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EVOLUTION EQUATIONS WITH ALMOST PERIODIC INITIAL DATA

Dedicated to Professor Hitoshi Ishii on the occasion of his 60th birthday

YOSHIKAZU GIGA
Graduate School of Mathematical Sciences
University of Tokyo, Komaba 3-8-1
Meguro, Tokyo, 153-8914, Japan

Abstract. It is rather clear that the solution is periodic if it is initially periodic provided that the evolution equation under study is well-posed and translation invariant. It is less obvious to see that almost periodicity is preserved under slightly stronger assumptions. We are interested in evolution of the frequency set when initial data is almost periodic. We consider such a problem for the Navier-Stokes equations and other evolution equations. A typical result is that an additive group generated by frequency set called module is contained in the module of initial data. We propose a method embedding almost periodic problems to periodic problems which yields such a result.

1 Introduction

We consider the Cauchy problem for an abstract evolution equation :

$$\frac{du}{dt} = A[u]\ ,\ u|_{t=0} = u_0, \tag{1.1}$$

where $u_0 = u_0(x)$ is defined on $\mathbf{R}^N$. It is rather clear that the solution $u(t) = u(t,x)$ is spatially periodic that (1.1) is invariant under translation in x and admits at most one solution.

In this note we are concerned with persistency of almost periodicity instead of periodicity. The problem is under what condition the solution u of (1.1) is almost periodic in x if u_0 is almost periodic. This is less obvious compared with periodic case. Roughly speaking, a typical sufficient condition for persistency is that the solution of (1.1) is continuous with respect to initial data in addition to the translation invariance of (1.1). We shall explain properties by giving examples of equations. Sometimes, we are forced to consider function spaces for u_0 smaller than $\mathrm{BUC}(\mathbf{R}^N)$, the space of all bounded, uniformly continuous functions in which classical almost periodic functions in the sense of Bochner and Bohr live. In this case extra work is necessary.

If u_0 is almost periodic, as well known it has a Fourier expansion, *i.e.*, $u_0 \sim \sum_{\lambda \in \Lambda} \alpha_\lambda e^{i\lambda x}$, where Λ is a countable set called a *frequency set* and α_λ is a complex (possibly vector-valued) amplitude. The element of Λ is called an *exponent* or *frequency*. See [AG], [Co] and [F] for details. We are interested in the frequency set $\Lambda(u(t))$ at time t. Let $< \Lambda >$ be an additive group generated by Λ. This set $< \Lambda >$ for the frequency set Λ is often called a *module* of u_0 and is denoted by $\mathrm{Mod}(u_0)$(cf.[F]). We shall give a sufficient condition such that $\mathrm{Mod}(u(t)) \subset \mathrm{Mod}(u_0)$ by embedding the problem to a periodic problem in a higher dimensional space.

Here is a rough idea. Suppose that the initial data $u_0 = a_1 e^{i\lambda_1 x} + a_2 e^{i\lambda_2 x}$ and λ_1 and λ_2 are linearly independent over the field $\mathbf{Q}$ of rational numbers so that u_0 is not periodic. We consider $U_0(x,y) = a_1 e^{i\lambda_1 x} + a_2 e^{i\lambda_2 y}$ and solve the Cauchy problem associated with (1.1) so that the solution $U(x,y,t)$ yields a solution u of (1.1) by setting $u(x,t) = U(x,x,t)$. Since U is expected to be periodic, $U(x,y,t)$ must be almost periodic and $< \Lambda(u(t)) > \subset < \Lambda(u_0) >$

; *i.e.*, $\mathrm{Mod}(u(t)) \subset \mathrm{Mod}(u_0)$. It turns out that such inclusion itself follows from classical properties of the module inclusion in [[F], Theorem 4.5] under the same condition of the persistency of almost periodicity. However, our approach provides a method to reduce an almost periodic problem to a periodic problem which is potentially useful if the original problem (1.1) is difficult to solve for general bounded data u_0 defined in $\mathbf{R}^N$. We are grateful to Professor Hitoshi Ishii for pointing out the reference [F].

