cup \{0\}$, we define

$$\begin{aligned} G(2\pi m\ell + \beta_k) &= \tilde{G}(2\pi\ell m + \beta_{k-1}) \quad \text{for } k \in \mathbb{N}, \\ G(2\pi m\ell + \beta_0) &= \tilde{G}(2\pi m\ell + \beta_0). \end{aligned}$$

We define $G: [2\pi m\ell, 2\pi m(\ell + 1)] \rightarrow \mathbb{R}$ by

$$\begin{aligned} G(s) &= \frac{G(2\pi m\ell + \beta_{k-1}) - G(2\pi m\ell + \beta_k)}{\beta_{k-1} - \beta_k} (s - 2\pi m\ell - \beta_k) \\ &\quad + G(2\pi m\ell + \beta_k) \quad \text{for } s \in [2\pi m\ell + \beta_k, 2\pi m\ell + \beta_{k-1}], \quad k \in \mathbb{N}. \end{aligned}$$

Thereby we have a continuous function G on $(0, \infty)$, since \tilde{G} is upper semicontinuous and nondecreasing, which yields that \tilde{G} is right continuous. By continuity of \tilde{G} at $s = 0$ we also have $\lim_{s \rightarrow +0} G(s) = 0$. We now extend G onto \mathbb{R} by

$$G(s) := 0 \quad \text{if } s \leq 0.$$

We then have $G \in C(\mathbb{R})$. It is easy to show (i)–(v) in this lemma by Lemma 3.6. \square

We are now in position to prove Theorem 2.4.

Proof of Theorem 2.4. We define

$$w(t, x) = [G(v - \theta) + \theta](t, x),$$

where $G: \mathbb{R} \rightarrow \mathbb{R}$ is a function defined as in Lemma 3.7, the right hand side is in the sense of Definition 3.2 with $\Theta(x) = \sum_{j=1}^N m_j(\Theta_j(x) + 2\pi k_j)$, and $k_j \in \mathbb{Z}$ is a constant satisfying $\inf_{[0, T] \times \overline{W}} (v - \Theta_*) \geq s_0$.

By the assumption (A2) and Lemma (3.7) (iii) we have

$$u_0(x) \leq w(0, x) \quad \text{for } x \in \overline{W},$$

since $(x, \Theta_*(x)) \in \mathfrak{X}$. Then we obtain

$$u(t, x) \leq w(t, x) \quad \text{for } (t, x) \in (0, T) \times \overline{W}$$

by the comparison principle in [17, Theorem 2.1]. Note that w is continuous on $[0, T] \times \overline{W}$ and $w(t, x) = G(v(t, x) - \Theta_*(x)) + \Theta_*(x)$. Then we have

$$u(t, x) - \Theta_*(x) \leq G(v(t, x) - \Theta_*(x)) \quad \text{for } (t, x) \in [0, T] \times \overline{W}.$$

Since $(x, \Theta_*(x)) \in \mathfrak{X}$, the above inequality implies

$$\begin{aligned} &\{(t, x, \xi) \in [0, T] \times \mathfrak{X}; \xi_j - 2\pi k_j \in [-\pi, \pi), \tilde{u}(t, x, \xi) > 0\} \\ &\subset \{(t, x, \xi) \in [0, T] \times \mathfrak{X}; \xi_j - 2\pi k_j \in [-\pi, \pi), \tilde{v}(t, x, \xi) > 0\}, \end{aligned}$$

here we have used the notation $\tilde{u}(t, x, \xi) = u(t, x) - \sum_{j=1}^N m_j \xi_j$ for a function $u(t, x)$.

For other branch of θ we consider

$$G(s; k) := G(s + 2\pi mk) - 2\pi mk,$$

where $k \in \mathbb{Z}$. We define $w_k(t, x) = [G(v - \theta; k) - \theta](t, x)$ in the sense of Definition 3.2. The same argument above leads us that

$$\begin{aligned} & \{(t, x, \xi) \in [0, T] \times \mathfrak{X}; \xi_j + 2\pi k_j \in [-\pi, \pi), \tilde{u}(t, x, \xi) > -2\pi mk\} \\ & \subset \{(t, x, \xi) \in [0, T] \times \mathfrak{X}; \xi_j + 2\pi k_j \in [-\pi, \pi), \tilde{v}(t, x, \xi) > -2\pi mk\} \end{aligned}$$

for any $k \in \mathbb{Z}$. This implies

$$\begin{aligned} & \{(t, x, \xi) \in [0, T] \times \mathfrak{X}; \tilde{u}(t, x, \xi) > 0\} \\ & \subset \{(t, x, \xi) \in [0, T] \times \mathfrak{X}; \tilde{v}(t, x, \xi) > 0\}. \end{aligned}$$

