ON THE DOUBLE CRITICAL-STATE MODEL FOR TYPE-II SUPERCONDUCTIVITY IN 3D *

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Abstract. In this paper we mathematically analyse an evolution variational inequality which formulates the double critical-state model for type-II superconductivity in 3D space and propose a finite element method to discretize the formulation. The double critical-state model originally proposed by Clem and Perez-Gonzalez is formulated as a model in 3D space which characterises the nonlinear relation between the electric field, the electric current, the perpendicular component of the electric current to the magnetic flux, and the parallel component of the current to the magnetic flux in bulk type-II superconductor. The existence of a solution to the variational inequality formulation is proved and the representation theorem of subdifferential for a class of energy functionals including our energy is established. The variational inequality formulation is discretized in time by a semi-implicit scheme and in space by the edge finite element of lowest order on a tetrahedral mesh. The fully discrete formulation is an unconstrained optimisation problem. The subsequence convergence property of the fully discrete solution is proved. Some numerical results computed under a rotating applied magnetic field are presented.

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1. Introduction

In this paper we analyse an evolution variational inequality mathematically and propose a finite element method to discretize the problem. The evolution variational inequality studied in this paper is a formulation of the double critical-state model for type-II superconductivity proposed by Clem and Perez-Gonzalez [12], [23], [24], [25] who developed the general critical-state theory by postulating that the electric field $\boldsymbol{\mathcal{E}}$ should be decomposed as

$$\boldsymbol{\mathcal{E}} = \rho_\perp \boldsymbol{\mathcal{J}}_\perp + \rho_\parallel \boldsymbol{\mathcal{J}}_\parallel, \tag{1.1}$$

where $\boldsymbol{\mathcal{J}}_\perp$ is the perpendicular component of the electric current density $\boldsymbol{\mathcal{J}}$ to the magnetic flux density $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{J}}_\parallel$ is the parallel component of the current density $\boldsymbol{\mathcal{J}}$ to $\boldsymbol{\mathcal{B}}$ and the resistivity $\rho_\perp$ and $\rho_\parallel$ satisfy the following relations:

$$\begin{aligned} &\text{If } |\boldsymbol{\mathcal{J}}_\perp| < \mathcal{J}_{c\perp}, \Rightarrow \rho_\perp = 0, \\ &\text{if } |\boldsymbol{\mathcal{J}}_\parallel| < \mathcal{J}_{c\parallel}, \Rightarrow \rho_\parallel = 0, \end{aligned}$$

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for the corresponding critical values $\mathcal{J}_{c\perp}$ and $\mathcal{J}_{c\parallel}$. In the series of their work [12], [23], [24], [25] Clem and Perez-Gonzalez considered the situation where the rotating parallel magnetic field is applied to a superconducting infinite slab so that the model can be formulated in a 1D problem in space.

In [3], [4], [5] Badía and López included the original double critical state theory by Clem and Perez-Gonzalez in their unifying theoretical framework to investigate the critical state problems by revealing the variational structure of the model. Along with their variational formulation, Badía and López demonstrated analysis of the double critical state model in 1D infinite slab geometry.

In this paper we modify the double critical-state model (1.1) by adding the term $\rho_0\boldsymbol{\mathcal{J}}$ with the resistivity ρ_0, which vanishes if the magnitude $|\boldsymbol{\mathcal{J}}|$ is smaller than certain critical value $\mathcal{J}_{c0}$, and propose the model

$$\boldsymbol{\mathcal{E}} = \rho_\perp \boldsymbol{\mathcal{J}}_\perp + \rho_\parallel \boldsymbol{\mathcal{J}}_\parallel + \rho_0 \boldsymbol{\mathcal{J}}. \tag{1.2}$$

The additional term $\rho_0\boldsymbol{\mathcal{J}}$ makes the energy density deriving the constitutive relation (1.2) coercive with respect to the current $\boldsymbol{\mathcal{J}}$ so that a solution to the variational inequality formulation of the problem exists and the convex optimisation problem derived as the fully discrete formulation admits the existence of its unique minimiser. By taking the critical current density $\mathcal{J}_{c0}$ and the resistivity ρ_0 relatively large, we consider the term $\rho_0\boldsymbol{\mathcal{J}}$ as the appearance of the high resistivity after the jump from the superconducting state to the normal state. In our numerical simulations in Section 5, however, we will confirm that the resistivity ρ_0 always vanishes and the term $\rho_0\boldsymbol{\mathcal{J}}$ does not affect the nature of the model.

The numerical analysis of the variational inequality formulation of macroscopic critical-state models for type-II superconductivity was initiated by Prigozhin [26], [27]. Prigozhin proposed the subdifferential formulation of the Bean critical-state model [7], proved the well-posedness of the formulation in [26] and intensively studied the numerical simulations in 2D in [27]. Elliott, Kay and Styles [14], [15] established error estimates for their finite element approximation of the variational inequality formulation of the Bean model in 2D. Barrett and Prigozhin [6] derived the dual formulation of the Bean model in terms of the electric field as the conjugate variable to the magnetic field and proved the convergence property of a practical finite element approximation of their dual formulation. Recently Elliott and Kashima [13] reported a 3D finite element analysis of the variational inequality formulation of the critical-state models governing the magnetic field and the current density around a bulk type-II superconductor. For more on the preceding mathematical work concerning the critical-state models see the introduction of [13] and the references therein.

The subdifferential formulation of the Bean critical-state model requires a restriction that the electric field has to be always parallel to the electric current. Though this condition holds true in some geometric configurations where the component $\boldsymbol{\mathcal{J}}_\parallel$ is predicted to vanish such as in axially symmetric superconductors (see [27]) or superconducting thin films (see [28]) under perpendicular applied field, the redistribution of the pinned magnetic flux driven by Lorentz force may induce the electric field which is not parallel to the current in general 3D geometry. See [11] or [27] for the limitation of the Bean model argued from a modelling perspective.

In order to investigate the macroscopic behaviour of the electromagnetic fields around a bulk type-II superconductor in general 3D configurations, we need to employ a mathematical model which deals with the full vectorial character of the current $\boldsymbol{\mathcal{J}}$ and develop a numerical method for discretization of the model. Hence we are motivated to formulate the double critical-state model (1.1) in a 3D configuration and propose a finite element method to carry out its numerical simulations.

For a class of the nonlinear Ohm's laws where the electric field $\boldsymbol{\mathcal{E}}$ is treated as a subdifferential of convex energy density at $\boldsymbol{\mathcal{J}}$ such as the Bean model [7], the modified Bean model proposed by Bossavit [9] and the power-law model [29], the corresponding variational inequality formulation in terms of the unknown magnetic field is a gradient system driven by the subdifferential operator of the convex energy functional and the existence of a unique solution to the evolution variational inequality is immediately ensured by applying the unique solvability theorem of nonlinear evolution system by Brezis [10] (see also [26], [13] for the proof of the well-posedness of these formulations). In our formulation of the model (1.2), however, the current density needs to be decomposed into parallel and perpendicular components to the magnetic flux. Accordingly, the energy functional depends not only on the electric current but also on the magnetic flux and is not convex with

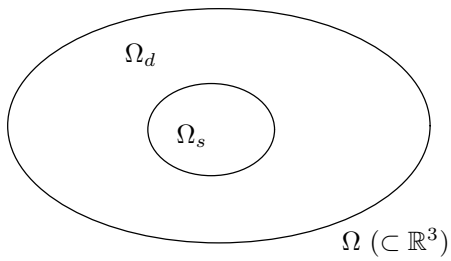

FIGURE 1. Geometry of the problem.

respect to the unknown magnetic field. The existence theorem by Brezis [10] does not apply in this case and the analysis for the existence of a solution differs from that of the preceding articles [26], [13]. We show the solvability by applying the Schauder fixed point theorem coupled with the unique existence theorem for nonlinear evolution system driven by time dependent subdifferentials (see [18], [20], [38]). We give the characterisation theorem of subdifferential for a class of energy functionals including the energy deriving our formulation and observe that Faraday's law and the nonlinear Ohm's law can be recovered in the superconductor in the sense of almost everywhere. The space discretization is carried out by means of the lowest order edge finite element by Nédélec [22] on a tetrahedral mesh. In the time discretization we employ a semi-implicit time-stepping scheme so the fully discrete formulation is an unconstrained minimisation problem. In order to handle the curl-free constraint imposed on the magnetic field outside the superconductor, we introduce a scalar magnetic potential and propose the magnetic field−scalar potential hybrid formulation, which is equivalent to the original formulation. This hybrid formulation was adopted to compute the nonlinear eddy current models in [13] by following [8].

The outline of this paper is as follows. In Section 2 we recall the eddy current models and the double critical-state model, formulate these models in evolution variational inequalities in terms of the unknown magnetic field and prove the existence of a solution to the formulations. In Section 3 we show the representation theorem of subdifferential operator in a general setting containing our case. In Section 4 we discretize our variational inequality formulation and prove the subsequence convergence property of the fully discrete solution. In Section 5 we report numerical simulations under a rotating applied magnetic field.

2. The models and the formulation

First we define the geometry. Throughout the paper the problem is analysed in a bounded simply connected Lipschitz domain Ω ($\subset \mathbb{R}^3$) with a connected boundary $\partial\Omega$. The bulk type-II superconductor Ω_s ($\subset \Omega$) is a simply connected Lipschitz domain with a connected boundary $\partial\Omega_s$ satisfying $\partial\Omega \cap \partial\Omega_s = \emptyset$. Let Ω_d denote the non-conducting region $\Omega \backslash \overline{\Omega_s}$. Note that in this setting Ω_d is simply connected (see Figure 1).

2.1. Maxwell equations

The electromagnetic fields are governed by the eddy current model, a version of Maxwell's equations with the displacement current neglected:

$$\partial_t \boldsymbol{\mathcal{B}} + \mathbf{curl}\, \boldsymbol{\mathcal{E}} = \mathbf{0} \text{ in } \Omega \times (0,T) \text{ (Faraday's law)}\ , \tag{2.1}$$

$$\mathbf{curl}\, \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}} \text{ in } \Omega \times (0,T) \text{ (Ampère's law)}\ , \tag{2.2}$$

$$\operatorname{div} \boldsymbol{\mathcal{B}} = 0 \text{ in } \Omega \times (0,T) \text{ (Gauss' law)}\ , \tag{2.3}$$

where $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{H}}$, $\boldsymbol{\mathcal{J}}$ denote the magnetic flux density, the electric field intensity, the magnetic field intensity, and the electric current density respectively and $\partial_t \boldsymbol{\mathcal{B}}$ denotes $\partial \boldsymbol{\mathcal{B}}/\partial t$. Note that Gauss' law (2.3) follows from Faraday's law (2.1) if the initial magnetic flux satisfies the divergence-free condition $\operatorname{div} \boldsymbol{\mathcal{B}}|_{t=0} = \mathbf{0}$ in Ω.

Since the region Ω_d is assumed to be non-conducting, the current $\boldsymbol{\mathcal{J}}$ vanishes in Ω_d:

$$\boldsymbol{\mathcal{J}} = \mathbf{0} \text{ in } \Omega_d \times (0,T). \tag{2.4}$$

With the piecewise constant magnetic permeability $\mu : \Omega \to \mathbb{R}$ defined by

$$\mu = \begin{cases} \mu_s & \text{in } \Omega_s, \\ \mu_d & \text{in } \Omega_d, \end{cases}$$

for positive constants $\mu_s, \mu_d > 0$, we assume the constitutive equation

$$\boldsymbol{\mathcal{B}} = \mu \boldsymbol{\mathcal{H}} \text{ in } \Omega \times (0,T). \tag{2.5}$$

We apply a time-varying external source magnetic field $\boldsymbol{\mathcal{H}}_s$ to the domain so that on the assumption that the boundary $\partial\Omega$ is far from the conductor Ω_s the magnetic field $\boldsymbol{\mathcal{H}}$ satisfies the boundary condition

$$\mathbf{n} \times \boldsymbol{\mathcal{H}} = \mathbf{n} \times \boldsymbol{\mathcal{H}}_s \text{ on } \partial\Omega \times (0,T), \tag{2.6}$$

where $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$.

We extend the external magnetic field $\boldsymbol{\mathcal{H}}_s$ into the inside of Ω. Since $\boldsymbol{\mathcal{H}}_s$ is induced by the source current supported outside the domain Ω, it satisfies

$$\mathbf{curl}\, \boldsymbol{\mathcal{H}}_s = \mathbf{0} \text{ in } \Omega \times [0,T]. \tag{2.7}$$

We introduce a new vector field $\widehat{\boldsymbol{\mathcal{H}}}$ by

$$\widehat{\boldsymbol{\mathcal{H}}} := \boldsymbol{\mathcal{H}} - \boldsymbol{\mathcal{H}}_s. \tag{2.8}$$

The boundary condition (2.6) yields the boundary condition for the field $\widehat{\boldsymbol{\mathcal{H}}}$:

$$\mathbf{n} \times \widehat{\boldsymbol{\mathcal{H}}} = \mathbf{0} \text{ on } \partial\Omega \times (0,T). \tag{2.9}$$

We assume that at the beginning of the time evolution the initial values $\widehat{\boldsymbol{\mathcal{H}}}_0$ of $\widehat{\boldsymbol{\mathcal{H}}}$ and $\boldsymbol{\mathcal{H}}_s(0)$ of $\boldsymbol{\mathcal{H}}_s$ satisfy the divergence-free condition:

$$\operatorname{div}(\mu\widehat{\boldsymbol{\mathcal{H}}}_0 + \mu\boldsymbol{\mathcal{H}}_s(0)) = 0 \text{ in } \Omega. \tag{2.10}$$

The condition (2.10) is satisfied, for example, if $\widehat{\boldsymbol{\mathcal{H}}}_0 \equiv \boldsymbol{\mathcal{H}}_s(0) \equiv \mathbf{0}$ in Ω, or $\mu_s = \mu_d$ and $\widehat{\boldsymbol{\mathcal{H}}}_0$, $\boldsymbol{\mathcal{H}}_s(0)$ are constant.

By combining (2.1)-(2.3), (2.5), (2.7), (2.8) we derive the following system:

$$\mu\partial_t\widehat{\boldsymbol{\mathcal{H}}} + \mu\partial_t\boldsymbol{\mathcal{H}}_s + \mathbf{curl}\,\boldsymbol{\mathcal{E}} = \mathbf{0} \text{ in } \Omega \times (0,T), \tag{2.11}$$

$$\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}} = \boldsymbol{\mathcal{J}} \text{ in } \Omega \times (0,T), \tag{2.12}$$

$$\operatorname{div}(\mu\widehat{\boldsymbol{\mathcal{H}}} + \mu\boldsymbol{\mathcal{H}}_s) = 0 \text{ in } \Omega \times (0,T). \tag{2.13}$$

Note that (2.13) can be also derived from (2.10) and (2.11).

We solve (2.11)-(2.13) in terms of $\widehat{\boldsymbol{\mathcal{H}}}$ under the initial boundary conditions (2.10) and (2.9). In order to close the system (2.11)-(2.13) we need a relation between $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{J}}$, which will be defined in the following subsection.

2.2. **The double critical-state model**

We define the critical-state $\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{J}}$ relation employed in this paper. Let the vector $\boldsymbol{\mathcal{J}}_\perp$ stand for the perpendicular component of $\boldsymbol{\mathcal{J}}$ to $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{J}}_\parallel$ stand for the parallel component of $\boldsymbol{\mathcal{J}}$ to $\boldsymbol{\mathcal{B}}$. We assume that if $|\boldsymbol{\mathcal{J}}_\perp|$ is smaller than the critical value $\mathcal{J}_{c\perp}$, then $\boldsymbol{\mathcal{J}}_\perp$ flows without resistivity, otherwise, the resistivity $\rho_\perp$ appears due to the magnetic flux depinning with energy dissipation. If $|\boldsymbol{\mathcal{J}}_\parallel|$ is smaller than the critical value

$\mathcal{J}_{c\|}$, then $\boldsymbol{\mathcal{J}}_\|$ flows without resistivity, otherwise, the resistivity $\rho_\|$ appears due to the flux line cutting with energy dissipation. These assumptions agree with the theory of the double critical-state model developed by Clem and Perez-Gonzalez [12], [23], [24], [25]. Moreover we assume that the resistivity ρ_0 appears if $|\boldsymbol{\mathcal{J}}|$ exceeds the critical value $\mathcal{J}_{c0}$. On these assumptions our nonlinear $\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{J}}$ relation is proposed as follows.

$$\boldsymbol{\mathcal{E}} = \rho_\perp \boldsymbol{\mathcal{J}}_\perp + \rho_\| \boldsymbol{\mathcal{J}}_\| + \rho_0 \boldsymbol{\mathcal{J}}, \tag{2.14}$$

where the resistivity $\rho_\perp, \rho_\|, \rho_0 \geq 0$ satisfy the relation

$$\rho_\perp = 0 \text{ if } |\boldsymbol{\mathcal{J}}_\perp| < \mathcal{J}_{c\perp},\ \rho_\| = 0 \text{ if } |\boldsymbol{\mathcal{J}}_\|| < \mathcal{J}_{c\|},\ \rho_0 = 0 \text{ if } |\boldsymbol{\mathcal{J}}| < \mathcal{J}_{c0}, \tag{2.15}$$

for the positive constants $\mathcal{J}_{c\perp}$, $\mathcal{J}_{c\|}$, $\mathcal{J}_{c0} > 0$.

The resistivity ρ_0 automatically vanishes if $|\boldsymbol{\mathcal{J}}| < \mathcal{J}_{c0}$ so the term $\rho_0 \boldsymbol{\mathcal{J}}$ does not affect the electric field $\boldsymbol{\mathcal{E}}$ in such region. By taking ρ_0 and $\mathcal{J}_{c0}$ relatively large, we consider that the term $\rho_0 \boldsymbol{\mathcal{J}}$ expresses the appearance of high resistivity after the jump from the superconducting state to the normal (or non-conducting) state.

Mathematically we define the vectors $\boldsymbol{\mathcal{J}}_\perp$ and $\boldsymbol{\mathcal{J}}_\|$ by

$$\boldsymbol{\mathcal{J}}_\perp = \widehat{\boldsymbol{\mathcal{B}}} \times \boldsymbol{\mathcal{J}} \times \widehat{\boldsymbol{\mathcal{B}}},\ \boldsymbol{\mathcal{J}}_\| = \langle \widehat{\boldsymbol{\mathcal{B}}}, \boldsymbol{\mathcal{J}} \rangle \widehat{\boldsymbol{\mathcal{B}}},$$

with

$$\widehat{\boldsymbol{\mathcal{B}}} := \frac{\boldsymbol{\mathcal{B}}}{\sqrt{|\boldsymbol{\mathcal{B}}|^2 + \varepsilon^2}},$$

where $\langle \cdot, \cdot \rangle$ denotes $\mathbb{R}^3$-inner product and $\varepsilon > 0$ is a small constant. Throughout the paper in order to define $\boldsymbol{\mathcal{J}}_\perp$ and $\boldsymbol{\mathcal{J}}_\|$ we use the vector $\widehat{\boldsymbol{\mathcal{B}}}$ defined with the small positive constant ε as above so that the vector $\widehat{\boldsymbol{\mathcal{B}}}$ approximates the unit direction vector of $\boldsymbol{\mathcal{B}}$ and the dependency of $\boldsymbol{\mathcal{J}}_\perp$ and $\boldsymbol{\mathcal{J}}_\|$ on $\boldsymbol{\mathcal{B}}$ has no discontinuity at $\boldsymbol{\mathcal{B}} = \mathbf{0}$.

Let us define the energy densities $\gamma_\perp(\cdot)$, $\gamma_\|(\cdot)$, $\gamma_0(\cdot) : \mathbb{R}^3 \to \mathbb{R}$ by

$$\gamma_\sharp(\mathbf{v}) := \begin{cases} 0 & \text{if } |\mathbf{v}| \leq \mathcal{J}_{c\sharp}, \\ \dfrac{\hat{\rho}_\sharp}{2}(|\mathbf{v}|^2 - \mathcal{J}_{c\sharp}^2) & \text{if } |\mathbf{v}| > \mathcal{J}_{c\sharp}, \end{cases} \tag{2.16}$$

with positive constants $\hat{\rho}_\sharp > 0$ ($\sharp = \perp, \|, 0$) and introduce the function $G(\cdot, \cdot) : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ by

$$G(\widehat{\boldsymbol{\mathcal{B}}}, \mathbf{v}) := \gamma_\perp(\widehat{\boldsymbol{\mathcal{B}}} \times \mathbf{v} \times \widehat{\boldsymbol{\mathcal{B}}}) + \gamma_\|(\langle \widehat{\boldsymbol{\mathcal{B}}}, \mathbf{v} \rangle \widehat{\boldsymbol{\mathcal{B}}}) + \gamma_0(\mathbf{v}).$$

Proposition 2.1. *The relation* (2.14)−(2.15) *holds for the resistivity* $\rho_\perp$, $\rho_\|$, ρ_0 *defined by*

$$\rho_\perp = \begin{cases} 0 & \text{if } |\boldsymbol{\mathcal{J}}_\perp| < \mathcal{J}_{c\perp}, \\ \dfrac{\hat{\rho}_\perp |\boldsymbol{\mathcal{B}}|^2}{|\boldsymbol{\mathcal{B}}|^2 + \varepsilon^2} & \text{if } |\boldsymbol{\mathcal{J}}_\perp| \geq \mathcal{J}_{c\perp}, \end{cases} \quad \rho_\| = \begin{cases} 0 & \text{if } |\boldsymbol{\mathcal{J}}_\|| < \mathcal{J}_{c\|}, \\ \dfrac{\hat{\rho}_\| |\boldsymbol{\mathcal{B}}|^2}{|\boldsymbol{\mathcal{B}}|^2 + \varepsilon^2} & \text{if } |\boldsymbol{\mathcal{J}}_\|| \geq \mathcal{J}_{c\|}, \end{cases} \quad \rho_0 = \begin{cases} 0 & \text{if } |\boldsymbol{\mathcal{J}}| < \mathcal{J}_{c0}, \\ \hat{\rho}_0 & \text{if } |\boldsymbol{\mathcal{J}}| \geq \mathcal{J}_{c0}, \end{cases}$$

if, and only if, the following inclusion holds.

$$\boldsymbol{\mathcal{E}} \in \partial G_{\widehat{\boldsymbol{\mathcal{B}}}}(\boldsymbol{\mathcal{J}}), \tag{2.17}$$

where $\partial G_{\widehat{\boldsymbol{\mathcal{B}}}}(\boldsymbol{\mathcal{J}})$ *is a subdifferential of* $G(\widehat{\boldsymbol{\mathcal{B}}}, \cdot)$ *at* $\boldsymbol{\mathcal{J}}$ *defined by*

$$\partial G_{\widehat{\boldsymbol{\mathcal{B}}}}(\boldsymbol{\mathcal{J}}) := \{\mathbf{u} \in \mathbb{R}^3 \mid \langle \mathbf{u}, \mathbf{v} \rangle + G(\widehat{\boldsymbol{\mathcal{B}}}, \boldsymbol{\mathcal{J}}) \leq G(\widehat{\boldsymbol{\mathcal{B}}}, \boldsymbol{\mathcal{J}} + \mathbf{v}),\ \forall \mathbf{v} \in \mathbb{R}^3\}. \tag{2.18}$$

Proof. Let us define the functions $f_\perp(\cdot)$, $f_\parallel(\cdot)$ $:\mathbb{R}^3\to\mathbb{R}$ by

$$f_\perp(\mathbf{v}) := \gamma_\perp(\widehat{\boldsymbol{\mathcal{B}}}\times\mathbf{v}\times\widehat{\boldsymbol{\mathcal{B}}}),\ f_\parallel(\mathbf{v}) := \gamma_\parallel(\langle\widehat{\boldsymbol{\mathcal{B}}},\mathbf{v}\rangle\widehat{\boldsymbol{\mathcal{B}}}).$$

Then we see that $f_\perp(\cdot)$, $f_\parallel(\cdot)$, $\gamma_0(\cdot)$ are convex and continuous in $\mathbb{R}^3$. Therefore by [16, Proposition 5.6, Chapter 1] we deduce that

$$\partial G_{\widehat{\boldsymbol{\mathcal{B}}}}(\boldsymbol{\mathcal{J}}) = \partial f_\perp(\boldsymbol{\mathcal{J}}) + \partial f_\parallel(\boldsymbol{\mathcal{J}}) + \partial\gamma_0(\boldsymbol{\mathcal{J}}). \tag{2.19}$$

Let us define the symmetric transformation $\Lambda_\perp$, $\Lambda_\parallel:\mathbb{R}^3\to\mathbb{R}^3$ by $\Lambda_\perp\mathbf{v} := \widehat{\boldsymbol{\mathcal{B}}}\times\mathbf{v}\times\widehat{\boldsymbol{\mathcal{B}}}$, $\Lambda_\parallel\mathbf{v} := \langle\widehat{\boldsymbol{\mathcal{B}}},\mathbf{v}\rangle\widehat{\boldsymbol{\mathcal{B}}}$. Since $\gamma_\perp(\cdot)$ and $\gamma_\parallel(\cdot)$ are continuous in $\mathbb{R}^3$, by [16, Proposition 5.7, Chapter 1] we obtain for $\sharp=\perp,\parallel$,

$$\partial f_\sharp(\boldsymbol{\mathcal{J}}) = \partial(\gamma_\sharp\circ\Lambda_\sharp)(\boldsymbol{\mathcal{J}}) = \Lambda_\sharp\partial\gamma_\sharp(\Lambda_\sharp\boldsymbol{\mathcal{J}}) = \Lambda_\sharp\partial\gamma_\sharp(\boldsymbol{\mathcal{J}}_\sharp). \tag{2.20}$$

In the same way as in [13, Proposition 2.2] we can characterise the subdifferentials $\partial\gamma_\perp(\boldsymbol{\mathcal{J}}_\perp)$, $\partial\gamma_\parallel(\boldsymbol{\mathcal{J}}_\parallel)$, $\partial\gamma_0(\boldsymbol{\mathcal{J}})$ as follows. For $\sharp=\perp,\parallel,0$,

$$\partial\gamma_\sharp(\boldsymbol{\mathcal{J}}_\sharp) = \left\{E\in\mathbb{R}^3\ \middle|\ E=\left\{\begin{array}{ll}\mathbf{0} & \text{if } |\boldsymbol{\mathcal{J}}_\sharp|<\mathcal{J}_{c\sharp},\\ \hat\rho_\sharp\boldsymbol{\mathcal{J}}_\sharp & \text{if } |\boldsymbol{\mathcal{J}}_\sharp|\geq\mathcal{J}_{c\sharp}.\end{array}\right.\right\}, \tag{2.21}$$

where $\boldsymbol{\mathcal{J}}_0 := \boldsymbol{\mathcal{J}}$. Also note that

$$\Lambda_\perp\boldsymbol{\mathcal{J}}_\perp = \frac{|\boldsymbol{\mathcal{B}}|^2}{|\boldsymbol{\mathcal{B}}|^2+\varepsilon^2}\boldsymbol{\mathcal{J}}_\perp,\ \Lambda_\parallel\boldsymbol{\mathcal{J}}_\parallel = \frac{|\boldsymbol{\mathcal{B}}|^2}{|\boldsymbol{\mathcal{B}}|^2+\varepsilon^2}\boldsymbol{\mathcal{J}}_\parallel. \tag{2.22}$$

By (2.19)−(2.22) we complete the characterisation of $\partial G_{\widehat{\boldsymbol{\mathcal{B}}}}(\boldsymbol{\mathcal{J}})$ as follows.

$$\partial G_{\widehat{\boldsymbol{\mathcal{B}}}}(\boldsymbol{\mathcal{J}}) = \left\{E_\perp+E_\parallel+E_0\in\mathbb{R}^3\middle| E_\sharp=\left\{\begin{array}{ll}\mathbf{0} & \text{if } |\boldsymbol{\mathcal{J}}_\sharp|<\mathcal{J}_{c\sharp},\\ \dfrac{\hat\rho_\sharp|\boldsymbol{\mathcal{B}}|^2\boldsymbol{\mathcal{J}}_\sharp}{|\boldsymbol{\mathcal{B}}|^2+\varepsilon^2} & \text{if } |\boldsymbol{\mathcal{J}}_\sharp|\geq\mathcal{J}_{c\sharp},\end{array}\right.(\sharp=\perp,\parallel), E_0=\left\{\begin{array}{ll}\mathbf{0} & \text{if } |\boldsymbol{\mathcal{J}}|<\mathcal{J}_{c0},\\ \hat\rho\boldsymbol{\mathcal{J}} & \text{if } |\boldsymbol{\mathcal{J}}|\geq\mathcal{J}_{c0}.\end{array}\right.\right\},$$

which concludes the proof. □

Remark 2.2. In order to prove the solvability of our formulation and perform the convergence analysis of the practical finite element approximation of (2.14) we need to introduce the regularised direction vector $\widehat{\boldsymbol{\mathcal{B}}}$. As a consequence, the resistivity $\rho_\sharp$ in the region $|\boldsymbol{\mathcal{J}}_\sharp|\geq\mathcal{J}_{c\sharp}$ ($\sharp=\perp,\parallel$) in Proposition 2.1 becomes dependent of the term $|\boldsymbol{\mathcal{B}}|^2/(|\boldsymbol{\mathcal{B}}|^2+\varepsilon^2)$, which is close to 1 when $|\boldsymbol{\mathcal{B}}|$ is relatively larger than ε. Note that if we use the discontinuous direction vector $\widehat{\boldsymbol{\mathcal{B}}}_0$ defined by $\widehat{\boldsymbol{\mathcal{B}}}_0 := \boldsymbol{\mathcal{B}}/|\boldsymbol{\mathcal{B}}|$ if $\boldsymbol{\mathcal{B}}\neq\mathbf{0}$, $\widehat{\boldsymbol{\mathcal{B}}}_0 := \mathbf{0}$ if $\boldsymbol{\mathcal{B}}=\mathbf{0}$ to define $\boldsymbol{\mathcal{J}}_\perp$ and $\boldsymbol{\mathcal{J}}_\parallel$ instead of $\widehat{\boldsymbol{\mathcal{B}}}$, the same statement as Proposition 2.1 without the term $|\boldsymbol{\mathcal{B}}|^2/(|\boldsymbol{\mathcal{B}}|^2+\varepsilon^2)$ holds true. This is seen as a limit case of Proposition 2.1 for $\varepsilon=0$. Actually, since $G(\widehat{\boldsymbol{\mathcal{B}}},\cdot)$ converges to $G(\widehat{\boldsymbol{\mathcal{B}}}_0,\cdot)$ in the sense of Mosco as $\varepsilon\searrow 0$, $\partial G_{\widehat{\boldsymbol{\mathcal{B}}}}$ converges to $\partial G_{\widehat{\boldsymbol{\mathcal{B}}}_0}$ in the sense of graph (see [1, Theorem 3.66]).

