# Facet bending driven by the planar crystalline curvature with a generic nonuniform forcing term


Yoshikazu Giga[1], Piotr Rybka[2]

[1] Graduate School of Mathematical Sciences
University of Tokyo
Komaba 3-8-1, Tokyo 153-8914, Japan
e-mail: labgiga@ms.u-tokyo.ac.jp

[2] Institute of Applied Mathematics and Mechanics, The University of Warsaw
ul. Banacha 2, 02-097 Warsaw, Poland
fax: +(48 22) 554 4300, e-mail: rybka@mimuw.edu.pl


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**Abstract**

We study crystalline driven curvature flow with spatially nonuniform driving force term. We assume special monotonicity properties of the driving term, which are motivated by our previous work on Berg's effect. We consider special initial data which we call 'bent rectangles'. We prove existence of solutions for a generic forcing term as well as generic subclass of bent rectangles. We show the initially flat facets may begin to bend, provided, loosely speaking, they are too large. Moreover, depending on the initial configuration we notice instantaneous loss of regularity of the moving curve.



**Key words:** singular energies, bending of facets, driven curvature flow, variational principle

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## 1 Introduction

In our previous paper [GR5] we initiated studies of evolution of special Lipschitz curves, which we called *bent rectangles* (see next section for the definition) by the driven singular weighted mean curvature,

$$\beta V = \kappa_\gamma + \sigma \quad \text{on } \Gamma(t). \tag{1.1}$$

Let us explain the rational for considering evolution of such curves. Many authors in the physical literature consider circular cylinders as an approximation to hexagonal prisms, which are the shapes of ice crystals. We also studied this topic in a series of papers, see

[GR1]-[GR4], where we adopted the Gibbs-Thomson law with kinetic undercooling on the cylindrical interface. We are interested in what happens when the interface loses stability, *i.e.,* it begins to bend. We expect that the cross-section of our deforming interface after the onset of instability is just a bent rectangle.

We defined in [GR5] the crucial notions and proved local in time existence for a restricted class of data. By data, we mean the curve itself as well as the driving force $\sigma$. Here, we permit a generic driving $\sigma$, conforming to the Berg's effect and symmetry conditions, *i.e.* for all $t \geq 0$

$$x_i \frac{\partial \sigma}{\partial x_i}(t, x_1, x_2) > 0 \quad \text{for } x_i \neq 0, \quad i = 1, 2, \tag{1.2}$$

and

$$\sigma(t, x_1, x_2) = \sigma(t, -x_1, x_2), \quad \sigma(t, x_1, x_2) = \sigma(t, x_1, -x_2). \tag{1.3}$$

Here, our assumptions on $\sigma$ are generic, in the sense that any $\sigma$, satisfying the symmetries above and (1.2) after a small perturbation (in the $C^1$-topology), not only conforms to the same restrictions but also fulfills the hypothesis of our Theorem 1.1 below. We exclude, however, one type of initial curves, because our methods are not applicable. We shall comment on that after Proposition 3.4.

On the way we discover new phenomena, they may be most easily explained in the terms of smoothness. We have already seen that the interfacial point, separating the flat part of $\Gamma(t)$ from its curved part, is the point where the solution may lose differentiability. To be precise we have seen in [GR5] that if the velocity of the interfacial point is not zero, then $\Gamma(t)$ is as smooth as the data $\Gamma_0$. On the other hand if this interfacial point is motionless, then this is the point of non-differentiability of $\Gamma(t)$ for $t > 0$, no matter how smooth the data were. This event of loss of differentiability was observed for the only type of interfacial curve considered in [GR5], which we call *tangency curves*.

This phenomenon is recorded to hold for the remaining type of the interfacial curve, which we discovered here. We call them *matching curves.* We show below that $\Gamma(t)$ is never differentiable at the *matching curves*, no matter how smooth $\Gamma_0$ is. This is presented in our existence result, Theorem 3.5. In other words, we exhibit an example of a parabolic equation, whose solutions suffer from a loss of regularity. This phenomenon is also observed if the parabolic equation degenerates in some directions, see [GSS], [G].

We have seen that for a convenient choice of $\beta$ equation (1.1) reduces to a system of ODE's, see (2.7) below. The resulting system of ODE's is closed if it is supplied with the evolution of the interfacial points, separating flat facets and the curved part of $\Gamma(t)$. In [GR5] we were able to close the system but for a restricted class of $\sigma$, satisfying what we called a working hypothesis, [GR5, eq. (3.12)]. The interfacial points moved along what we call here, *tangency curves*. In the generic case the interfacial points move also along another type of curves called here *matching curves*.

One of our main results is Theorem 1.1 exhibiting existence of solutions to (1.1) for special Lipschitz curves called *bent rectangles*, which are defined in the next section. In order to state Theorem 1.1, we need quantities $\Sigma_0^\Lambda$, $\Sigma_1^\Lambda$, $\Sigma_0^R$, $\Sigma_1^R$. They are defined by formulas (3.14), (3.26) and (3.41). Then, we can state the main existence result as follows.