2 Persistency of almost periodicity

We begin by recalling a definition of almost periodicity. For a function f defined in $\mathbf{R}^N$ let τ_ξ with $\xi \in \mathbf{R}^N$ denote a shift operator defined by

$$(\tau_\xi f)(x) = f(x+\xi) \ , \ x \in \mathbf{R}^N.$$

We say that $f \in \mathrm{BUC}(\mathbf{R}^N)$ is *almost periodic* (in the sense of Bochner) if $\Sigma_f := \{\tau_\xi f | \xi \in \mathbf{R}^N\}$ is relatively compact in $\mathrm{BUC}(\mathbf{R}^N)$ ($\subset L^\infty(\mathbf{R}^N)$) equipped with the supremum norm. The totality of almost periodic functions is denoted by $\mathrm{AP}(\mathbf{R}^N)$. By definition it is clear that $\mathrm{AP}(\mathbf{R}^N)$ is a closed subspace of $\mathrm{BUC}(\mathbf{R}^N)$.

The next proposition is easy to prove but fundamental to prove the persistency of almost periodicity.

Proposition 2.1. *Let S be a continuous mapping from $\mathrm{BUC}(\mathbf{R}^N)$ into itself. Assume that S commutes with all translations τ_ξ , $\xi \in \mathbf{R}^N$. Then S maps from $AP(\mathbf{R}^N)$ into $AP(\mathbf{R}^N)$.*

Proof. Since $S\tau_\xi = \tau_\xi S$, S maps from Σ_f into Σ_{Sf}. By continuity S maps $\overline{\Sigma}_f$ into $\overline{\Sigma}_{Sf}$. If $\overline{\Sigma}_f$ is compact, then the image by a continuous mapping is also compact so $\overline{\Sigma}_{Sf}$ is compact. Thus S maps $\mathrm{AP}(\mathbf{R}^N)$ into $\mathrm{AP}(\mathbf{R}^N)$.

In practice we apply this property of S for the solution operator $S(t)$ of (1.1) *i.e.*, $u(t,x) = S(t)u_0$ is the solution of (1.1) to get the persistency of almost periodicity. Here is a direct application of Proposition 2.1.

Proposition 2.2. *Assume that (1.1) is translation invariant and admits a unique solution $u(t) = u(t,\cdot) \in \mathrm{BUC}(\mathbf{R}^N)$ for $t > 0$ and $u_0 \in AP(\mathbf{R}^N)$. If $u(t)$ depends on u_0 continuously in $\mathrm{BUC}(\mathbf{R}^N)$, then $u(t)$ is almost periodic i.e., $u(t) \in AP(\mathbf{R}^N)$.*

For applications to PDEs or more general evolution equations we encouter at least two technical difficulties listed below.

(a) The problem (1.1) may not be well-posed in $\mathrm{BUC}(\mathbf{R}^N)$. It may be solvable in a smaller spaces.

(b) The solution of (1.1) may be local in time.

To overcome (a) we define a special class of function spaces.

Definition 2.3. Let X be a Banach space contained in $\mathrm{BUC}(\mathbf{R}^N)$. Assume that the inclusion $X \subset \mathrm{BUC}(\mathbf{R}^N)$ is continuous and that the norm in X is translation invariant. We say that X is *admissible* if the convergence of $\{\tau_{\xi_j} f\}_{j=1}^{\infty}$ in $\mathrm{BUC}(\mathbf{R}^N)$ as $j \to \infty$ implies its convergence in X for $f \in X$, where $\{\xi_j\}_{j=1}^{\infty} \subset \mathbf{R}^N$.

Evidently, the compatness of $\overline{\Sigma}_f$ in X and $\mathrm{BUC}(\mathbf{R}^N)$ is equivalent if X is admissible. The space $\mathrm{AP}_X(\mathbf{R}^N) := \mathrm{AP}(\mathbf{R}^N) \cap X$ is a closed subspace of X. The next lemma is as trivial variant of Proposition 2.1. It is stated to overcome the difficulty (a).

Lemma 2.4. *Let X be an admissible subspace of $\mathrm{BUC}(\mathbf{R}^N)$. Assume that S is continuous from X into $\mathrm{BUC}(\mathbf{R}^N)$ and that S commutes with all translations. Then S maps from $\mathrm{AP}_X(\mathbf{R}^N)$ into $\mathrm{AP}(\mathbf{R}^N)$.*

To overcome (b) we need a kind of stability of solutions. In an abstract setting the situation (b) corresponds to the case that S is defined only on Σ_f.