Throughout the paper we employ the inclusion (2.17) as a formulation of the $\boldsymbol{\mathcal{E}}-\boldsymbol{\mathcal{J}}$ relation (2.14)−(2.15). Let us generalise the energy densities $\gamma_\perp$, $\gamma_\parallel$, γ_0 by introducing a class of energy densities as follows.

$$G(\widehat{\boldsymbol{\mathcal{B}}},\mathbf{v}) := g_\perp(|\widehat{\boldsymbol{\mathcal{B}}}\times\mathbf{v}\times\widehat{\boldsymbol{\mathcal{B}}}|) + g_\parallel(|\langle\widehat{\boldsymbol{\mathcal{B}}},\mathbf{v}\rangle\widehat{\boldsymbol{\mathcal{B}}}|) + g_0(|\mathbf{v}|),$$

where $g_\perp, g_\parallel, g_0$ satisfy the following properties.

$$
\begin{aligned}
&g_\sharp : \mathbb{R} \to \mathbb{R}_{\geq 0} \text{ is convex,} \\
&g_\sharp(x) = \begin{cases} 0 & \text{if } x \leq \mathcal{J}_{c\sharp}, \\ > 0 & \text{if } x > \mathcal{J}_{c\sharp}, \end{cases} \\
&A_{\sharp 1} x^2 - A_{\sharp 2} \leq g_\sharp(x) \leq A_{\sharp 3} x^2 + A_{\sharp 4}, \forall x \in \mathbb{R}_{\geq 0},
\end{aligned}
\tag{2.23}
$$

where $A_{\sharp 1}, A_{\sharp 3} > 0$ are positive constants and $A_{\sharp 2}, A_{\sharp 4} \geq 0$ are nonnegative constants ($\sharp = \perp, \|, 0$). Note that $\gamma_\perp(\cdot)$, $\gamma_\|(\cdot)$, $\gamma_0(\cdot)$ are examples of these $g_\perp(|\cdot|)$, $g_\|(|\cdot|)$, $g_0(|\cdot|)$.

We couple the critical-state constitutive relation (2.17) for this generalised energy density $G(\widehat{\boldsymbol{\mathcal{B}}}, \mathbf{v})$ with the eddy current model (2.11)−(2.13) and the initial boundary conditions (2.10), (2.9) to derive the evolution variational inequality for the unknown field $\widehat{\boldsymbol{\mathcal{H}}}$ in Section 2.4.

2.3. **Function spaces**

Let us define the function spaces used in our analysis.

$$
\begin{aligned}
H(\mathbf{curl}; \Omega) &:= \{\boldsymbol{\phi} \in L^2(\Omega; \mathbb{R}^3) \mid \mathbf{curl}\, \boldsymbol{\phi} \in L^2(\Omega; \mathbb{R}^3)\}, \\
H^1(\mathbf{curl}; \Omega) &:= \{\boldsymbol{\phi} \in H^1(\Omega; \mathbb{R}^3) \mid \mathbf{curl}\, \boldsymbol{\phi} \in H^1(\Omega; \mathbb{R}^3)\},
\end{aligned}
$$

with the norms

$$
\begin{aligned}
\|\boldsymbol{\phi}\|_{H(\mathbf{curl};\Omega)} &:= (\|\boldsymbol{\phi}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\,\mathbf{curl}\, \boldsymbol{\phi}\|^2_{L^2(\Omega;\mathbb{R}^3)})^{1/2}, \\
\|\boldsymbol{\phi}\|_{H^1(\mathbf{curl};\Omega)} &:= (\|\boldsymbol{\phi}\|^2_{H^1(\Omega;\mathbb{R}^3)} + \|\,\mathbf{curl}\, \boldsymbol{\phi}\|^2_{H^1(\Omega;\mathbb{R}^3)})^{1/2}.
\end{aligned}
$$

Let us next define the trace spaces. For all $\boldsymbol{\phi} \in H^1(\Omega; \mathbb{R}^N)$ $(N = 1, 3)$ $\boldsymbol{\phi}|_{\partial\Omega} \in H^{1/2}(\partial\Omega; \mathbb{R}^N)$, where $H^{1/2}(\partial\Omega; \mathbb{R}^N)$ is a Sobolev space with the norm

$$
\|\boldsymbol{\phi}\|_{H^{1/2}(\partial\Omega;\mathbb{R}^N)} := \left(\|\boldsymbol{\phi}\|^2_{L^2(\partial\Omega;\mathbb{R}^N)} + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|\boldsymbol{\phi}(\mathbf{x}) - \boldsymbol{\phi}(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^3} dA(\mathbf{x}) dA(\mathbf{y}) \right)^{1/2}.
$$

Let $H^{-1/2}(\partial\Omega; \mathbb{R}^N)$ denote the dual space of $H^{1/2}(\partial\Omega; \mathbb{R}^N)$ with the norm

$$
\|\boldsymbol{\phi}\|_{H^{-1/2}(\partial\Omega;\mathbb{R}^N)} := \sup_{\boldsymbol{\psi} \in H^{1/2}(\partial\Omega;\mathbb{R}^N)} \frac{|\langle\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle\rangle|}{\|\boldsymbol{\psi}\|_{H^{1/2}(\partial\Omega;\mathbb{R}^N)}},
$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ is the inner product of the duality between $H^{1/2}(\partial\Omega; \mathbb{R}^N)$ and $H^{-1/2}(\partial\Omega; \mathbb{R}^N)$. For all $\boldsymbol{\phi} \in H(\mathbf{curl}; \Omega)$, the trace $\mathbf{n} \times \boldsymbol{\phi}$ on $\partial\Omega$ is well-defined in $H^{-1/2}(\partial\Omega; \mathbb{R}^3)$, where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$, in the sense that

$$
\langle \mathbf{n} \times \boldsymbol{\phi}, \boldsymbol{\psi} \rangle := \langle\, \mathbf{curl}\, \boldsymbol{\phi}, \boldsymbol{\psi} \rangle_{L^2(\Omega;\mathbb{R}^3)} - \langle \boldsymbol{\phi}, \mathbf{curl}\, \boldsymbol{\psi} \rangle_{L^2(\Omega;\mathbb{R}^3)},
$$

for all $\boldsymbol{\psi} \in H^1(\Omega; \mathbb{R}^3)$.

Define the subspace $V(\Omega)$ of $H(\mathbf{curl}; \Omega)$ by

$$
V(\Omega) := \{\boldsymbol{\phi} \in H(\mathbf{curl}; \Omega) \mid \mathbf{curl}\, \boldsymbol{\phi} = \mathbf{0} \text{ in } \Omega_d,\ \mathbf{n} \times \boldsymbol{\phi} = \mathbf{0} \text{ on } \partial\Omega\}.
$$

Define the Hilbert space $B(\Omega)$ by $B(\Omega) := V(\Omega) \cap H^1(\mathbf{curl}; \Omega)$ with the norm $\|\boldsymbol{\phi}\|_{B(\Omega)} := \|\boldsymbol{\phi}\|_{H^1(\mathbf{curl};\Omega)}$ and the dual space $(B(\Omega))^*$ of $B(\Omega)$ with the norm

$$
\|\boldsymbol{\phi}\|_{(B(\Omega))^*} = \sup_{\boldsymbol{\psi} \in B(\Omega)} \frac{|\langle\langle\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle\rangle\rangle|}{\|\boldsymbol{\psi}\|_{B(\Omega)}},
$$

where $\langle\langle\langle\cdot,\cdot\rangle\rangle\rangle$ is the inner product of the duality between $B(\Omega)$ and $(B(\Omega))^*$.

The subspace $X^{(\mu)}(\Omega)$ of $H(\mathbf{curl};\Omega)$ consisting of divergence-free functions for the magnetic permeability μ is defined by

$$X^{(\mu)}(\Omega) := \{\boldsymbol{\phi} \in H(\mathbf{curl};\Omega) \mid \operatorname{div}(\mu\boldsymbol{\phi}) = 0 \text{ in } \mathscr{D}'(\Omega)\},$$

where $\mathscr{D}'(\Omega)$ denotes the space of Schwartz distributions.

Define the Hilbert space $Y^{(\mu)}(\Omega)$ by

$$Y^{(\mu)}(\Omega) := \{\boldsymbol{\phi} \in X^{(\mu)}(\Omega) \mid \mathbf{n}\times\boldsymbol{\phi} \in L^2(\partial\Omega;\mathbb{R}^3)\},$$

with the norm $\|\boldsymbol{\phi}\|_{Y^{(\mu)}(\Omega)} := \Big(\|\boldsymbol{\phi}\|^2_{H(\mathbf{curl};\Omega)} + \|\mathbf{n}\times\boldsymbol{\phi}\|^2_{L^2(\partial\Omega;\mathbb{R}^3)}\Big)^{1/2}$.

Let us state two lemmas which will be used in this section and Section 4. Lemma 2.3 requires our assumption on Ω and μ.

Lemma 2.3. *(see [21, Theorem 4.7, Corollary 4.8]) The space $Y^{(\mu)}(\Omega)$ is compactly imbedded in $L^2(\Omega;\mathbb{R}^3)$. Moreover, there exists a constant $C>0$ such that for all $\boldsymbol{\phi}\in Y^{(\mu)}(\Omega)$*

$$\|\boldsymbol{\phi}\|_{L^2(\Omega;\mathbb{R}^3)} \le C(\|\,\mathbf{curl}\,\boldsymbol{\phi}\|_{L^2(\Omega;\mathbb{R}^3)} + \|\mathbf{n}\times\boldsymbol{\phi}\|_{L^2(\partial\Omega;\mathbb{R}^3)}).$$

Lemma 2.4. *For positive constants C_1, $C_2>0$ a set defined by*

$$\Big\{\boldsymbol{\phi}:[0,T]\to L^2(\Omega;\mathbb{R}^3)\ \Big|\ \|\boldsymbol{\phi}\|_{L^\infty(0,T;Y^{(\mu)}(\Omega))}\le C_1,\ \|\partial_t\boldsymbol{\phi}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}\le C_2.\Big\}$$

is relatively compact in $C([0,T];L^2(\Omega;\mathbb{R}^3))$.

Proof. By the compactness property of $Y^{(\mu)}(\Omega)$ this is an immediate consequence of [36, Corollary 4]. □

2.4. Variational inequality formulation of the magnetic field $\widehat{\boldsymbol{\mathcal{H}}}$

By coupling Faraday's law (2.11) and Ampère's law (2.12) with the subdifferential formulation (2.17) we can derive the following evolution variational inequality (see [26], [27], [13] for the derivation of similar variational inequalities).

$$\begin{aligned}&\int_\Omega \mu\langle\partial_t\widehat{\boldsymbol{\mathcal{H}}}(\mathbf{x},t)+\partial_t\boldsymbol{\mathcal{H}}_s(\mathbf{x},t),\boldsymbol{\phi}(\mathbf{x})-\widehat{\boldsymbol{\mathcal{H}}}(\mathbf{x},t)\rangle d\mathbf{x}\\&\quad+\int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}(\mathbf{x},t),\mathbf{curl}\,\boldsymbol{\phi}(\mathbf{x}))d\mathbf{x}-\int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}(\mathbf{x},t),\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}(\mathbf{x},t))d\mathbf{x}\ge 0,\end{aligned}\tag{2.24}$$

for any function $\boldsymbol{\phi}:\Omega\to\mathbb{R}^3$ with $\mathbf{curl}\,\boldsymbol{\phi}=\mathbf{0}$ in Ω_d and $\mathbf{n}\times\boldsymbol{\phi}=\mathbf{0}$ on $\partial\Omega$. If we take $\boldsymbol{\phi}:\Omega\times[0,T]\to\mathbb{R}^3$ satisfying $\mathbf{curl}\,\boldsymbol{\phi}=\mathbf{0}$ in $\Omega_d\times[0,T]$ and $\mathbf{n}\times\boldsymbol{\phi}=\mathbf{0}$ on $\partial\Omega\times[0,T]$ in (2.24), we can eliminate $\partial_t\widehat{\boldsymbol{\mathcal{H}}}$ by integrating over $[0,T]$ by parts as follows.

$$\begin{aligned}&\int_\Omega \mu\langle\widehat{\boldsymbol{\mathcal{H}}}(T),\boldsymbol{\phi}(T)\rangle d\mathbf{x}-\int_\Omega \mu\langle\widehat{\boldsymbol{\mathcal{H}}}(0),\boldsymbol{\phi}(0)\rangle d\mathbf{x}-\int_0^T\int_\Omega \mu\langle\widehat{\boldsymbol{\mathcal{H}}},\partial_t\boldsymbol{\phi}\rangle d\mathbf{x}dt+\int_0^T\int_\Omega \mu\langle\partial_t\boldsymbol{\mathcal{H}}_s,\boldsymbol{\phi}-\widehat{\boldsymbol{\mathcal{H}}}\rangle d\mathbf{x}dt\\&\quad+\int_0^T\int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}},\mathbf{curl}\,\boldsymbol{\phi})d\mathbf{x}dt+\frac12\int_\Omega \mu|\widehat{\boldsymbol{\mathcal{H}}}(0)|^2d\mathbf{x}\ge\int_0^T\int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}},\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}})d\mathbf{x}dt+\frac12\int_\Omega \mu|\widehat{\boldsymbol{\mathcal{H}}}(T)|^2d\mathbf{x},\end{aligned}\tag{2.25}$$

where we have used the equality

$$\int_0^T\int_\Omega \mu\langle\partial_t\widehat{\boldsymbol{\mathcal{H}}},\widehat{\boldsymbol{\mathcal{H}}}\rangle d\mathbf{x}dt=\frac12\int_\Omega \mu|\widehat{\boldsymbol{\mathcal{H}}}(T)|^2d\mathbf{x}-\frac12\int_\Omega \mu|\widehat{\boldsymbol{\mathcal{H}}}(0)|^2d\mathbf{x}.$$

Now we propose our mathematical formulations of the problem. Let us assume that the initial value $\widehat{\boldsymbol{\mathcal{H}}}_0$ and the external source magnetic field $\boldsymbol{\mathcal{H}}_s$ satisfy the regularity

$$\widehat{\boldsymbol{\mathcal{H}}}_0 \in V(\Omega) \cap H^1(\mathbf{curl};\Omega),\ \boldsymbol{\mathcal{H}}_s \in C^1([0,T];H^1(\mathbf{curl};\Omega)), \tag{2.26}$$

the divergence-free condition

$$\widehat{\boldsymbol{\mathcal{H}}}_0 + \boldsymbol{\mathcal{H}}_s(0) \in X^{(\mu)}(\Omega), \tag{2.27}$$

and $\boldsymbol{\mathcal{H}}_s$ satisfies the curl-free condition (2.7). Note that the regularity of $\widehat{\boldsymbol{\mathcal{H}}}_0$ and $\boldsymbol{\mathcal{H}}_s$ in space is required especially to define the interpolation operator of the finite element space on these vector field later in Section 4 and is not essentially needed in the argument in this section and Section 3. The inequality (2.24) is formulated mathematically as follows.

(**P1**) Find $\widehat{\boldsymbol{\mathcal{H}}} \in H^1(0,T;L^2(\Omega;\mathbb{R}^3))$ such that $\widehat{\boldsymbol{\mathcal{H}}}(t) \in V(\Omega)$ for all $t \in [0,T]$,

$$\begin{aligned}&\int_\Omega \mu\langle \partial_t \widehat{\boldsymbol{\mathcal{H}}}(\mathbf{x},t) + \partial_t \boldsymbol{\mathcal{H}}_s(\mathbf{x},t), \boldsymbol{\phi}(\mathbf{x}) - \widehat{\boldsymbol{\mathcal{H}}}(\mathbf{x},t)\rangle d\mathbf{x} \\ &\quad + \int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}(\mathbf{x},t), \mathbf{curl}\,\boldsymbol{\phi}(\mathbf{x}))d\mathbf{x} - \int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}(\mathbf{x},t), \mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}(\mathbf{x},t))d\mathbf{x} \geq 0\end{aligned} \tag{2.28}$$

holds for a.e. $t \in (0,T)$, for all $\boldsymbol{\phi} \in V(\Omega)$ and $\widehat{\boldsymbol{\mathcal{H}}}|_{t=0} = \widehat{\boldsymbol{\mathcal{H}}}_0$, where $\widehat{\boldsymbol{\mathcal{B}}} \in L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))$ is defined by

$$\widehat{\boldsymbol{\mathcal{B}}}(\mathbf{x},t) = \frac{\mu(\mathbf{x})\widehat{\boldsymbol{\mathcal{H}}}(\mathbf{x},t) + \mu(\mathbf{x})\boldsymbol{\mathcal{H}}_s(\mathbf{x},t)}{\sqrt{|\mu(\mathbf{x})\widehat{\boldsymbol{\mathcal{H}}}(\mathbf{x},t) + \mu(\mathbf{x})\boldsymbol{\mathcal{H}}_s(\mathbf{x},t)|^2 + \varepsilon^2}}.$$

The mathematical formulation of the inequality (2.25) is stated as follows.

(**P1'**) Find $\widehat{\boldsymbol{\mathcal{H}}} \in L^2(0,T;H(\mathbf{curl};\Omega))$ with $\partial_t \widehat{\boldsymbol{\mathcal{H}}} \in L^2(0,T;(B(\Omega))^*)$ and $\widehat{\boldsymbol{\mathcal{H}}}^T \in L^2(\Omega;\mathbb{R}^3)$ such that $\widehat{\boldsymbol{\mathcal{H}}}(t) \in V(\Omega)$ for a.e. $t \in (0,T)$,

$$\begin{aligned}&\int_\Omega \mu\langle \widehat{\boldsymbol{\mathcal{H}}}^T(\mathbf{x}), \boldsymbol{\phi}(\mathbf{x},T)\rangle d\mathbf{x} - \int_\Omega \mu\langle \widehat{\boldsymbol{\mathcal{H}}}_0(\mathbf{x}), \boldsymbol{\phi}(\mathbf{x},0)\rangle d\mathbf{x} - \int_0^T\int_\Omega \mu\langle \widehat{\boldsymbol{\mathcal{H}}}(\mathbf{x},t), \partial_t\boldsymbol{\phi}(\mathbf{x},t)\rangle d\mathbf{x}dt \\ &\quad + \int_0^T\int_\Omega \mu\langle \partial_t\boldsymbol{\mathcal{H}}_s(\mathbf{x},t), \boldsymbol{\phi}(\mathbf{x},t) - \widehat{\boldsymbol{\mathcal{H}}}(\mathbf{x},t)\rangle d\mathbf{x}dt + \int_0^T\int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}(\mathbf{x},t), \mathbf{curl}\,\boldsymbol{\phi}(\mathbf{x},t))d\mathbf{x}dt + \frac{1}{2}\int_\Omega \mu|\widehat{\boldsymbol{\mathcal{H}}}_0(\mathbf{x})|^2 d\mathbf{x} \\ &\quad \geq \int_0^T\int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}(\mathbf{x},t), \mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}(\mathbf{x},t))d\mathbf{x}dt + \frac{1}{2}\int_\Omega \mu|\widehat{\boldsymbol{\mathcal{H}}}^T(\mathbf{x})|^2 d\mathbf{x}\end{aligned} \tag{2.29}$$

holds for all $\boldsymbol{\phi} \in C^1([0,T];L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T;H(\mathbf{curl};\Omega))$ with $\boldsymbol{\phi}(t) \in V(\Omega)$ for all $t \in [0,T]$, and the equality

$$\begin{aligned}\int_0^T\int_\Omega \langle \partial_t\widehat{\boldsymbol{\mathcal{H}}}(\mathbf{x},t), \boldsymbol{\psi}(\mathbf{x},t)\rangle d\mathbf{x}dt &= \int_\Omega \langle \widehat{\boldsymbol{\mathcal{H}}}^T(\mathbf{x}), \boldsymbol{\psi}(\mathbf{x},T)\rangle d\mathbf{x} - \int_\Omega \langle \widehat{\boldsymbol{\mathcal{H}}}_0(\mathbf{x}), \boldsymbol{\psi}(\mathbf{x},0)\rangle d\mathbf{x} \\ &\quad - \int_0^T\int_\Omega \langle \widehat{\boldsymbol{\mathcal{H}}}(\mathbf{x},t), \partial_t\boldsymbol{\psi}(\mathbf{x},t)\rangle d\mathbf{x}dt\end{aligned} \tag{2.30}$$

holds for all $\boldsymbol{\psi} \in C^1([0,T];B(\Omega))$.

A combination of the unique solvability theorem of nonlinear evolution system by [18], [20], [38] with the Schauder fixed point theorem shows the existence of a solution to (**P1**).

Theorem 2.5. *There exists a solution $\widehat{\boldsymbol{\mathcal{H}}}$ to (**P1**) which satisfies that $\widehat{\boldsymbol{\mathcal{H}}} : [0,T] \to L^2(\Omega;\mathbb{R}^3)$ is absolutely continuous and $\widehat{\boldsymbol{\mathcal{H}}}(t) + \boldsymbol{\mathcal{H}}_s(t) \in X^{(\mu)}(\Omega)$ for all $t \in [0,T]$. Moreover, the following energy inequality holds.*

$$\frac{1}{2}\int_{t_1}^{t_2}\int_\Omega \mu|\partial_t \widehat{\boldsymbol{\mathcal{H}}}|^2 d\mathbf{x}dt + \int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}(t_2), \mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}(t_2))d\mathbf{x} \le \frac{1}{2}\int_{t_1}^{t_2}\int_\Omega \mu|\partial_t \boldsymbol{\mathcal{H}}_s|^2 d\mathbf{x}dt + \int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}(t_1), \mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}(t_1))d\mathbf{x} \tag{2.31}$$

for any $t_1, t_2 \in [0,T]$ with $t_1 \le t_2$.

Proof. Let $L^2_\mu(\Omega;\mathbb{R}^3)$ denote the Hilbert space $L^2(\Omega;\mathbb{R}^3)$ equipped with the inner product $\langle \mu\cdot,\cdot\rangle_{L^2(\Omega;\mathbb{R}^3)}$. For all $\mathbf{u} \in C([0,T];L^2(\Omega;\mathbb{R}^3))$, let us define $\widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}} \in L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))$ by

$$\widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}}(\mathbf{x},t) = \frac{\mu(\mathbf{x})\mathbf{u}(\mathbf{x},t) + \mu(\mathbf{x})\boldsymbol{\mathcal{H}}_s(\mathbf{x},t)}{\sqrt{|\mu(\mathbf{x})\mathbf{u}(\mathbf{x},t) + \mu(\mathbf{x})\boldsymbol{\mathcal{H}}_s(\mathbf{x},t)|^2 + \varepsilon^2}}.$$

and the functional $E^t_{\mathbf{u}} : L^2_\mu(\Omega;\mathbb{R}^3) \to \mathbb{R}\cup\{+\infty\}$ by

$$E^t_{\mathbf{u}}(\boldsymbol{\phi}) := \begin{cases} \displaystyle\int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}}(t), \mathbf{curl}\,\boldsymbol{\phi})d\mathbf{x} & \text{if } \boldsymbol{\phi} \in V(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then we see that the functional $E^t_{\mathbf{u}}$ is convex, lower semi-continuous, and not identically $+\infty$ in $L^2_\mu(\Omega;\mathbb{R}^3)$ for all $t \in [0,T]$. Also note that $\partial_t \boldsymbol{\mathcal{H}}_s \in L^2(0,T;L^2_\mu(\Omega;\mathbb{R}^3))$ and the effective domain $D(E^t_{\mathbf{u}})$ of $E^t_{\mathbf{u}}$ defined by $D(E^t_{\mathbf{u}}) := \{\boldsymbol{\phi} \in L^2_\mu(\Omega;\mathbb{R}^3) \mid E^t_{\mathbf{u}}(\boldsymbol{\phi}) < +\infty\}$ does not depend on time variable. These properties are sufficient to apply the unique existence theorem of evolution equation with time dependent subdifferential operator summarised in [18, Theorem 2.1] which is based on the preceding results by [20], [38] to ensure that there exists a unique $\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}} \in H^1(0,T;L^2(\Omega;\mathbb{R}^3))$ such that $\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}(t) \in V(\Omega)$ for all $t \in [0,T]$, $\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}(\cdot) : [0,T] \to L^2_\mu(\Omega;\mathbb{R}^3)$ is absolutely continuous,

$$\begin{cases} d_t\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}(t) + \partial_t\boldsymbol{\mathcal{H}}_s(t) \in -\partial E^t_{\mathbf{u}}(\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}(t)) \text{ a.e. } t \in (0,T), \\ \widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}(0) = \widehat{\boldsymbol{\mathcal{H}}}_0, \end{cases} \tag{2.32}$$

and the energy inequality

$$\frac{1}{2}\int_{t_1}^{t_2}\int_\Omega \mu|\partial_t\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}|^2 d\mathbf{x}dt + E^{t_2}_{\mathbf{u}}(\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}(t_2)) \le \frac{1}{2}\int_{t_1}^{t_2}\int_\Omega \mu|\partial_t\boldsymbol{\mathcal{H}}_s|^2 d\mathbf{x}dt + E^{t_1}_{\mathbf{u}}(\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}(t_1)) \tag{2.33}$$

holds for any $t_1, t_2 \in [0,T]$ with $t_1 \le t_2$.

The inclusion (2.32) leads to the inequality

$$\int_\Omega \mu\langle \partial_t\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}(t) + \partial_t\boldsymbol{\mathcal{H}}_s(t), \boldsymbol{\phi} - \widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}(t)\rangle d\mathbf{x} + E^t_{\mathbf{u}}(\boldsymbol{\phi}) - E^t_{\mathbf{u}}(\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}(t)) \ge 0 \tag{2.34}$$

for a.e. $t \in (0,T)$ and all $\boldsymbol{\phi} \in V(\Omega)$. Let us substitute $\boldsymbol{\phi} = \nabla f + \widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}(t)$ ($f \in \mathscr{D}(\Omega)$) into (2.34). By the absolutely continuity of $\mu\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}(t) + \mu\boldsymbol{\mathcal{H}}_s(t) : [0,T] \to L^2(\Omega;\mathbb{R}^3)$ and the assumption (2.27) we can integrate (2.34) over $(0,t)$ by parts to obtain

$$\int_\Omega \mu\langle \widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}(t) + \boldsymbol{\mathcal{H}}_s(t), \nabla f\rangle d\mathbf{x} = 0, \tag{2.35}$$

for all $f \in \mathscr{D}(\Omega)$ and all $t \in [0,T]$.

On the other hand, by using the inequalities

$$E^t_{\mathbf{u}}(\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}(t)) \geq A_{01}\int_{\Omega_s} |\,\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}(t)|^2 d\mathbf{x} - (A_{\perp 2}+A_{\|2}+A_{02})|\Omega_s|,$$
$$E^t_{\mathbf{u}}(\widehat{\boldsymbol{\mathcal{H}}}_0) \leq (A_{\perp 3}+A_{\|3}+A_{03})\int_{\Omega_s} |\,\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}_0|^2 d\mathbf{x} + A_{\perp 4}+A_{\|4}+A_{04},$$

and the energy inequality (2.33), we obtain

$$\int_0^T\int_\Omega |\partial_t \widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}|^2 d\mathbf{x}dt \leq C_1,\ \int_{\Omega_s} |\,\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}(t)|^2 d\mathbf{x} \leq C_2, \tag{2.36}$$

for all $t\in[0,T]$, where we have set

$$\begin{aligned} C_1 :=& \frac{\max\{\mu_s,\mu_d\}}{\min\{\mu_s,\mu_d\}}\int_0^T\int_\Omega |\partial_t \boldsymbol{\mathcal{H}}_s|^2 d\mathbf{x}dt \\ &+ \frac{2}{\min\{\mu_s,\mu_d\}}\left((A_{\perp 3}+A_{\|3}+A_{03})\int_{\Omega_s} |\,\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}_0|^2 d\mathbf{x} + A_{\perp 4}+A_{\|4}+A_{04}\right), \\ C_2 :=& \frac{(A_{\perp 2}+A_{\|2}+A_{02})|\Omega_s|}{A_{01}} + \frac{\max\{\mu_s,\mu_d\}}{2A_{01}}\int_0^T\int_\Omega |\partial_t \boldsymbol{\mathcal{H}}_s|^2 d\mathbf{x}dt \\ &+ \frac{1}{A_{01}}\left((A_{\perp 3}+A_{\|3}+A_{03})\int_{\Omega_s} |\,\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}_0|^2 d\mathbf{x} + A_{\perp 4}+A_{\|4}+A_{04}\right). \end{aligned}$$

The inequalities (2.36) and Lemma 2.3 yield

$$\|\partial_t(\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}+\boldsymbol{\mathcal{H}}_s)\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \leq C_1^{1/2} + \|\partial_t \boldsymbol{\mathcal{H}}_s\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}, \tag{2.37}$$
$$\|\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}+\boldsymbol{\mathcal{H}}_s\|_{L^\infty(0,T;Y^{(\mu)}(\Omega))} \leq (2C^2+1)^{1/2}(C_2 + \|\mathbf{n}\times\boldsymbol{\mathcal{H}}_s\|^2_{L^\infty(0,T;L^2(\partial\Omega;\mathbb{R}^3))})^{1/2}, \tag{2.38}$$

where $C>0$ is the constant which appears in the inequality in Lemma 2.3.