**Theorem 1.1** *Let us suppose that $\sigma$ is of class $C^1$ on $[0, T_*) \times \mathbb{R}^2$, it satisfies (1.2) as well as (1.3), $\beta$ is given by (2.7) and $\gamma$ is defined by the formula below,*

$$\gamma(p_1, p_2) = \gamma_\Lambda |p_1| + \gamma_T |p_2|. \tag{1.4}$$

*If the initial curve $\Gamma_0$ is a bent rectangle (possibly a rectangle) defined by (BR) below, $l_{00} < l_{10}$ and none of the quantities $\Sigma_0^\Lambda$, $\Sigma_1^\Lambda$, $\Sigma_0^R$, $\Sigma_1^R$ are zero, then there exists a unique local-in-time variational solution to (1.1) on a time interval $[0, T)$, $0 < T \le T_*$.*

In fact, the inequality $l_{00} < l_{10}$ is a technical restrictions. These two quantities are introduced at the beginning of Section 3.

We have to explain the appearance of $\Sigma_j^i$, $i = \Lambda, R$, $j = 0, 1$. Roughly speaking, $\Sigma_0^\Lambda > 0$ (respectively, $\Sigma_0^\Lambda < 0$) forces the flat portion (facet), which crosses the $x_1$-axis to expand (respectively, to shrink). One sufficient condition for $\Sigma_0^\Lambda > 0$ is that $\frac{\partial^2 \sigma}{\partial x_1 \partial x_2} < 0$ for $\sigma$ independent of time. In this case, at the end of a flat facet the tangency condition fails for $t > 0$ and its motion is a matching curve.

Theorem 1.1 is proved in Section 3. It is the content of Theorem 3.7, which specifies what exactly happens according to the signs of $\Sigma_i^k$, $i = 0, 1$, $k = \Lambda, R$.

Another principal result is Theorem 3.8 about bending of initially flat facets. It is much more general than similar result [GR5, Theorem 3.1]. Once we understood the evolution of bent rectangles, this result seems easier, because the structure of $\Gamma(t)$, $t \in (0, T)$ is more transparent in the general case. The proof is given is Subsection 3.2. Finally, we present a few examples in §3.3.

In Section 4, we show uniqueness of our solutions. This result is based on the monotonicity argument presented in the proof of [GR5, Theorem 3.2]. We adapt it here to handle the new phenomenon.

It is important to study stability or continuous dependence of solutions upon initial data and the driving force $\sigma$. Once this is proved, we would be able to show the strong containment principle as in [GGu]. But we do not touch this topic here.

# 2 Setting up the problem

Here, we recall the notions we used in [GR5]. We consider evolution of bent rectangles, as defined in [GR5]. The case of graphs is simpler and may be easily derived from the present one, thus it is omitted here.

After some slight improvement in comparison with the original definition (see [GR5, §3.1]), we shall call a Lipschitz closed curve $\Gamma$ a *bent rectangle* if the following conditions are satisfied:

> There exist even, Lipschitz continuous functions $d^R$, $d^\Lambda : \mathbb{R} \to \mathbb{R}_+$, which are non-decreasing for positive arguments and there are positive numbers $L_1$, $R_1$ such that
> $$d^\Lambda(L_1) = R_1, \qquad d^R(R_1) = L_1.$$
> In addition $d^\Lambda$ is constant in a neighborhood of zero and $L_1$ (respectively, $d^R$ is constant in a neighborhood of zero and $R_1$), furthermore

$$(BR) \qquad \Gamma = \partial\{(x_1, x_2) : \ |x_1| \le d^\Lambda(x_2)\ , |x_2| \le d^R(x_1)\}.$$

We shall call the points $(\pm R_1, \pm L_1)$ vertexes of $\Gamma$. Thus, after we set

$$S_\Lambda^\pm = \{(x_1, x_2) \in \Gamma : \ x_1 = \pm d^\Lambda(x_2),\ x_2 \in [-L_1, L_1]\},$$
$$S_R^\pm = \{(x_1, x_2) \in \Gamma : \ x_2 = \pm d^R(x_1),\ x_1 \in [-R_1, R_1]\}$$

we notice that the graphs of $\pm d^\Lambda$, $\pm d^R$ make up the whole $\Gamma(t)$, *i.e.*

$$\Gamma = S_R^- \cup S_R^+ \cup S_\Lambda^- \cup S_\Lambda^+.$$

We will collectively write $S_R$ for $S_R^\pm$ and $S_\Lambda$ for $S_\Lambda^\pm$. We will call them *sides* of $\Gamma(t)$. Vertexes of $\Gamma$ are the intersections $S_R^\pm \cap S_\Lambda^\pm$. Moreover, the sides meet at vertexes at the right angle.

We will denote by $\mathbf{n}$ the outer normal to $\Gamma$ and more specifically

$$\mathbf{n}_\Lambda = (1,0), \qquad \text{(resp.} \quad \mathbf{n}_R = (0,1))$$

are normals to the faceted regions of $S_\Lambda$, (resp. $S_R$). A rigorous definition of the notion of faceted regions is given later just before formula (2.8) in this section.