To change the role of u and v , we thus conclude

$$\begin{aligned} & \{(t, x) \in [0, T] \times \overline{W}; u(t, x) - \theta \equiv 0 \pmod{2\pi m\mathbb{Z}}\} \\ & = \{(t, x) \in [0, T] \times \overline{W}; v(t, x) - \theta \equiv 0 \pmod{2\pi m\mathbb{Z}}\}, \end{aligned}$$

which completes the proof of Theorem 2.4. \square

4. EXISTENCE OF A GENERALIZED MOTION

In this section, we only discuss on a way to construct $u_0 \in C(\overline{W})$ satisfying

$$(4.1) \quad \Gamma_0 = \{x \in \overline{W}; u_0(x) - \theta(x) \equiv 0 \pmod{2\pi m\mathbb{Z}}\},$$

for given a bunch of spirals Γ_0 in order to complete a level set formulation for spirals. If we have such an initial data u_0 , a generalized motion of spirals by (2.2)–(2.3) is established so that we can conclude Theorem 2.5.

For a usual level set formulation, a signed distance function from Γ_0 is one of candidates of u_0 . However, it does not work for our problem because spiral curves do not divide a domain into two regions. Because of this difficulty we construct u_0 without using a signed distance function; the construction is divided into three steps.

- Step 1.* Define $\varphi: \overline{W} \setminus \Gamma_0 \rightarrow \mathbb{R}$ satisfying $\varphi \in C(\overline{W} \setminus \Gamma_0)$ and $\varphi \equiv \theta \pmod{2\pi m\mathbb{Z}}$.
Step 2. Consider a tubular neighborhood $\Gamma_0^\delta = \{x \in \overline{W}; \text{dist}(x, \Gamma_0) < \delta\}$ of Γ_0 for some $\delta > 0$, and define $\psi: \overline{\Gamma_0^\delta} \rightarrow \mathbb{R}$ satisfying $\psi \in C(\overline{\Gamma_0^\delta})$, $\psi = \varphi$ on $\partial\Gamma_0^\delta \cap W$, and $\psi \equiv \theta + \pi m \pmod{2\pi m\mathbb{Z}}$ only on Γ_0 ,
Step 3. Define u_0 by

$$u_0(x) := \begin{cases} \varphi(x) + \pi m & \text{if } x \in \overline{W} \setminus \overline{\Gamma_0^\delta}, \\ \psi(x) + \pi m & \text{if } x \in \overline{\Gamma_0^\delta}. \end{cases}$$

We shall show Step 1 in next sections; §4.1 is for the only one spiral case, and §4.2 is for the two or more spiral case. We here demonstrate to construct ψ as in Step 2 from φ as in Step 1. We first determine the orientation of the unit normal vector field. For a principal spiral $\{P(s); s \in [0, \ell]\}$ we define

$$(4.2) \quad \vec{n}(P(s)) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} P_s(s),$$

where s is the arclength parameter. Since Γ_0 is a bunch of spirals, for any $x \in \Gamma_0$ there exists a principal spiral Γ such that $x \in \Gamma_0$. We denote the unit normal vector field of Γ_0 by $\vec{n}(\cdot)$ with the orientation on above.

We here choose $\delta > 0$ small enough satisfying $\delta < \min\{\rho_1, \dots, \rho_N\}$ and (Γ4) in Definition 2.1 (i). Then, we observe that Γ_0 divide Γ_0^δ into two domains. We now denote them by $\Gamma_{0,+}^\delta$ and $\Gamma_{0,-}^\delta$, where $\Gamma_{0,-}^\delta$ satisfies

$$\left\{ y \in W; y = x + \frac{\delta}{2} \vec{n}(x) \text{ for some } x \in \Gamma_0 \right\} \subset \Gamma_{0,-}^\delta,$$

and $\Gamma_{0,+}^\delta = \Gamma_0^\delta \setminus \overline{\Gamma_{0,-}^\delta}$. Let $\varphi_\pm: \overline{\Gamma_0^\delta} \rightarrow \mathbb{R}$ be functions satisfying $\varphi_\pm = \varphi$ on $\Gamma_{0,\pm}^\delta$ and

