The energy decay of divergence-free displacements for elastic waves with Neumann boundary condition *

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1 Introduction

It is well-known that the linear equation of displacements $u = u(t,x)$ for isotropic, elastic waves is

$$u_{tt} - \mathrm{div}\,\left(\lambda(\mathrm{div}\ u)I + \mu(\nabla u + {}^t\nabla u)\right) = 0, \tag{1.1}$$

where $\nabla u = (\partial_j u^i)$ is the gradient matrix and λ, μ are Lamé constants which satisfy $3\lambda + 2\mu > 0,\ \mu > 0$ by physical requirements. The boundary condition for the traction problem is

$$\boldsymbol{n} \cdot \left(\lambda(\mathrm{div}\ u)I + \mu(\nabla u + {}^t\nabla u)\right) = 0, \tag{1.2}$$

where $\boldsymbol{n}$ is an outer unit normal to the boundary.

In this paper, we investigate divergence-free displacements [1] of the form;

$$u(t,x) = {}^t\left(x_2\varphi(t,r), -x_1\varphi(t,r), 0\right),\ r = |x|,\ x = (x_1, x_2, x_3).$$

In the exterior domain $\{r \geq b\}$ with a constant $b > 0$, we find from (1.1) and (1.2) that φ satisfies the wave equation with a propergation speed $c_2 =$

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$\sqrt{\mu}$, that is the equation of the transeverse wave, and Neumann boundary condition;

$$\begin{cases} \varphi_{tt} - \dfrac{4c_2^2}{r}\varphi_r - c_2^2\varphi_{rr} = 0 & \text{in } [0,\infty)\times[b,\infty), \\ \varphi_r = 0 & \text{on } [0,\infty)\times\{b\}. \end{cases} \tag{1.3}$$

Here and hereafter we set the initial condition;

$$\varphi(0,r) = \varphi_0(r), \ \ \varphi_t(0,r) = \varphi_1(r), \tag{1.4}$$

where φ_0 and φ_1 are given functions of bounded support.

We also define the energy for (1.3) involving boundary integral;

$$\begin{aligned} E(\varphi,R,t) = & \int_{b\le|x|\le R} \left\{(r\varphi_t)^2 + c_2^2(2\varphi + r\varphi_r)^2 + 2c_2^2\varphi^2\right\} dx \\ & + \int_{|x|=b} 2b\mu\varphi^2 dS_x. \end{aligned} \tag{1.5}$$

Then we can prove

Theorem 1 *Let φ be a solution of (1.3).*

1. *(energy conservation) The following equality is valid for all $t > 0$.*

$$E(\varphi,\infty,t) = E(\varphi,\infty,0). \tag{1.6}$$

2. *(local energy decay) For any $R > b$, there exists a positive constant $C = C(R)$ such that the following inequality is valid for all $t > 0$.*

$$tE(\varphi,R,t) \le CE(\varphi,\infty,0). \tag{1.7}$$

Since Huygens' principle holds for (1.1), Morawetz argument in [2] leads to the exponential decay of the local energy;

$$E(\varphi,R,t) \le \widetilde{C}e^{-\alpha t}E(\varphi,\infty,0)$$

with some positive constants $\widetilde{C}$ and α. This is the final goal of this paper. The method of the proof of the second part of Theorem1 is based on another work of Morawetz [1].

2 Energy conservation

In this section, we shall prove the energy conservation property stated in Theorem1. First of all, we note that the solution φ of (1.3) satsifies

$$L(r^2\varphi) = 0,$$

where a differential operator L is defined by

$$L = \partial_t^2 - c_2^2\partial_r^2 + 2c_2^2 r^{-2}.$$

Here we make use of

Lemma 1 *Let ψ be a smooth function of (t, r). Then the following identity is valid.*

$$\partial_t \left(Q(\psi)\right) + \partial_r \left(Q_1(\psi)\right) = 2\psi_t L(\psi), \tag{2.1}$$

where Q and Q_1 are defined by

$$\begin{aligned} Q(\psi) &= \psi_t^2 + c_2^2\psi_r^2 + 2c_2^2 r^{-2}\psi^2, \\ Q_1(\psi) &= -2c_2^2\psi_t\psi_r. \end{aligned}$$

Proof. This is a usual divergence form of space-time. Lemma follows from direct computations. □

Introducing a notation

$$E(\psi)(t) = \int_b^\infty Q(\psi)(t, r)dr,$$

we obtain

Proposition 1 *Let φ be a solution of (1.3). Then the following equality is valid for all $T > 0$.*

$$E(r^2\varphi)(T) + 2\mu b^3\varphi(T, b)^2 = E(r^2\varphi)(0) + 2\mu b^3\varphi(0, b)^2 \tag{2.2}$$