The curvature, $\kappa_\gamma$, appearing in (1.1) is defined by

$$\kappa_\gamma = -\mathrm{div}_S \gamma(\mathbf{n}),$$

where $\mathbf{n}$ is the outer normal to $\Gamma$ and $\gamma$ is a surface energy function. In our case vector $\mathbf{n}$ is defined only $\mathcal{H}^1$-a.e. The physical examples we have in mind, see [GR1], give us the motivation to consider

$$\gamma(p_1, p_2) = \gamma_\Lambda |p_1| + \gamma_T |p_2|. \tag{2.1}$$

We notice that the flat parts with normals belonging to the set of normals of the Wulff shape $W_\gamma$ are energetically preferred. For the sake of completeness, we recall the definition of $W_\gamma$

$$W_\gamma = \bigcap_{|m|=1} \{x \in \mathbb{R}^2 : m \cdot x \le \gamma(m)\}$$

and the surface energy, $E(S)$,

$$E(S) = \int_S \gamma(\mathbf{n})\, d\mathcal{H}^1.$$

In our problem $W_\gamma$ is a rectangle of the following form,

$$W_\gamma = \{x = (x_1, x_2) \in \mathbb{R}^2 : \ |x_1| \le \gamma(\mathbf{n}_\Lambda), |x_2| \le \gamma(\mathbf{n}_R)\}.$$

It is a well-known fact that $W_\gamma$ minimizes $E$ under the volume constraint. Now, the fundamental problem is apparent: $\nabla\gamma(\mathbf{n})$ is not defined on bent rectangles on sets of positive $\mathcal{H}^1$-measure. In the case of the rectangle with its side parallel to the sides of $W_\gamma$, the situation is even worse, because $\nabla\gamma(\mathbf{n})$ is nowhere defined.

In order to resolve this issue we apply a variational principle, which we used in [GR5]. A similar approach was implemented by the Italian school, [BNP1]-[BNP3], reflecting the idea of [FG]. Namely, we replace the gradient $\nabla\gamma$ by the subdifferential $\partial\gamma$. This is justified by convexity of $\gamma$. Since in general $\partial\gamma$ is not a singleton, this leaves us with a necessity to select the proper Cahn-Hoffman vector field $\xi(x) \in \partial\gamma(\mathbf{n}(x))$. We note that this task is obvious on curved parts of $S^\Lambda$, $S^R$, because $\partial\gamma$ is a singleton there, while it is not trivial on flat facets.

We impose quite natural constraints on $\xi$, see [GR5],

$$\mathrm{div}_S \xi \in L^2(S).$$

This implies that $\xi \cdot \nu$ has a trace, where $\nu \in T_x S_i$ is a normal vector to $S_i$, $i = R, \Lambda$. If we combine it with

$$\partial\gamma(\mathbf{n}_R) \cap \partial\gamma(\mathbf{n}_\Lambda) = \{p\},$$

then we see that $\xi$ satisfies a boundary condition

$$\xi|_{vertex} = p.$$

The necessity of selecting $\xi$ implies that in order to define a solution to (1.1), we need to specify not only a curve $\Gamma(t)$ but also $\xi(t, \cdot)$. After [GR5], we recall the notion of solution. Namely, we call by *a solution to* (1.1) a family of couples $(\Gamma(t), \xi(t))$, $t \in [0, T)$, such that for some $T > 0$, the following conditions are satisfied:

(a) For each $t \in [0, T)$ the curve $\Gamma(t)$ is a bent rectangle and $d^\Lambda$, $d^R$ are continuous functions of its arguments, for each $x$, $d^j(\cdot, x)$, $j = \Lambda, R$ are Lipschitz continuous and for each $t \in [0, T)$ the functions $d^j(t, \cdot)$, $j = \Lambda, R$ are Lipschitz continuous;

(b) $\xi : \bigcup_{t\in[0,T)} \{t\} \times \Gamma(t) \to \mathbb{R}^2$ is at each time instant a Cahn-Hoffman vector. If $M := \sup_{t\in[0,T)} \max\{L_1(t), R_1(t)\} + 1$, and if for $j = \Lambda$, $R$ we set

$$\tilde{\xi}^R(t,x) = \begin{cases} (-\gamma(\mathbf{n}_\Lambda), \gamma(\mathbf{n}_R)) & x \in [-M, -R(t)], \\ \xi(t, (x, d^R(t,x))) & x \in [-R(t), R(t)], \\ (\gamma(\mathbf{n}_\Lambda), \gamma(\mathbf{n}_R)) & x \in [R(t), M]; \end{cases}$$
$$\tilde{\xi}^\Lambda(t,x) = \begin{cases} (-\gamma(\mathbf{n}_\Lambda), -\gamma(\mathbf{n}_R)) & x \in [-M, -L(t)], \\ \xi(t, (d^\Lambda(t,x), x)) & x \in [-L(t), L(t)], \\ (-\gamma(\mathbf{n}_\Lambda), \gamma(\mathbf{n}_R)) & x \in [L(t), M]; \end{cases} \tag{2.2}$$

then we assume that $t \mapsto \tilde{\xi}^j(t, \cdot) \in L^\infty(0, T; L^2(-M, M))$, $j = \Lambda, R$;

(c) Equation (1.1) is satisfied in the $L^2$ sense for *a.e.* $t \geq 0$.