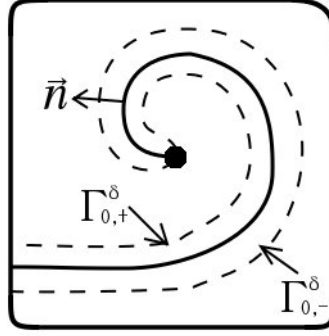


FIGURE 3. $\Gamma_{0,-}^\delta$ and $\Gamma_{0,+}^\delta$

$\varphi_\pm \equiv \theta \pmod{2\pi m\mathbb{Z}}$ on $\overline{\Gamma_0^\delta}$. Set

$$d_0(x) = \begin{cases} \text{dist}(x, \Gamma_0) & \text{if } x \in \overline{\Gamma_{0,+}^\delta}, \\ -\text{dist}(x, \Gamma_0) & \text{if } x \in \Gamma_0^\delta \setminus \overline{\Gamma_{0,+}^\delta}. \end{cases}$$

We now define

$$\psi(x) = \frac{\delta + d_0(x)}{2\delta} \varphi_+(x) + \frac{\delta - d_0(x)}{2\delta} \varphi_-(x).$$

This is the desired function in Step 2.

See also [18] for a construction of an initial datum for numerical computations.

4.1. Sheet structure function for one spiral. We first consider the situation that there is only one principal spiral on a domain. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^2 boundary. Let $a \in \Omega$ be a center of a spiral and $W = \Omega \setminus \overline{B_\rho(a)}$. We take $\rho > 0$ small enough so that ∂W is C^2 . We now define $\arg(x - a)$ whose discontinuity arises on Γ instead of $\{a + y \in \mathbb{R}^2; y \in (-\infty, 0] \times \{0\}\}$.

Lemma 4.1. *Let Γ be a principal spiral on W . There exists $\varphi \in C^\infty(W \setminus \Gamma) \cap C(\overline{W} \setminus \Gamma)$ satisfying*

$$\varphi(x) - \arg(x - a) \equiv 0 \pmod{2\pi\mathbb{Z}} \text{ for } x \in \overline{W} \setminus \Gamma.$$

Proof. We may assume that $a = 0$ and $P(0) = (\rho, 0)$ without loss of generality.

We introduce the polar coordinate. Let $\Psi: (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}^2$ be a map defined by $\Psi(r, \tau) = (r \cos \tau, r \sin \tau)$. Set

$$\mathcal{D} = \{(r, \tau) \in (0, \infty) \times \mathbb{R}; \Psi(r, \tau) \in W\},$$

$$\tilde{\Gamma} = \{(r, \tau) \in \overline{\mathcal{D}}; \Psi(r, \tau) \in \Gamma\}.$$

We find infinite curves $\tilde{\Gamma}_k$ on $\bar{\mathcal{D}}$ for $k \in \mathbb{Z}$ and the boundaries of W , i.e.,

$$\begin{aligned}\tilde{\Gamma}_k &= \{(r(s), \tau_k(s)); P(s) = \Psi(r(s), \tau_k(s)) \text{ for } s \in [0, \ell], \tau_k(0) = 2\pi k\}, \\ \mathcal{C}_1 &= \{(\rho, \tau); \tau \in \mathbb{R}\}, \\ \mathcal{C}_2 &= \{Q(s) = (p(s), q(s)); \Psi(p(s), q(s)) \in \partial\Omega\}.\end{aligned}$$

We observe that \mathcal{C}_1 , \mathcal{C}_2 , $\tilde{\Gamma}_k$ and $\tilde{\Gamma}_{k+1}$ yield a enclosed bounded domain, which we

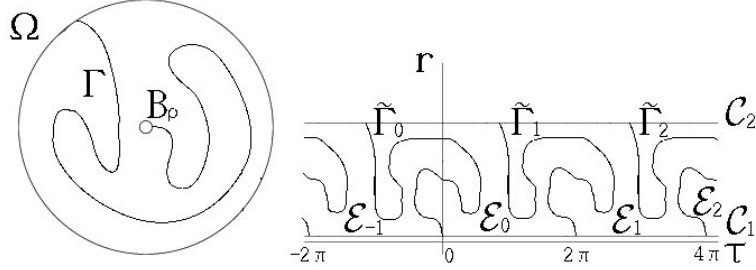


FIGURE 4. A domain and a spiral by the polar coordinate.

denote by \mathcal{E}_k . We define $\tilde{\varphi}: \bar{\mathcal{D}} \setminus \tilde{\Gamma} \rightarrow \mathbb{R}$ by

$$\tilde{\varphi}(r, \tau) = \tau - 2\pi k \quad \text{if } (r, \tau) \in \mathcal{E}_k$$

for $k \in \mathbb{Z}$, and define

$$\varphi(x) = \tilde{\varphi}(|x|, \text{Arg}(x)),$$

where $\text{Arg}(x) \in (-\pi, \pi]$ is the principal value of the argument of $x \in \mathbb{R}^2 \setminus \{0\}$.