Proof. Integrating (2.1) with $\psi = r^2\varphi$ over $[0, T] \times [b, \infty)$, and making use of the boundary condition, we have this proposition because of

$$\begin{aligned} Q_1(r^2\varphi)(t, b) &= -2c_2^2 b^2\varphi_t(t, b)\left(2b\varphi(t, b) + b^2\varphi_r(t, b)\right) \\ &= -2c_2^2 b^3\partial_t\left(\varphi(t, b)^2\right). \end{aligned}$$

□

The energy coservation in Theorem1 immediately follows from this proposition. Because the polar coordinate shows that

$$E(\varphi, \infty, t) = 4\pi\left(E(\varphi)(t) + 2\mu b^3\varphi(t, b)^2\right).$$

3 Local energy decay

Now we shall prove the local energy decay property stated in Theorem1. First we set the boundedness of the support of the initial data to be more clear as

$$\mathrm{supp}\varphi_0,\ \mathrm{supp}\varphi_1 \subset \{b \le r \le k\},$$

where k is a fixed positive constant greater than b. Making use of the multiplier similar to Morawetz [1], we have

Lemma 2 *Let ψ be a smooth function of (t,r). Then the following identity is valid.*

$$\begin{aligned}&2(t\psi_t + r\psi_r + 2\psi)\left(r^2 L(r^2\psi)\right)\\ &= \partial_t\left\{tQ(r^2\psi) + 2r\partial_r(r^2\psi)\partial_t(r^2\psi)\right\} + \partial_r\left(tQ_1(r^2\psi)\right)\\ &\quad -\partial_r\left\{c_2^2 r\left(\partial_r(r^2\psi)\right)^2 + r\left(\partial_t(r^2\psi)\right)^2 - 2c_2^2 r^3\psi^2\right\}\end{aligned} \tag{3.1}$$

Proof. This lemma is proved by direct computations as follows. Setting $\Psi = r^2\psi$, one can see that

$$\begin{aligned}&2t\psi_t\left(r^2 L(r^2\psi)\right)\\ &= 2t\Psi_t(\Psi_{tt} - c_2^2\Psi_{rr} + 2c_2^2\psi)\\ &= \partial_t(t\Psi_t^2) - \Psi_t^2 - \partial_r(2c_2^2 t\Psi_t\Psi_r) + 2c_2^2 t\Psi_{tr}\Psi_r + \partial_t(2c_2^2 tr^2\psi^2) - 2c_2^2 r^2\psi^2\\ &= \partial_t\left(tQ(\Psi)\right) - \Psi_t^2 - c_2^2\Psi_r^2 - 2c_2^2 r^2\psi^2 + \partial_r\left(tQ_1(\Psi)\right).\end{aligned}$$

Moreover $\Psi_r = r^2\psi_r + 2r\psi$ yields that

$$\begin{aligned}&2(r\psi_r + 2\psi)\left(r^2 L(r^2\psi)\right)\\ &= 2r\Psi_r(\Psi_{tt} - c_2^2\Psi_{rr} + 2c_2^2\psi)\\ &= \partial_t(2r\Psi_r\Psi_t) - 2r\Psi_{rt}\Psi_t - \partial_r(c_2^2 r\Psi_r^2) + c_2^2\Psi_r^2 + 4c_2^2 r^3\psi_r\psi + 8c_2^2 r^2\psi^2\\ &= \partial_t(2r\Psi_r\Psi_t) - \partial_r(c_2^2 r\Psi_r^2 + r\Psi_t^2) + \Psi_t^2 + c_2^2\Psi_r^2 + \partial_r(2c_2^2 r^3\psi^2) + 2c_2^2 r^2\psi^2.\end{aligned}$$

Therefore lemma is proved. □

Integrationg (3.1) with $\psi = \varphi$ over the domain

$$\{(t,r)\ :\ b \le r \le c_2 t + k,\ 0 \le t \le T\}$$

we have, by $L(r^2\varphi) = 0$, that

$$\begin{aligned}&\int_b^{c_2T+k}\left\{TQ(r^2\varphi)(T,r) + 2r\partial_r(r^2\varphi)\partial_t(r^2\varphi)\big|_{t=T}\right\}dr\\ &-\int_b^k 2r\partial_r(r^2\varphi)\partial_t(r^2\varphi)\big|_{t=0}\,dr + I = 0,\end{aligned} \tag{3.2}$$

where

$$I=\int_0^T \left\{-tQ_1(r^2\varphi)+c_2^2 r\left(\partial_r(r^2\varphi)\right)^2+r\left(\partial_t(r^2\varphi)\right)^2-2c_2^2r^3\varphi^2\right\}\Big|_{r=b}\,dt.$$