A remark on the argument of $\xi$ is in order. In principle the Cahn-Hoffman vector depends upon time $t$ and $(x_1, x_2) \in \Gamma(t)$. However, we shall frequently suppress $t$ and write $\xi(x)$ when the meaning of the second spacial argument is clear from the context, e.g. on the sides.

We also distinguished variational solutions based on a specific way to select $\xi$. In order to define them, we introduce two convenient energy functionals,

$$\mathcal{E}_j(\xi) = \frac{1}{2}\int_{S_j} |\sigma - \mathrm{div}_S \xi|^2 \, d\mathcal{H}^1, \quad j = R, \Lambda. \tag{2.3}$$

Their natural domains of definition are the sets of Cahn-Hoffman vectors, satisfying all the above constraints,

$$\begin{aligned} \mathcal{D}_\Lambda &= \{\xi \in L^\infty(S_\Lambda) : \ \xi(x) \in \partial\gamma(\mathbf{n}(x)), \ \mathrm{div}_S \xi \in L^2(S_\Lambda), \ (2.5) \text{ holds}\}, \\ \mathcal{D}_R &= \{\xi \in L^\infty(S_R) : \ \xi(x) \in \partial\gamma(\mathbf{n}(x)), \ \mathrm{div}_S \xi \in L^2(S_R), \ (2.5) \text{ holds}\}. \end{aligned} \tag{2.4}$$

where

$$\xi(\pm R_1, \pm L_1) \in \partial\gamma(\pm\mathbf{n}_\Lambda) \cap \partial\gamma(\pm\mathbf{n}_R). \tag{2.5}$$

We recall that $\{(\Gamma(t), \xi(t))\}$, $t \in [0, T)$, a solution to (1.1), was called a *variational solution* if in addition for each $t \in [0, T)$ $\xi|_{S_j}(t) \in L^2(S_j)$ is a solution to

$$\mathcal{E}_j(\xi) = \min\{\mathcal{E}_j(\zeta) : \ \zeta \in \mathcal{D}_j\}, \quad j = R, \Lambda. \tag{2.6}$$

It is worthwhile to remark that a minimizer is essentially unique. Indeed, since the problem is convex, we at least observe that the surface divergence $\mathrm{div}_S\xi$ of a minimizer is unique. But in our one dimensional setting it turns out that $\xi$ itself must be unique as well.

The rationale for this definition is that equation (1.1) is the Euler-Lagrange equation of $\mathcal{E}_R$ on $S_R$ (resp. $\mathcal{E}_\Lambda$ on $S_\Lambda$) on its flat parts, *i.e.*, pre-images of faceted regions, see below for a precise definition. This definition is in line with the notion of solution, we used earlier, see [GGM], [GGa], [GR3], [GR4], [GR5] and by the Italian school, [BNP1]-[BNP3].

We have to recall another assumption, we made in [GR5, (1.5)], which simplified the analysis, *i.e.*,

$$\beta(n_1, n_2) = \frac{1}{\max(|n_1|, |n_2|)}, \tag{2.7}$$

where $n_1^2 + n_2^2 = 1$.

In order to deal with different parts of $\Gamma$, we introduced in [GR5] a number of auxiliary notions. Here is the first one. Let us consider an open line segment $I$ in the plane, *i.e.* $I = (a, b) \equiv \{x = at + b(1-t),\ t \in (0, 1)\}$, where $a, b \in \mathbb{R}^2$. We shall say that $I \subset \Gamma$, having a normal equal to $\mathbf{n}_\Lambda$ or $\mathbf{n}_R$, is *a faceted region* of $\Gamma$ if it is maximal (with respect to inclusion) and it satisfies

$$(\sigma - \mathrm{div}_S\xi)|_I = const., \tag{2.8}$$

where $\xi$ is a solution to (2.6).

However, $S^\pm_\Lambda(t)$ and $S^\pm_R(t)$ are graphs, *e.g.* $S^+_R$ is the image of segment $[-R(t), R(t)]$ under the function

$$x \mapsto (x, d^R(t, x)) =: \tilde{d}^R(t, x). \tag{2.9}$$

Frequently it is more convenient to work with the inverse image of a faceted region $I$, $(\alpha, \beta) = \tilde{d}^{-1}(I)$. We stress that this definition permits $S^\pm_j(t)$, $j = R, \Lambda$ being a line segment which has more than one faceted region.