Let us define a subset S of $C([0,T];L^2(\Omega;\mathbb{R}^3))$ by

$$S := \left\{\boldsymbol{\phi}\in C([0,T];L^2(\Omega;\mathbb{R}^3)) \mid \|\partial_t(\boldsymbol{\phi}+\boldsymbol{\mathcal{H}}_s)\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \leq C_3,\ \|\boldsymbol{\phi}+\boldsymbol{\mathcal{H}}_s\|_{L^\infty(0,T;Y^{(\mu)}(\Omega))} \leq C_4\right\},$$

where we have set

$$C_3 := C_1^{1/2} + \|\partial_t \boldsymbol{\mathcal{H}}_s\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))},\ C_4 := (2C^2+1)^{1/2}(C_2 + \|\mathbf{n}\times\boldsymbol{\mathcal{H}}_s\|^2_{L^\infty(0,T;L^2(\partial\Omega;\mathbb{R}^3))})^{1/2}.$$

Let $\overline{S}$ denote the closure of S in $C([0,T];L^2(\Omega;\mathbb{R}^3))$. Lemma 2.4 implies that $\overline{S}$ is a compact, convex set in $C([0,T];L^2(\Omega;\mathbb{R}^3))$. We define the map $F:\overline{S}\to\overline{S}$ by $F\mathbf{u} := \widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}$ where $\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}}$ is the unique solution of the evolution system (2.32) for $\mathbf{u}\in\overline{S}$. In order to apply the Schauder fixed point theorem (see, e.g. [33]) we need to show that $F:\overline{S}\to\overline{S}$ is continuous in $C([0,T];L^2(\Omega;\mathbb{R}^3))$.

Assume that $\mathbf{u}_n\to\mathbf{u}$ strongly in $C([0,T];L^2(\Omega;\mathbb{R}^3))$ as $n\to+\infty$, where $\mathbf{u}_n\in\overline{S}$ for all $n\in\mathbb{N}$. We see $\mathbf{u}\in\overline{S}$. We will prove that $\lim_{n\to+\infty}F\mathbf{u}_n = F\mathbf{u}$.

By the definition of F there is $\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}_n}\in\overline{S}$ solving (2.32) such that $F\mathbf{u}_n = \widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}_n}$. Since $\overline{S}$ is compact, by taking a subsequence still denoted by $\{\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}_n}\}_{n=1}^\infty$ we see that

$$\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}_n} \to \widehat{\boldsymbol{\mathcal{H}}} \text{ strongly in } C([0,T];L^2(\Omega;\mathbb{R}^3)) \tag{2.39}$$

as $n \to +\infty$. We can show by using the bounds (2.36) that $\widehat{\boldsymbol{\mathcal{H}}}(t) \in V(\Omega)$ for all $t \in [0,T]$, $\widehat{\boldsymbol{\mathcal{H}}}(0) = \widehat{\boldsymbol{\mathcal{H}}}_0$, and $\widehat{\boldsymbol{\mathcal{H}}} : [0,T] \to L^2_\mu(\Omega;\mathbb{R}^3)$ is absolutely continuous. Moreover, we observe by taking a subsequence if necessary that

$$\begin{aligned} &\partial_t \widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}_n} \rightharpoonup \partial_t \widehat{\boldsymbol{\mathcal{H}}} \text{ weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \\ &\mathbf{curl}\, \widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}_n} \rightharpoonup \mathbf{curl}\, \widehat{\boldsymbol{\mathcal{H}}} \text{ weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \end{aligned} \tag{2.40}$$

as $n \to +\infty$. The assumption that $\mathbf{u}_n \to \mathbf{u}$ strongly in $C([0,T];L^2(\Omega;\mathbb{R}^3))$ ensures by taking a subsequence if necessary that

$$\widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}_n} \to \widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}} \text{ strongly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{2.41}$$

as $n \to +\infty$. The second convergence property of (2.40) and (2.41) yield that

$$\begin{aligned} &\widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}_n} \times \mathbf{curl}\, \widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}_n} \times \widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}_n} \rightharpoonup \widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}} \times \mathbf{curl}\, \widehat{\boldsymbol{\mathcal{H}}} \times \widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}} \text{ weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \\ &\langle \widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}_n}, \mathbf{curl}\, \widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}_n} \rangle \widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}_n} \rightharpoonup \langle \widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}}, \mathbf{curl}\, \widehat{\boldsymbol{\mathcal{H}}} \rangle \widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}} \text{ weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \\ &\widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}_n} \times \mathbf{curl}\, \boldsymbol{\phi} \times \widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}_n} \rightharpoonup \widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}} \times \mathbf{curl}\, \boldsymbol{\phi} \times \widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}} \text{ strongly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \\ &\langle \widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}_n}, \mathbf{curl}\, \boldsymbol{\phi} \rangle \widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}_n} \rightharpoonup \langle \widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}}, \mathbf{curl}\, \boldsymbol{\phi} \rangle \widehat{\boldsymbol{\mathcal{B}}}_{\mathbf{u}} \text{ strongly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \end{aligned} \tag{2.42}$$

as $n \to +\infty$ for all $\boldsymbol{\phi} \in L^2(0,T;H(\mathbf{curl};\Omega))$. By the second convergence property of (2.40) and (2.42), the properties (2.23) of the energy densities $g_\perp(|\cdot|)$, $g_\parallel(|\cdot|)$, $g_0(|\cdot|)$, and the Lebesgue convergence theorem, we see that

$$\liminf_{n\to+\infty} \int_0^T E^t_{\mathbf{u}_n}(\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}_n}(t))dt \geq \int_0^T E^t_{\mathbf{u}}(\widehat{\boldsymbol{\mathcal{H}}}(t))dt, \quad \lim_{n\to+\infty} \int_0^T E^t_{\mathbf{u}_n}(\boldsymbol{\phi}(t))dt = \int_0^T E^t_{\mathbf{u}}(\boldsymbol{\phi}(t))dt, \tag{2.43}$$

for all $\boldsymbol{\phi} \in L^2(0,T;H(\mathbf{curl};\Omega))$ with $\boldsymbol{\phi}(t) \in V(\Omega)$ for a.e. $t \in (0,T)$. The convergence properties (2.39), (2.40), (2.43) ensure that

$$\begin{aligned} &\int_0^T \int_\Omega \mu \langle \partial_t \widehat{\boldsymbol{\mathcal{H}}} + \partial_t \boldsymbol{\mathcal{H}}_s, \boldsymbol{\phi} - \widehat{\boldsymbol{\mathcal{H}}} \rangle d\mathbf{x}dt + \int_0^T E^t_{\mathbf{u}}(\boldsymbol{\phi})dt \\ &\quad = \lim_{n\to+\infty} \int_0^T \int_\Omega \mu \langle \partial_t \widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}_n} + \partial_t \boldsymbol{\mathcal{H}}_s, \boldsymbol{\phi} - \widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}_n} \rangle d\mathbf{x}dt + \lim_{n\to+\infty} \int_0^T E^t_{\mathbf{u}_n}(\boldsymbol{\phi})dt \\ &\quad \geq \liminf_{n\to+\infty} \int_0^T E^t_{\mathbf{u}_n}(\widehat{\boldsymbol{\mathcal{H}}}_{\mathbf{u}_n})dt \geq \int_0^T E^t_{\mathbf{u}}(\widehat{\boldsymbol{\mathcal{H}}})dt, \end{aligned} \tag{2.44}$$

for all $\boldsymbol{\phi} \in L^2(0,T;H(\mathbf{curl};\Omega))$ with $\boldsymbol{\phi}(t) \in V(\Omega)$, which is equivalent to the inequality

$$\int_\Omega \mu \langle \partial_t \widehat{\boldsymbol{\mathcal{H}}}(t) + \partial_t \boldsymbol{\mathcal{H}}_s(t), \boldsymbol{\phi} - \widehat{\boldsymbol{\mathcal{H}}}(t) \rangle d\mathbf{x} + E^t_{\mathbf{u}}(\boldsymbol{\phi}) - E^t_{\mathbf{u}}(\widehat{\boldsymbol{\mathcal{H}}}(t)) \geq 0, \tag{2.45}$$

for a.e. $t \in (0,T)$ and all $\boldsymbol{\phi} \in V(\Omega)$. The absolutely continuity of $\widehat{\boldsymbol{\mathcal{H}}}(\cdot)$ implies that $\widehat{\boldsymbol{\mathcal{H}}}$ solves (2.32). Thus, $F\mathbf{u} = \widehat{\boldsymbol{\mathcal{H}}}$.

The uniqueness of a solution of the evolution variational inequality (2.32) ensures the convergence property (2.39) without extracting any subsequence. Therefore, we have proved that $\lim_{n\to+\infty} F\mathbf{u}_n = \widehat{\boldsymbol{\mathcal{H}}} = F\mathbf{u}$, which implies that $F : \overline{S} \to \overline{S}$ is continuous in $C([0,T];L^2(\Omega;\mathbb{R}^3))$. Thus, the Schauder fixed point theorem proves the existence of a fixed point $\widehat{\boldsymbol{\mathcal{H}}} \in \overline{S}$ such that $F\widehat{\boldsymbol{\mathcal{H}}} = \widehat{\boldsymbol{\mathcal{H}}}$, which shows the existence of a solution to (**P1**) with the desired properties. □

Corollary 2.6. *A solution $\widehat{\boldsymbol{\mathcal{H}}}$ of (**P1**) whose existence was proved in Theorem 2.5 is a solution of (**P1'**).*

Proof. Since $\widehat{\boldsymbol{\mathcal{H}}} : [0,T] \to L^2(\Omega;\mathbb{R}^3)$ is absolutely continuous, we can derive the inequality (2.29) by integrating (2.28) by parts and the equality (2.30) for $\widehat{\boldsymbol{\mathcal{H}}}^T = \widehat{\boldsymbol{\mathcal{H}}}(T)$. □

Remark 2.7. In Section 4 we will prove that by taking a subsequence the fully discrete approximation of (**P1**) converges to a solution $\widehat{\boldsymbol{\mathcal{H}}}$ to (**P1'**).

2.5. Formulation with the magnetic scalar potential

One difficulty in the practical computation of the eddy current model is the curl-free constraint imposed in the non-conductive region Ω_d. We handle this constraint by expressing the magnetic field as a gradient of magnetic scalar potential in Ω_d in the same manner as in [13].

For $\mathbf{u_1} \in L^2(\Omega_s;\mathbb{R}^3)$ and $\mathbf{u_2} \in L^2(\Omega_d;\mathbb{R}^3)$, $(\mathbf{u_1}|\mathbf{u_2}) \in L^2(\Omega;\mathbb{R}^3)$ is defined by

$$(\mathbf{u_1}|\mathbf{u_2}) := \begin{cases} \mathbf{u_1} & \text{in } \Omega_s, \\ \mathbf{u_2} & \text{in } \Omega_d. \end{cases}$$

Using this notation, we define the Hilbert space $W(\Omega)$ by

$$W(\Omega) := \{(\boldsymbol{\phi}|\nabla v) \in L^2(\Omega;\mathbb{R}^3) \mid (\boldsymbol{\phi}, v) \in L^2(\Omega_s;\mathbb{R}^3) \times H^1(\Omega_d),\ (\boldsymbol{\phi}|\nabla v) \in H(\mathbf{curl};\Omega), v = 0 \text{ on } \partial\Omega\}$$

equipped with the inner product of $H(\mathbf{curl};\Omega)$.

The following proposition enables us to replace the space $V(\Omega)$ with the curl-free constraint by the space $W(\Omega)$ with the scalar potential in our formulations above. To prove the proposition below requires that Ω_d is simply connected, which is our case.

Proposition 2.8. *([13, Proposition 2.7]) The space $W(\Omega)$ is isomorphic to $V(\Omega)$ as a Hilbert space.*

Let us propose the hybrid formulations (**P2**) and (**P2'**). We take the initial value $(\boldsymbol{\psi}_0|\nabla u_0) \in W(\Omega)$ to satisfy the following divergence-free condition.

$$(\boldsymbol{\psi}_0|\nabla u_0) + \boldsymbol{\mathcal{H}}_s(0) \in X^{(\mu)}(\Omega).$$

The problem (**P1**) can be rewritten as follows.

(**P2**) Find $\boldsymbol{\psi} : [0,T] \to H(\mathbf{curl};\Omega_s)$ and $u : [0,T] \to H^1(\Omega_d)$ such that $(\boldsymbol{\psi}|\nabla u) \in H^1(0,T;L^2(\Omega;\mathbb{R}^3))$, $(\boldsymbol{\psi}|\nabla u)(t) \in W(\Omega)$ for all $t \in [0,T]$,

$$\int_{\Omega_s} \mu_s \langle \partial_t \boldsymbol{\psi}(\mathbf{x},t) + \partial_t \boldsymbol{\mathcal{H}}_s(\mathbf{x},t), \boldsymbol{\phi}(\mathbf{x}) - \boldsymbol{\psi}(\mathbf{x},t)\rangle d\mathbf{x} + \int_{\Omega_d} \mu_d \langle \partial_t \nabla u(\mathbf{x},t) + \partial_t \boldsymbol{\mathcal{H}}_s(\mathbf{x},t), \nabla v(\mathbf{x}) - \nabla u(\mathbf{x},t)\rangle d\mathbf{x}$$
$$+ \int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}'(\mathbf{x},t), \mathbf{curl}\,\boldsymbol{\phi}(\mathbf{x}))d\mathbf{x} - \int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}'(\mathbf{x},t), \mathbf{curl}\,\boldsymbol{\psi}(\mathbf{x},t))d\mathbf{x} \geq 0$$

holds for a.e. $t \in (0,T)$, for all $(\boldsymbol{\phi}|\nabla v) \in W(\Omega)$ and $(\boldsymbol{\psi}|\nabla u)|_{t=0} = (\boldsymbol{\psi}_0|\nabla u_0)$, where $\widehat{\boldsymbol{\mathcal{B}}}' \in L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))$ is defined by

$$\widehat{\boldsymbol{\mathcal{B}}}'(\mathbf{x},t) = \frac{\mu(\mathbf{x})(\boldsymbol{\psi}|\nabla u)(\mathbf{x},t) + \mu(\mathbf{x})\boldsymbol{\mathcal{H}}_s(\mathbf{x},t)}{\sqrt{|\mu(\mathbf{x})(\boldsymbol{\psi}|\nabla u)(\mathbf{x},t) + \mu(\mathbf{x})\boldsymbol{\mathcal{H}}_s(\mathbf{x},t)|^2 + \varepsilon^2}}.$$

We can rewrite the problem (**P1'**) as follows.

(**P2'**) Find $\boldsymbol{\psi} : [0,T] \to H(\mathbf{curl};\Omega_s)$, $u : [0,T] \to H^1(\Omega_d)$, and $\widehat{\boldsymbol{\mathcal{H}}}^T \in L^2(\Omega;\mathbb{R}^3)$ such that $(\boldsymbol{\psi}|\nabla u) \in L^2(0,T;H(\mathbf{curl};\Omega))$, $\partial_t(\boldsymbol{\psi}|\nabla u) \in L^2(0,T;(B(\Omega))^*)$, $(\boldsymbol{\psi}|\nabla u)(t) \in W(\Omega)$ for a.e. $t \in (0,T)$,

$$
\begin{aligned}
&\int_\Omega \mu\langle \widehat{\boldsymbol{\mathcal{H}}}^T(\mathbf{x}), (\boldsymbol{\phi}|\nabla v)(\mathbf{x},T)\rangle d\mathbf{x} - \int_\Omega \mu\langle (\boldsymbol{\psi}_0|\nabla u_0)(\mathbf{x}), (\boldsymbol{\phi}|\nabla v)(\mathbf{x},0)\rangle d\mathbf{x} \\
&\quad - \int_0^T\int_\Omega \mu\langle (\boldsymbol{\psi}|\nabla u)(\mathbf{x},t), \partial_t(\boldsymbol{\phi}|\nabla v)(\mathbf{x},t)\rangle d\mathbf{x}dt + \int_0^T\int_\Omega \mu\langle \partial_t\boldsymbol{\mathcal{H}}_s(\mathbf{x},t), (\boldsymbol{\phi}|\nabla v)(\mathbf{x},t) - (\boldsymbol{\psi}|\nabla u)(\mathbf{x},t)\rangle d\mathbf{x}dt \\
&\quad + \int_0^T\int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}'(\mathbf{x},t), \mathbf{curl}\,\boldsymbol{\phi}(\mathbf{x},t))d\mathbf{x}dt + \frac{1}{2}\int_\Omega \mu|(\boldsymbol{\psi}_0|\nabla u_0)(\mathbf{x})|^2 d\mathbf{x} \\
&\quad \geq \int_0^T\int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}'(\mathbf{x},t), \mathbf{curl}\,\boldsymbol{\psi}(\mathbf{x},t))d\mathbf{x}dt + \frac{1}{2}\int_\Omega \mu|\widehat{\boldsymbol{\mathcal{H}}}^T(\mathbf{x})|^2 d\mathbf{x},
\end{aligned}
$$

holds for all $(\boldsymbol{\phi}|\nabla v) : [0,T] \to W(\Omega)$ satisfying $(\boldsymbol{\phi}|\nabla v) \in C^1([0,T];L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T;H(\mathbf{curl};\Omega))$, and the equality

$$
\begin{aligned}
&\int_0^T\int_\Omega \langle \partial_t(\boldsymbol{\psi}|\nabla u)(\mathbf{x},t), \boldsymbol{\xi}(\mathbf{x},t)\rangle d\mathbf{x}dt \\
&\quad = \int_\Omega \langle \widehat{\boldsymbol{\mathcal{H}}}^T(\mathbf{x}), \boldsymbol{\xi}(\mathbf{x},T)\rangle d\mathbf{x} - \int_\Omega \langle (\boldsymbol{\psi}_0|\nabla u_0)(\mathbf{x}), \boldsymbol{\xi}(\mathbf{x},0)\rangle d\mathbf{x} - \int_0^T\int_\Omega \langle (\boldsymbol{\psi}|\nabla u)(\mathbf{x},t), \partial_t\boldsymbol{\xi}(\mathbf{x},t)\rangle d\mathbf{x}dt
\end{aligned}
$$

holds for all $\boldsymbol{\xi} \in C^1([0,T];B(\Omega))$.

Theorem 2.5, Corollary 2.6 and Proposition 2.8 ensure the existence of a solution to (**P2**) and (**P2'**) immediately. In Section 4, it will be proved that the fully discretization of (**P2**) converges to a solution to (**P2'**) by taking a subsequence.

3. Characterisation of the subdifferentials in $L^2(\Omega;\mathbb{R}^3)$

In this section we characterise subdifferentials of a class of convex energies defined in $L^2(\Omega;\mathbb{R}^3)$. By applying the characterisation theorem proved below to the energy functional deriving the problem (**P1**) we will recover Faraday's law $\partial_t\boldsymbol{\mathcal{B}} + \mathbf{curl}\,\boldsymbol{\mathcal{E}} = \mathbf{0}$ and the nonlinear Ohm's law $\boldsymbol{\mathcal{E}} \in \partial G_{\widehat{\boldsymbol{\mathcal{B}}}}(\boldsymbol{\mathcal{J}})$ from our variational inequality formulation (**P1**) in the superconductor. The argument in this section follows the theory developed in [2] with some revision.

Throughout this section we assume $p \geq 2$. We define the Banach space $V_p(\Omega)$ by

$$
V_p(\Omega) := \{\boldsymbol{\phi} \in H(\mathbf{curl};\Omega) \mid \mathbf{curl}\,\boldsymbol{\phi} = \mathbf{0} \text{ in } \Omega_d,\ \mathbf{curl}\,\boldsymbol{\phi}|_{\Omega_s} \in L^p(\Omega_s;\mathbb{R}^3),\ \mathbf{n}\times\boldsymbol{\phi} = \mathbf{0} \text{ on } \partial\Omega\}
$$

with the norm $\|\boldsymbol{\phi}\|_{V_p(\Omega)} := (\|\boldsymbol{\phi}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\mathbf{curl}\,\boldsymbol{\phi}\|^2_{L^p(\Omega;\mathbb{R}^3)})^{1/2}$.

We consider the function $Q(\mathbf{x},\mathbf{r}) : \Omega\times\mathbb{R}^3 \to \mathbb{R}$ satisfying the following properties.

(1) $Q(\cdot,\mathbf{r}) : \Omega \to \mathbb{R}$ is measurable for all $\mathbf{r} \in \mathbb{R}^3$,
(2) $Q(\mathbf{x},\cdot) : \mathbb{R}^3 \to \mathbb{R}$ is convex for a.e. $\mathbf{x} \in \Omega$,
(3) There exist constants $C_1, C_2 > 0$ and $d_1, d_2 \in L^1(\Omega)$ such that

$$
C_1|\mathbf{r}|^p + d_1(\mathbf{x}) \leq Q(\mathbf{x},\mathbf{r}) \leq C_2|\mathbf{r}|^p + d_2(\mathbf{x}),
$$

for all $\mathbf{r} \in \mathbb{R}^3$, a.e. $\mathbf{x} \in \Omega$.

Note that by the conditions (1) and (2) the function $Q(\cdot,\mathbf{u}(\cdot))$ is measurable in Ω for any measurable function $\mathbf{u} : \Omega \to \mathbb{R}^3$ (see, e.g. [30, p. 529, Corollary, p. 531, Corollary] or [31]), by the condition (3), $Q(\cdot,\mathbf{u}(\cdot)) \in L^1(\Omega)$ for all $\mathbf{u} \in L^p(\Omega;\mathbb{R}^3)$.

Let us define the convex functional $F : L^2(\Omega;\mathbb{R}^3) \to \mathbb{R}\cup\{+\infty\}$ by

$$F(\boldsymbol{\phi}) = \begin{cases} \displaystyle\int_\Omega Q(\mathbf{x}, \mathbf{curl}\,\boldsymbol{\phi}(\mathbf{x}))d\mathbf{x} & \text{if } \boldsymbol{\phi}\in V_p(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

The functional F is convex, lower semi-continuous and not identically $+\infty$ in $L^2(\Omega;\mathbb{R}^3)$.

In the argument below $Q^*(\mathbf{x},\cdot)$ denotes the convex conjugate of $Q(\mathbf{x},\cdot)$ in $\mathbb{R}^3$ and $F^*(\cdot)$ denotes the convex conjugate of $F(\cdot)$ in $L^2(\Omega;\mathbb{R}^3)$ respectively defined by

$$Q^*(\mathbf{x},\mathbf{s}) = \sup_{\mathbf{r}\in\mathbb{R}^3}\{\langle\mathbf{s},\mathbf{r}\rangle - Q(\mathbf{x},\mathbf{r})\},\ F^*(\mathbf{v}) = \sup_{\mathbf{u}\in L^2(\Omega;\mathbb{R}^3)}\{\langle\mathbf{v},\mathbf{u}\rangle_{L^2(\Omega;\mathbb{R}^3)} - F(\mathbf{u})\}.$$

Note that $Q^*(\mathbf{x},\cdot) : \mathbb{R}^3 \to \mathbb{R}$ is convex and lower semi-continuous for a.e. $\mathbf{x}\in\Omega$ and $F^*(\cdot) : L^2(\Omega;\mathbb{R}^3) \to \mathbb{R}\cup\{+\infty\}$ is convex and lower semi-continuous and not identically $+\infty$ (see [16]).

Lemma 3.1. *The following inequality holds. For a.e.* $\mathbf{x}\in\Omega$, *all* $\mathbf{s}\in\mathbb{R}^3$

$$\frac{C_2(p-1)}{(C_2p)^{p/(p-1)}}|\mathbf{s}|^{p/(p-1)} - d_2(\mathbf{x}) \le Q^*(\mathbf{x},\mathbf{s}) \le \frac{C_1(p-1)}{(C_1p)^{p/(p-1)}}|\mathbf{s}|^{p/(p-1)} - d_1(\mathbf{x}). \tag{3.1}$$

Proof. By the first inequality of the condition (3) we see that

$$\begin{aligned} Q^*(\mathbf{x},\mathbf{s}) &\le \sup_{\mathbf{r}\in\mathbb{R}^3}\{\langle\mathbf{s},\mathbf{r}\rangle - C_1|\mathbf{r}|^p - d_1(\mathbf{x})\} = \sup_{\delta\ge0}\sup_{\mathbf{r}\in\mathbb{R}^3,|\mathbf{r}|=\delta}\{\langle\mathbf{s},\mathbf{r}\rangle - C_1|\mathbf{r}|^p - d_1(\mathbf{x})\} \\ &= \sup_{\delta\ge0}\{|\mathbf{s}|\delta - C_1\delta^p - d_1(\mathbf{x})\} = \frac{C_1(p-1)}{(C_1p)^{p/(p-1)}}|\mathbf{s}|^{p/(p-1)} - d_1(\mathbf{x}). \end{aligned}$$

By using the second inequality of (3), the first inequality of (3.1) can be proved in the same way. □

Note that $Q^*(\cdot,\mathbf{u}(\cdot))$ is measurable in Ω for any measurable function $\mathbf{u}:\Omega\to\mathbb{R}^3$ (see [30, p. 529, Lemma 3]). By the inequalities of (3.1), we see that $Q^*(\cdot,\mathbf{u}(\cdot))\in L^1(\Omega)$ for all $\mathbf{u}\in L^{p/(p-1)}(\Omega;\mathbb{R}^3)$.

We next characterise the conjugate $F^*(\cdot)$. In the proof of the characterisation of F^* below we consider the reflexive Banach space $L^2(\Omega;\mathbb{R}^3)\times L^p(\Omega;\mathbb{R}^3)$ equipped with the norm

$$\|(\mathbf{u},\mathbf{v})\|_{L^2(\Omega;\mathbb{R}^3)\times L^p(\Omega;\mathbb{R}^3)} := (\|\mathbf{u}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\mathbf{v}\|^2_{L^p(\Omega;\mathbb{R}^3)})^{1/2},\ (\mathbf{u},\mathbf{v})\in L^2(\Omega;\mathbb{R}^3)\times L^p(\Omega;\mathbb{R}^3).$$

Note that the dual space of $L^2(\Omega;\mathbb{R}^3)\times L^p(\Omega;\mathbb{R}^3)$ is $L^2(\Omega;\mathbb{R}^3)\times L^{p/(p-1)}(\Omega;\mathbb{R}^3)$.

Let $(V_p(\Omega))^*$ denote the dual space of $V_p(\Omega)$. Note that for $\mathbf{w}\in L^2(\Omega;\mathbb{R}^3)$ and $\mathbf{u}\in L^{p/(p-1)}(\Omega;\mathbb{R}^3)$ we define $\mathbf{w}$, $\mathbf{curl}\,\mathbf{u}\in(V_p(\Omega))^*$ by

$$\langle\mathbf{w},\boldsymbol{\phi}\rangle := \langle\mathbf{w},\boldsymbol{\phi}\rangle_{L^2(\Omega;\mathbb{R}^3)},\ \langle\mathbf{curl}\,\mathbf{u},\boldsymbol{\phi}\rangle := \langle\mathbf{u},\mathbf{curl}\,\boldsymbol{\phi}\rangle_{L^2(\Omega;\mathbb{R}^3)},$$

for all $\boldsymbol{\phi}\in V_p(\Omega)$.

Proposition 3.2. *For all* $\mathbf{w}\in L^2(\Omega;\mathbb{R}^3)$ *with* $F^*(\mathbf{w}) < +\infty$,

$$F^*(\mathbf{w}) = \min_{\substack{\mathbf{u}\in L^{p/(p-1)}(\Omega,\mathbb{R}^3)\\ \mathbf{w}-\mathbf{curl}\,\mathbf{u}=0\ in\ (V_p(\Omega))^*}} \int_\Omega Q^*(\mathbf{x},\mathbf{u}(\mathbf{x}))d\mathbf{x}.$$

Proof. Let us define the function $P : \Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ by $P(\mathbf{x}, \mathbf{r}, \mathbf{s}) := Q(\mathbf{x}, \mathbf{s})$ and the functionals $R, \theta : L^2(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ by

$$R(\mathbf{v}_1, \mathbf{v}_2) := \begin{cases} 0 & \text{if } \mathbf{v}_1 \in V_p(\Omega), \mathbf{v}_2 = \mathbf{curl}\, \mathbf{v}_1, \\ +\infty & \text{otherwise,} \end{cases} \quad \theta(\mathbf{v}_1, \mathbf{v}_2) := \int_\Omega P(\mathbf{x}, \mathbf{v}_1(\mathbf{x}), \mathbf{v}_2(\mathbf{x}))d\mathbf{x}.$$

Then we see that R and θ are convex, lower semi-continuous, and not identically $+\infty$ in the reflexive Banach space $L^2(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{R}^3)$. Moreover, we observe that for all $(\mathbf{w}_1, \mathbf{w}_2) \in L^2(\Omega; \mathbb{R}^3) \times L^{p/(p-1)}(\Omega; \mathbb{R}^3)$

$$\begin{aligned} (\theta + R)^*(\mathbf{w}_1, \mathbf{w}_2) &= \sup_{(\mathbf{v}_1, \mathbf{v}_2) \in L^2(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{R}^3)} \{\langle \mathbf{w}_1, \mathbf{v}_1 \rangle_{L^2(\Omega; \mathbb{R}^3)} + \langle \mathbf{w}_2, \mathbf{v}_2 \rangle_{L^2(\Omega; \mathbb{R}^3)} - (\theta + R)(\mathbf{v}_1, \mathbf{v}_2)\} \\ &= \sup_{\mathbf{v} \in V_p(\Omega)} \{\langle \mathbf{w}_1, \mathbf{v} \rangle_{L^2(\Omega; \mathbb{R}^3)} + \langle \mathbf{w}_2, \mathbf{curl}\, \mathbf{v} \rangle_{L^2(\Omega; \mathbb{R}^3)} - F(\mathbf{v})\}, \end{aligned}$$

where $(\theta + R)^*$ is the conjugate functional in $L^2(\Omega; \mathbb{R}^3) \times L^{p/(p-1)}(\Omega; \mathbb{R}^3)$ of $(\theta + R)$ with respect to the L^2-inner product. Thus, for all $\mathbf{w} \in L^2(\Omega; \mathbb{R}^3)$

$$(\theta + R)^*(\mathbf{w}, 0) = F^*(\mathbf{w}). \tag{3.2}$$

Let $D(\theta)$, $D(R) \subset L^2(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{R}^3)$ denote the effective domains of the functionals θ and R respectively. Then we see that

$$D(\theta) - D(R) = L^2(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{R}^3),$$

especially $D(\theta) - D(R)$ is a neighbourhood of the origin in $L^2(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{R}^3)$. Thus, by the formula of inf-convolution [1, p. 268, Proposition 3.4] we deduce that for all $(\mathbf{w}_1, \mathbf{w}_2) \in L^2(\Omega; \mathbb{R}^3) \times L^{p/(p-1)}(\Omega; \mathbb{R}^3)$

$$(\theta + R)^*(\mathbf{w}_1, \mathbf{w}_2) = \inf_{(\mathbf{u}_1, \mathbf{u}_2) \in L^2(\Omega; \mathbb{R}^3) \times L^{p/(p-1)}(\Omega; \mathbb{R}^3)} \{\theta^*((\mathbf{w}_1, \mathbf{w}_2) - (\mathbf{u}_1, \mathbf{u}_2)) + R^*(\mathbf{u}_1, \mathbf{u}_2)\}, \tag{3.3}$$

where θ^*, R^* denote the conjugate functionals in $L^2(\Omega; \mathbb{R}^3) \times L^{p/(p-1)}(\Omega; \mathbb{R}^3)$ of θ and R with respect to the L^2-inner product, respectively.