Lemma 2.5. *Let X be an admissible subspace of $\mathrm{BUC}(\mathbf{R}^N)$. (The space X can be $\mathrm{BUC}(\mathbf{R}^N)$.) For $f \in \mathrm{AP}_X(\mathbf{R}^N)$ assume that S maps Σ_f into $\mathrm{BUC}(\mathbf{R}^N)$ and that S is continuous on Σ_f in the topology of X. Assume that $S\tau_\xi f = \tau_\xi Sf$ for all $\xi \in \mathbf{R}^N$. If S is continuously extendable to $\overline{\Sigma}_f$, then Sf is almost periodic.*

This is a variant of Lemma 2.4 and is stated to overcome the difficulty (b).

Remark 2.6. In above statement X and $\mathrm{BUC}(\mathbf{R}^N)$ can be interpreted as either a complex or a real Banach space. It may consists of vector-valued functions. Unless it causes any confusion we rather write $\mathrm{BUC}(\mathbf{R}^N)$ instead of writing $\mathrm{BUC}(\mathbf{R}^N, \mathbf{R}^m)$ or $(\mathrm{BUC}(\mathbf{R}^N))^m$ even if it consists of $\mathbf{R}^m$-valued functions.

3 Example

We give a couple of important classes of equations to which results in Section 2 applies.

3.1 Levet set equations

We consider a second order evolution equation of $u = u(t,x)$ of the form

$$u_t + F(\nabla u\ ,\ \nabla^2 u) = 0 \text{ in } (0,\infty)\times \mathbf{R}^N. \tag{3.1}$$

Here we assume that

(i) F is a real-valued continuous function in $(\mathbf{R}^N\backslash\{0\})\times \mathbf{S}^N$, where $\mathbf{S}^N$ denotes the space of all real symmetric matrices.

(ii) F is *degenerate elliptic* in the sense that $F(p,X) \geq F(p,Y)$ whenever $X \leq Y$ for all $p \in \mathbf{R}^N\backslash\{0\}$. Here $X \leq Y$ means that $Y - X$ is a nonegative definite matrix, i.e. ${}^t\xi(Y-X)\xi \leq 0$ for all $\xi \in \mathbf{R}^N$.

(iii) F is *geometric* in the sense that

$$F(\lambda p\ ,\ \lambda X + \sigma p\otimes p) = \lambda F(p,X)$$

$$\text{for all } \lambda > 0\ ,\ \sigma\in\mathbf{R}\ ,\ p\in\mathbf{R}^N\backslash\{0\}\ ,\ X\in\mathbf{S}^N.$$

This class of equations is closely related to a parabolic surface evolution equation

$$V = f(\mathbf{n}\ ,\ -\mathbf{A}) \tag{3.2}$$

for an evolving hypersurface $\{\Gamma_t\}_{t>0}$ in $\mathbf{R}^N$, where V is the normal velocity of Γ_t in the direction of the unit normal $\mathbf{n}$ of Γ_t and $\mathbf{A}$ is the second fundamental form of Γ_t; f is a given function. Typical example includes the mean curvature flow equation $V = \mathrm{tr}\mathbf{A}$ as well as a wave front equations $V = 1$. If we require the equation (3.1) so that *each* level set of u solves (3.2), F is of the form

$$F(p,X) = -|p|f(-p, -Q_p(X)/|p|) \tag{3.3}$$

$$Q_p(X) = R_pXR_p\ ,\ R_p = I - p\otimes p/|p|^2$$

with an orientation convention $\mathbf{n} = -p/|p|$. As observed in [GG] we are able to say more.

Lemma 3.1. *If F fulfills (i)-(iii), there is unique continuous f satisfying (3.3) so that (3.2) is a (degenerate) parabolic equation. Conversely, if f is continuous and (3.2) is degenerate parabolic, F determined by (3.2) fulfills (i)-(iii).*

This result justifies to say that (3.1) satisfying (i)-(iii) is the *level set equation* of a (degenerate) parabolic surface evolution equation (3.2).