In order to make the presentation more clear we propose to use the notion of a *curved part* of side to denote the (relative) interior of the subset of $\Gamma$, where normal $\mathbf{n}$ is such that $\partial\gamma(\mathbf{n})$ is a singleton. In particular, it may happen that a line segment of $\Gamma$ will be called a curved part if its normal is different from $\mathbf{n}_R$, $\mathbf{n}_\Lambda$.

Before proceeding, we mention that in principle it is possible to consider a more direct approach to defining the weighted mean curvature for singular $\gamma$. We have in mind the results of [FG], [GPR] and independent and quite different ones in a recent paper [MR], which is however restricted to one-dimension.

# 3 Existence of solutions

Here, we continue the studies which we initiated in [GR5]. There, we considered two cases: (a) evolution of a graph under (1.1) for driving terms independent of time and (b) evolution of a bent rectangle for special time dependent $\sigma$.

Case (a) was much simpler and it helped to develop the notions needed for the analysis of (b). In the case (b), we studied evolution of bent rectangles, which are not rectangles. Moreover, we could also analyze the case of rectangles, as initial data, if $\sigma$ satisfied the so-called working hypothesis, [GR5, eq. (3.12)]. Here, we remove this restriction on the driving term.

We consider only the evolution of (bent) rectangles, the evolution of graphs is easier and may be easily deduced from our analysis. In order to make the presentation more transparent, we consider separately the cases of evolution of bent rectangles and the process of facet bending of rectangles. We will present it in subsection 3.1 and 3.2, respectively.

We want to write (1.1) in the local coordinate system with the help of functions $d^j$, $j = \Lambda, R$. We will recall the basic steps from [GR5], but we carefully explain the final form of equations because, contrary to what we did in [GR5], we do not want to impose any extra assumptions on solutions to (2.6). We do, however assume that $(\Gamma, \xi)$ is a variational solution to (1.1), such that each side $S_j$ has exactly three faceted regions, their pre-images are

$$(-L_1, -l_1), \quad (-l_0, l_0), \quad (l_1, L_1), \qquad (-R_1, -r_1), \quad (-r_0, r_0), \quad (r_1, R_1).$$

Moreover, the functions $d^\Lambda|_{[0,L_1]}$, $d^R|_{[0,R_1]}$ are increasing.

We recall a useful object in the process of writing the equation in the local coordinates,

$$E_Z^\Lambda(t) = \{x \in (-l_1, -l_0) \cup (l_0, l_1) : \ \tfrac{\partial d^\Lambda}{\partial x}(t,x) = 0 \text{ or } d^\Lambda(t,\cdot) \text{ is not differentiable at } x\},$$

$$E_Z^R(t) = \{x \in (-r_1, -r_0) \cup (r_0, r_1) : \ \tfrac{\partial d^R}{\partial x}(t,x) = 0 \text{ or } d^R(t,\cdot) \text{ is not differentiable at } x\}.$$

We have shown (see [GR5, Proposition 2.1]).

**Proposition 3.1** *Let us suppose that $\sigma$ is of $C^1$-class on $[0, T_*) \times \mathbb{R}^2$, it satisfies (1.2) and (1.3), moreover $(\Gamma, \xi)$ is a variational solution to (1.1) and $d^\Lambda$, $d^R$ satisfy the restrictions above. In addition, we assume*

$$\mathcal{H}^1(E_Z^j \setminus \mathrm{int}\, E_Z^j) = 0, \quad j = R, \Lambda, \tag{3.1}$$

*where $\mathrm{int}\, E$ denotes the interior of $E$. Then, $\xi$ is constant over each component of the complement of the faceted regions.*

If we assume that $(\Gamma, \xi)$ is a variational solution satisfying condition (3.1), then we may repeat the reasoning from [GR5, §2.1] to deduce that on $S_\Lambda$ vector field $\xi$ must have the form, $\xi = (\gamma(\mathbf{n}_\Lambda), \xi_2)$, (see[GR5, eq. (2.28)]), where

$$\xi_2(t,x) = \begin{cases} x\left(\displaystyle{-\!\!\!\!\!\!\int_0^x} \sigma(t, R_0, s)\, ds - {-\!\!\!\!\!\!\int_0^{l_0}} \sigma(t, R_0, s)\, ds\right) + \dfrac{x}{l_0}\, \xi_2(t, l_0) & x \in [0, l_0] \\ \xi_2(t, l_0) & x \in (l_0, l_1) \\ (L - x)\left(\displaystyle{-\!\!\!\!\!\!\int_{l_1}^L} \sigma(t, R_1, s)\, ds - {-\!\!\!\!\!\!\int_x^L} \sigma(t, R_1, s)\, ds\right) + \gamma(\mathbf{n}_R) & \\ \qquad + \dfrac{L - x}{L - l_1}(\xi_2(t, l_1) - \gamma(\mathbf{n}_R)) & x \in [l_1, L]. \end{cases} \tag{3.2}$$

We have to determine the values of $\xi_2(t, l_0)$ and $\xi_2(t, l_1)$.

A similar formula for $\xi_1$ on $S_R$ is valid and we have to determine the values of $\xi_1(t, r_0)$, $\xi_1(t, r_1)$.