Let $P^*(\mathbf{x}, \cdot, \cdot)$ denote the conjugate function of $P(\mathbf{x}, \cdot, \cdot)$ in $\mathbb{R}^6$ for fixed $\mathbf{x} \in \Omega$. By definition we see that

$$\begin{aligned} P^*(\mathbf{x}, \mathbf{s}_1, \mathbf{s}_2) &= \sup_{(\mathbf{r}_1, \mathbf{r}_2) \in \mathbb{R}^6} \{\langle \mathbf{s}_1, \mathbf{r}_1 \rangle + \langle \mathbf{s}_2, \mathbf{r}_2 \rangle - P(\mathbf{x}, \mathbf{r}_1, \mathbf{r}_2)\} \\ &= \begin{cases} Q^*(\mathbf{x}, \mathbf{s}_2) & \text{if } \mathbf{s}_1 = \mathbf{0}, \\ +\infty & \text{otherwise.} \end{cases} \end{aligned} \tag{3.4}$$

Thus we see that for any $\mathbf{v} \in L^{p/(p-1)}(\Omega; \mathbb{R}^3)$ $P^*(\cdot, \mathbf{0}, \mathbf{v}(\cdot)) \in L^1(\Omega)$. Therefore, we can apply the theorem on conjugate convex integrals [30, p. 532, Theorem 2] to deduce that

$$\theta^*(\mathbf{w}_1, \mathbf{w}_2) = \int_\Omega P^*(\mathbf{x}, \mathbf{w}_1(\mathbf{x}), \mathbf{w}_2(\mathbf{x}))d\mathbf{x}, \tag{3.5}$$

for all $(\mathbf{w}_1, \mathbf{w}_2) \in L^2(\Omega; \mathbb{R}^3) \times L^{p/(p-1)}(\Omega; \mathbb{R}^3)$.

Also we have that for all $(\mathbf{u}_1, \mathbf{u}_2) \in L^2(\Omega; \mathbb{R}^3) \times L^{p/(p-1)}(\Omega; \mathbb{R}^3)$

$$\begin{aligned} R^*(\mathbf{u}_1, \mathbf{u}_2) &= \sup_{(\mathbf{v}_1, \mathbf{v}_2) \in L^2(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{R}^3)} \{\langle \mathbf{u}_1, \mathbf{v}_1 \rangle_{L^2(\Omega; \mathbb{R}^3)} + \langle \mathbf{u}_2, \mathbf{v}_2 \rangle_{L^2(\Omega; \mathbb{R}^3)} - R(\mathbf{v}_1, \mathbf{v}_2)\} \\ &= \sup_{\mathbf{v} \in V_p(\Omega)} \{\langle \mathbf{u}_1, \mathbf{v} \rangle_{L^2(\Omega; \mathbb{R}^3)} + \langle \mathbf{u}_2, \mathbf{curl}\, \mathbf{v} \rangle_{L^2(\Omega; \mathbb{R}^3)}\} \\ &= \begin{cases} 0 & \text{if } \mathbf{u}_1 + \mathbf{curl}\, \mathbf{u}_2 = 0 \text{ in } (V_p(\Omega))^*, \\ +\infty & \text{otherwise.} \end{cases} \end{aligned} \tag{3.6}$$

By combining (3.6) with (3.3) we deduce that for all $(\mathbf{w}_1, \mathbf{w}_2) \in L^2(\Omega;\mathbb{R}^3) \times L^{p/(p-1)}(\Omega;\mathbb{R}^3)$

$$(\theta + R)^*(\mathbf{w}_1, \mathbf{w}_2) = \inf_{\substack{(\mathbf{u}_1,\mathbf{u}_2)\in L^2(\Omega;\mathbb{R}^3)\times L^{p/(p-1)}(\Omega;\mathbb{R}^3)\\ \mathbf{u}_1+\mathbf{curl}\,\mathbf{u}_2=0 \text{ in } (V_p(\Omega))^*}} \theta^*((\mathbf{w}_1, \mathbf{w}_2) - (\mathbf{u}_1, \mathbf{u}_2)). \tag{3.7}$$

By (3.2), (3.4), (3.5), and (3.7), we see that for all $\mathbf{w} \in L^2(\Omega;\mathbb{R}^3)$

$$\begin{aligned} F^*(\mathbf{w}) &= \inf_{\substack{(\mathbf{u}_1,\mathbf{u}_2)\in L^2(\Omega;\mathbb{R}^3)\times L^{p/(p-1)}(\Omega;\mathbb{R}^3)\\ \mathbf{u}_1+\mathbf{curl}\,\mathbf{u}_2=0 \text{ in } (V_p(\Omega))^*}} \int_\Omega P^*(\mathbf{x}, \mathbf{w}(\mathbf{x}) - \mathbf{u}_1(\mathbf{x}), -\mathbf{u}_2(\mathbf{x}))d\mathbf{x} \\ &= \inf_{\substack{\mathbf{u}\in L^{p/(p-1)}(\Omega;\mathbb{R}^3)\\ \mathbf{w}-\mathbf{curl}\,\mathbf{u}=0 \text{ in } (V_p(\Omega))^*}} \int_\Omega Q^*(\mathbf{x}, \mathbf{u}(\mathbf{x}))d\mathbf{x}. \end{aligned}$$

If $F^*(\mathbf{w}) < +\infty$, by using the first inequality of (3.1) we can show that the functional $\int_\Omega Q^*(\mathbf{x}, \mathbf{u}(\mathbf{x}))d\mathbf{x}$ takes its minimum in the closed convex set $\{\mathbf{u} \in L^{p/(p-1)}(\Omega;\mathbb{R}^3) \mid \mathbf{w} - \mathbf{curl}\,\mathbf{u} = 0 \text{ in } (V_p(\Omega))^*\}$, which completes the proof. □

We can characterise the subdifferential of F as follows.

Theorem 3.3. *For all* $\mathbf{u} \in L^2(\Omega;\mathbb{R}^3)$ *with* $\partial F(\mathbf{u}) \neq \emptyset$,

$$\partial F(\mathbf{u}) = \left\{ \mathbf{w} \in L^2(\Omega;\mathbb{R}^3) \;\middle|\; \begin{array}{l} \textit{There exists } \boldsymbol{\phi} \in L^{p/(p-1)}(\Omega;\mathbb{R}^3) \textit{ such that } \mathbf{w} - \mathbf{curl}\,\boldsymbol{\phi} = 0 \textit{ in } (V_p(\Omega))^*, \\ \boldsymbol{\phi}(\mathbf{x}) \in \partial Q(\mathbf{x}, \mathbf{curl}\,\mathbf{u}(\mathbf{x})) \textit{ a.e. } \mathbf{x} \in \Omega. \end{array} \right\},$$

where $\partial Q(\mathbf{x}, \mathbf{r})$ *is the subdifferential of* $Q(\mathbf{x}, \cdot)$ *for fixed* $\mathbf{x} \in \Omega$.

Proof. (‘ $\subset$’): Let $\mathbf{w} \in \partial F(\mathbf{u})$. This inclusion is equivalent to the equality

$$F(\mathbf{u}) + F^*(\mathbf{w}) = \langle \mathbf{u}, \mathbf{w} \rangle_{L^2(\Omega;\mathbb{R}^3)} \tag{3.8}$$

(see, e.g. [16]). By Proposition 3.2 there exists $\boldsymbol{\phi} \in L^{p/(p-1)}(\Omega;\mathbb{R}^3)$ such that $\mathbf{w} - \mathbf{curl}\,\boldsymbol{\phi} = 0$ in $(V_p(\Omega))^*$ and

$$F^*(\mathbf{w}) = \int_\Omega Q^*(\mathbf{x}, \boldsymbol{\phi}(\mathbf{x}))d\mathbf{x}. \tag{3.9}$$

By (3.8) and (3.9) we obtain

$$\int_\Omega (Q(\mathbf{x}, \mathbf{curl}\,\mathbf{u}(\mathbf{x})) + Q^*(\mathbf{x}, \boldsymbol{\phi}(\mathbf{x})) - \langle \mathbf{u}(\mathbf{x}), \mathbf{w}(\mathbf{x}) \rangle)d\mathbf{x} = 0. \tag{3.10}$$

Since $\langle \mathbf{u}, \mathbf{w} \rangle_{L^2(\Omega;\mathbb{R}^3)} = \langle \mathbf{curl}\,\mathbf{u}, \boldsymbol{\phi} \rangle_{L^2(\Omega;\mathbb{R}^3)}$, the equality (3.10) leads to

$$\int_\Omega (Q(\mathbf{x}, \mathbf{curl}\,\mathbf{u}(\mathbf{x})) + Q^*(\mathbf{x}, \boldsymbol{\phi}(\mathbf{x})) - \langle \mathbf{curl}\,\mathbf{u}(\mathbf{x}), \boldsymbol{\phi}(\mathbf{x}) \rangle)d\mathbf{x} = 0. \tag{3.11}$$

By the definition of $Q^*(\mathbf{x}, \boldsymbol{\phi}(\mathbf{x}))$ we see that

$$Q(\mathbf{x}, \mathbf{curl}\,\mathbf{u}(\mathbf{x})) + Q^*(\mathbf{x}, \boldsymbol{\phi}(\mathbf{x})) - \langle \mathbf{curl}\,\mathbf{u}(\mathbf{x}), \boldsymbol{\phi}(\mathbf{x}) \rangle \geq 0 \tag{3.12}$$

for a.e. $\mathbf{x} \in \Omega$. Thus, by (3.11) and (3.12) we deduce that

$$Q(\mathbf{x}, \mathbf{curl}\,\mathbf{u}(\mathbf{x})) + Q^*(\mathbf{x}, \boldsymbol{\phi}(\mathbf{x})) - \langle \mathbf{curl}\,\mathbf{u}(\mathbf{x}), \boldsymbol{\phi}(\mathbf{x}) \rangle = 0$$

for a.e. $\mathbf{x} \in \Omega$, which implies that $\boldsymbol{\phi}(\mathbf{x}) \in \partial Q(\mathbf{x}, \mathbf{curl}\,\mathbf{u}(\mathbf{x}))$ a.e. $\mathbf{x} \in \Omega$.

(‘ $\supset$’): Let $\mathbf{w} \in L^2(\Omega;\mathbb{R}^3)$ and $\boldsymbol{\phi} \in L^{p/(p-1)}(\Omega;\mathbb{R}^3)$ satisfy $\mathbf{w} - \mathbf{curl}\,\boldsymbol{\phi} = 0$ in $(V_p(\Omega))^*$, and

$$\boldsymbol{\phi}(\mathbf{x}) \in \partial Q(\mathbf{x}, \mathbf{curl}\,\mathbf{u}(\mathbf{x})) \text{ for a.e. } \mathbf{x} \in \Omega. \tag{3.13}$$

The inclusion (3.13) is equivalent to the equality that

$$Q(\mathbf{x}, \mathbf{curl}\,\mathbf{u}(\mathbf{x})) + Q^*(\mathbf{x}, \boldsymbol{\phi}(\mathbf{x})) - \langle\, \mathbf{curl}\,\mathbf{u}(\mathbf{x}), \boldsymbol{\phi}(\mathbf{x})\rangle = 0$$

for a.e. $\mathbf{x} \in \Omega$. This equality yields

$$F(\mathbf{u}) + \int_\Omega Q^*(\mathbf{x}, \boldsymbol{\phi}(\mathbf{x}))d\mathbf{x} = \langle\, \mathbf{curl}\,\mathbf{u}, \boldsymbol{\phi}\rangle_{L^2(\Omega;\mathbb{R}^3)}.$$

By the condition $\mathbf{w} - \mathbf{curl}\,\boldsymbol{\phi} = 0$ in $(V_p(\Omega))^*$, we have

$$F(\mathbf{u}) + \int_\Omega Q^*(\mathbf{x}, \boldsymbol{\phi}(\mathbf{x}))d\mathbf{x} = \langle \mathbf{u}, \mathbf{w}\rangle_{L^2(\Omega;\mathbb{R}^3)}.$$

By Proposition 3.2 we deduce that

$$F(\mathbf{u}) + F^*(\mathbf{w}) \le \langle \mathbf{u}, \mathbf{w}\rangle_{L^2(\Omega;\mathbb{R}^3)}. \tag{3.14}$$

By the definition of $F^*(\mathbf{w})$ it is immediately shown that

$$F(\mathbf{u}) + F^*(\mathbf{w}) \ge \langle \mathbf{u}, \mathbf{w}\rangle_{L^2(\Omega;\mathbb{R}^3)}. \tag{3.15}$$

By (3.14) and (3.15) we obtain $F(\mathbf{u}) + F^*(\mathbf{w}) = \langle \mathbf{u}, \mathbf{w}\rangle_{L^2(\Omega;\mathbb{R}^3)}$, or $\mathbf{w} \in \partial F(\mathbf{u})$. □

Let us apply this characterisation theorem to the formulation (**P1**).

Corollary 3.4. *Let $\widehat{\boldsymbol{\mathcal{H}}} \in H^1(0,T;L^2(\Omega;\mathbb{R}^3))$ be a solution to the problem* (**P1**)*. For a.e. $t \in (0,T)$ there exists $\boldsymbol{\mathcal{E}}^t \in H(\mathbf{curl};\Omega_s)$ such that*

$$\begin{aligned}
&\mu\partial_t\widehat{\boldsymbol{\mathcal{H}}}(\mathbf{x},t) + \mu\partial_t\boldsymbol{\mathcal{H}}_s(\mathbf{x},t) + \mathbf{curl}\,\boldsymbol{\mathcal{E}}^t(\mathbf{x}) = \mathbf{0} \ \textit{a.e.}\ \mathbf{x} \in \Omega_s,\\
&\boldsymbol{\mathcal{E}}^t(\mathbf{x}) \in \partial G_{\widehat{\boldsymbol{\mathcal{B}}}(\mathbf{x},t)}(\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}(\mathbf{x},t)) \ \textit{a.e.}\ \mathbf{x} \in \Omega_s.
\end{aligned}$$

Proof. Let us fix a.e. $t \in (0,T)$. The energy density $Q(\mathbf{x},\mathbf{r}) := G(\widehat{\boldsymbol{\mathcal{B}}}(\mathbf{x},t),\mathbf{r}) : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ satisfies the conditions (1)−(3) for $p = 2$. Thus we can apply Theorem 3.3 to the energy

$$F(\boldsymbol{\phi}) = \begin{cases} \displaystyle\int_\Omega Q(\mathbf{x}, \mathbf{curl}\,\boldsymbol{\phi}(\mathbf{x}))d\mathbf{x} & \text{if } \boldsymbol{\phi} \in V(\Omega),\\ +\infty & \text{otherwise.}\end{cases}$$

Since the inclusion $\mu\partial_t\widehat{\boldsymbol{\mathcal{H}}}(t) + \mu\partial_t\boldsymbol{\mathcal{H}}_s(t) \in -\partial F(\widehat{\boldsymbol{\mathcal{H}}}(t))$ holds, there exists $\mathbf{w}^t \in \partial F(\widehat{\boldsymbol{\mathcal{H}}}(t))$ such that $\mu\partial_t\widehat{\boldsymbol{\mathcal{H}}}(\mathbf{x},t) + \mu\partial_t\boldsymbol{\mathcal{H}}_s(\mathbf{x},t) + \mathbf{w}^t(\mathbf{x}) = \mathbf{0}$ a.e. $\mathbf{x} \in \Omega$. Note that if $\mathbf{w}^t - \mathbf{curl}\,\boldsymbol{\phi} = 0$ in $(V(\Omega))^*$ for some $\boldsymbol{\phi} \in L^2(\Omega;\mathbb{R}^3)$, then $\mathbf{w}^t|_{\Omega_s} = \mathbf{curl}\,\boldsymbol{\phi}|_{\Omega_s}$ in $\mathscr{D}'(\Omega_s)$. Thus, the characterisation of $\partial F(\widehat{\boldsymbol{\mathcal{H}}}(t))$ yields the result. □

4. Finite element approximation

In this section we discretize our variational inequality formulations (**P1**) and (**P2**) by a semi-implicit time stepping scheme and by the edge finite element of lowest order on a tetrahedral mesh and prove the subsequence convergence property of the fully discrete solutions.

In order to mesh the domains Ω_s and Ω_d by tetrahedrons, additionally let us assume that Ω and Ω_s are bounded simply connected Lipschitz polyhedrons with the connected boundaries $\partial\Omega$ and $\partial\Omega_s$, respectively. Moreover we assume that Ω and Ω_s are star-shaped for a point $\mathbf{y}_0 \in \Omega_s$ in the sense that

$$\alpha(\mathbf{x}-\mathbf{y}_0)+\mathbf{y}_0 \in \Omega,\ \alpha(\mathbf{z}-\mathbf{y}_0)+\mathbf{y}_0 \in \Omega_s,\ \forall \alpha \in [0,1), \forall \mathbf{x} \in \overline{\Omega}, \forall \mathbf{z} \in \overline{\Omega_s}. \tag{4.1}$$

From now on we add the following differentiability condition to the energy densities $g_\perp(\cdot)$, $g_\parallel(\cdot)$, $g_0(\cdot)$.

$$\begin{aligned} &g_\perp(\cdot), g_\parallel(\cdot), g_0(\cdot) : \mathbb{R} \to \mathbb{R} \text{ are differentiable,} \\ &|g'_\perp(x)| \le A_{\perp 5}|x|,\ |g'_\parallel(x)| \le A_{\parallel 5}|x|,\ |g'_0(x)| \le A_{05}|x|, \end{aligned} \tag{4.2}$$

for all $x \in \mathbb{R}$, where $A_{\perp 5}, A_{\parallel 5}, A_{05} > 0$ are positive constants.

4.1. **Finite element method**

We define finite element spaces and prepare some lemmas used in our convergence analysis. Let τ_h be a tetrahedral mesh covering Ω, where $h = \max\{h_K \mid K \in \tau_h\}$ and h_K is the diameter of the smallest sphere containing $\overline{K}$. We assume that each element $K \in \tau_h$ belongs either to $\overline{\Omega_s}$ or to $\overline{\Omega_d}$ and the mesh τ_h is regular in the sense that there are constants $C > 0$ and $h_0 > 0$ such that

$$\frac{h_K}{\rho_K} \le C, \forall K \in \tau_h, \forall h \in (0, h_0], \tag{4.3}$$

where ρ_K is the diameter of the largest sphere contained in $\overline{K}$. Moreover, the mesh τ_h is assumed to satisfy the property that there is a constant $\hat{C} > 0$ such that

$$\frac{h}{h_K} \le \hat{C}, \forall K \in \tau_h, K \subset \overline{\Omega_s}, \forall h \in (0, h_0], \tag{4.4}$$

and the quasi-uniform property on $\partial\Omega$ that there is a constant $C' > 0$ such that

$$\frac{h}{h_f} \le C' \text{ for any face } f \subset \partial\Omega \text{ and any } h \in (0, h_0], \tag{4.5}$$

where h_f is the diameter of the smallest circle containing f. Note that the property (4.4) is weaker than the quasi-uniform property over Ω.

Next let us define the finite element spaces. The curl-conforming finite element space $U_h(\Omega)$ of the lowest order by Nédélec [22] is defined by

$$U_h(\Omega) := \{\boldsymbol{\phi}_h \in H(\mathbf{curl};\Omega) \mid \boldsymbol{\phi}_h|_K \in R_1, \forall K \in \tau_h\},$$

where $R_1 := \{\mathbf{a} + \mathbf{b} \times \mathbf{x} \mid \mathbf{a}, \mathbf{b} \in \mathbb{R}^3\}$. The degrees of freedom of the space $U_h(\Omega)$ are

$$M_e(\boldsymbol{\phi}_h) := \int_e \langle \boldsymbol{\phi}_h, \boldsymbol{\tau} \rangle ds,$$

where e is an edge of $K \in \tau_h$ and $\boldsymbol{\tau}$ is a unit tangent to e. Let $r_h(\boldsymbol{\phi}) \in U_h(\Omega)$ denote the interpolant of a sufficiently smooth function $\boldsymbol{\phi}$. In this paper, we only consider $r_h(\boldsymbol{\phi})$ for $\boldsymbol{\phi} \in H^1(\mathbf{curl};\Omega)$, which is well-defined (see [21, p. 134]).

The finite-dimensional subspace $V_h(\Omega)$ of $V(\Omega)$ is defined by

$$V_h(\Omega) := \{\boldsymbol{\phi}_h \in U_h(\Omega) \mid \mathbf{curl}\,\boldsymbol{\phi}_h = \mathbf{0} \text{ in } \Omega_d,\ \mathbf{n}\times\boldsymbol{\phi}_h = \mathbf{0} \text{ on } \partial\Omega\}.$$

Note that the boundary condition $\mathbf{n}\times\boldsymbol{\phi}_h = \mathbf{0}$ on $\partial\Omega$ is attained by taking all the degrees of freedom associated with the edges on $\partial\Omega$ to be zero.

The H^1-conforming finite element space $Z_h(\Omega)$ of the lowest order is defined by

$$Z_h(\Omega) := \{f_h \in H^1(\Omega) \mid f_h|_K \in P_1, \forall K \in \tau_h\},$$

where $P_1 := \{a_0 + a_1x + a_2y + a_3z \mid a_i \in \mathbb{R}, i = 0,1,2,3\}$. The degrees of freedom $m_v(f_h)$ of $Z_h(\Omega)$ are defined by

$$m_v(f_h) := f_h(\mathbf{x}_v),$$

where $\mathbf{x}_v$ ($\in \mathbb{R}^3$) is the coordinate of the vertex v. Let $\pi_h(f) \in Z_h(\Omega)$ denote the interpolant of a sufficiently smooth function f. In this paper, we only consider $\pi_h(f)$ for $f \in H^2(\Omega)$, which is well-defined (see [21, p. 144]). The subspace $Z_{0,h}(\Omega)$ of $Z_h(\Omega)$ is defined by

$$Z_{0,h}(\Omega) := \{f_h \in Z_h(\Omega) \mid f_h|_{\partial\Omega} = 0\}.$$

The boundary condition $f_h|_{\partial\Omega} = 0$ is achieved by taking $m_v(f_h)$ for each vertex v on $\partial\Omega$ to be zero.

The space of discrete divergence-free functions $X_h^{(\mu)}(\Omega)$ is defined by

$$X_h^{(\mu)}(\Omega) := \{\boldsymbol{\phi}_h \in U_h(\Omega) \mid \langle \mu\boldsymbol{\phi}_h, \nabla f_h\rangle_{L^2(\Omega;\mathbb{R}^3)} = 0, \forall f_h \in Z_{0,h}(\Omega)\}.$$

The discrete subspace $W_h(\Omega)$ of the hybrid space $W(\Omega)$ is defined by

$$W_h(\Omega) := \{(\boldsymbol{\phi}_h|\nabla u_h) \in L^2(\Omega;\mathbb{R}^3) \mid (\boldsymbol{\phi}_h, u_h) \in U_h(\Omega_s)\times Z_h(\Omega_d),\ (\boldsymbol{\phi}_h|\nabla u_h) \in U_h(\Omega),\ u_h|_{\partial\Omega} = 0\},$$

where $U_h(\Omega_s) := \{\boldsymbol{\phi}_h|_{\Omega_s} \mid \boldsymbol{\phi}_h \in U_h(\Omega)\}$ and $Z_h(\Omega_d) := \{u_h|_{\Omega_d} \mid u_h \in Z_h(\Omega)\}$.

Thanks to the equivalence stated below, we can deal with the curl-free constraint in the nonconducting region Ω_d by implementing $W_h(\Omega)$ in practise.

Proposition 4.1. *([13, Proposition 3.1]) The space $W_h(\Omega)$ is isomorphic to $V_h(\Omega)$ as a Hilbert space.*

Remark 4.2. Note that the space $W_h(\Omega)$ can be equivalently rewritten as

$$W_h(\Omega) = \{(\boldsymbol{\phi}_h|\nabla u_h) \in L^2(\Omega;\mathbb{R}^3) \mid (\boldsymbol{\phi}_h, u_h) \in U_h(\Omega_s)\times Z_h(\Omega_d),\ \mathbf{n}\times\boldsymbol{\phi}_h = \mathbf{n}\times\nabla u_h \text{ on } \partial\Omega_s,\ u_h|_{\partial\Omega} = 0\},$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega_s$. A point to construct $W_h(\Omega)$ in practical computation is to fulfil the tangential continuity condition

$$\mathbf{n}\times\boldsymbol{\phi}_h = \mathbf{n}\times\nabla u_h \text{ on } \partial\Omega_s. \tag{4.6}$$

The condition (4.6) is equivalent to the relation that

$$M_e(\boldsymbol{\phi}_h) = m_{v_1}(u_h) - m_{v_0}(u_h) \tag{4.7}$$

for each edge $e \subset \partial\Omega_s$, where v_0 is the initial vertex of e and v_1 is the terminal vertex of e. To implement the linear relation (4.7) all along $\partial\Omega_s$ ensures the tangential continuity condition (4.6) in $W_h(\Omega)$.

Below we list a couple of estimates from [17], [21] which will be needed in our analysis.

Lemma 4.3. *([17, Chapter III, Theorem 5.4], [21, Theorem 5.41]) There is a constant $C > 0$ such that for any $\boldsymbol{\phi} \in H^1(\mathbf{curl};\Omega)$*

$$\|\boldsymbol{\phi} - r_h(\boldsymbol{\phi})\|_{H(\mathbf{curl};\Omega)} \le Ch\|\boldsymbol{\phi}\|_{H^1(\mathbf{curl};\Omega)}.$$

By following the proof of [21, Lemma 5.52] we see that the following estimate holds.

Lemma 4.4. *([21, Lemma 5.52]) There is a constant $C > 0$ such that for any $\boldsymbol{\phi} \in H^1(\mathbf{curl};\Omega)$*

$$\|\boldsymbol{\phi} - r_h(\boldsymbol{\phi})\|_{L^2(\partial\Omega;\mathbb{R}^3)} \le Ch^{1/2}\|\boldsymbol{\phi}\|_{H^1(\mathbf{curl};\Omega)}.$$

The assumption (4.1) is required to show the following lemma, which enables us to approximate a function $\boldsymbol{\phi} : [0,T] \to V(\Omega)$ by a sequence of smooth functions.

Lemma 4.5. *(1) For any $\boldsymbol{\phi} \in L^2(0,T;H(\mathbf{curl};\Omega))$ with $\boldsymbol{\phi}(t) \in V(\Omega)$ for a.e. $t \in (0,T)$ there exists a sequence $\{\boldsymbol{\phi}_l\}_{l=1}^{\infty} \subset L^2(0,T;W^{p,q}(\Omega;\mathbb{R}^3))$ for all $p \in \mathbb{N}\cup\{0\}$ and $1 \le q \le +\infty$ with $\boldsymbol{\phi}_l(t) \in V(\Omega)\cap C_0^\infty(\Omega;\mathbb{R}^3)$ for a.e. $t \in (0,T)$ such that as $l \to +\infty$*

$$\boldsymbol{\phi}_l \to \boldsymbol{\phi} \text{ strongly in } L^2(0,T;H(\mathbf{curl};\Omega)).$$

(2) For any $\boldsymbol{\phi} \in C^1([0,T];L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T;H(\mathbf{curl};\Omega))$ with $\boldsymbol{\phi}(t) \in V(\Omega)$ for all $t \in [0,T]$ there exists a sequence $\{\boldsymbol{\phi}_l\}_{l=1}^{\infty} \subset C^1([0,T];W^{p,q}(\Omega;\mathbb{R}^3))$ for all $p \in \mathbb{N}\cup\{0\}$ and $1 \le q \le +\infty$ with $\boldsymbol{\phi}_l(t) \in V(\Omega)\cap C_0^\infty(\Omega;\mathbb{R}^3)$ for all $t \in [0,T]$ such that as $l \to +\infty$

$$\begin{aligned}
\boldsymbol{\phi}_l &\to \boldsymbol{\phi} \text{ strongly in } L^2(0,T;H(\mathbf{curl};\Omega)),\\
\partial_t\boldsymbol{\phi}_l &\to \partial_t\boldsymbol{\phi} \text{ strongly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)),\\
\boldsymbol{\phi}_l(t) &\to \boldsymbol{\phi}(t) \text{ strongly in } L^2(\Omega;\mathbb{R}^3), \forall t \in [0,T].
\end{aligned}$$

Proof. The proof is essentially same as that of [13, Lemma 3.4]. We give the proof for (1) for completeness. The statement (2) can be proved in the same way.

Take any $\boldsymbol{\phi} \in L^2(0,T;H(\mathbf{curl};\Omega))$ with $\boldsymbol{\phi}(t) \in V(\Omega)$ and fix a.e. $t \in (0,T)$. Since $\mathbf{n}\times\boldsymbol{\phi}(t) = \mathbf{0}$ on $\partial\Omega$, we can define $\hat{\boldsymbol{\phi}}(t) \in H(\mathbf{curl};\mathbb{R}^3)$ by $\hat{\boldsymbol{\phi}}(t) := \boldsymbol{\phi}(t)$ in Ω, $\hat{\boldsymbol{\phi}}(t) := \mathbf{0}$ in $\mathbb{R}^3\backslash\Omega$. For $\theta \in (0,1)$, define $\hat{\boldsymbol{\phi}}_\theta(t) \in H(\mathbf{curl};\mathbb{R}^3)$ by $\hat{\boldsymbol{\phi}}_\theta(\mathbf{x},t) := \theta\hat{\boldsymbol{\phi}}((\mathbf{x}-\mathbf{y}_0)/\theta + \mathbf{y}_0, t)$, where $\mathbf{y}_0 \in \Omega_s$ is the point in the assumption (4.1).