A level set equation is by now a standard tool to extend the solution Γ_t of surface evolution equations after it develops singularities. Its analytic foundation is established by [CGG] and [ES] in 1991 by adjusting the notion of viscosity solutions. Note that the parabolicity of (3.1) is always degenerate in the direction of ∇u by the geometricity of F so a smooth solution is not expected to exist even if initial data is smooth. Thus we need a weak notion of solution like a viscosity solution. By a further development of analysis including [IS] (see also [GG]) we now observe that (3.1) is solvable in BUC space. For details of the theory of level set equations the reader is refereed to a recent book [Gi], where the theory of viscosity solutions are also given. (As the reader may find, the extension to unbounded domain is one of important contributions of H. Ishii to the theory of viscosity solutions.)

Theorem 3.2. *For a given initial data $u_0 \in \mathrm{BUC}(\mathbf{R}^N)$ there is a unique viscosity solution $u \in \mathrm{BUC}([0,T] \times \mathbf{R}^N)$ of the level set equation (3.1) (satisfying (i)-(ii)) with $u(0,x) = u_0(x)$ for all $T > 0$. Let $S(t)$ be the solution operator, i.e., $u(t,x) = (S(t)u_0)(x)$. Then $S(t)$ is a contruction in the sense that*

$$||S(t)u_0 - S(t)v_0||_\infty \leq ||u_0 - v_0||_\infty$$

for all u_0 , $v_0 \in \mathrm{BUC}(\mathbf{R}^N)$, where $||f||_\infty$ denotes the norm in $\mathrm{BUC}(\mathbf{R}^N)$.

The unique existence of the viscosity solution is by now well establised ; see e.g. [Gi]. The contraction propoerty follows from a comparison principle. Indeed, since

$$u_0 \leq v_0 + ||u_0 - v_0||_\infty,$$

the comparison principle and commutativity of $S(t)$ with an additive constant yields

$$S(t)u_0 \leq S(t)(v_0 + ||u_0 - v_0||_\infty) = S(t)v_0 + ||u_0 - v_0||_\infty.$$

Changing the role of u_0 and v_0 yields

$$S(t)v_0 \leq S(t)u_0 + ||u_0 - v_0||_\infty.$$

We thus observe that $||S(t)u_0 - S(t)v_0||_\infty \leq ||u_0 - v_0||_\infty$. □

This is an ideal situation to apply Proposition 2.1 to $S(t)$ so that almost periodicity is preserved.

Corollary 3.3. *If u_0 is almost periodic, then the solution u of (3.1) with initial data u_0 is almost periodic in space variable for all $t > 0$.*

Remark 3.4. We are fully aware of the importance to know persistance of almost periodicity of the level set of u. However, we do not pursue this problem here in this paper.

3.2 Navier-Stoeks equations with the Coriolis force.

We consider the Navier-Stokes equations with or without the Coriolis force in $\mathbf{R}^3$:

$$u_t - \Delta u + (u, \nabla)u + \Omega e_3 \times u + \nabla p = 0 \quad \text{in} \quad (0,T) \times \mathbf{R}^3, \tag{3.4}$$

$$\mathrm{div} u = 0 \quad \text{in} \quad (0,T) \times \mathbf{R}^3, \tag{3.5}$$

with initial data

$$u(0,x) = u_0(x). \tag{3.6}$$

Here $e_3 = {}^t(0,0,1)$ and $\times$ denotes the exterior product of vectors in $\mathbf{R}^3$. As usual $(u, \nabla) = \sum_{i=1}^3 u^i \partial/\partial x_i$ and Ω is a real parameter. This equation is locally well-posed for $u_0 \in \mathrm{BUC}(\mathbf{R}^N)$ with $\mathrm{div} u_0 = 0$ when there is no Coriolis force, *i.e.*, $\Omega = 0$; see [GIM]. (The solvability for BUC_σ is not trivial unless one notices

$$||\nabla e^{t\Delta} P f||_\infty \leq C t^{-1/2} ||f||_\infty \tag{3.7}$$

with a constant C independent of t and f, where P is the Leray-Helmoltz projection *i.e.*, $P = (\delta_{ij} + R_i R_j)_{1 \leq i\ ,\ j \leq 3}$ with the Riesz operator R_i despite the fact that R_i is not bounded in $L^\infty(\mathbf{R}^N)$. A simple proof of (3.7) is given in [GJMY].) By choosing

$$X = \mathrm{BUC}_\sigma = \{f \in \mathrm{BUC}(\mathbf{R}^N) | \mathrm{div}\ f = 0\}$$

it is proved in [GMN] that the solution operator $S(t)$ fulfills the assumption of Lemma 2.5 ; in this case admissibility of X is trivial since X is a closed subspace of $\mathrm{BUC}(\mathbf{R}^N)$. Thus the almost periodicity is preserved.