**Proposition 3.2** *Let us suppose that $\sigma$ satisfies the conditions stated in Proposition 3.1 and $(\Gamma, \xi)$ is a variational solution, then $\xi_2(t, l_0) = -\gamma(\mathbf{n}_R)$ and $\xi_1(t, r_0) = -\gamma(\mathbf{n}_\Lambda)$. If in addition (3.1) is satisfied, then $\xi_2(t, l_0) = \xi_2(t, l_1) = -\gamma(\mathbf{n}_R)$ and $\xi_1(t, r_0) = \xi_1(t, r_1) = -\gamma(\mathbf{n}_\Lambda)$.*

Let us remark that earlier we proved Proposition 3.2 under an additional assumption that the so called tangency condition holds, *i.e.*,

$$\frac{\partial \xi_2}{\partial x}(t,l_i)=0, \quad \frac{\partial \xi_1}{\partial x}(t,r_i)=0, \qquad i=0,1, \tag{3.3}$$

(see [GR5, (3.11)]). Here, we make no such hypothesis.

In order to prove this Proposition we make few remarks on (3.3) which will be useful in the future too. We shall talk only about $l_i$, $i=0,1$ since considering $r_i$, $i=0,1$ requires no changes. If $\xi$ is a solution to the variational problem (2.6), then we see that

$$\begin{aligned} \sigma(t,R_0,x)-\frac{\partial}{\partial x_2}\xi_2(t,R_0,x) &\equiv V_0=\dot R_0=const, \quad \text{for } x\in(-l_0,l_0) \\ \sigma(t,R_1,x)-\frac{\partial}{\partial x_2}\xi_2(t,R_1,x) &\equiv V_1=\dot R_1=const, \quad \text{for } x\in(-L_1,-l_1)\cup(l_1,L_1). \end{aligned} \tag{3.4}$$

However, $V_0$, $V_1$ are different, in fact we showed in [GR5, Corollary 2.2] that Berg's effect (1.2) and the tangency condition (3.3) imply $V_1>V_0$.

If we write

$$G(t,x)=\int_0^x \sigma(t,d^\Lambda(t,s),s)\,ds,$$

then we can rewrite (3.4) as

$$\begin{gathered} G(t,x)-\xi_2(t,R_0,x)=V_0x, \quad x\in(0,l_0), \\ G(t,x)-\xi_2(t,R_1,x)=V_1(x-L_1)+G(t,L_1)-\gamma(\mathbf{n}_R), \quad x\in(l_1,L_1), \end{gathered}$$

where we used $\xi_2(t,R_0,0)=0$ and $\xi_2(t,R_1,L_1)=-\gamma(\mathbf{n}_R)$. In other words,

$$G(t,x)-V_ix-b_i=\xi_2(t,R_i,x), \quad i=0,1, \tag{3.5}$$

where $b_0=0$, $b_1=V_1L_1-G(t,L_1)+\gamma(\mathbf{n}_R)$. We recall that $\xi_2(t,R_i,x)\in[-\gamma(\mathbf{n}_R),\gamma(\mathbf{n}_R)]$, $i=0,1$. In particular, the line $V_ix+b_i$ is below $G(t,x)+\gamma(\mathbf{n}_R)$ and above $G(t,x)-\gamma(\mathbf{n}_R)$, *i.e.*,

$$\begin{aligned} G(t,x)-\gamma(\mathbf{n}_R)\le V_0x\le G(t,x)+\gamma(\mathbf{n}_R) \quad x\in(0,l_0) \\ G(t,x)-\gamma(\mathbf{n}_R)\le V_1x+b_1\le G(t,x)+\gamma(\mathbf{n}_R) \quad x\in(l_1,L_1). \end{aligned} \tag{3.6}$$

We will use this geometric insight in the argument below.

*Proof of Proposition 3.2.* We know that at $x=l_0$ the solution $\xi$ touches the constraint, *i.e.*, $\xi_2(t,l_0)=\pm\gamma(\mathbf{n}_R)$. We have to determine the sign. We differentiate (3.4) with respect to $x$. Hence, $\frac{\partial}{\partial x_2}\sigma(t,R_0,x)=\frac{\partial^2}{\partial x_2^2}\xi_2(t,R_0,x)$ for $x\in[0,l_0)$. Due to monotonicity of $\sigma$ implied by Berg's effect, we deduce that the function $x\mapsto\xi_2(t,R_0,x)$ is strictly convex and it has at most one critical point.

We have exactly three possible values of $V$: (a) $V_0=\sigma(t,R_0,l_0)$, (b) $V_0>\sigma(t,R_0,l_0)$, (c) $V_0<\sigma(t,R_0,l_0)$. In case (a) occurs, we see that $\frac{\partial}{\partial x_2}\sigma(t,R_0,l_0)=0$, *i.e.*, $\xi_2(t,R_0,\cdot)$ has a minimum there. As a result, $\xi_2(l_0)=-\gamma(\mathbf{n}_R)$. Comparing this with (3.3), we see that the tangency condition is satisfied.