We show that $\mathrm{supp}(\hat{\boldsymbol{\phi}}_\theta(t)) \subset \Omega$. Assume $\mathrm{supp}(\hat{\boldsymbol{\phi}}_\theta(t)) \ne \emptyset$. For all $\hat{\mathbf{x}} \in \mathrm{supp}(\hat{\boldsymbol{\phi}}_\theta(t))$ there exists a sequence $\{\mathbf{x}_n\}_{n=1}^{\infty} \subset \mathbb{R}^3$ such that $\mathbf{x}_n \to \hat{\mathbf{x}}$ as $n \to +\infty$ and $\hat{\boldsymbol{\phi}}_\theta(\mathbf{x}_n,t) \ne \mathbf{0}$. By the definition of $\hat{\boldsymbol{\phi}}_\theta(t)$, $(\mathbf{x}_n-\mathbf{y}_0)/\theta+\mathbf{y}_0 \in \Omega$ for all $n \in \mathbb{N}$. By $n \to +\infty$ we obtain $(\hat{\mathbf{x}}-\mathbf{y}_0)/\theta + \mathbf{y}_0 \in \overline{\Omega}$. By the assumption (4.1) we see that

$$\hat{\mathbf{x}} = \theta\left(\frac{\hat{\mathbf{x}}-\mathbf{y}_0}{\theta} + \mathbf{y}_0 - \mathbf{y}_0\right) + \mathbf{y}_0 \in \Omega.$$

Thus, $\mathrm{supp}(\hat{\boldsymbol{\phi}}_\theta(t)) \subset \Omega$. The inclusion $\mathrm{supp}(\mathbf{curl}\,\hat{\boldsymbol{\phi}}_\theta(t)) \subset \Omega_s$ is similarly proved by using (4.1). Therefore, we have $\hat{\boldsymbol{\phi}}_\theta(t)|_\Omega \in V(\Omega)$.

We can choose $\varepsilon = \varepsilon(\theta) > 0$ sufficiently small so that $\rho_\varepsilon * \hat{\boldsymbol{\phi}}_\theta(t)|_\Omega \in V(\Omega)\cap C_0^\infty(\Omega;\mathbb{R}^3)$, where $\rho_\varepsilon \in C_0^\infty(\Omega)$ is a mollifier. By the standard properties of mollifier we see that $\rho_\varepsilon * \hat{\boldsymbol{\phi}}_\theta(t)|_\Omega \to \boldsymbol{\phi}$ strongly in $L^2(0,T;H(\mathbf{curl};\Omega))$ as $\theta \nearrow 1$, $\varepsilon(\theta) \searrow 0$. Moreover, for any multi-index $\alpha \in (\mathbb{N}\cup\{0\})^3$

$$|\frac{\partial^\alpha}{\partial\mathbf{x}^\alpha}(\rho_\varepsilon * \hat{\boldsymbol{\phi}}_\theta)(\mathbf{x},t)| \le C(\varepsilon,\alpha)\|\hat{\boldsymbol{\phi}}_\theta(t)\|_{L^2(\Omega;\mathbb{R}^3)} = C(\varepsilon,\alpha)\theta^{5/2}\|\boldsymbol{\phi}(t)\|_{L^2(\Omega;\mathbb{R}^3)},$$

which shows that $\rho_\varepsilon * \hat{\boldsymbol{\phi}}_\theta|_\Omega \in L^2(0,T;W^{p,q}(\Omega;\mathbb{R}^3))$ for all $p \in \mathbb{N}\cup\{0\}$ and $1 \le q \le +\infty$. $\square$

Take $N \in \mathbb{N}$ and set $\Delta t := T/N$. By using the function $\boldsymbol{\phi}_l$ in Lemma 4.5 (2), we define a piecewise linear in time function $\widetilde{\boldsymbol{\phi}}_{l,h} : [0,T] \to V_h(\Omega)$ and a piecewise constant in time function $\overline{\boldsymbol{\phi}}_{l,h} : [0,T] \to V_h(\Omega)$ by

$$\widetilde{\boldsymbol{\phi}}_{l,h}(t) := \frac{t-(n-1)\Delta t}{\Delta t} r_h(\boldsymbol{\phi}_l(n\Delta t)) + \frac{n\Delta t - t}{\Delta t} r_h(\boldsymbol{\phi}_l((n-1)\Delta t)) \text{ in } [(n-1)\Delta t, n\Delta t],\ (n = 1, \cdots, N), \quad (4.8)$$

$$\overline{\boldsymbol{\phi}}_{l,h}(t) := \begin{cases} r_h(\boldsymbol{\phi}_l(n\Delta t)) & \text{in } ((n-1)\Delta t, n\Delta t],\ (n = 1, \cdots, N), \\ r_h(\boldsymbol{\phi}_l(0)) & \text{on } \{t = 0\}. \end{cases} \quad (4.9)$$

From now let Λ denote a subset of $(0, h_0]$ which has the only accumulation point 0.

Lemma 4.6. *For the function $\boldsymbol{\phi}_l$ in Lemma 4.5 (2) and the functions $\widetilde{\boldsymbol{\phi}}_{l,h}$ and $\overline{\boldsymbol{\phi}}_{l,h}$ defined in* (4.8), (4.9)*, the following convergence properties hold as $h \searrow 0$, $h \in \Lambda$ and $\Delta t \searrow 0$.*

$$\begin{aligned} \widetilde{\boldsymbol{\phi}}_{l,h} &\to \boldsymbol{\phi}_l \text{ strongly in } C([0,T]; L^\infty(\Omega;\mathbb{R}^3)), \\ \mathbf{curl}\, \widetilde{\boldsymbol{\phi}}_{l,h} &\to \mathbf{curl}\, \boldsymbol{\phi}_l \text{ strongly in } C([0,T]; L^\infty(\Omega;\mathbb{R}^3)), \\ \partial_t \widetilde{\boldsymbol{\phi}}_{l,h} &\to \partial_t \boldsymbol{\phi}_l \text{ strongly in } L^\infty(0,T; L^\infty(\Omega;\mathbb{R}^3)), \\ \overline{\boldsymbol{\phi}}_{l,h} &\to \boldsymbol{\phi}_l \text{ strongly in } L^\infty(0,T; L^\infty(\Omega;\mathbb{R}^3)), \\ \mathbf{curl}\, \overline{\boldsymbol{\phi}}_{l,h} &\to \mathbf{curl}\, \boldsymbol{\phi}_l \text{ strongly in } L^\infty(0,T; L^\infty(\Omega;\mathbb{R}^3)). \end{aligned}$$

Proof. By a similar argument as in [17, Chapter III, Theorem 5.4], [21, Theorem 5.41], we can prove that there exists a constant $C > 0$ depending only on the constant in (4.3) such that for any $\boldsymbol{\phi} \in C^2(\overline{\Omega};\mathbb{R}^3)$

$$\|\boldsymbol{\phi} - r_h(\boldsymbol{\phi})\|_{L^\infty(\Omega;\mathbb{R}^3)} \le Ch\|\nabla \boldsymbol{\phi}\|_{L^\infty(\Omega;\mathbb{R}^9)},\ \|\mathbf{curl}\, \boldsymbol{\phi} - \mathbf{curl}\, r_h(\boldsymbol{\phi})\|_{L^\infty(\Omega;\mathbb{R}^3)} \le Ch\|\nabla\, \mathbf{curl}\, \boldsymbol{\phi}\|_{L^\infty(\Omega;\mathbb{R}^9)}.$$

The desired convergence properties are proved by using these estimates and the property of $\boldsymbol{\phi}_l$. □

Let us derive the inverse inequality for the edge finite element of lowest order on a tetrahedral mesh. The proof uses the regular condition (4.3).

Lemma 4.7. *There is a constant $C > 0$ depending only on the constant in* (4.3) *such that*

$$\int_K |\,\mathbf{curl}\, \boldsymbol{\phi}_h|^2 d\mathbf{x} \le \frac{C}{\rho_K^2} \int_K |\boldsymbol{\phi}_h|^2 d\mathbf{x}$$

for all $\boldsymbol{\phi}_h \in U_h(\Omega)$, $K \in \tau_h$ and $h \in \Lambda$.

Proof. (Step1) First we derive an inequality of the form

$$\int_{\hat{K}} |\,\mathbf{curl}\, \hat{\boldsymbol{\phi}}|^2 d\hat{\mathbf{x}} \le C \int_{\hat{K}} |\hat{\boldsymbol{\phi}}|^2 d\hat{\mathbf{x}}$$

for any edge finite element function $\hat{\boldsymbol{\phi}}$ on the reference element $\hat{K}$. Let $\hat{\boldsymbol{\phi}}_i$ $(i = 1, \cdots, 6)$ be the basis functions on $\hat{K}$ associated with the edges $\hat{e_i}$ $(i = 1, \cdots, 6)$ respectively, where $\hat{e_1}$ is from v_1 to v_2, $\hat{e_2}$ is from v_1 to v_3, $\hat{e_3}$ is from v_1 to v_4, $\hat{e_4}$ is from v_2 to v_3, $\hat{e_5}$ is from v_2 to v_4, and $\hat{e_6}$ is from v_3 to v_4 for the vertexes $v_1 = (0,0,0)$, $v_2 = (1,0,0)$, $v_3 = (0,1,0)$, $v_4 = (0,0,1)$.

Let A, B be 6×6 symmetric matrices defined by

$$A := (\langle\, \mathbf{curl}\, \hat{\boldsymbol{\phi}}_i, \mathbf{curl}\, \hat{\boldsymbol{\phi}}_j \rangle_{L^2(\hat{K};\mathbb{R}^3)})_{1 \le i,j \le 6},\ B := (\langle \hat{\boldsymbol{\phi}}_i, \hat{\boldsymbol{\phi}}_j \rangle_{L^2(\hat{K};\mathbb{R}^3)})_{1 \le i,j \le 6}.$$

An explicit calculation shows that the eigen values of A are $0, 2/3, 8/3$ and those of B are $1/60, 1/24, 1/6$. Thus we obtain $\sup_{\mathbf{x}\in\mathbb{R}^6} \langle A\mathbf{x},\mathbf{x}\rangle/|\mathbf{x}|^2 = 8/3$, $\inf_{\mathbf{x}\in\mathbb{R}^6} \langle B\mathbf{x},\mathbf{x}\rangle/|\mathbf{x}|^2 = 1/60$, or $\langle A\mathbf{x},\mathbf{x}\rangle \le 160\langle B\mathbf{x},\mathbf{x}\rangle$ $\forall \mathbf{x}\in\mathbb{R}^6$, which implies that

$$\int_{\hat{K}} |\,\mathbf{curl}\,\hat{\boldsymbol{\phi}}|^2 d\hat{\mathbf{x}} \le 160 \int_{\hat{K}} |\hat{\boldsymbol{\phi}}|^2 d\hat{\mathbf{x}} \tag{4.10}$$

for any edge finite element function $\hat{\boldsymbol{\phi}} : \hat{K} \to \mathbb{R}^3$.

(Step2) We derive inequalities of the form

$$\int_K |\,\mathbf{curl}\,\boldsymbol{\phi}_h|^2 d\mathbf{x} \le \frac{C}{\rho_K}\int_{\hat{K}} |\,\mathbf{curl}\,\hat{\boldsymbol{\phi}}_h|^2 d\hat{\mathbf{x}},\ \int_{\hat{K}} |\hat{\boldsymbol{\phi}}_h|^2 d\hat{\mathbf{x}} \le \frac{C}{\rho_K}\int_K |\boldsymbol{\phi}_h|^2 d\mathbf{x}$$

with a constant $C > 0$ depending only on the constant in (4.3). Note that for an edge finite element function $\boldsymbol{\phi}_h : K \to \mathbb{R}^3$ the functions $\hat{\boldsymbol{\phi}}_h$, $\mathbf{curl}\,\hat{\boldsymbol{\phi}}_h : \hat{K} \to \mathbb{R}^3$ are defined by the relations (see, e.g. [21, p.77, p.78])

$$\boldsymbol{\phi}_h(B_K\hat{\mathbf{x}} + b_K) = B_K^{-T}\hat{\boldsymbol{\phi}}_h(\hat{\mathbf{x}}),\ \mathbf{curl}\,\boldsymbol{\phi}_h(B_K\hat{\mathbf{x}} + b_K) = \frac{1}{\det(B_K)} B_K(\mathbf{curl}\,\hat{\boldsymbol{\phi}}_h)(\hat{\mathbf{x}}),$$

with the non-singular 3×3 matrix B_K and the vector b_K such that $\hat{\mathbf{x}} \mapsto B_K\hat{\mathbf{x}} + b_K$ is a bijective map from $\hat{K}$ to K.

Let the norm $|B_K|$ be defined by $|B_K| := \sup_{\mathbf{v}\in\mathbb{R}^3\setminus\{\mathbf{0}\}} |B_K\mathbf{v}|/|\mathbf{v}|$. Then we deduce that

$$\int_K |\,\mathbf{curl}\,\boldsymbol{\phi}_h(\mathbf{x})|^2 d\mathbf{x} \le \frac{|B_K|^2}{|\det(B_K)|}\int_{\hat{K}} |\,\mathbf{curl}\,\hat{\boldsymbol{\phi}}_h(\hat{\mathbf{x}})|^2 d\hat{\mathbf{x}} \le \frac{|\hat{K}|h_K^2}{|K|\rho_{\hat{K}}^2}\int_{\hat{K}} |\,\mathbf{curl}\,\hat{\boldsymbol{\phi}}_h(\hat{\mathbf{x}})|^2 d\hat{\mathbf{x}} \le \frac{C}{\rho_K}\int_{\hat{K}} |\,\mathbf{curl}\,\hat{\boldsymbol{\phi}}_h(\hat{\mathbf{x}})|^2 d\hat{\mathbf{x}}, \tag{4.11}$$

$$\int_{\hat{K}} |\hat{\boldsymbol{\phi}}_h(\hat{\mathbf{x}})|^2 d\hat{\mathbf{x}} \le \frac{|B_K^T|^2}{|\det(B_K)|}\int_K |\boldsymbol{\phi}_h(\mathbf{x})|^2 d\mathbf{x} \le \frac{|\hat{K}|h_K^2}{|K|\rho_{\hat{K}}^2}\int_K |\boldsymbol{\phi}_h(\mathbf{x})|^2 d\mathbf{x} \le \frac{C}{\rho_K}\int_K |\boldsymbol{\phi}_h(\mathbf{x})|^2 d\mathbf{x}, \tag{4.12}$$

where we have used the facts that $|B_K| \le h_K/\rho_{\hat{K}}$, $|\det(B_K)| = |K|/|\hat{K}|$, $h_K/\rho_K \le C$ $(\forall K \in \tau_h)$. By combining (4.11) and (4.12) with (4.10), we complete the proof. □

Let $P_h : L^2(\Omega;\mathbb{R}^3) \to V_h(\Omega)$ denote the L^2-projection. The condition (4.4) is assumed and Lemma 4.7 was prepared in order to prove the following lemma, which will be used to establish a stability bound for the time derivative of the discrete solution.

Lemma 4.8. *The following inequalities hold.*

$$\|P_h(\boldsymbol{\phi})\|_{L^2(\Omega;\mathbb{R}^3)} \le \|\boldsymbol{\phi}\|_{L^2(\Omega;\mathbb{R}^3)} \tag{4.13}$$

for all $\boldsymbol{\phi} \in L^2(\Omega;\mathbb{R}^3)$.

$$\|\,\mathbf{curl}\,P_h(\boldsymbol{\phi})\|_{L^2(\Omega;\mathbb{R}^3)} \le C\|\boldsymbol{\phi}\|_{B(\Omega)} \tag{4.14}$$

for all $\boldsymbol{\phi} \in B(\Omega)$*, where* $C > 0$ *is a constant independent of* $\boldsymbol{\phi}, h$.

Proof. Take $\boldsymbol{\phi} \in L^2(\Omega;\mathbb{R}^3)$. The function $\boldsymbol{\phi}$ can be decomposed as $\boldsymbol{\phi} = P_h(\boldsymbol{\phi}) + \boldsymbol{\psi}$, $\boldsymbol{\psi} \in V_h(\Omega)^\perp$. Then we see that

$$\|P_h(\boldsymbol{\phi})\|^2_{L^2(\Omega;\mathbb{R}^3)} = \langle \boldsymbol{\phi}-\boldsymbol{\psi}, \boldsymbol{\phi}-\boldsymbol{\psi}\rangle_{L^2(\Omega;\mathbb{R}^3)} = \|\boldsymbol{\phi}\|^2_{L^2(\Omega;\mathbb{R}^3)} - \|\boldsymbol{\psi}\|^2_{L^2(\Omega;\mathbb{R}^3)} \le \|\boldsymbol{\phi}\|^2_{L^2(\Omega;\mathbb{R}^3)},$$

which is (4.13).

We show the inequality (4.14). Note that for $\boldsymbol{\phi} \in B(\Omega)$ $r_h(\boldsymbol{\phi}) \in V_h(\Omega)$ (see [21, Remark 5.42, Lemma 5.44]). For any $\boldsymbol{\phi} \in B(\Omega)$

$$\begin{aligned}
\|\operatorname{\mathbf{curl}} P_h(\boldsymbol{\phi})\|_{L^2(\Omega;\mathbb{R}^3)} &\le \|\operatorname{\mathbf{curl}} P_h(\boldsymbol{\phi}) - \operatorname{\mathbf{curl}} r_h(\boldsymbol{\phi})\|_{L^2(\Omega_s;\mathbb{R}^3)} + \|\operatorname{\mathbf{curl}} r_h(\boldsymbol{\phi})\|_{L^2(\Omega;\mathbb{R}^3)} \\
&\le \left(\sum_{K\in\tau_h, K\subset\overline{\Omega_s}} \frac{C}{\rho_K^2} \int_K |P_h(\boldsymbol{\phi}) - r_h(\boldsymbol{\phi})|^2 d\mathbf{x} \right)^{1/2} + \|\operatorname{\mathbf{curl}} r_h(\boldsymbol{\phi})\|_{L^2(\Omega;\mathbb{R}^3)} \\
&\le \frac{C}{h} \|P_h(\boldsymbol{\phi}) - r_h(\boldsymbol{\phi})\|_{L^2(\Omega_s;\mathbb{R}^3)} + \|\operatorname{\mathbf{curl}} r_h(\boldsymbol{\phi})\|_{L^2(\Omega;\mathbb{R}^3)} \\
&\le \frac{C}{h} \|r_h(\boldsymbol{\phi}) - \boldsymbol{\phi}\|_{L^2(\Omega;\mathbb{R}^3)} + \|\operatorname{\mathbf{curl}} r_h(\boldsymbol{\phi})\|_{L^2(\Omega;\mathbb{R}^3)} \\
&\le C\|\boldsymbol{\phi}\|_{H^1(\mathbf{curl};\Omega)} + Ch\|\boldsymbol{\phi}\|_{H^1(\mathbf{curl};\Omega)} + \|\operatorname{\mathbf{curl}} \boldsymbol{\phi}\|_{L^2(\Omega;\mathbb{R}^3)} \\
&\le C\|\boldsymbol{\phi}\|_{B(\Omega)},
\end{aligned}$$

where we have used Lemma 4.7, (4.3), (4.4), the inequality $\|P_h(\boldsymbol{\phi}) - \boldsymbol{\phi}\|_{L^2(\Omega;\mathbb{R}^3)} \le \|r_h(\boldsymbol{\phi}) - \boldsymbol{\phi}\|_{L^2(\Omega;\mathbb{R}^3)}$, which is ensured by the definition of P_h, and Lemma 4.3. □

4.2. **Compactness property**

We prepare a compactness property on which the proof of the strong convergence property of our fully discrete solution is based. By the assumptions on μ, Ω, and the conditions (4.3) and (4.5) on τ_h, we can apply the following discrete compactness result proved in [21, Chapter 7]. In particular, the quasi-uniform property (4.5) of τ_h on $\partial\Omega$ is required only to apply this lemma.

Lemma 4.9. *([21, Chapter 7]) Let $\{\boldsymbol{\phi}_h\}_{h\in\Lambda}$ satisfy $\boldsymbol{\phi}_h \in X_h^{(\mu)}(\Omega)$ for all $h \in \Lambda$. The following statements hold.*

(1) If there is a constant $C > 0$ such that $\|\boldsymbol{\phi}_h\|_{H(\mathbf{curl};\Omega)} \le C$ for all $h \in \Lambda$, there exist a subsequence $\{\boldsymbol{\phi}_{h_n}\}_{n=1}^{\infty} \subset \{\boldsymbol{\phi}_h\}_{h\in\Lambda}$ and $\boldsymbol{\phi} \in X^{(\mu)}(\Omega)$ such that as $n \to +\infty$

$$\boldsymbol{\phi}_{h_n} \to \boldsymbol{\phi} \text{ strongly in } L^2(\Omega;\mathbb{R}^3),\ \boldsymbol{\phi}_{h_n} \rightharpoonup \boldsymbol{\phi} \text{ weakly in } H(\mathbf{curl};\Omega).$$

(2) There is a constant $\hat{C} > 0$ such that for any $h \in \Lambda$,

$$\|\boldsymbol{\phi}_h\|_{L^2(\Omega;\mathbb{R}^3)} \le \hat{C}(\|\operatorname{\mathbf{curl}} \boldsymbol{\phi}_h\|_{L^2(\Omega;\mathbb{R}^3)} + \|\mathbf{n} \times \boldsymbol{\phi}_h\|_{L^2(\partial\Omega;\mathbb{R}^3)}).$$

We need to couple the discrete compactness property Lemma 4.9 with certain compactness theorem for time dependent function spaces in order to extract a strong converging sequence from our discrete solutions. To answer this purpose we apply the compactness theorem [32, Theorem 4.1]. Let us rewrite the statement of [32, Theorem 4.1] to be suitable for our problem as follows.

Proposition 4.10. *([32, Theorem 4.1]) Let B_1 and B_2 be separable Banach spaces satisfying that B_1 is continuously imbedded in B_2. Let $\mathscr{L}$ and $\mathscr{B}$ denote the σ-algebras of the Lebesgue measurable subsets of $(0,T)$ and that of the Borel subsets of B_1, respectively, and $\mathscr{L} \times \mathscr{B}$ denote the product σ-algebra in $(0,T) \times B_1$. Let $p \in [1,\infty)$. Let U be a subset of $L^p(0,T;B_1)$ satisfying the following conditions.*

(1) There exists a function $\mathcal{F} : (0,T) \times B_1 \to \mathbb{R}_{\ge 0} \cup \{+\infty\}$ satisfying that
  (i) $\mathcal{F}$ is $\mathscr{L} \times \mathscr{B}$-measurable on $(0,T) \times B_1$,
  (ii) $v \mapsto \mathcal{F}(t,v)$ is lower semi-continuous on B_1 for a.e. $t \in (0,T)$,
  (iii) $\{v \in B_1 \mid \mathcal{F}(t,v) \le \lambda\}$ is compact for all $\lambda \ge 0$, a.e. $t \in (0,T)$,

and

$$\sup_{u\in U}\int_0^T \mathcal{F}(t,u(t))dt < +\infty,$$

(2)

$$\lim_{\delta\searrow 0}\sup_{u\in U}\int_0^{T-\delta} \|u(t+\delta)-u(t)\|_{B_2}dt = 0,$$

(3)

$$\lim_{|J|\searrow 0, J\subset(0,T)}\sup_{u\in U}\int_J \|u(t)\|_{B_1}^p dt = 0.$$

Then, U is relatively compact in $L^p(0,T;B_1)$.

A practical application of Proposition 4.10 to our problem is stated as follows.

Proposition 4.11. *Let $\{\mathbf{H}_h\}_{h\in\Lambda}\subset L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$ satisfy that $\mathbf{H}_h(t)\in X_h^{(\mu)}(\Omega)$ for a.e. $t\in(0,T)$ and all $h\in\Lambda$, $\{\mathbf{H}_h\}_{h\in\Lambda}$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$, $\{\mathbf{curl}\,\mathbf{H}_h\}_{h\in\Lambda}$ is bounded in $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$, $\{\mathbf{n}\times\mathbf{H}_h\}_{h\in\Lambda}$ is bounded in $L^2(0,T;L^2(\partial\Omega;\mathbb{R}^3))$ and*

$$\lim_{\delta\searrow 0}\sup_{h\in\Lambda}\int_0^{T-\delta} \|\mathbf{H}_h(t+\delta)-\mathbf{H}_h(t)\|_{(B(\Omega))^*}dt = 0. \tag{4.15}$$

Then, $\{\mathbf{H}_h\}_{h\in\Lambda}$ is relatively compact in $L^p(0,T;L^2(\Omega;\mathbb{R}^3))$ for all $p\in[1,\infty)$.

Proof. First we construct a separable Banach space in which $L^2(\Omega;\mathbb{R}^3)$ is continuously imbedded. Since $\dim B(\Omega)=+\infty$, we can take an orthonormal system $\{\boldsymbol{\psi}_i\}_{i=1}^\infty\subset B(\Omega)$, i.e. $\|\boldsymbol{\psi}_i\|_{B(\Omega)}=1$ and $\langle\boldsymbol{\psi}_i,\boldsymbol{\psi}_j\rangle_{B(\Omega)}=0$ $(i\neq j)$ for all $i,j\in\mathbb{N}$. Let us define the Hilbert space $K(\subset B(\Omega))$ by $K:=\{\sum_{i=1}^\infty\alpha_i\boldsymbol{\psi}_i \mid \sum_{i=1}^\infty|\alpha_i|^2<+\infty,\alpha_i\in\mathbb{R}\}$ equipped with the inner product of $B(\Omega)$. Let K^* denote the dual space of K. Since K is separable, K^* is separable. Moreover, we see that $L^2(\Omega;\mathbb{R}^3)$ is continuously imbedded in K^*. We consider $L^2(\Omega;\mathbb{R}^3)$ and K^* as B_1 and B_2 in Proposition 4.10 respectively. Since $(B(\Omega))^*$ is continuously imbedded in K^*, the condition (2) of Proposition 4.10 is satisfied for $B_2=K^*$ by the assumption (4.15).

(Step 1) We check the condition (1) of Proposition 4.10. Define a set A by

$$A:=\bigcup_{h\in\Lambda}X_h^{(\mu)}(\Omega)\cup Y^{(\mu)}(\Omega).$$

Define $\mathcal{F}:(0,T)\times L^2(\Omega;\mathbb{R}^3)\to\mathbb{R}_{\geq 0}\cup\{+\infty\}$ by

$$\mathcal{F}(t,\mathbf{v}):=\begin{cases}\|\,\mathbf{curl}\,\mathbf{v}\|_{L^2(\Omega;\mathbb{R}^3)}^2+\|\mathbf{n}\times\mathbf{v}\|_{L^2(\partial\Omega;\mathbb{R}^3)}^2 & \text{if } \mathbf{v}\in A,\\ +\infty & \text{otherwise.}\end{cases}$$

We show that for all $\lambda\geq 0$ and a.e. $t\in(0,T)$ the set

$$\{\boldsymbol{\phi}\in L^2(\Omega;\mathbb{R}^3)\mid\mathcal{F}(t,\boldsymbol{\phi})\leq\lambda\}(=\{\boldsymbol{\phi}\in A\mid\|\,\mathbf{curl}\,\boldsymbol{\phi}\|_{L^2(\Omega;\mathbb{R}^3)}^2+\|\mathbf{n}\times\boldsymbol{\phi}\|_{L^2(\partial\Omega;\mathbb{R}^3)}^2\leq\lambda\}) \tag{4.16}$$

is compact in $L^2(\Omega;\mathbb{R}^3)$. Assume that a sequence $\{\boldsymbol{\phi}_n\}_{n=1}^\infty\subset A$ satisfies

$$\|\,\mathbf{curl}\,\boldsymbol{\phi}_n\|_{L^2(\Omega;\mathbb{R}^3)}^2+\|\mathbf{n}\times\boldsymbol{\phi}_n\|_{L^2(\partial\Omega;\mathbb{R}^3)}^2\leq\lambda \tag{4.17}$$

for all $n\in\mathbb{N}$. Since $\boldsymbol{\phi}_n\in\bigcup_{h\in\Lambda}X_h^{(\mu)}(\Omega)$ or $\boldsymbol{\phi}_n\in Y^{(\mu)}(\Omega)$ we can define a map $n\mapsto\eta_n$ by

$$\eta_n:=\begin{cases}h_n & \text{if } \boldsymbol{\phi}_n\in X_{h_n}^{(\mu)}(\Omega),\\ 0 & \text{if } \boldsymbol{\phi}_n\in Y^{(\mu)}(\Omega).\end{cases}$$

If $\liminf_{n\to+\infty}\eta_n \geq C > 0$, then for a small $\xi \in (0, C)$ there is $M \in \mathbb{N}$ such that

$$\boldsymbol{\phi}_n \in \bigcup_{h\in\Lambda, h\geq C-\xi} X_h^{(\mu)}(\Omega)$$

for all $n \geq M$. By the discrete Friedrichs inequality Lemma 4.9 (2) and (4.17) we see that $\{\boldsymbol{\phi}_n\}_{n=1}^{\infty}$ is bounded in $H(\mathbf{curl};\Omega)$. Since $\Sigma_{h\in\Lambda,h\geq C-\xi}\dim X_h^{(\mu)}(\Omega) < +\infty$, by choosing a subsequence we have $\boldsymbol{\phi}_n \to \boldsymbol{\phi}$ strongly in $L^2(\Omega;\mathbb{R}^3)$ with some $\boldsymbol{\phi} \in \bigcup_{h\in\Lambda,h\geq C-\xi} X_h^{(\mu)}(\Omega) \subset A$.

If $\liminf_{n\to+\infty}\eta_n = 0$, then by noting the discrete compactness property Lemma 4.9 (1) and the compactness property Lemma 2.3, we can choose a subsequence $\boldsymbol{\phi}_n$ so that $\boldsymbol{\phi}_n \to \boldsymbol{\phi}$ strongly in $L^2(\Omega;\mathbb{R}^3)$ with some $\boldsymbol{\phi} \in Y^{(\mu)}(\Omega) \subset A$.

The limit $\boldsymbol{\phi}$ satisfies the inequality (4.17). Therefore, the set (4.16) is compact.