Theorem 3.5 ([GMN]). *Assume that $\Omega = 0$. Assume that $u_0 \in \mathrm{BUC}_\sigma(\mathbf{R}^N)$. Then the local-in-time unique solution $u \in C([0,T], \mathrm{BUC}_\sigma(\mathbf{R}^N))$ of (3.4)-(3.6) is almost periodic in x provided that u_0 is almost periodic.*

Unfortunately, if $\Omega \neq 0$ then the problem is not well-posed in BUC_σ as proved in [GIMM1]. It is well-posed in a Besov space $\dot{B}^0_{\infty,1}$ ([S], [HS], [GIMM1]) and FM_0 space [GIMM2], [GIMS]. The latter space is the Fourier image of the space of all finite Radon measures having no point mass at the origin. For this space the existence time can be estimated from below uniformly in $\Omega \in \mathbf{R}$. The admissibility of $\dot{B}^0_{\infty,1}$ is proved in [GMN]. For FM_0 space its admissibility is explicitly proved in [GJMY]. We are able to apply Lemma 2.5 for this setting to conclude the persistence of almost periodicity.

Theorem 3.6([GMN], [GIMM2], [GJMY]). *Assume that either $u_0 \in \dot{B}^0_{\infty,1}(\mathbf{R}^N)$ or $u_0 \in FM_0(\mathbf{R}^N)$ with $\mathrm{div} u_0 = 0$ for $\Omega \in \mathbf{R}$. Then the local-in-time unique solution $u \in C([0,T]\ ,\ \dot{B}^0_{\infty,1})$ (or respectively $u \in C([0,T]\ ,\ FM_0)$ of (3.4)-(3.6) is almost periodic in x provided that u_0 is almost periodic.*

Remma 3.7. (i) It is an open problem to construct a global weak solution for (3.4)-(3.6) for nondecaying initial data ; only a loal-in-time weak solution is constructed in [L]. We do not know its existence even for non-periodic almost periodic initial data.
(ii) At least for $\Omega = 0$ it is proved in [MT] that almost periodicity is preserved even if initial data is an L^p-type almost periodic function not neccesarity continuous.

4 Embedding into periodic problems

We begin with definition of a frequency set of an almost periodic function. Let us first recall averaging operator for a function in $\mathbf{R}^N$:

$$M[h] = \lim_{R \to \infty} \frac{1}{|C_R|} \int_{C_R} h(x) dx,$$

where C_R is a cube of the form

$$C_R = \{x = (x_1, \cdots, x_N) | \ |x_i| \leq R, 1 \leq i \leq N\}.$$

If $f \in \mathrm{BUC}(\mathbf{R}^N)$ is almost periodic, its average $\mathrm{M}[f]$ always exists [Co].

Definition 4.1. Let $f \in \mathrm{BUC}(\mathbf{R}^N)$ be an almost periodic function. We say that $\lambda \in \mathbf{R}^N$ is a *frequency* or *exponent* of f if

$$\mathrm{M}[fe^{-i\lambda x}] = a_\lambda \neq 0.$$

The set of all frequencies is denoted by $\Lambda(f) \subset \mathbf{R}^N$ and is called the *frequency set* of f. The value a_λ is called a complex *amplitude* corresponding to λ. It is known that $\Lambda = \Lambda(f)$ is a countable set and f has an Fourier series type representation

$$f \sim \sum_{\lambda \in \Lambda} a_\lambda e^{i\lambda \cdot x}$$

as discussed in [Co] and [F]. Unfortunately, the right hand side may not converge in uniform sense. However, it is known (cf [Co], [F]) that its Fejér type sum converges uniformly to f if $f \in \mathrm{BUC}(\mathbf{R}^N)$. In particular, there is a series of trigonometric polynomial whose exponents are contained in $\Lambda(f)$ such that it converges to f in $\mathrm{BUC}(\mathbf{R}^N)$.