By the definition of φ we obtain $\varphi - \arg x \equiv 0 \pmod{2\pi\mathbb{Z}}$. Therefore it suffices to see that $\varphi \in C(\bar{W} \setminus \Gamma)$. Since it is easy to see that φ is continuous in $\bar{W} \setminus (\Gamma \cup \mathcal{L}_0)$, where $\mathcal{L}_0 = (-\infty, 0) \times \{0\}$, it suffices to see the continuity of φ on $\mathcal{L}_0 \cap \bar{W} \setminus \Gamma$.

To prove a continuity of φ , we first verify that

$$(4.3) \quad (\hat{r}, \hat{\tau} \pm 2\pi) \in \mathcal{E}_{k \pm 1} \quad \text{if } (\hat{r}, \hat{\tau}) \in \mathcal{E}_k.$$

Since the proof is parallel, we only prove that $(\hat{r}, \hat{\tau} + 2\pi) \in \mathcal{E}_{k+1}$. Since \mathcal{E}_k is connected, there exists a continuous curve

$$\mathcal{C} = \{\tilde{Q}(s) \in \bar{\mathcal{E}}_k \setminus \tilde{\Gamma}; s \in [0, 1], \tilde{Q}(0) = (\hat{r}, \hat{\tau}), \tilde{Q}(1) = (\rho, (2k+1)\pi)\}.$$

Let us consider a curve

$$\mathcal{C}_+ = \{\tilde{Q}(s) + (0, 2\pi) \in \bar{\mathcal{D}}; s \in [0, 1]\}.$$

If we assume that $\tilde{Q}(1) = (\hat{r}, \hat{\tau} + 2\pi) \notin \mathcal{E}_{k+1}$, then there exists $s_0 \in (0, 1)$ such that $\tilde{Q}_0 := \tilde{Q}(s_0) + (0, 2\pi) \in \tilde{\Gamma} \cap \mathcal{C}_+$, since $\tilde{Q}(0) + (0, 2\pi) = (0, (2k+3)\pi) \in \mathcal{E}_{k+1}$. This yields that $\tilde{Q}_0 \in \tilde{\Gamma} \cap \mathcal{C}$ by the definition of $\tilde{\Gamma}$. This contradicts to the definition of \mathcal{C} .

Let $x \in \mathcal{L}_0 \cap \bar{W} \setminus \Gamma$. Since there exists $r > 0$ such that $B_r(x) \cap \Gamma = \emptyset$, there exists \mathcal{E}_k such that

$$\tilde{B}_+(x) \cup \tilde{B}_-(x) \subset \mathcal{E}_k,$$

where

$$\begin{aligned}\tilde{B}_\pm(x) &= \{(|y|, \text{Arg}(y)) \in \overline{\mathcal{D}}; y \in B_r(x)_\pm\}, \\ B_r(x)_+ &= \{y = (y_1, y_2) \in B_r(x); y_2 \geq 0\}, \\ B_r(x)_- &= \{y = (y_1, y_2) \in B_r(x); y_2 < 0\}.\end{aligned}$$

By the definition of φ and $\text{Arg}(x)$ we obtain

$$\lim_{y \rightarrow x} \varphi(y) = \text{Arg}(x) - 2\pi k = \varphi(x).$$

□

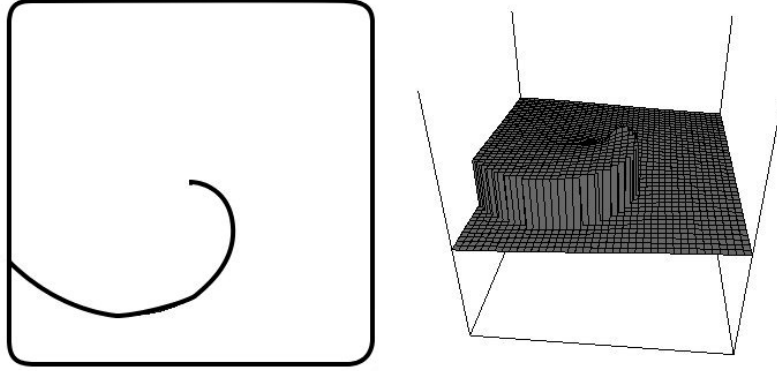


FIGURE 5. Sample graph of initial datum for a principal spiral.