Since the boundary condition yields that

$$\partial_r(r^2\varphi)=2b\varphi+b^2\varphi_r=2b\varphi,$$

we find by $Q_1(r^2\varphi)(t,b)=-2c_2^2b^2\varphi_t\cdot 2b\varphi$ that

$$I=c_2^2b^3\int_0^T \left(4t\varphi_t\varphi+4\varphi^2-2\varphi^2\right)\big|_{r=b}\,dt+b^5\int_0^T \varphi_t^2\big|_{r=b}\,dt.$$

Making use of

$$4t\varphi_t\varphi=2\partial_t(t\varphi^2)-2\varphi^2,$$

we obtain

$$I=2c_2^2b^3T\varphi(T,b)^2+b^5\int_0^T \varphi_t(t,b)^2dt. \tag{3.3}$$

Note that

$$\left|\int_b^k 2r\partial_r(r^2\varphi)\partial_t(r^2\varphi)\big|_{t=0}\,dr\right|\le c_2^{-1}kE(\varphi)(0). \tag{3.4}$$

Hence it follows from (3.2), (3.3) and (3.4) that, for $R\ge k$,

$$\begin{aligned}&\int_b^R \left\{TQ(r^2\varphi)+2r\partial_r(r^2\varphi)\partial_t(r^2\varphi)\right\}\big|_{t=T}\,dr+2c_2^2b^3T\varphi(T,b)^2\\ &\le c_2^{-1}kE(\varphi)(0)-\int_R^M \left\{TQ(r^2\varphi)+2r\partial_r(r^2\varphi)\partial_t(r^2\varphi)\right\}\big|_{t=T}\,dr,\end{aligned} \tag{3.5}$$

where we set

$$M=\max\{c_2T+k,R\}.$$

Then, by energy conservation, we have

$$\begin{aligned}&\left|\int_R^M 2r\partial_r(r^2\varphi)\partial_t(r^2\varphi)\big|_{t=T}\,dr\right|\\ &\le 2(c_2T+k)\int_R^M |\partial_r(r^2\varphi)||\partial_t(r^2\varphi)|\big|_{t=T}\,dr\\ &\le 2c_2T\int_R^M |\partial_r(r^2\varphi)||\partial_t(r^2\varphi)|\big|_{t=T}\,dr+kE(\varphi)(0).\end{aligned} \tag{3.6}$$

Here we find that

$$
\begin{aligned}
& -\int_R^M Q(r^2\varphi)dr + 2c_2^2\int_R^M |\partial_r(r^2\varphi)||\partial_t(r^2\varphi)|dr \\
& \le -\int_R^M \left\{(\partial_t(r^2\varphi))^2 + c_2^2(\partial_r(r^2\varphi))^2\right\} dr \\
& \quad +2c_2\left(\int_R^M (\partial_t(r^2\varphi))^2 dr\right)^{1/2}\left(\int_R^M c_2^2(\partial_r(r^2\varphi))^2 dr\right)^{1/2} \\
& = -\left\{\left(\int_R^M (\partial_t(r^2\varphi))^2 dr\right)^{1/2} - c_2\left(\int_R^M c_2^2(\partial_r(r^2\varphi))^2 dr\right)^{1/2}\right\}^2 .
\end{aligned}
\tag{3.7}
$$

Moreover the energy conservation again yields that

$$
\begin{aligned}
\left|\int_b^R 2r\partial_r(r^2\varphi)\partial_t(r^2\varphi)\big|_{t=T} dr\right| & \le 2R\int_b^R \left|\partial_r(r^2\varphi)||\partial_t(r^2\varphi)|\right|_{t=T} dr \\
& \le c_2^{-1}RE(\varphi)(0).
\end{aligned}
\tag{3.8}
$$

Therefore, summing up all the estimates (3.6), (3.7) and (3.8), we obtain by (3.5) that

$$
T\left(\int_b^R Q(r^2\varphi)(T,r)dr + 2\mu b^3\varphi(T,b)^2\right) \le CE(\varphi)(0)
$$

holds for all $T > 0$, where a positive constant C can be taken as

$$
C = k + c_2^{-1}(k+R).
$$

The proof of Theorem1 is ended by this inequality.

References

[1] C.S.Morawetz, *The decay of solutions of the exterior initial-boundary value problem for the wave equation*, Comm. Pure Appl. Math., **XIV**(1961), 561-568.

[2] C.S.Morawetz, *Exponential decay of solutions of the wave equation*, Comm. Pure Appl. Math., **XIX**(1966), 439-444.