It is known [GJMY] that an almost periodic function f belong to $FM(\mathbf{R}^N)$ if and only of $f = \sum_{\lambda \in \Lambda} a_\lambda e^{i\lambda x}$ with $\sum_{\lambda \in \Lambda} |a_\lambda| < \infty$.

Definition 4.2. Let $< \Lambda >$ denote the additive group generated by $\Lambda \subset \mathbf{R}^N$, *i.e.*, $< \Lambda >$ is the smallest additive group containing Λ. We often call $< \Lambda(f) >$ a *module* of f and denote it by $\mathrm{Mod}(f)$. Since $\Lambda(f)$ is coutable, there is a countable set $\{\lambda_j\}_{j=1}^\infty$ which generates $\mathrm{Mod}(f)$ (We may take such a set linearly independent over the field of rational numbers.) We call $\{\lambda_j\}_{j=1}^\infty$ is a *generating system* of $\mathrm{Mod}(f)$. If there is a generating system consisting of finite elements, $\mathrm{Mod}(f)$ is called finitely generated.

Theorem 4.3. *Let S be a continuous mapping from $\mathrm{BUC}(\mathbf{R}^N)$ into itself. Assume that S commutes with all translations. Then $\mathrm{Mod}(Sf) \subset \mathrm{Mod}(f)$ is $f \in \mathrm{AP}(\mathbf{R}^N)$. In other words a generating system of $< \Lambda(Sf) >$ is contained in that of $< \Lambda(f) >$.*

We first give a proof by embedding this problem to a periodic problem.

Proof. We may assume that $< \Lambda(f) >$ is finitely generated by approximation. Assume that $\{\lambda_j\}_{j=0}^l$ is a generating system of $< \Lambda(f) >$. We may assume that f is a trigonometric polynomial of the form by approximation:

$$f(x) = \sum_{|m| \leq K} a_m \exp(\sum_{j=0}^{l} m_j \lambda_j \cdot x),$$

where $m = (m_1, \cdots, m_l) \in \mathbf{Z}^{l+1}$ and $|m| = \max_{0 \leq j \leq l} |m_j|$. We associate a periodic function F of $N(l+1)$ variables of the form

$$F(x, y) = \sum_{|m| \leq K} a_m \exp(m_0 \lambda_0 \cdot x) \exp(\sum_{j=1}^{l} m_j \lambda_j \cdot y_j)$$

where $y = (y_1, \cdots, y_l) \in \mathbf{R}^{Nl}$. Evidently, $F(x, x, \cdots, x) = f(x)$. By definition F is periodic in the direction of $\sigma_j = (\delta_{ij} \lambda_j)_{i=1}^{l+1} = (0, \cdots, 0, \lambda_j, 0, \cdots, 0)$ and constant orthogonal to the space spanned by $\sigma_0, \cdots, \sigma_l$ in $\mathbf{R}^{(l+1)}$. It is convenient to introduce new independent variables

$$z = x, \quad \eta_j = y_j - x \quad (1 \leq j \leq l).$$

The function F is expressed as

$$\overline{f}(z, \eta) := \sum_{|m| \leq K} a_m \exp(\sum_{j=0}^{l} m_j \lambda_j \cdot z) \exp(\sum_{j=1}^{l} m_j \lambda_j \cdot \eta_j) \ (= F(x, y)),$$

where $\eta = (\eta_1, \cdots, \eta_l) \in \mathbf{R}^{Nl}$ so that

$$\overline{f}(z, 0) = f(z).$$

Of course, F is periodic and its fundamental domain is not rectangle but a parallerogram. Let $\overline{S}$ be the operator acting on $\mathrm{BUC}(\mathbf{R}^{N(l+1)})$ defined by

$$(\overline{S}h)(z, \eta) := (Sh(\cdot, \eta))(z),$$

where $h \in \mathrm{BUC}(\mathbf{R}^{N(l+1)})$ is regarded as a family $\{h(\cdot, \eta)\}$ of elements of $\mathrm{BUC}(\mathbf{R}^N)$ parametrized by η. Since S commutes with tranlation in $\mathbf{R}^N$, $\overline{S}$ commutes with all translations in $\mathbf{R}^{N(l+1)}$. Thus the image $\overline{S}\ \overline{f}$ of periodic function $\overline{f}$ by $\overline{S}$ is also periodic. The set of all periodic function $\overline{S}\ \overline{f}$ is contained in that of $\overline{f}$.