Note that for all $\lambda \in \mathbb{R}$

$$\{(t,\mathbf{v}) \in (0,T)\times L^2(\Omega;\mathbb{R}^3) \mid \mathcal{F}(t,\mathbf{v}) \leq \lambda\} = (0,T)\times\{\mathbf{v}\in A \mid \|\,\mathbf{curl}\,\mathbf{v}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\mathbf{n}\times\mathbf{v}\|^2_{L^2(\partial\Omega;\mathbb{R}^3)} \leq \lambda\}. \quad (4.18)$$

We have seen that the set

$$\{\mathbf{v}\in A \mid \|\,\mathbf{curl}\,\mathbf{v}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\mathbf{n}\times\mathbf{v}\|^2_{L^2(\partial\Omega;\mathbb{R}^3)} \leq \lambda\}$$

is compact, especially a Borel set of $L^2(\Omega;\mathbb{R}^3)$. Thus by (4.18) we see that $\mathcal{F}$ is $\mathscr{L}\times\mathscr{B}$-measurable on $(0,T)\times L^2(\Omega;\mathbb{R}^3)$.

The lower semi-continuity of $\boldsymbol{\phi}\mapsto\mathcal{F}(t,\boldsymbol{\phi})$ in $L^2(\Omega;\mathbb{R}^3)$ can be confirmed in the same way as above.

By assumption we see that $\{\mathbf{H}_h(t)\}_{h\in\Lambda}\subset A$ for a.e. $t\in(0,T)$ and

$$\sup_{h\in\Lambda}\int_0^T \mathcal{F}(t,\mathbf{H}_h(t))dt < +\infty.$$

(Step 2) We need to check the condition (3) of Proposition 4.10 to complete the proof. For any measurable set $J\subset(0,T)$,

$$\int_J \|\mathbf{H}_h(t)\|^p_{L^2(\Omega;\mathbb{R}^3)}dt \leq |J|\|\mathbf{H}_h\|^p_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} \leq C_p|J|,$$

where $C_p > 0$ is a constant independent of h, J, depending on p. This inequality implies that the condition (3) is satisfied. □

Remark 4.12. As the proof shows, the same result still holds if the $(B(\Omega))^*$-norm of the condition (4.15) is replaced by that of any separable Banach space in which $L^2(\Omega;\mathbb{R}^3)$ is continuously imbedded.

4.3. **Fully discrete problems**

Let us propose the fully discrete formulations of our problems (**P1**) and (**P2**), establish the stability bounds for the discrete solutions and show the subsequence convergence property of these solutions.

For $N\in\mathbb{N}$, we define the time step size Δt by $\Delta t := T/N$. By the assumption (2.26), the interpolant $r_h(\boldsymbol{\mathcal{H}}_s(t))$ are well-defined for all $t\in[0,T]$ (see, e.g. [21, Lemma 5.38]). Let us define the piecewise linear in time function $\widetilde{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t}:[0,T]\to U_h(\Omega)$ and the piecewise constant in time functions $\overline{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t}, \underline{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t}:[0,T]\to U_h(\Omega)$ by

$$\widetilde{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t}(t) := \frac{t-(n-1)\Delta t}{\Delta t}\boldsymbol{\mathcal{H}}_{s,h,n} + \frac{n\Delta t - t}{\Delta t}\boldsymbol{\mathcal{H}}_{s,h,n-1} \text{ in } [(n-1)\Delta t, n\Delta t],$$

$$\overline{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t}(t) := \begin{cases} \boldsymbol{\mathcal{H}}_{s,h,n} & \text{in } ((n-1)\Delta t, n\Delta t], \\ \boldsymbol{\mathcal{H}}_{s,h,0} & \text{on } \{t=0\}, \end{cases} \quad \underline{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t}(t) := \begin{cases} \boldsymbol{\mathcal{H}}_{s,h,n-1} & \text{in } [(n-1)\Delta t, n\Delta t), \\ \boldsymbol{\mathcal{H}}_{s,h,N} & \text{on } \{t=T\}, \end{cases}$$

where $\boldsymbol{\mathcal{H}}_{s,h,n} = r_h(\boldsymbol{\mathcal{H}}_s(n\Delta t))$ for $n = 0, \cdots, N$. We can show the following properties in the same way as in [13, Lemma 4.1].

Lemma 4.13. *The following estimate holds.*

$$\|\partial_t \widetilde{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \le Ch\|\partial_t \boldsymbol{\mathcal{H}}_s\|_{L^2(0,T;H^1(\Omega;\mathbb{R}^3))} + C\|\partial_t \boldsymbol{\mathcal{H}}_s\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}, \tag{4.19}$$

where the constant $C > 0$ is independent of h and Δt, depending only on the constant in Lemma 4.3. Moreover, the following convergence properties hold as $h \searrow 0$ and $\Delta t \searrow 0$.

$$\widetilde{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t} \to \boldsymbol{\mathcal{H}}_s \text{ strongly in } C([0,T]; L^2(\Omega;\mathbb{R}^3)), \tag{4.20}$$

$$\overline{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t} \to \boldsymbol{\mathcal{H}}_s \text{ strongly in } L^\infty(0,T; L^2(\Omega;\mathbb{R}^3)), \tag{4.21}$$

$$\underline{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t} \to \boldsymbol{\mathcal{H}}_s \text{ strongly in } L^\infty(0,T; L^2(\Omega;\mathbb{R}^3)), \tag{4.22}$$

$$\partial_t \widetilde{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t} \to \partial_t \boldsymbol{\mathcal{H}}_s \text{ strongly in } L^\infty(0,T; L^2(\Omega;\mathbb{R}^3)). \tag{4.23}$$

In order to propose our fully discrete formulations, we additionally assume that the interpolations $\widehat{\boldsymbol{\mathcal{H}}}_{h,0} := r_h(\widehat{\boldsymbol{\mathcal{H}}}_0)$ and $\boldsymbol{\mathcal{H}}_{s,h,0} := r_h(\boldsymbol{\mathcal{H}}_s(0))$ satisfy the discrete divergence-free condition

$$\widehat{\boldsymbol{\mathcal{H}}}_{h,0} + \boldsymbol{\mathcal{H}}_{s,h,0} \in X_h^{(\mu)}(\Omega), \tag{4.24}$$

which is the discrete analogue of the assumption (2.27). Note that the condition (4.24) holds, for example, if $\widehat{\boldsymbol{\mathcal{H}}}_0 \equiv \boldsymbol{\mathcal{H}}_s(0) \equiv \mathbf{0}$, or μ, $\widehat{\boldsymbol{\mathcal{H}}}_0$ and $\boldsymbol{\mathcal{H}}_s(0)$ are constant over Ω.

Let us define the functional $F_{h,n}$ $(n = 1, \cdots, N)$ on the finite element space $U_h(\Omega)$ by

$$F_{h,n}(\boldsymbol{\phi}_h) := \frac{1}{2\Delta t}\int_\Omega \mu|\boldsymbol{\phi}_h|^2 d\mathbf{x} + \frac{1}{\Delta t}\int_\Omega \mu\langle -\widehat{\boldsymbol{\mathcal{H}}}_{h,n-1} + \boldsymbol{\mathcal{H}}_{s,h,n} - \boldsymbol{\mathcal{H}}_{s,h,n-1}, \boldsymbol{\phi}_h\rangle d\mathbf{x} + \int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}, \mathbf{curl}\,\boldsymbol{\phi}_h) d\mathbf{x},$$

where $\widehat{\boldsymbol{\mathcal{B}}}_{h,n-1} = \boldsymbol{\mathcal{B}}_{h,n-1}/\sqrt{|\boldsymbol{\mathcal{B}}_{h,n-1}|^2 + \varepsilon^2}$ $(\varepsilon > 0)$ and $\boldsymbol{\mathcal{B}}_{h,n-1} = \mu(\widehat{\boldsymbol{\mathcal{H}}}_{h,n-1} + \boldsymbol{\mathcal{H}}_{s,h,n-1})$.

We discretize the problem (**P1**) in time and in space by the curl-conforming element to obtain the following unconstrained minimisation problem.

($\mathbf{P1}_{h,\Delta t}$) On the assumption (4.24), for $n = 1 \to N$, find $\widehat{\boldsymbol{\mathcal{H}}}_{h,n} \in V_h(\Omega)$ such that

$$F_{h,n}(\widehat{\boldsymbol{\mathcal{H}}}_{h,n}) = \min_{\boldsymbol{\phi}_h \in V_h(\Omega)} F_{h,n}(\boldsymbol{\phi}_h).$$

Note that at each time step we take $\widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}$ from the previous time step so that the problem ($\mathbf{P1}_{h,\Delta t}$) is a convex optimisation problem.

Proposition 4.14. *There exists a unique minimiser $\widehat{\boldsymbol{\mathcal{H}}}_{h,n} \in V_h(\Omega)$ of ($\mathbf{P1}_{h,\Delta t}$). Moreover, the discrete divergence-free condition*

$$\widehat{\boldsymbol{\mathcal{H}}}_{h,n} + \boldsymbol{\mathcal{H}}_{s,h,n} \in X_h^{(\mu)}(\Omega) \tag{4.25}$$

holds. The minimiser satisfies the following discrete variational inequality. For all $\boldsymbol{\phi}_h \in V_h(\Omega)$

$$\begin{aligned}\int_\Omega \mu\langle(\widehat{\boldsymbol{\mathcal{H}}}_{h,n} - \widehat{\boldsymbol{\mathcal{H}}}_{h,n-1} + \boldsymbol{\mathcal{H}}_{s,h,n} - \boldsymbol{\mathcal{H}}_{s,h,n-1})/\Delta t, \boldsymbol{\phi}_h - \widehat{\boldsymbol{\mathcal{H}}}_{h,n}\rangle d\mathbf{x} \\ + \int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}, \mathbf{curl}\,\boldsymbol{\phi}_h) d\mathbf{x} - \int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}, \mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}_{h,n}) d\mathbf{x} \ge 0.\end{aligned} \tag{4.26}$$

Proof. We check that $F_{h,n} : V_h(\Omega) \to \mathbb{R}$ is coercive. By the property of $g_\perp(|\cdot|)$, $g_\|(|\cdot|)$, $g_0(|\cdot|)$,

$$F_{h,n}(\boldsymbol{\phi}_h) \geq \frac{\min\{\mu_s,\mu_d\}}{2\Delta t}\int_\Omega |\boldsymbol{\phi}_h|^2 d\mathbf{x} - \frac{\max\{\mu_s,\mu_d\}}{\Delta t}\int_\Omega |-\widehat{\boldsymbol{\mathcal{H}}}_{h,n-1}+\boldsymbol{\mathcal{H}}_{s,h,n}-\boldsymbol{\mathcal{H}}_{s,h,n-1}||\boldsymbol{\phi}_h| d\mathbf{x}$$
$$+ A_{01}\int_{\Omega_s} |\,\mathbf{curl}\,\boldsymbol{\phi}_h|^2 d\mathbf{x} - (A_{\perp 2}+A_{\|2}+A_{02})|\Omega_s|,$$

which implies that $F_{h,n}$ is coercive on $V_h(\Omega)$ with respect to the $H(\mathbf{curl};\Omega)$-norm. Since $F_{h,n}$ is strictly convex, the minimiser uniquely exists.

To derive (4.26) is standard. By noting the fact that $\nabla Z_{0,h}(\Omega) \subset V_h(\Omega)$, using the assumption of induction that $\widehat{\boldsymbol{\mathcal{H}}}_{h,n-1}+\boldsymbol{\mathcal{H}}_{s,h,n-1} \in X_h^{(\mu)}(\Omega)$ and (4.24) and substituting $\boldsymbol{\phi}_h = \nabla f_h + \widehat{\boldsymbol{\mathcal{H}}}_{h,n}$ ($f_h \in Z_{0,h}(\Omega)$) into (4.26) we can deduce the condition (4.25). □

We define the piecewise linear in time functions $\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t}$, $\widetilde{\boldsymbol{\mathcal{H}}}_{s,\Delta t}$, and the piecewise constant in time functions $\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t}$, $\underline{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t}$, $\underline{\widehat{\boldsymbol{\mathcal{B}}}}_{h,\Delta t}$ by

$$\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t}(t) := \frac{t-(n-1)\Delta t}{\Delta t}\widehat{\boldsymbol{\mathcal{H}}}_{h,n} + \frac{n\Delta t - t}{\Delta t}\widehat{\boldsymbol{\mathcal{H}}}_{h,n-1} \text{ in } [(n-1)\Delta t, n\Delta t],$$
$$\widetilde{\boldsymbol{\mathcal{H}}}_{s,\Delta t}(t) := \frac{t-(n-1)\Delta t}{\Delta t}\boldsymbol{\mathcal{H}}_s(n\Delta t) + \frac{n\Delta t - t}{\Delta t}\boldsymbol{\mathcal{H}}_s((n-1)\Delta t) \text{ in } [(n-1)\Delta t, n\Delta t],$$
$$\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t}(t) := \begin{cases} \widehat{\boldsymbol{\mathcal{H}}}_{h,n} & \text{in } ((n-1)\Delta t, n\Delta t], \\ \widehat{\boldsymbol{\mathcal{H}}}_{h,0} & \text{on } \{t=0\}, \end{cases} \quad \underline{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t}(t) := \begin{cases} \widehat{\boldsymbol{\mathcal{H}}}_{h,n-1} & \text{in } [(n-1)\Delta t, n\Delta t), \\ \widehat{\boldsymbol{\mathcal{H}}}_{h,N} & \text{on } \{t=T\}, \end{cases}$$
$$\underline{\widehat{\boldsymbol{\mathcal{B}}}}_{h,\Delta t}(t) := \begin{cases} \widehat{\boldsymbol{\mathcal{B}}}_{h,n-1} & \text{in } [(n-1)\Delta t, n\Delta t), \\ \widehat{\boldsymbol{\mathcal{B}}}_{h,N} & \text{on } \{t=T\}, \end{cases}$$

for $n = 1, \cdots, N$, where $\widehat{\boldsymbol{\mathcal{H}}}_{h,n}$ is the minimiser of $(\mathbf{P1}_{h,\Delta t})$.

By definition we see that $\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t}(t), \overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t}(t), \underline{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t}(t) \in V_h(\Omega)$ for all $t \in [0,T]$ and the discrete divergence-free condition holds in the sense that $\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t}(t) + \widetilde{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t}(t)$, $\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t}(t) + \overline{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t}(t)$, $\underline{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t}(t) + \underline{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t}(t) \in X_h^{(\mu)}(\Omega)$ for all $t \in [0,T]$ by (4.25).

Proposition 4.15. *Take any $\tau \in (0,1)$. The following bounds hold. For all $h \in \Lambda$, $\Delta t \in (0,\tau]$*

$$\|\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t}\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} \leq \frac{C}{1-\tau}e^{T/(1-\tau)}\frac{\max\{\mu_d,\mu_s\}}{\min\{\mu_d,\mu_s\}}(h^2\|\partial_t\boldsymbol{\mathcal{H}}_s\|^2_{L^2(0,T;H^1(\mathbf{curl};\Omega))} + \|\partial_t\boldsymbol{\mathcal{H}}_s\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}$$
$$+ h^2\|\widehat{\boldsymbol{\mathcal{H}}}_0\|^2_{H^1(\mathbf{curl};\Omega)} + \|\widehat{\boldsymbol{\mathcal{H}}}_0\|^2_{L^2(\Omega;\mathbb{R}^3)}), \tag{4.27}$$

$$\int_0^T\int_{\Omega_s} G(\underline{\widehat{\boldsymbol{\mathcal{B}}}}_{h,\Delta t}, \mathbf{curl}\,\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t})d\mathbf{x}dt \leq C\max\{\mu_d,\mu_s\}\left(1+\frac{Te^{T/(1-\tau)}}{(1-\tau)\min\{\mu_d,\mu_s\}}\right)$$
$$(h^2\|\partial_t\boldsymbol{\mathcal{H}}_s\|^2_{L^2(0,T;H^1(\mathbf{curl};\Omega))} + \|\partial_t\boldsymbol{\mathcal{H}}_s\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} + h^2\|\widehat{\boldsymbol{\mathcal{H}}}_0\|^2_{H^1(\mathbf{curl};\Omega)} + \|\widehat{\boldsymbol{\mathcal{H}}}_0\|^2_{L^2(\Omega;\mathbb{R}^3)}), \tag{4.28}$$

where $C > 0$ is a positive constant independent of $h, \Delta t, \mu$.

Proof. Substituting $\boldsymbol{\phi}_h = \mathbf{0}$ into (4.26) and noting an equality $\langle \mathbf{p}-\mathbf{q}, \mathbf{p}\rangle = |\mathbf{p}-\mathbf{q}|^2/2 + (|\mathbf{p}|^2 - |\mathbf{q}|^2)/2$, we have

$$\begin{aligned}\frac{\Delta t}{2}\int_\Omega \mu|(\widehat{\boldsymbol{\mathcal{H}}}_{h,n} - \widehat{\boldsymbol{\mathcal{H}}}_{h,n-1})/\Delta t|^2 d\mathbf{x} + \frac{1}{2\Delta t}\int_\Omega \mu|\widehat{\boldsymbol{\mathcal{H}}}_{h,n}|^2 d\mathbf{x} - \frac{1}{2\Delta t}\int_\Omega \mu|\widehat{\boldsymbol{\mathcal{H}}}_{h,n-1}|^2 d\mathbf{x} \\ + \int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}, \mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}_{h,n}) d\mathbf{x} \le \frac{1}{2}\int_\Omega \mu|(\boldsymbol{\mathcal{H}}_{s,h,n} - \boldsymbol{\mathcal{H}}_{s,h,n-1})/\Delta t|^2 d\mathbf{x} + \frac{1}{2}\int_\Omega \mu|\widehat{\boldsymbol{\mathcal{H}}}_{h,n}|^2 d\mathbf{x}.\end{aligned} \tag{4.29}$$

Multiplying (4.29) by $2\Delta t$ and summing over $n = 1 \to m(\le N)$, we have

$$\int_\Omega \mu|\widehat{\boldsymbol{\mathcal{H}}}_{h,m}|^2 d\mathbf{x} \le \int_0^{\Delta t m}\int_\Omega \mu|\partial_t \widetilde{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t}|^2 d\mathbf{x}dt + \int_\Omega \mu|\widehat{\boldsymbol{\mathcal{H}}}_{h,0}|^2 d\mathbf{x} + \sum_{n=0}^{m}\Delta t\int_\Omega \mu|\widehat{\boldsymbol{\mathcal{H}}}_{h,n}|^2 d\mathbf{x}. \tag{4.30}$$

By the bound (4.19), Lemma 4.3 and applying the discrete Gronwall's inequality (see, e.g. [37, Lemma 10.5]) to (4.30) we obtain (4.27).

On the other hand, by multiplying (4.29) by $2\Delta t$ and summing over $n = 1 \to N$, we see that

$$\int_0^T\int_{\Omega_s} G(\underline{\widehat{\boldsymbol{\mathcal{B}}}}_{h,\Delta t}, \mathbf{curl}\,\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t}) d\mathbf{x}dt \le \int_0^T\int_\Omega \mu|\partial_t \widetilde{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t}|^2 d\mathbf{x}dt + \int_\Omega \mu|\widehat{\boldsymbol{\mathcal{H}}}_{h,0}|^2 d\mathbf{x} + \int_0^T\int_\Omega \mu|\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t}|^2 d\mathbf{x}dt. \tag{4.31}$$

Combining (4.31) with (4.27) we obtain (4.28). □

We can also establish a bound for $\partial_t \widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t}$. For this purpose, we have prepared the properties of the projection $P_h : L^2(\Omega;\mathbb{R}^3) \to V_h(\Omega)$ in Lemma 4.8.

Proposition 4.16. *There is a constant $C > 0$ independent of $h, \Delta t$ such that*

$$\|\partial_t \widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t}\|_{L^2(0,T;(B(\Omega))^*)} \le C.$$

Proof. The optimisation problem $(\mathbf{P1}_{h,\Delta t})$ leads to the following weak form. For all $\boldsymbol{\phi}_h \in V_h(\Omega)$

$$\begin{aligned}&\langle \mu(\widehat{\boldsymbol{\mathcal{H}}}_{h,n} - \widehat{\boldsymbol{\mathcal{H}}}_{h,n-1})/\Delta t, \boldsymbol{\phi}_h\rangle_{L^2(\Omega;\mathbb{R}^3)} + \langle \mu(\boldsymbol{\mathcal{H}}_{s,h,n} - \boldsymbol{\mathcal{H}}_{s,h,n-1})/\Delta t, \boldsymbol{\phi}_h\rangle_{L^2(\Omega;\mathbb{R}^3)} \\ &+ \int_{\Omega_s} \frac{g'_\perp(|\widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}\times \mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}_{h,n}\times \widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}|)}{|\widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}\times \mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}_{h,n}\times \widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}|}\langle \widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}\times \mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}_{h,n}\times \widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}, \widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}\times \mathbf{curl}\,\boldsymbol{\phi}_h\times \widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}\rangle d\mathbf{x} \\ &+ \int_{\Omega_s} \frac{g'_\parallel(|\langle \widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}, \mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}_{h,n}\rangle \widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}|)}{|\langle \widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}, \mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}_{h,n}\rangle \widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}|}\langle \langle \widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}, \mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}_{h,n}\rangle \widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}, \langle \widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}, \mathbf{curl}\,\boldsymbol{\phi}_h\rangle \widehat{\boldsymbol{\mathcal{B}}}_{h,n-1}\rangle d\mathbf{x} \\ &+ \int_{\Omega_s} \frac{g'_0(|\,\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}_{h,n}|)}{|\,\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}_{h,n}|}\langle \mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}_{h,n}, \mathbf{curl}\,\boldsymbol{\phi}_h\rangle d\mathbf{x} = 0.\end{aligned} \tag{4.32}$$

Take any $\boldsymbol{\phi} \in B(\Omega)$. By using (4.32), the properties of the energy densities (4.2) and the L^2-projection $P_h(\cdot)$ (4.13), (4.14) we see that

$$\begin{aligned}|\langle \mu(\widehat{\boldsymbol{\mathcal{H}}}_{h,n} - \widehat{\boldsymbol{\mathcal{H}}}_{h,n-1})/\Delta t, \boldsymbol{\phi}\rangle_{L^2(\Omega;\mathbb{R}^3)}| &= |\langle \mu(\widehat{\boldsymbol{\mathcal{H}}}_{h,n} - \widehat{\boldsymbol{\mathcal{H}}}_{h,n-1})/\Delta t, P_h(\boldsymbol{\phi})\rangle_{L^2(\Omega;\mathbb{R}^3)}| \\ &\le \max\{\mu_s, \mu_d\}\|(\boldsymbol{\mathcal{H}}_{s,h,n} - \boldsymbol{\mathcal{H}}_{s,h,n-1})/\Delta t\|_{L^2(\Omega;\mathbb{R}^3)}\|P_h(\boldsymbol{\phi})\|_{L^2(\Omega;\mathbb{R}^3)} \\ &\quad + (A_{\perp 5} + A_{\parallel 5} + A_{05})\|\,\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}_{h,n}\|_{L^2(\Omega;\mathbb{R}^3)}\|\,\mathbf{curl}\,P_h(\boldsymbol{\phi})\|_{L^2(\Omega;\mathbb{R}^3)} \\ &\le \max\{\mu_s, \mu_d\}\|(\boldsymbol{\mathcal{H}}_{s,h,n} - \boldsymbol{\mathcal{H}}_{s,h,n-1})/\Delta t\|_{L^2(\Omega;\mathbb{R}^3)}\|\boldsymbol{\phi}\|_{L^2(\Omega;\mathbb{R}^3)} \\ &\quad + C(A_{\perp 5} + A_{\parallel 5} + A_{05})\|\,\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}_{h,n}\|_{L^2(\Omega;\mathbb{R}^3)}\|\boldsymbol{\phi}\|_{B(\Omega)}.\end{aligned} \tag{4.33}$$

The inequality (4.33) implies that

$$\|(\widehat{\boldsymbol{\mathcal{H}}}_{h,n}-\widehat{\boldsymbol{\mathcal{H}}}_{h,n-1})/\Delta t\|_{(B(\Omega))^*}\le C(\|(\boldsymbol{\mathcal{H}}_{s,h,n}-\boldsymbol{\mathcal{H}}_{s,h,n-1})/\Delta t\|_{L^2(\Omega;\mathbb{R}^3)}+\|\,\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}_{h,n}\|_{L^2(\Omega;\mathbb{R}^3)}).$$

Moreover,

$$\int_0^T\|\partial_t\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t}\|^2_{(B(\Omega))^*}dt\le C\left(\int_0^T\|\partial_t\widetilde{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t}\|^2_{L^2(\Omega;\mathbb{R}^3)}dt+\int_0^T\|\,\mathbf{curl}\,\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t}\|^2_{L^2(\Omega;\mathbb{R}^3)}dt\right)\le C,$$

where we have used the bounds (4.19), (4.28) coupled with the property of the energy densities (2.23). □

In order to show the convergence of our discrete solutions we assume that Δt depends on h and satisfies that

$$\sup_{h\in\Lambda}\Delta t(h)<1,\ \lim_{h\searrow 0,h\in\Lambda}\Delta t(h)=0. \tag{4.34}$$

By applying Proposition 4.11 for $p\ge 2$ we observe the following convergence properties.

Proposition 4.17. *For any $p\in[2,\infty)$ there are a subsequence $\{h_n\}_{n=1}^\infty\subset\Lambda$ and $\widehat{\boldsymbol{\mathcal{H}}}\in L^2(0,T;H(\mathbf{curl};\Omega))\cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$ with $\partial_t\widehat{\boldsymbol{\mathcal{H}}}\in L^2(0,T;(B(\Omega))^*)$ and $\widehat{\boldsymbol{\mathcal{H}}}(t)\in V(\Omega)$ a.e. $t\in(0,T)$ such that*

$$\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}\to\widehat{\boldsymbol{\mathcal{H}}}\ \textit{strongly in}\ L^p(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{4.35}$$

$$\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}\rightharpoonup\widehat{\boldsymbol{\mathcal{H}}}\ \textit{weak}*\ \textit{in}\ L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{4.36}$$

$$\mathbf{curl}\,\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}\rightharpoonup\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}\ \textit{weakly in}\ L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{4.37}$$

$$\partial_t\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}\rightharpoonup\partial_t\widehat{\boldsymbol{\mathcal{H}}}\ \textit{weakly in}\ L^2(0,T;(B(\Omega))^*), \tag{4.38}$$

$$\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}\to\widehat{\boldsymbol{\mathcal{H}}}\ \textit{strongly in}\ L^p(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{4.39}$$

$$\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}\rightharpoonup\widehat{\boldsymbol{\mathcal{H}}}\ \textit{weak}*\ \textit{in}\ L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{4.40}$$

$$\mathbf{curl}\,\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}\rightharpoonup\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}\ \textit{weakly in}\ L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{4.41}$$

$$\underline{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}\to\widehat{\boldsymbol{\mathcal{H}}}\ \textit{strongly in}\ L^p(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{4.42}$$

$$\underline{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}\rightharpoonup\widehat{\boldsymbol{\mathcal{H}}}\ \textit{weak}*\ \textit{in}\ L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{4.43}$$

$$\mathbf{curl}\,\underline{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}\rightharpoonup\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}\ \textit{weakly in}\ L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{4.44}$$

$$\underline{\widehat{\boldsymbol{\mathcal{B}}}}_{h,\Delta t(h_n)}\to\widehat{\boldsymbol{\mathcal{B}}}\ \textit{strongly in}\ L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{4.45}$$

as $n\to+\infty$, where $\widehat{\boldsymbol{\mathcal{B}}}:=(\mu\widehat{\boldsymbol{\mathcal{H}}}+\mu\boldsymbol{\mathcal{H}}_s)/\sqrt{|\mu\widehat{\boldsymbol{\mathcal{H}}}+\mu\boldsymbol{\mathcal{H}}_s|^2+\varepsilon^2}$.

Proof. We check that $\{\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t(h)}+\widetilde{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t(h)}\}_{h\in\Lambda}$ satisfies the conditions of Proposition 4.11. We set $\widetilde{\Xi}_h:=\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t(h)}+\widetilde{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t(h)}$. By the bounds proved in Proposition 4.15 and Proposition 4.16 we see that $\{\widetilde{\Xi}_h\}_{h\in\Lambda}$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$, $\{\mathbf{curl}\,\widetilde{\Xi}_h\}_{h\in\Lambda}$ is bounded in $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$, and $\widetilde{\Xi}_h(t)\in X_h^{(\mu)}(\Omega)$ for all $t\in[0,T]$. We see by Lemma 4.4 and the continuous imbedding $H^1(\Omega;\mathbb{R}^3)\hookrightarrow L^2(\partial\Omega;\mathbb{R}^3)$ that

$$\begin{aligned}\|\mathbf{n}\times\widetilde{\Xi}_h(t)\|_{L^2(\partial\Omega;\mathbb{R}^3)}&\le\|\mathbf{n}\times(\widetilde{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t(h)}(t)-\widetilde{\boldsymbol{\mathcal{H}}}_{s,\Delta t(h)}(t))\|_{L^2(\partial\Omega;\mathbb{R}^3)}+\|\mathbf{n}\times\widetilde{\boldsymbol{\mathcal{H}}}_{s,\Delta t(h)}(t)\|_{L^2(\partial\Omega;\mathbb{R}^3)}\\&\le Ch^{1/2}\|\widetilde{\boldsymbol{\mathcal{H}}}_{s,\Delta t(h)}(t)\|_{H^1(\Omega;\mathbb{R}^3)}+\|\widetilde{\boldsymbol{\mathcal{H}}}_{s,\Delta t(h)}(t)\|_{L^2(\partial\Omega;\mathbb{R}^3)}\\&\le C(h^{1/2}+1)\|\boldsymbol{\mathcal{H}}_s\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^3))}.\end{aligned} \tag{4.46}$$

The inequality (4.46) ensures that $\{\mathbf{n}\times\widetilde{\Xi}_h\}_{h\in\Lambda}$ is bounded in $L^2(0,T;L^2(\partial\Omega;\mathbb{R}^3))$. Moreover, we deduce by using the bound obtained in Proposition 4.16 that

$$\begin{aligned}\int_0^{T-\delta}\|\widetilde{\Xi}_h(t+\delta)-\widetilde{\Xi}_h(t)\|_{(B(\Omega))^*}dt &\le \int_0^{T-\delta}\int_t^{t+\delta}\|\partial_t\widetilde{\Xi}_h(\tau)\|_{(B(\Omega))^*}d\tau dt\\ &\le \delta^{1/2}\int_0^{T-\delta}\left(\int_t^{t+\delta}\|\partial_t\widetilde{\Xi}_h(\tau)\|^2_{(B(\Omega))^*}d\tau\right)^{1/2}dt\le C\delta^{1/2},\end{aligned}$$

where $C>0$ is independent of h,δ. This inequality shows that (4.15) is satisfied.