Remark 4.2. (i) We can extend a principal spiral Γ on \overline{W} to a principal spiral S on $B_R(a) \setminus B_\rho(a)$ such that $S \supset \Gamma$, where R is large enough so that $B_R(a) \supset \overline{W}$. Therefore we can define the function $\varphi(x + h\vec{n}(x))$ for $x \in \Gamma$ and $h > 0$ small enough.

(ii) We observe that

$$(4.4) \quad \lim_{h \rightarrow +0} \varphi(x + h\vec{n}(x)) + 2\pi = \lim_{h \rightarrow +0} \varphi(x - h\vec{n}(x))$$

for $x \in \Gamma$. In fact, for enough small $h > 0$, the curves

$$\begin{aligned}\tilde{C}_\pm &= \{(|P_\pm^h(s)|, \varphi(P_\pm^h(s))) \in \overline{\mathcal{D}}; s \in [0, \ell]\}, \\ P_\pm^h(s) &= P(s) \pm h\vec{n}(P(s))\end{aligned}$$

give continuous curves on $\overline{\mathcal{D}}$, where $P(s)$ is a map such that $\Gamma = \{P(s); s \in [0, \ell]\}$. By (4.3) we have $\tilde{C}_- \subset \mathcal{E}_{-1}$ and $\tilde{C}_+ \subset \mathcal{E}_0$, which yields (4.4).

4.2. Sheet structure function for several spiral. We next consider the case that there exist two screw dislocations. In this case we can find two types of a bunch of spirals, two principal spirals or a connecting spiral. In this case we can make φ as in the Step 1 of how to make an initial datum by combining Lemma 4.1.

Lemma 4.3. (i) *Let $\Gamma = \Gamma_1 \cup \Gamma_2$ with principal spirals Γ_1 on $\overline{W_1}$ and Γ_2 on $\overline{W_2}$, respectively. Assume that $\Gamma_1 \cap \Gamma_2 = \emptyset$. Then there exists $\varphi \in C^\infty(W \setminus \Gamma) \cap C(\overline{W} \setminus \Gamma)$ satisfying*

$$\varphi(x) - (m_1 \arg(x - a_1) + m_2 \arg(x - a_2)) \equiv 0 \pmod{2\pi\mathbb{Z}},$$

for $x \in \overline{W} \setminus \Gamma$, where m_1 and m_2 are constants satisfying $|m_j| = 1$ for $j = 1, 2$.

(ii) *Let Γ be a connecting spiral between a_1 and a_2 . Then there exists $\varphi \in C^\infty(W \setminus \Gamma) \cap C(\overline{W} \setminus \Gamma)$ satisfying*

$$\varphi(x) - m_1(\arg(x - a_1) - \arg(x - a_2)) \equiv 0 \pmod{2\pi\mathbb{Z}},$$

for $x \in \overline{W} \setminus \Gamma$, where m_1 is a constant satisfying $|m_1| = 1$.

Proof. Let φ_j be a function on $\overline{W_j} \setminus \Gamma_j$ obtained by applying Lemma 4.1, where $W_j = \Omega \setminus \overline{B_{\rho_j}(a_j)}$ for $j = 1, 2$. We define

$$\varphi(x) := m_1\varphi_1(x) + m_2\varphi_2(x)$$

for $x \in \overline{W} \setminus (\Gamma_1 \cap \Gamma_2)$. For the proof of (ii), we take $m_2 = -m_1$ and define

$$\varphi(x) := \lim_{y \rightarrow x} \varphi(x)$$

for $x \in \Gamma_1 \cap \Gamma_2$. It is easy to see that this is a desired function by direct calculation with Remark 4.2 (ii). \square

We finally consider the situation that there exist N times screw dislocations. Let Γ be a bunch of spirals, which is described by

$$\Gamma = \left(\bigcup_{j=1}^N \bigcup_{k=1}^{M_j} \Gamma_{j,k} \right) \cup \bigcup_{n=1}^{M_c} \Gamma_n,$$

where $M_j \in [0, |m_j|/m] \cap \mathbb{N}$, $M_c \in \mathbb{N}$, $\Gamma_{j,k}$ are principal spirals, and Γ_n are connecting spirals satisfying

- (i) $\Gamma_{j,k} \cap \Gamma_{l,n} = \emptyset$ if $j \neq l$ or $k \neq n$,
- (ii) $\Gamma_{j,k} \cap \Gamma_n = \emptyset$ for any j, k, n .