Since S is continuous in $\mathrm{BUC}(\mathbf{R}^N)$ and since $\overline{f}(\cdot,\eta)$ is continuous in η with values in $\mathrm{BUC}(\mathbf{R}^N)$, $\overline{S}\ \overline{f}$ is in $\mathrm{BUC}(\mathbf{R}^{N(l+1)})$. Using the variables (x,y) we see that $\overline{S}\ \overline{f}$ has a Fourier series expansion of the form

$$\overline{S}\ \overline{f}(x, y_1 - x, \cdots, y_l - x) \sim \sum_{m \in \mathbf{Z}^{l+1}} b_m \exp(m_0 \lambda_0 \cdot x) \exp(\sum_{j=1}^{l} m_j \lambda_j \cdot y_j).$$

Precisely, the left hand side is uniform limit of Fejér type trigonometric polynomial whose periods are the same as F. If one restricts $\overline{S}\ \overline{f}(x, y_1 - x, \cdots, y_l - x)$ on $\mathbf{R}^N$ by setting $y_1 = \cdots = y_l = x$, we obtain a Fourier type approximation of almost periodic functions whose exponents are contained in $< \{\lambda_j\}_{j=0}^{l} >$. The proof is now complete. $\square$

It turns out that this can be proved in a short way if one admits several properties of almost periodidc functions. We give here a shorter proof.

Alternate proof. In [F] it is shown among other results that
(a) $\mathrm{Mod}(f) \supset \mathrm{Mod}(g)$ is equivalent to that

is equivalent to that

(b) the convergence of $\tau_{\xi_j} f$ in $\mathrm{BUC}(\mathbf{R}^N)$ as $j \to \infty$ provided that $f, g \in \mathrm{AP}(\mathbf{R}^N)$.

It is stated in [[F], Theorem 4.5] for $N = 1$; however, its extension to higher dimension is straightforward. We take $g = Sf$. By commutativity of τ_ξ and S and continuity of S it is straightforward to see that (b) holds. Thus $\mathrm{Mod}(f) \subset \mathrm{Mod}(Sf)$ which is the desired result. $\square$

Remark 4.4. (i) From the alternate proof it is rather clear to observe that a similar statement holds for $f \in \mathrm{AP}_X(\mathbf{R}^N)$ even if S is continuous from X into $\mathrm{BUC}(\mathbf{R}^N)$ provided that X is admissible. To carry out the first proof in addition to these assumptions it suffices to assume that $\overline{f}(\cdot,\eta)$ is continuous in η with values in X; we do not know whether our proof actually requires such an extra assumption.
(ii) Although the second proof is more transparent, the idea to embedd an almost periodic function to a periodic function in higher dimension is usuful especially when it is very difficult to solve original problems (1.1) for

general BUC($\mathbf{R}^N$) initial data. Our procedure reduces the problem to periodic boundary problems with expense of higher dimensionality as well as the degeneracy of the equation in η direction.

Remark 4.5. (i) It is clear that the solution operator $S(t)$ in Theorem 3.2 fulfills the assumption of Theorem 4.3 to conclude $< \Lambda(S(t)u_0 > \subset < \Lambda(u_0) >$. The idea to embedd to a periodic problem seems to be useful to solve a degenerate parabolic problem for almost periodic initial data without knowing the solvability for general BUC initial data.
(ii) For the Navier-Stokes equations with or without the Coriolis force by a structure of nonlinearity one is able to claim a stronger statement for the solution operator $S = S(t)$ than stated in Theorem 4.3 (with modification by Remark 4.4 and by Lemma 2.5). In fact, in [GJMY] it is proved that

$$< \Lambda(S(t)u_0) >_{\mathrm{semi}} \subset < \Lambda(u_0) >_{\mathrm{semi}},$$

where $< \Lambda >_{\mathrm{semi}}$ denotes the additive semigroup generated by Λ.

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