Thus by Proposition 4.11 we see by extracting a subsequence that

$$\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}+\widetilde{\boldsymbol{\mathcal{H}}}_{s,h_n,\Delta t(h_n)}\to w \text{ strongly in } L^p(0,T;L^2(\Omega;\mathbb{R}^3))$$

as $n\to+\infty$ for some $w\in L^p(0,T;L^2(\Omega;\mathbb{R}^3))$. By setting $\widehat{\boldsymbol{\mathcal{H}}}:=w-\boldsymbol{\mathcal{H}}_s$ and the convergence (4.20), we obtain the convergence (4.35). The weak(*) convergences (4.36)-(4.38) are consequences of the bounds proved in Proposition 4.15 and Proposition 4.16.

In order to show the strong convergence (4.39), we apply Proposition 4.11 to the sequence $\{\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t(h)}+\overline{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t(h)}\}_{h\in\Lambda}$. In particular, we need to prove that

$$\lim_{\delta\searrow 0}\sup_{h\in\Lambda}\int_0^{T-\delta}\|\overline{\Xi}_h(t+\delta)-\overline{\Xi}_h(t)\|_{(B(\Omega))^*}dt=0, \tag{4.47}$$

where we have set $\overline{\Xi}_h=\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t(h)}+\overline{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t(h)}$. The other conditions required in Proposition 4.11 can be confirmed in the same way as we checked for $\{\widetilde{\Xi}_h\}_{h\in\Lambda}$ above.

If $\delta<\Delta t(h)$,

$$\begin{aligned}\int_0^{T-\delta}\|\overline{\Xi}_h(t+\delta)-\overline{\Xi}_h(t)\|_{(B(\Omega))^*}dt &= \sum_{i=1}^{N-1}\int_{i\Delta t-\delta}^{i\Delta t}\|\overline{\Xi}_h(t+\delta)-\overline{\Xi}_h(t)\|_{(B(\Omega))^*}dt\\ &=\sum_{i=1}^{N-1}\int_{i\Delta t-\delta}^{i\Delta t}\|\overline{\Xi}_h((i+1)\Delta t)-\overline{\Xi}_h(i\Delta t)\|_{(B(\Omega))^*}dt=\delta\sum_{i=1}^{N-1}\Delta t\|(\overline{\Xi}_h((i+1)\Delta t)-\overline{\Xi}_h(i\Delta t))/\Delta t\|_{(B(\Omega))^*}\\ &\le\delta\int_0^T\|\partial_t\widetilde{\Xi}_h(t)\|_{(B(\Omega))^*}dt\le C\delta,\end{aligned} \tag{4.48}$$

where $C>0$ is a constant independent of h,δ.

Assume $\delta\ge\Delta t(h)$ and fix $t\in(0,T-\delta]$. Then there are $l,m\in\{1,\cdots,N\}$ with $l<m$ such that $t\in((l-1)\Delta t,l\Delta t]$, $t+\delta\in((m-1)\Delta t,m\Delta t]$.

$$\begin{aligned}\|\overline{\Xi}_h(t+\delta)-\overline{\Xi}_h(t)\|_{(B(\Omega))^*} &= \|\int_{l\Delta t}^{m\Delta t}\partial_t\widetilde{\Xi}_h(t)dt\|_{(B(\Omega))^*}\\ &\le(\Delta t(m-l))^{1/2}\left(\int_{l\Delta t}^{m\Delta t}\|\partial_t\widetilde{\Xi}_h(t)\|^2_{(B(\Omega))^*}dt\right)^{1/2}\le(2\delta)^{1/2}\|\partial_t\widetilde{\Xi}_h\|_{L^2(0,T;(B(\Omega))^*)},\end{aligned} \tag{4.49}$$

where we have used the inequality $\Delta t(m-l) \le t+\delta+\Delta t-t \le 2\delta$. By (4.49) we have

$$\int_0^{T-\delta} \|\overline{\Xi}_h(t+\delta)-\overline{\Xi}_h(t)\|_{(B(\Omega))^*}dt \le C\delta^{1/2}, \tag{4.50}$$

where $C>0$ is independent of h,δ. The inequalities (4.48) and (4.50) yield (4.47).

Therefore, Proposition 4.11 shows that there exists $\overline{\boldsymbol{\mathcal{H}}} \in L^p(0,T;L^2(\Omega;\mathbb{R}^3))$ such that as $n\to+\infty$,

$$\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)} \to \overline{\boldsymbol{\mathcal{H}}} \text{ strongly in } L^p(0,T;L^2(\Omega;\mathbb{R}^3)).$$

Let us show that $\widehat{\boldsymbol{\mathcal{H}}}=\overline{\boldsymbol{\mathcal{H}}}$. By Lemma 4.5 (1) there is $\boldsymbol{\phi}_l \in L^2(0,T;B(\Omega))$ such that $\boldsymbol{\phi}_l \to \widehat{\boldsymbol{\mathcal{H}}}-\overline{\boldsymbol{\mathcal{H}}}$ strongly in $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$ as $l\to+\infty$. By using the bound in Proposition 4.16 we deduce that

$$\begin{aligned}\left|\int_0^T\int_\Omega \langle \widehat{\boldsymbol{\mathcal{H}}}-\overline{\boldsymbol{\mathcal{H}}},\boldsymbol{\phi}_l\rangle d\mathbf{x}dt\right| &= \lim_{n\to+\infty}\left|\int_0^T\int_\Omega \langle \widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}-\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)},\boldsymbol{\phi}_l\rangle d\mathbf{x}dt\right| \\ &\le \lim_{n\to+\infty}\Delta t(h_n)\int_0^T \|\partial_t\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}(t)\|_{(B(\Omega))^*}\|\boldsymbol{\phi}_l(t)\|_{B(\Omega)}dt=0.\end{aligned} \tag{4.51}$$

By sending $l\to+\infty$ in (4.51) we obtain $\widehat{\boldsymbol{\mathcal{H}}}-\overline{\boldsymbol{\mathcal{H}}}=\mathbf{0}$. Thus, the convergence properties (4.39)-(4.41) hold.

The convergences (4.42)-(4.44) can be confirmed in the same way as above. By using the Lebesgue's convergence theorem we can prove (4.45). □

Corollary 4.18. *For the converging sequences $\{\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}\}_{n\in\mathbb{N}}$, $\{\underline{\widehat{\boldsymbol{\mathcal{B}}}}_{h_n,\Delta t(h_n)}\}_{n\in\mathbb{N}}$ in Proposition 4.17, the following convergence properties hold as $n\to+\infty$.*

$$\begin{aligned}&\underline{\widehat{\boldsymbol{\mathcal{B}}}}_{h_n,\Delta t(h_n)}\times\mathbf{curl}\,\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}\times\underline{\widehat{\boldsymbol{\mathcal{B}}}}_{h_n,\Delta t(h_n)} \rightharpoonup \widehat{\boldsymbol{\mathcal{B}}}\times\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}\times\widehat{\boldsymbol{\mathcal{B}}} \text{ weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)),\\ &\langle\underline{\widehat{\boldsymbol{\mathcal{B}}}}_{h_n,\Delta t(h_n)},\mathbf{curl}\,\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}\rangle\underline{\widehat{\boldsymbol{\mathcal{B}}}}_{h_n,\Delta t(h_n)} \rightharpoonup \langle\widehat{\boldsymbol{\mathcal{B}}},\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}\rangle\widehat{\boldsymbol{\mathcal{B}}} \text{ weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)).\end{aligned}$$

By using the convergence properties (4.35) and (4.38) we can characterise the boundary value of the limit $\widehat{\boldsymbol{\mathcal{H}}}$ at $t=0,T$. By the bound (4.27), there exists $\widehat{\boldsymbol{\mathcal{H}}}^T \in L^2(\Omega;\mathbb{R}^3)$ such that

$$\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}(T) \rightharpoonup \widehat{\boldsymbol{\mathcal{H}}}^T \text{ weakly in } L^2(\Omega;\mathbb{R}^3) \tag{4.52}$$

as $n\to+\infty$. For this limit $\widehat{\boldsymbol{\mathcal{H}}}^T$ we observe the following property.

Corollary 4.19. *The following equality holds. For all $\boldsymbol{\phi}\in C^1([0,T];B(\Omega))$,*

$$\int_0^T\int_\Omega\langle\partial_t\widehat{\boldsymbol{\mathcal{H}}},\boldsymbol{\phi}\rangle d\mathbf{x}dt=\int_\Omega\langle\widehat{\boldsymbol{\mathcal{H}}}^T,\boldsymbol{\phi}(T)\rangle d\mathbf{x}-\int_\Omega\langle\widehat{\boldsymbol{\mathcal{H}}}_0,\boldsymbol{\phi}(0)\rangle d\mathbf{x}-\int_0^T\int_\Omega\langle\widehat{\boldsymbol{\mathcal{H}}},\partial_t\boldsymbol{\phi}\rangle d\mathbf{x}dt.$$

In order to prove that the limit $\widehat{\boldsymbol{\mathcal{H}}}$ solves the formulation (**P1'**), we need one more proposition.

Corollary 4.20. *Let $\{\overline{\boldsymbol{\phi}}_{l,h}\}_{h\in\Lambda}$ be the sequence in Lemma 4.6. The following convergence property holds.*

$$\lim_{n\to+\infty}\int_0^T\int_\Omega\mu\langle\partial_t\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)},\overline{\boldsymbol{\phi}}_{l,h_n}\rangle d\mathbf{x}dt=\int_\Omega\mu\langle\widehat{\boldsymbol{\mathcal{H}}}^T,\boldsymbol{\phi}_l(T)\rangle d\mathbf{x}-\int_\Omega\mu\langle\widehat{\boldsymbol{\mathcal{H}}}_0,\boldsymbol{\phi}_l(0)\rangle d\mathbf{x}-\int_0^T\int_\Omega\mu\langle\widehat{\boldsymbol{\mathcal{H}}},\partial_t\boldsymbol{\phi}_l\rangle d\mathbf{x}dt.$$

Proof. A calculation shows that

$$\begin{aligned}&\int_0^T\int_\Omega \mu\langle\partial_t\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)},\overline{\boldsymbol{\phi}}_{l,h_n}\rangle d\mathbf{x}dt\\&=\int_\Omega \mu\langle\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}(T),\widetilde{\boldsymbol{\phi}}_{l,h_n}(T)\rangle d\mathbf{x}-\int_\Omega \mu\langle\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}(0),\widetilde{\boldsymbol{\phi}}_{l,h_n}(0)\rangle d\mathbf{x}-\int_0^T\int_\Omega \mu\langle\underline{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)},\partial_t\widetilde{\boldsymbol{\phi}}_{l,h_n}\rangle d\mathbf{x}dt.\end{aligned}\tag{4.53}$$

By Lemma 4.6, Proposition 4.17 and sending $n\to+\infty$ in (4.53) we obtain the result. □

Finally we show that the limit $\widehat{\boldsymbol{\mathcal{H}}}$ is a solution to (**P1'**).

Theorem 4.21. *The limit $\widehat{\boldsymbol{\mathcal{H}}}$ obtained in Proposition 4.17 is a solution to (**P1'**).*

Proof. Let $\boldsymbol{\phi}_l:[0,T]\to V(\Omega)\cap C_0^\infty(\Omega;\mathbb{R}^3)$ be a function in Lemma 4.5 (2). The inequality (4.26) yields

$$\begin{aligned}&\int_\Omega \mu\langle(\widehat{\boldsymbol{\mathcal{H}}}_{h,n}-\widehat{\boldsymbol{\mathcal{H}}}_{h,n-1}+\boldsymbol{\mathcal{H}}_{s,h,n}-\boldsymbol{\mathcal{H}}_{s,h,n-1})/\Delta t,\boldsymbol{\phi}_{l,h}(n\Delta t)\rangle d\mathbf{x}\\&\quad-\int_\Omega \mu\langle(\boldsymbol{\mathcal{H}}_{s,h,n}-\boldsymbol{\mathcal{H}}_{s,h,n-1})/\Delta t,\widehat{\boldsymbol{\mathcal{H}}}_{h,n}\rangle d\mathbf{x}+\int_{\Omega_s}G(\widehat{\boldsymbol{\mathcal{B}}}_{h,n-1},\mathbf{curl}\,\boldsymbol{\phi}_{l,h}(n\Delta t))d\mathbf{x}\\&\geq\int_{\Omega_s}G(\widehat{\boldsymbol{\mathcal{B}}}_{h,n-1},\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}}_{h,n})d\mathbf{x}+\frac{1}{2\Delta t}\int_\Omega\mu|\widehat{\boldsymbol{\mathcal{H}}}_{h,n}|^2d\mathbf{x}-\frac{1}{2\Delta t}\int_\Omega\mu|\widehat{\boldsymbol{\mathcal{H}}}_{h,n-1}|^2d\mathbf{x},\end{aligned}\tag{4.54}$$

where we have used the inequality

$$\langle(\widehat{\boldsymbol{\mathcal{H}}}_{h,n}-\widehat{\boldsymbol{\mathcal{H}}}_{h,n-1})/\Delta t,\widehat{\boldsymbol{\mathcal{H}}}_{h,n}\rangle\geq(|\widehat{\boldsymbol{\mathcal{H}}}_{h,n}|^2-|\widehat{\boldsymbol{\mathcal{H}}}_{h,n-1}|^2)/(2\Delta t).$$

Multiplying (4.54) by Δt and summing over $n=1\to N$, we obtain

$$\begin{aligned}&\int_0^T\int_\Omega\mu\langle\partial_t\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t(h)}+\partial_t\widetilde{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t(h)},\overline{\boldsymbol{\phi}}_{l,h}\rangle d\mathbf{x}dt-\int_0^T\int_\Omega\mu\langle\partial_t\widetilde{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t(h)},\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t(h)}\rangle d\mathbf{x}dt\\&\quad+\int_0^T\int_{\Omega_s}G(\underline{\widehat{\boldsymbol{\mathcal{B}}}}_{h,\Delta t(h)},\mathbf{curl}\,\overline{\boldsymbol{\phi}}_{l,h})d\mathbf{x}dt+\frac{1}{2}\int_\Omega\mu|\widehat{\boldsymbol{\mathcal{H}}}_{h,\Delta t(h)}(0)|^2d\mathbf{x}\\&\geq\int_0^T\int_{\Omega_s}G(\underline{\widehat{\boldsymbol{\mathcal{B}}}}_{h,\Delta t(h)},\mathbf{curl}\,\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t(h)})d\mathbf{x}dt+\frac{1}{2}\int_\Omega\mu|\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h,\Delta t(h)}(T)|^2d\mathbf{x}.\end{aligned}\tag{4.55}$$

By the convergence properties given in Lemma 4.13, Proposition 4.17 and Corollary 4.20, we can choose a subsequence $\{h_n\}_{n=1}^\infty\subset\Lambda$ so that by $n\to+\infty$ in (4.55)

$$\begin{aligned}&\int_\Omega\mu\langle\widehat{\boldsymbol{\mathcal{H}}}^T,\boldsymbol{\phi}_l(T)\rangle d\mathbf{x}-\int_\Omega\mu\langle\widehat{\boldsymbol{\mathcal{H}}}_0,\boldsymbol{\phi}_l(0)\rangle d\mathbf{x}-\int_0^T\int_\Omega\mu\langle\widehat{\boldsymbol{\mathcal{H}}},\partial_t\boldsymbol{\phi}_l\rangle d\mathbf{x}dt+\int_0^T\int_\Omega\mu\langle\partial_t\boldsymbol{\mathcal{H}}_s,\boldsymbol{\phi}_l-\widehat{\boldsymbol{\mathcal{H}}}\rangle d\mathbf{x}dt\\&+\int_0^T\int_{\Omega_s}G(\widehat{\boldsymbol{\mathcal{B}}},\mathbf{curl}\,\boldsymbol{\phi}_l)d\mathbf{x}dt+\frac{1}{2}\int_\Omega\mu|\widehat{\boldsymbol{\mathcal{H}}}_0|^2d\mathbf{x}\geq\int_0^T\int_{\Omega_s}G(\widehat{\boldsymbol{\mathcal{B}}},\mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}})d\mathbf{x}dt+\frac{1}{2}\int_\Omega\mu|\widehat{\boldsymbol{\mathcal{H}}}^T|^2d\mathbf{x}.\end{aligned}\tag{4.56}$$

Here we have used the facts that

$$\lim_{n\to+\infty}\int_0^T\int_{\Omega_s} G(\widehat{\underline{\boldsymbol{\mathcal{B}}}}_{h_n,\Delta t(h_n)}, \mathbf{curl}\,\overline{\boldsymbol{\phi}}_{l,h_n})d\mathbf{x} = \int_0^T\int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}, \mathbf{curl}\,\boldsymbol{\phi}_l)d\mathbf{x}dt,$$
$$\liminf_{n\to+\infty}\int_0^T\int_{\Omega_s} G(\widehat{\underline{\boldsymbol{\mathcal{B}}}}_{h_n,\Delta t(h_n)}, \mathbf{curl}\,\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)})d\mathbf{x}dt \geq \int_0^T\int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}, \mathbf{curl}\,\widehat{\boldsymbol{\mathcal{H}}})d\mathbf{x}dt,$$
$$\lim_{n\to+\infty}\int_\Omega \mu|\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}(0)|^2 d\mathbf{x} = \int_\Omega \mu|\widehat{\boldsymbol{\mathcal{H}}}_0|^2 d\mathbf{x},\ \liminf_{n\to+\infty}\int_\Omega \mu|\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)}(T)|^2 d\mathbf{x} \geq \int_\Omega \mu|\widehat{\boldsymbol{\mathcal{H}}}^T|^2 d\mathbf{x},$$

which can be proved by using Lemma 4.6, the strong convergence (4.45), the properties of the energy densities (2.23), the Lebesgue convergence theorem, the weak convergence properties obtained in Corollary 4.18, the property of the interpolation $r_h(\widehat{\boldsymbol{\mathcal{H}}}_0)$, and the weak convergence (4.52).

In order to pass $l \to +\infty$ in (4.56), let us note the following convergence property.

$$\int_0^T\int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}, \mathbf{curl}\,\boldsymbol{\phi})d\mathbf{x}dt = \lim_{l\to+\infty}\int_0^T\int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}, \mathbf{curl}\,\boldsymbol{\phi}_l)d\mathbf{x}dt. \tag{4.57}$$

Indeed, by the convexity of $g_\sharp(\cdot)$ and (4.2) we deduce that for all $x, y \in \mathbb{R}$

$$|g_\sharp(x) - g_\sharp(y)| \leq \max\{|g'_\sharp(x)|, |g'_\sharp(y)|\}|x-y| \leq A_{\sharp 5}(|x|+|y|)|x-y|, (\sharp = \perp, \|, 0). \tag{4.58}$$

By (4.58) and Lemma 4.5 (2), we see that as $l \to +\infty$

$$\left|\int_0^T\int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}, \mathbf{curl}\,\boldsymbol{\phi})d\mathbf{x}dt - \int_0^T\int_{\Omega_s} G(\widehat{\boldsymbol{\mathcal{B}}}, \mathbf{curl}\,\boldsymbol{\phi}_l)d\mathbf{x}dt\right|$$
$$\leq (A_{\perp 5} + A_{\| 5} + A_{05})\int_0^T\int_{\Omega_s} (|\,\mathbf{curl}\,\boldsymbol{\phi}_l| + |\,\mathbf{curl}\,\boldsymbol{\phi}|)|\,\mathbf{curl}\,\boldsymbol{\phi}_l - \mathbf{curl}\,\boldsymbol{\phi}|d\mathbf{x}dt \to 0.$$

Thus, by sending $l \to +\infty$ in (4.56), noting (4.57) and Lemma 4.5 (2) we obtain the desired inequality. The equality (2.30) has been already given in Corollary 4.19. □

In our practical computation we solve the problem in the hybrid space $W_h(\Omega)$. Let us propose the fully discrete hybrid optimisation problem $(\mathbf{P2}_{h,\Delta t})$. We assume that the initial value $(\boldsymbol{\psi}_0|\nabla u_0) \in W(\Omega)$ satisfies the regularities $\boldsymbol{\psi}_0 \in H^1(\mathbf{curl};\Omega_s)$ and $u_0 \in H^2(\Omega_d)$ so that the interpolations $\boldsymbol{\psi}_{0,h} := r_h(\boldsymbol{\psi}_0)$ and $u_{0,h} := \pi_h(u_0)$ are well-defined and satisfy the discrete divergence-free condition

$$(\boldsymbol{\psi}_{0,h}|\nabla u_{0,h}) + \boldsymbol{\mathcal{H}}_{s,h,0} \in X_h^{(\mu)}(\Omega). \tag{4.59}$$

$(\mathbf{P2}_{h,\Delta t})$ On the assumption (4.59), for $n = 1 \to N$, find $(\boldsymbol{\psi}_{h,n}|\nabla u_{h,n}) \in W_h(\Omega)$ such that

$$F_{h,n}((\boldsymbol{\psi}_{h,n}|\nabla u_{h,n})) = \min_{(\boldsymbol{\phi}_h|\nabla v_h)\in W_h(\Omega)} F_{h,n}((\boldsymbol{\phi}_h|\nabla v_h)).$$

The existence of a unique minimiser of the problem $(\mathbf{P2}_{h,\Delta t})$ immediately follows Proposition 4.1 and Proposition 4.14.

Let us define the discrete functions $\widetilde{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t} \in C([0,T];H(\mathbf{curl};\Omega))$ and $\overline{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t}, \underline{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t} \in L^\infty(0,T;H(\mathbf{curl};\Omega))$ consisting of the minimisers of $(\mathbf{P2}_{h,\Delta t})$ by

$$\widetilde{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t}(t) := \frac{t-(n-1)\Delta t}{\Delta t}(\boldsymbol{\psi}_{h,n}|\nabla u_{h,n}) + \frac{n\Delta t - t}{\Delta t}(\boldsymbol{\psi}_{h,n-1}|\nabla u_{h,n-1}) \text{ in } [(n-1)\Delta t, n\Delta t],$$

$$\overline{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t}(t) := \begin{cases} (\boldsymbol{\psi}_{h,n}|\nabla u_{h,n}) & \text{in } ((n-1)\Delta t, n\Delta t], \\ (\boldsymbol{\psi}_{h,0}|\nabla u_{h,0}) & \text{on } \{t=0\}, \end{cases}$$

$$\underline{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t}(t) := \begin{cases} (\boldsymbol{\psi}_{h,n-1}|\nabla u_{h,n-1}) & \text{in } [(n-1)\Delta t, n\Delta t), \\ (\boldsymbol{\psi}_{h,N}|\nabla u_{h,N}) & \text{on } \{t=T\}, \end{cases} \quad \underline{\widehat{\boldsymbol{\mathcal{B}}}}'_{h,\Delta t}(t) := \begin{cases} \widehat{\boldsymbol{\mathcal{B}}}'_{h,n-1} & \text{in } [(n-1)\Delta t, n\Delta t), \\ \widehat{\boldsymbol{\mathcal{B}}}'_{h,N} & \text{on } \{t=T\}, \end{cases}$$

for $n = 1, \cdots, N$, where $(\boldsymbol{\psi}_{h,n}|\nabla u_{h,n}) \in W_h(\Omega)$ is the minimiser of $(\mathbf{P2}_{h,\Delta t})$ for n, $\widehat{\boldsymbol{\mathcal{B}}}'_{h,n-1}$ is defined by $\widehat{\boldsymbol{\mathcal{B}}}'_{h,n-1} = \boldsymbol{\mathcal{B}}'_{h,n-1}/\sqrt{|\boldsymbol{\mathcal{B}}'_{h,n-1}|^2 + \varepsilon^2}$ for $\varepsilon > 0$ and $\boldsymbol{\mathcal{B}}'_{h,n-1} = \mu((\boldsymbol{\psi}_{h,n-1}|\nabla u_{h,n-1}) + \boldsymbol{\mathcal{H}}_{s,h,n-1})$.

We see that the discrete divergence-free condition holds in the sense that $\widetilde{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t}(t) + \widetilde{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t}(t)$, $\overline{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t}(t) + \overline{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t}(t)$, $\underline{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t}(t) + \underline{\boldsymbol{\mathcal{H}}}_{s,h,\Delta t}(t) \in X_h^{(\mu)}(\Omega)$ for all $t \in [0,T]$.

Proposition 4.1 and Theorem 4.21 ensure the following result.

Corollary 4.22. *For any $p \in [2,\infty)$ there are a subsequence $\{h_n\}_{n=1}^\infty \subset \Lambda$ and a solution $(\boldsymbol{\psi}|\nabla u) \in L^2(0,T;H(\mathbf{curl};\Omega))$ to the hybrid formulation (**P2'**) such that the discrete functions $\widetilde{(\boldsymbol{\psi}|\nabla u)}_{h_n,\Delta t(h_n)}$, $\overline{(\boldsymbol{\psi}|\nabla u)}_{h_n,\Delta t(h_n)}$, $\underline{(\boldsymbol{\psi}|\nabla u)}_{h_n,\Delta t(h_n)}$, and $\underline{\widehat{\boldsymbol{\mathcal{B}}}}'_{h_n,\Delta t(h_n)}$ converge in the same sense as (4.35)−(4.45) for $\widetilde{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)} = \widetilde{(\boldsymbol{\psi}|\nabla u)}_{h_n,\Delta t(h_n)}$, $\overline{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)} = \overline{(\boldsymbol{\psi}|\nabla u)}_{h_n,\Delta t(h_n)}$, $\underline{\widehat{\boldsymbol{\mathcal{H}}}}_{h_n,\Delta t(h_n)} = \underline{(\boldsymbol{\psi}|\nabla u)}_{h_n,\Delta t(h_n)}$, $\underline{\widehat{\boldsymbol{\mathcal{B}}}}_{h_n,\Delta t(h_n)} = \underline{\widehat{\boldsymbol{\mathcal{B}}}}'_{h_n,\Delta t(h_n)}$, $\widehat{\boldsymbol{\mathcal{H}}} = (\boldsymbol{\psi}|\nabla u)$ and $\widehat{\boldsymbol{\mathcal{B}}} = \mu((\boldsymbol{\psi}|\nabla u) + \boldsymbol{\mathcal{H}}_s)/\sqrt{|\mu((\boldsymbol{\psi}|\nabla u) + \boldsymbol{\mathcal{H}}_s)|^2 + \varepsilon^2}$ as $n \to +\infty$.*

Remark 4.23. If a solution of (**P1'**) is unique, the convergence results in Proposition 4.17 and Corollary 4.22 hold without extracting any subsequence.

5. Numerical results

In this section we solve the discrete optimisation problem $(\mathbf{P2}_{h,\Delta t})$ in the space $W_h(\Omega)$ numerically by means of Newton's method coupled with the conjugate gradient method. The hybrid space $W_h(\Omega)$ is implemented by following Remark 4.2. The code was developed in ALBERTA [34] platform and based on the documentation on the practical implementation of the edge finite element of lowest order reported in [19, Chapter 4].

Let us first set the parameters, the external magnetic field $\boldsymbol{\mathcal{H}}_s$, the initial value $(\boldsymbol{\psi}_0|\nabla u_0)$ and the energy densities $g_\perp(\cdot)$, $g_\parallel(\cdot)$, $g_0(\cdot)$ used throughout this section. We assume that $\mu_d = \mu_s = 1$ and $\varepsilon = 10^{-6}$ which is used to define $\widehat{\boldsymbol{\mathcal{B}}}$.

We apply the external magnetic field $\boldsymbol{\mathcal{H}}_s$ defined by

$$\boldsymbol{\mathcal{H}}_s(t) := at(\cos t, 0, \sin t), \tag{5.1}$$

where $a > 0$ is a positive constant. The magnetic field $\boldsymbol{\mathcal{H}}_s$ is uniform in space and parallel to $x - z$ plane. The direction of $\boldsymbol{\mathcal{H}}_s$ rotates and the magnitude of $\boldsymbol{\mathcal{H}}_s$ increases as time goes.

The initial value $(\boldsymbol{\psi}_0|\nabla u_0)$ is taken to be uniformly zero so that the discrete divergence-free condition (4.59) is naturally satisfied.