If (b) occurs, it follows that $\frac{\partial}{\partial x_2}\xi(t,R_0,l_0)=\sigma(t,R_0,l_0)-V_0<0$, hence $\xi_2(t,l_0)=-\gamma(\mathbf{n}_R)$ is the only possibility.

It turns out that (c) is not possible at all. Let us suppose it does happen. Then, due to (3.5), the line connecting $(0,0)$ to $(l_0,G(t,l_0)+\gamma(\mathbf{n}_R))$ has the slope equal to $V_0$. However, by assumption $V_0 < \sigma(t, R_0, l_0)$, the line $V_0 x$ must intersect the graph of $G(t, x) + \gamma(\mathbf{n}_R)$ at $\lambda < l_0$. *i.e.*, the constraint $\xi \in \partial\gamma(\mathbf{n})$ is violated. The case (c) cannot happen. $\square$

We may now write equation (1.1) in the local coordinates, while keeping in mind the conclusions of Proposition 3.1. Namely, we showed, see [GR5, (3.11)], that (1.1) for variational solution takes the following form,

$$
\begin{aligned}
\dot{R}_0 &= \fint_0^{l_0} \sigma(t, R_0, s)\, ds + \frac{\gamma(\mathbf{n}_R)}{l_0} && \text{on} \quad [0, l_0],\\
&\frac{\partial}{\partial t} d^\Lambda = \sigma(t, d^\Lambda, x_2) && \text{on} \quad [l_0, l_1],\\
\dot{R}_1 &= \fint_{l_1}^{L_1} \sigma(t, R_1, s)\, ds - \frac{2\gamma(\mathbf{n}_R)}{L_1 - l_1} && \text{on} \quad [l_1, L_1],\\
\dot{L}_0 &= \fint_0^{r_0} \sigma(t, s, L_0)\, ds + \frac{\gamma(\mathbf{n}_\Lambda)}{r_0} && \text{on} \quad [0, r_0],\\
&\frac{\partial}{\partial t} d^R = \sigma(t, x_1, d^R) && \text{on} \quad [r_0, r_1],\\
\dot{L}_1 &= \fint_{r_1}^{R_1} \sigma(t, s, L_1)\, ds - \frac{2\gamma(\mathbf{n}_\Lambda)}{R_1 - r_1} && \text{on} \quad [r_1, R_1],
\end{aligned}
\tag{3.7}
$$

augmented with the following initial conditions,

$$
\begin{aligned}
l_0(0) = l_{00}, \quad l_1(0) = l_{10}, \quad r_0(0) = r_{00}, \quad r_1(0) = r_{10},\\
R_0(0) = R_{00}, \quad R_1(0) = R_{10}, \quad L_0(0) = L_{00}, \quad L_1(0) = L_{10},\\
d^R(0, x_1) = d_0^R(x_1), \quad d^\Lambda(0, x_2) = d_0^\Lambda(x_2).
\end{aligned}
\tag{3.8}
$$

We stress that the points $l_i(t)$, $r_i(t)$, $i = 0, 1$, are unknown, hence they may be called interfacial points.

## 3.1 Evolution of bent rectangles

We should explain the importance of the interfacial points $l_i(t)$, $r_i(t)$, $i = 0, 1$. We shall concentrate our attention on $l_0(t)$, $l_1(t)$, because the analysis of $r_0$, $r_1$ requires no changes. They are points separating pre-images of faceted regions from the pre-images of the curved parts of sides. If $\xi$ is a solution to the variational problem with the constraint $\xi \in \mathcal{D}$, then

$$
\xi_2(t, l_i) = -\gamma(\mathbf{n}_R). \tag{3.9}
$$

It is, however, possible that (3.9) is implied by the fact that $\partial\gamma(\mathbf{n}(x, d^\Lambda(t, x)))$, is a singleton for $x \in (l_0, l_1)$. Then, (3.9) is just a boundary condition for the minimization problem (2.6).

A more interesting case occurs if $d^\Lambda(t, x) = R_0$ on $(-\lambda_0, \lambda_0)$, $\lambda_0 > l_0$ and $d^\Lambda(t, x) > R_0$ for $x > \lambda_0$ or respectively $d^\Lambda(t, x) = R_1$ on $(-L_1, -\lambda_1) \cup (\lambda_1, L_1)$, $\lambda_1 < l_1$ and $d^\Lambda(t, x) < R_1$ for $x \in [0, \lambda_1)$. Then the minimization problem is of obstacle type, hence $l_0$ or $l_1$ is a free boundary, *i.e.*, the coincidence set in the obstacle problem, and it is a part of a solution. In these cases by the general theory

$$
\frac{\partial}{\partial x} \xi(t, l_0) = 0 \quad \text{or} \quad \frac{\partial}{\partial x} \xi(t, l_1) = 0.
$$

These are the *tangency conditions*.