To clarify the orientation of Γ we see curves in some neighborhood of $B_{\rho_j}(a_j)$. Then, we find $\ell_j \in \mathbb{Z}$ pieces of principal spirals which touch $\partial B_{\rho_j}(a_j)$. For these principal spirals we define the unit normal vector field \vec{n} by (4.2), where $P(s)$ in (4.2) is a map to a curve whose direction is as (4.2). We now assume that the orientations of the normal velocity \vec{n} for pieces of principal spirals in a neighborhood of $B_{\rho_j}(a_j)$ are same. Then, we have $\vec{n} = \pm \vec{n}$. We now set

$$m_j = \begin{cases} \ell_j & \text{if } \vec{n} = \vec{n} \text{ on } \partial B_{\rho_j}(a_j), \\ -\ell_j & \text{otherwise.} \end{cases}$$

and $\theta(x) = \sum_{j=1}^N m_j \arg(x - a_j)$. We note that, if there exists a connecting spiral Γ between a_j and a_k , then $\text{sgn}(m_j) = -\text{sgn}(m_k)$. In fact, let P_j and P_k be maps for the definition of principal spirals in a neighborhood of $B_{\rho_j}(a_j)$ and $B_{\rho_k}(a_k)$, respectively. If we extend maps P_j and P_k such that they describe Γ completely,

then we have $P_j(s) = P_k(\ell - s)$ for $s \in [0, \ell]$, where ℓ is the length of Γ . Therefore we have $\vec{n}_j(x) = -\vec{n}_k(x)$ for $x \in \Gamma$, where \vec{n}_j and \vec{n}_k are unit normal vector fields of Γ defined by (4.2) with P_j and P_k , respectively.

By applying the method as in Lemma 4.1 and 4.3 (ii), we obtain

Theorem 4.4. *Let $\Gamma_{j,k}$ for $j = 1, \dots, N$ and $k = 1, \dots, M_j$, and Γ_n for $n = 1, \dots, M_c$ be on above. Let*

$$\Gamma = \left(\bigcup_{j=1}^N \bigcup_{k=1}^{M_j} \Gamma_{j,k} \right) \cup \bigcup_{n=1}^{M_c} \Gamma_n.$$

Then there exists $\varphi \in C^\infty(W \setminus \Gamma) \cap C(\overline{W} \setminus \Gamma)$ satisfying

$$\varphi(x) - \sum_{j=1}^N m_j \arg(x - a_j) \equiv 0 \pmod{2\pi\mathbb{Z}}.$$

Proof. Let $\varphi_{j,k}$ be a function on $\overline{W} \setminus \Gamma_{j,k}$ which is constructed in Lemma 4.1. For a connecting spiral Γ_n between a_j and a_ℓ we construct φ_n on $\overline{W} \setminus \Gamma_n$ by Lemma 4.3 (ii) satisfying

$$\varphi_n(x) \equiv \operatorname{sgn}(m_j) \arg(x - a_j) + \operatorname{sgn}(m_\ell) \arg(x - a_\ell) \pmod{2\pi\mathbb{Z}}.$$

By using $\varphi_{j,k}$ and φ_n we define

$$\varphi(x) = \sum_{j=1}^N \sum_{k=1}^{M_j} \operatorname{sgn}(m_j) \varphi_{j,k}(x) + \sum_{n=1}^{M_c} \varphi_n(x).$$

It is easy to see φ is a desired function. □

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(Shun'ichi Goto) HOKKAIDO UNIVERSITY OF EDUCATION, SAPPORO CAMPUS, AINOSATO 5-3-1, SAPPORO 002-8502, JAPAN

E-mail address: gotos@sap.hokkyodai.ac.jp

(Maki Nakagawa) DEPARTMENT OF COMPUTATIONAL SCIENCE, FACULTY OF SCIENCE, KANAZAWA UNIVERSITY, KANAZAWA 920-1192, JAPAN(FORMER ADDRESS)

(Takeshi Ohtsuka) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY, SOPHIA UNIVERSITY, 7-1, KIOI-CHO, CHIYODA-KU, TOKYO 102-8554, JAPAN

E-mail address: ohtsuka@mm.sophia.ac.jp