We define the energy densities $g_\perp(\cdot)$, $g_\parallel(\cdot)$, $g_0(\cdot) \in C^2(\mathbb{R})$ by $g_\perp(x) := f(x^2 - \mathcal{J}_{c\perp}^2)$, $g_\parallel(x) := f(x^2 - \mathcal{J}_{c\parallel}^2)$, $g_0(x) := f(x^2 - \mathcal{J}_{c0}^2)$ by using $f \in C^2(\mathbb{R})$ defined by

h	Δt	d.o.f	$t=\frac{30\pi}{125}$	$t=\frac{60\pi}{125}$	$t=\frac{90\pi}{125}$	$t=\frac{120\pi}{125}$	$t=\frac{150\pi}{125}$	$t=\frac{180\pi}{125}$	$t=\frac{210\pi}{125}$	$t=\frac{240\pi}{125}$
1/2	$2\pi/125$	52	0.01796	0.03600	0.05402	0.07182	0.08881	0.10559	0.12236	0.13916
1/4	$2\pi/250$	632	0.01477	0.02955	0.04432	0.05887	0.07255	0.08601	0.09951	0.11301
1/8	$2\pi/500$	6064	0.01080	0.02163	0.03244	0.04301	0.05255	0.06202	0.07154	0.08119
1/16	$2\pi/1000$	52832	0.00754	0.01512	0.02267	0.02996	0.03619	0.04277	0.04872	0.05532

TABLE 1. Measures of $\|(\widetilde{\boldsymbol{\psi}|\nabla u})_{h,\Delta t}(t) - (\widetilde{\boldsymbol{\psi}|\nabla u})_{\hat{h},\hat{\Delta t}}(t)\|_{L^2(\Omega;\mathbb{R}^3)}$.

$$f(x) = \begin{cases} 0 & \text{in } (-\infty, 0] \\ 5x^3/3 & \text{in } [0, 0.1], \\ x^2/2 - x/20 + 1/600 & \text{in } [0.1, 0.5], \\ -x^3/3 + x^2 - 3x/10 + 13/300 & \text{in } [0.5, 1], \\ 7x/10 - 29/100 & \text{in } [1, +\infty). \end{cases}$$

These energy densities $g_\perp(\cdot)$, $g_\|(\cdot)$,$g_0(\cdot)$ satisfy the required properties (2.23) and (4.2). The shape of $g_0(x)$ and $g_0'(x)$, for example, is drawn as in Figure 2. Let us consider $g_\perp(|\mathbf{v}|)$, $g_\|(|\mathbf{v}|)$, $g_0(|\mathbf{v}|)$ ($\mathbf{v} \in \mathbb{R}^3$) as approximation

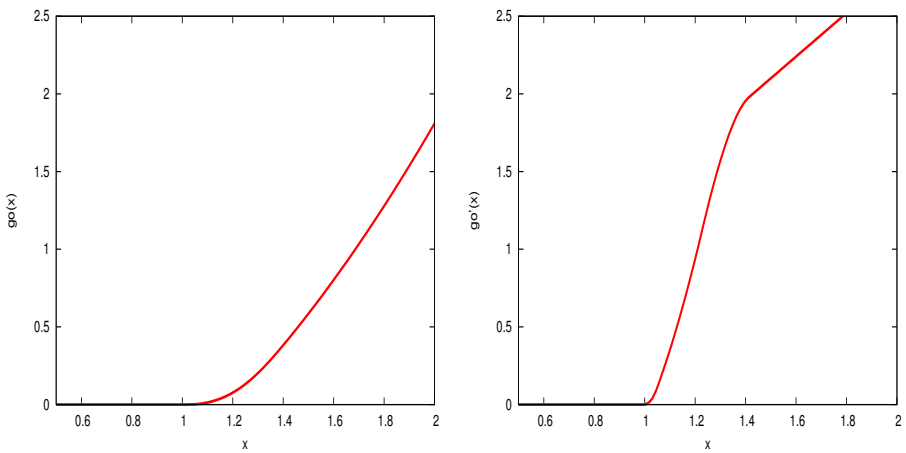

FIGURE 2. The graph of $g_0(x)$ (left) and the graph of $g_0'(x)$ (right).

of $\gamma_\perp(\mathbf{v})$, $\gamma_\|(\mathbf{v})$, $\gamma_0(\mathbf{v})$ defined in (2.16). In particular, its gradient $\nabla(g_0(|\mathbf{v}|)) = g_0'(|\mathbf{v}|)\mathbf{v}/|\mathbf{v}|$ is seen as a regularisation of $\partial\gamma_0(\mathbf{v})$ with jump discontinuity at $|\mathbf{v}| = \mathcal{J}_{c0}$ (see (2.21)).

5.1. Convergence rate of the discrete solution with respect to the $L^2(\Omega;\mathbb{R}^3)$-norm

We compute the rate of convergence of the sequence of the discrete solution $(\widetilde{\boldsymbol{\psi}|\nabla u})_{h,\Delta t}$ solving the hybrid problem ($\mathbf{P2}_{h,\Delta t}$). Here we consider the problem for $\mathcal{J}_{c0} = 2$, $\mathcal{J}_{c\perp} = \mathcal{J}_{c\|} = 1$, $a = 0.01$ in (5.1) in the cubic domains $\Omega = (-2, 2)^3$ and $\Omega_s = (-1, 1)^3$. We discretize these domains by uniform mesh. Let us fix a relatively small mesh size $\hat{h}$ and a small time step $\hat{\Delta t}$ as $\hat{h} \approx 1/32$ and $\hat{\Delta t} = 2\pi/2000$. For this mesh size $\hat{h}$ our computation involves 440512 degrees of freedom. For various mesh size h and time step Δt we measure the error between $(\widetilde{\boldsymbol{\psi}|\nabla u})_{h,\Delta t}$ and $(\widetilde{\boldsymbol{\psi}|\nabla u})_{\hat{h},\hat{\Delta t}}$ with respect to the $L^2(\Omega;\mathbb{R}^3)$-norm. The result is summarised in Table 1, which suggests that the convergence rate is consistent with the order $O(h^{1/2})$.

	Case 1	Case 2	Case 3
$\Vert\, \mathbf{curl}\, (\widetilde{\boldsymbol{\psi}\vert\nabla u})_{h,\Delta t}\Vert_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))}$	0.856	0.735	0.743

TABLE 2. The maximum current density.

5.2. **The electric current and the magnetic flux in a spherical domain**

We present pictures showing the distribution of the electric current density $\boldsymbol{\mathcal{J}}$ and the magnetic flux density $\boldsymbol{\mathcal{B}} = \mu(\widehat{\boldsymbol{\mathcal{H}}} + \boldsymbol{\mathcal{H}}_s)$ around a bulk superconductor. Here we assume that the domain Ω and the superconductor Ω_s are sphere-like polyhedrons whose centre is $(0,0,0)$ and radiuses are 4 and 1 respectively. We mesh Ω by tetrahedrons whose size h_K is around 0.5 in the neighbour of $\partial\Omega$ and is around 0.01 in the neighbour of $\partial\Omega_s$. The mesh was generated by TetGen [35]. Figure 3 shows the mesh on $\partial\Omega$, $\partial\Omega_s$ and the cross-section of Ω cut by the plane $y = 0$. Using this mesh, our computation involves 734123 degrees of freedom.

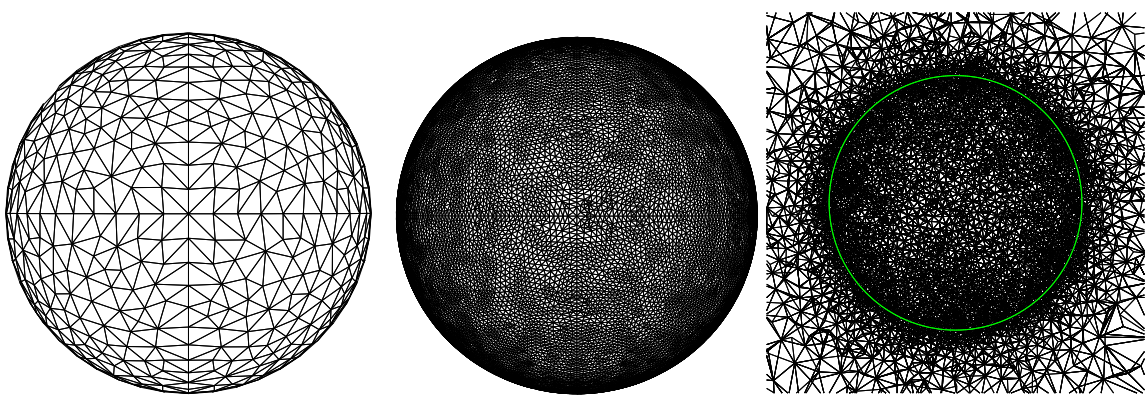

FIGURE 3. The mesh on $\partial\Omega$ (left), on $\partial\Omega_s$ (centre) and the cross-section of Ω (right).

We fix the time interval $T = \pi$ and the constant $a = 0.1$ in (5.1). The time step size Δt is fixed as $\Delta t = \pi/1000$ so that the direction of $\boldsymbol{\mathcal{H}}_s$ becomes opposite to the initial direction $(1,0,0)$ after 1000 time steps when $t = \pi$. The computation is carried out for the following three different sets of the critical current densities $\mathcal{J}_{c0}$, $\mathcal{J}_{c\perp}$ and $\mathcal{J}_{c\|}$, respectively.

$$\text{Case 1} : \mathcal{J}_{c0} = 1,\ \mathcal{J}_{c\perp} = \mathcal{J}_{c\|} = 0.5, \quad \text{Case 2} : \mathcal{J}_{c0} = 1,\ \mathcal{J}_{c\perp} = 0.25 \text{ and } \mathcal{J}_{c\|} = 0.5,$$
$$\text{Case 3} : \mathcal{J}_{c0} = 1,\ \mathcal{J}_{c\perp} = 0.5 \text{ and } \mathcal{J}_{c\|} = 0.25.$$

Table 2 shows the $L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))$-norm of the electric current $\mathbf{curl}\,(\widetilde{\boldsymbol{\psi}|\nabla u})_{h,\Delta t}$ for each case and suggests that the magnitude of the current does not exceed $\mathcal{J}_{c0}$ and the last term $\rho_0\boldsymbol{\mathcal{J}}$ in the $\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{J}}$-relation (2.14) always vanishes in these numerical simulations.

Figure 4 shows the magnitude of the current $\boldsymbol{\mathcal{J}}$ and its vector field on $\partial\Omega_s$ viewed from an angle in 3D space at $t = \pi$ for each case.

Figures 5-7 show the magnitude of the current $\boldsymbol{\mathcal{J}}$ on the cross-sections of Ω_s cut by the plane $y = 0$ and the plane $z = 0$ for each case.

Figures 8-10 show the magnitude of the magnetic flux $\boldsymbol{\mathcal{B}}$ on the cross-sections of Ω cut by the plane $y = 0$ and the plane $z = 0$ for each case.

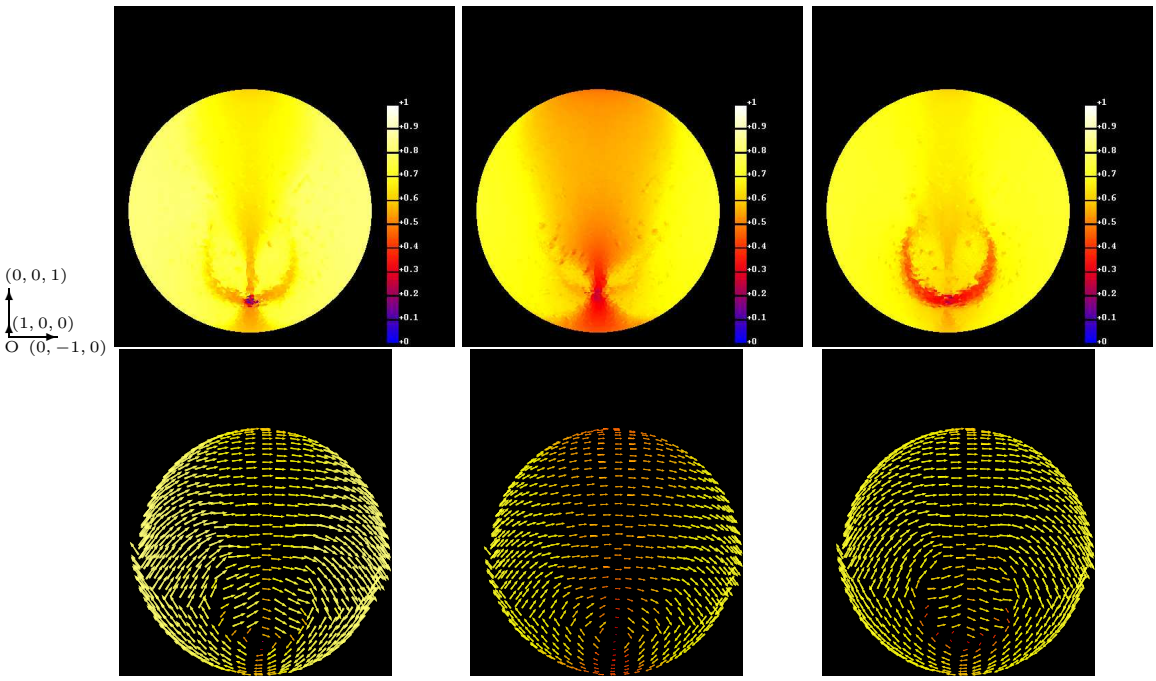

FIGURE 4. The electric current $\mathcal{J}$ on the surface $\partial\Omega_s$ at $t = \pi$. The first row and the second row show the magnitude and the direction, respectively: Case 1 (left), Case 2 (centre), Case 3 (right).

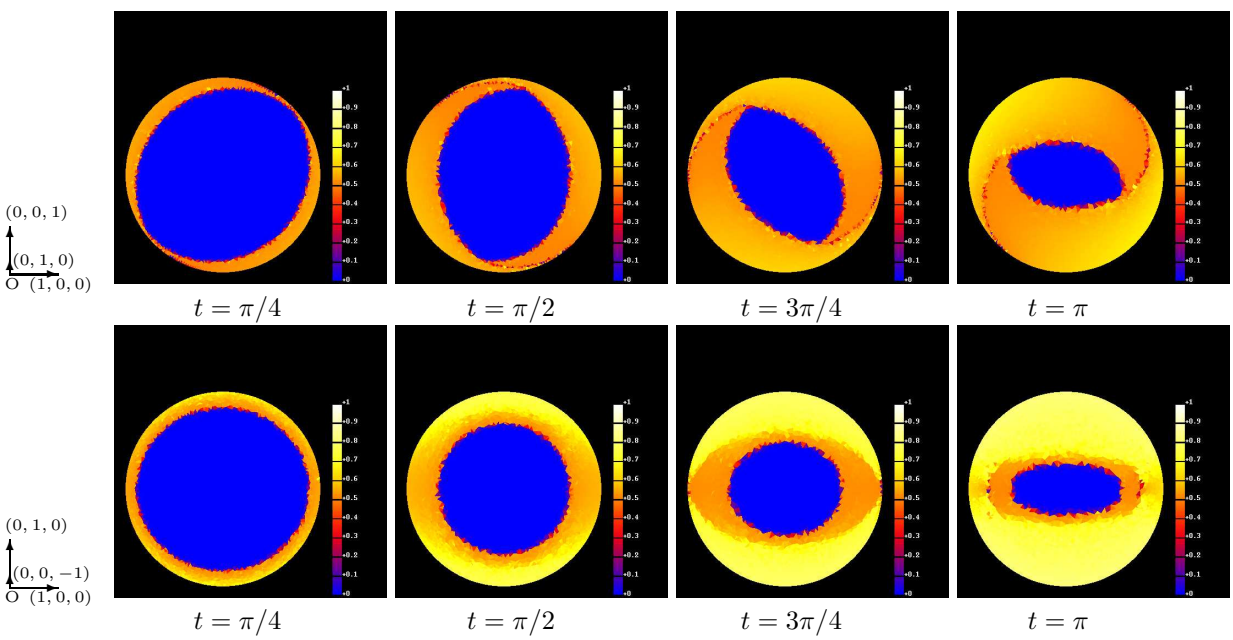

FIGURE 5. The magnitude of the electric current $\mathcal{J}$ on the cross-sections of Ω_s cut by the plane $y = 0$ (first row), by the plane $z = 0$ (second row) for Case 1.

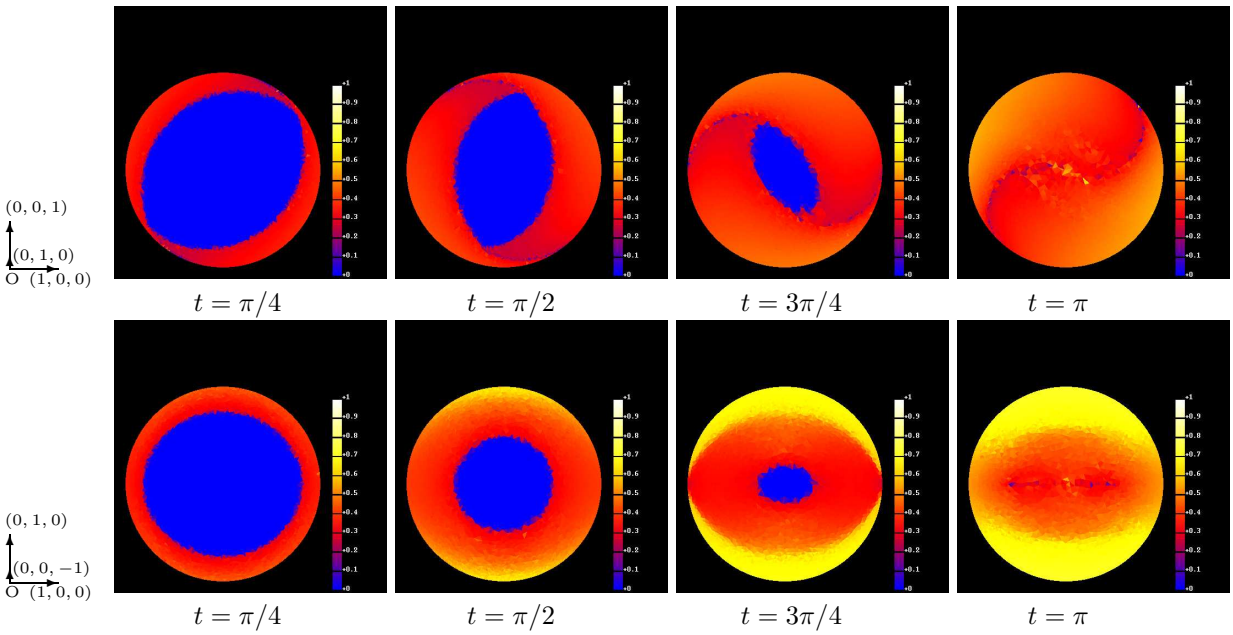

FIGURE 6. The magnitude of the electric current $\mathcal{J}$ on the cross-sections of Ω_s cut by the plane $y = 0$ (first row), by the plane $z = 0$ (second row) for Case 2.

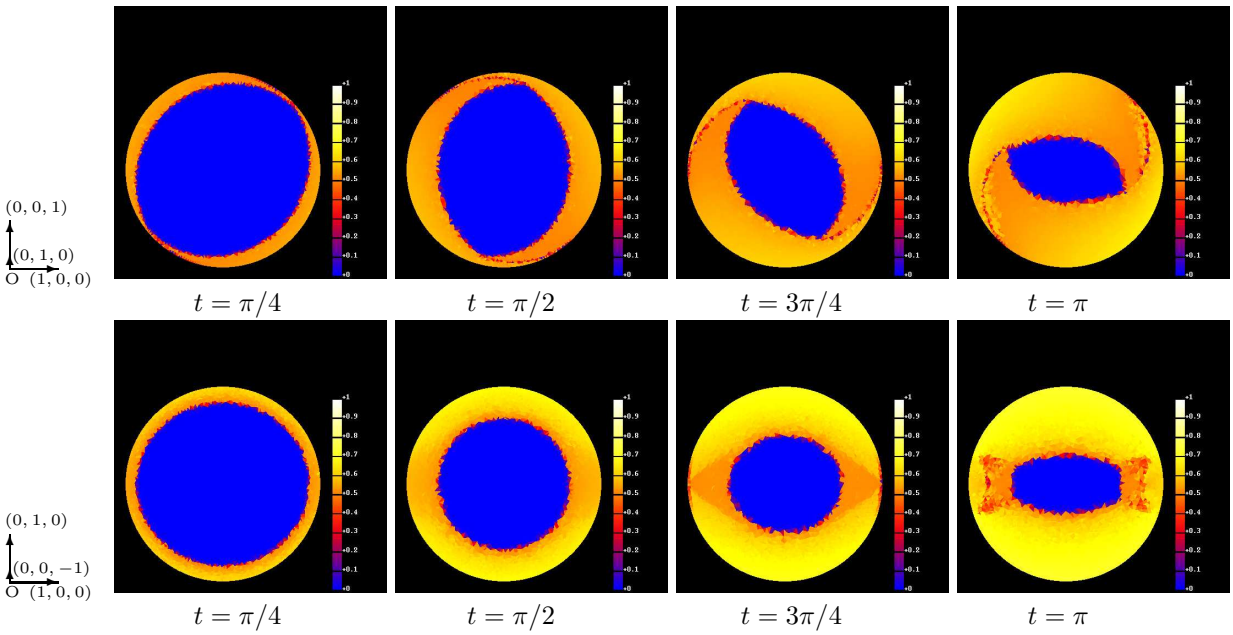

FIGURE 7. The magnitude of the electric current $\mathcal{J}$ on the cross-sections of Ω cut by the plane $y = 0$ (first row), by the plane $z = 0$ (second row) for Case 3.

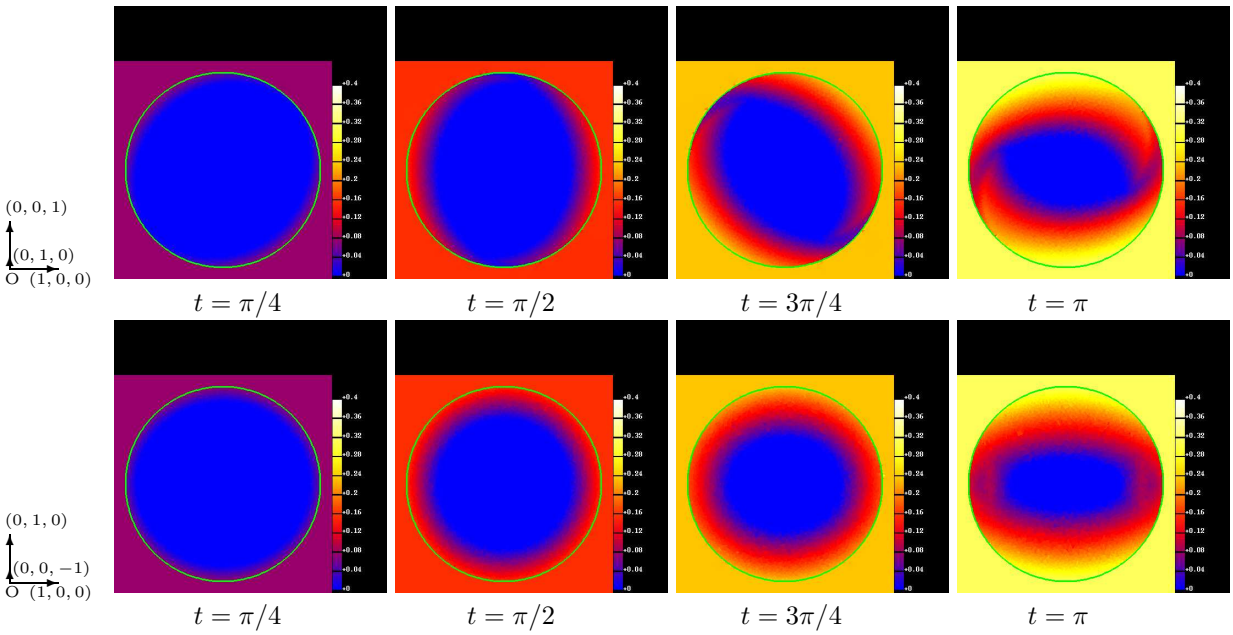

FIGURE 8. The magnitude of the magnetic flux $\boldsymbol{\mathcal{B}}$ on the cross-sections of Ω cut by the plane $y = 0$ (first row), by the plane $z = 0$ (second row) for Case 1.

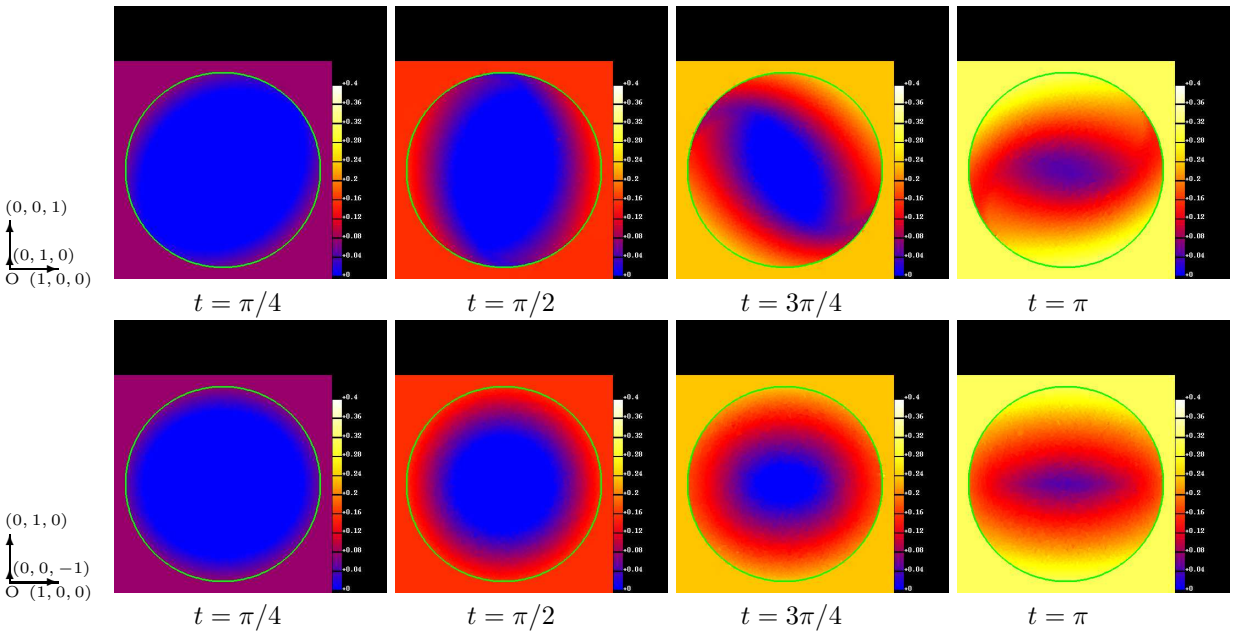

FIGURE 9. The magnitude of the magnetic flux $\boldsymbol{\mathcal{B}}$ on the cross-sections of Ω cut by the plane $y = 0$ (first row), by the plane $z = 0$ (second row) for Case 2.

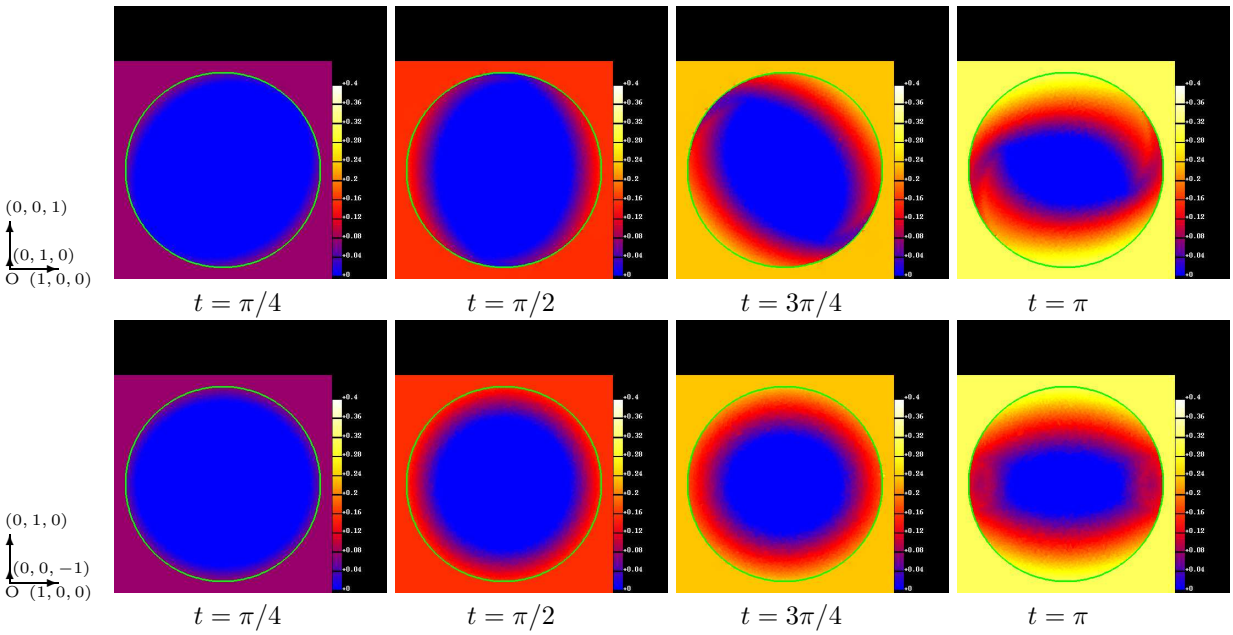

FIGURE 10. The magnitude of the magnetic flux $\boldsymbol{\mathcal{B}}$ on the cross-sections of Ω cut by the plane $y = 0$ (first row), by the plane $z = 0$ (second row) for Case 3.

Figure 11 shows the direction of the magnetic flux $\boldsymbol{\mathcal{B}}$ on the cross-section of Ω cut by the plane $y = 0$ at $t = \pi$.

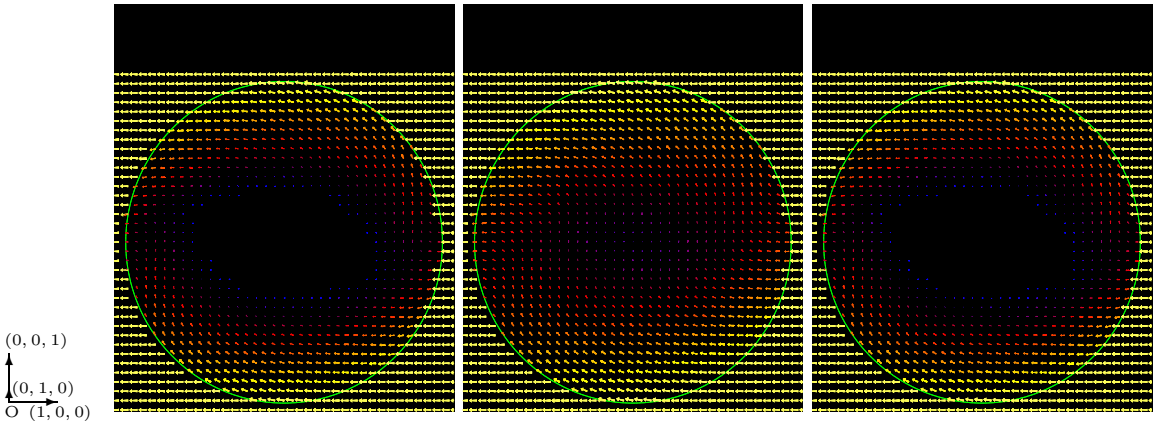

FIGURE 11. The direction of the magnetic flux $\boldsymbol{\mathcal{B}}$ on the cross-section of Ω cut by the plane $y = 0$ at $t = \pi$: Case 1 (left), Case 2 (centre), Case 3 (right). The colour scale is same as that of Figures 8-10 at $t = \pi$.

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