A more convenient version of (3.3) and the above equations follow (see [GR5, Proposition 2.1] and [GR5, (3.10)]),

$$
\begin{aligned}
\sigma(t, R_0(t), l_0(t)) &= \fint_0^{l_0(t)} \sigma(t, R_0(t), s)\, ds + \frac{\gamma(\mathbf{n}_R)}{l_0(t)}, \quad \xi_2(l_i(t)) = -\gamma(\mathbf{n}_R),\ i = 0, 1, \\
\sigma(t, R_1(t), l_1(t)) &= \fint_{l_1(t)}^{L_1(t)} \sigma(t, R_1(t), s)\, ds - \frac{2\gamma(\mathbf{n}_R)}{L_1(t) - l_1(t)}, \\
\sigma(t, r_0(t), L_0(t)) &= \fint_0^{r_0(t)} \sigma(t, s, L_0(t))\, ds + \frac{\gamma(\mathbf{n}_\Lambda)}{r_0(t)}, \quad \xi_1(r_i(t)) = -\gamma(\mathbf{n}_\Lambda),\ i = 0, 1, \\
\sigma(t, r_1(t), L_1(t)) &= \fint_{r_1(t)}^{R_1(t)} \sigma(t, s, L_1(t))\, ds - \frac{2\gamma(\mathbf{n}_\Lambda)}{R_1(t) - r_1(t)}.
\end{aligned}
\tag{3.10}
$$

In general, for any given bent rectangle and any driving term $\sigma$ is it unrealistic to expect that the tangency condition is satisfied at the interfacial points $l_i$, $r_i$, $i = 0, 1$. In [GR5], we set the working hypothesis implying the expected behavior of the interfacial points, namely, $\dot{l}_0 \leq 0$ (resp. $\dot{r}_0 \leq 0$) and $\dot{l}_1 \geq 0$ (resp. $\dot{r}_1 \geq 0$). Here, we give up such a constraint in favor of a generic condition on $\sigma$. It means that we admit both possibilities, $\dot{l}_0 > 0$ or $\dot{l}_1 < 0$.

We also stated the *matching condition* for solutions of (3.7),

$$
d^\Lambda(t, l_i(t)) = R_i, \qquad d^R(t, r_i(t)) = L_i, \qquad i = 0, 1. \tag{3.11}
$$

These are simple statements of continuity of $d^j$, $j = R, \Lambda$, thus they are more fundamental than the tangency conditions. Here, we shall see the profound implications of (3.11). We start with a simple observation.

**Proposition 3.3** *Let us suppose that $\sigma$ is given of class $C^1$ on $[0, T_*) \times \mathbb{R}^2$, it satisfies the Berg's effect (1.2) and (1.3), $\Gamma_0$ is a bent rectangle with $l_0 < l_1$, moreover $l_0(\cdot)$ and $l_1(\cdot)$ are $C^1$ curves.*
*(a) If the tangency as well as matching conditions are satisfied at $l_0(t)$ for all $t \in [0, \epsilon)$, then $l_0(\cdot)$ is decreasing.*
*(b) If the tangency as well as matching conditions are satisfied at $l_1(t)$ for all $t \in [0, \epsilon)$, then $l_1(\cdot)$ is increasing.*

**Remark.** As long as it does not lead into confusion, we shall suppress the superscript $\Lambda$, $R$ in $d^\Lambda$, $d^R$ and we shall simply write $d$.

*Proof of Proposition 3.3.* It is sufficient to consider only $l_0$.

Let us assume the contrary, *i.e.*, there are $t_1 > t_0 \geq 0$ such that for all $s \in (t_0, t_1)$ we have $l_0(s) < l_0(t_1)$. We note that the matching condition implies $d(t_0, l_0(t_0)) = R_0(t_0)$, as well as

$$
d(t_0, l_0(t_1)) + \int_{t_0}^{t_1} d_t(s, l_0(t_1))\, ds = R_0(t_0) + \int_{t_0}^{t_1} \dot{R}_0\, ds. \tag{3.12}
$$

Thus, by the definition of $\dot{R}_0$ we conclude that

$$
\begin{aligned}
&d_0(l_0(t_1)) + \int_{t_0}^{t_1} \sigma(s, d(s, l_0(t_1)), l_0(t_1))\, ds \\
&\quad = R_0(t_0) + \int_{t_0}^{t_1} \left( \fint_0^{l_0(s)} \sigma(s, R_0(t_1), y)\, dy + \frac{\gamma(\mathbf{n}_R)}{l_0(s)} \right) ds.
\end{aligned}
\tag{3.13}
$$

If (3.10) held, then we would have

$$d_0(l_0(t_1)) - R_0(t_0) + \int_{t_0}^{t_1} (\sigma(s, d(s, l_0(t_1)), l_0(t_1)) - \sigma(s, R_0(s), l_0(s)))\, ds = 0.$$

But $d_0(l_0(t_1)) - R_0(t_0) \geq 0$, moreover if $l_0(s) < l_0(t_1)$ for all $s \in (t_0, t_1)$ and Berg's effect holds, then

$$\begin{aligned}
&\sigma(s, d(s, l_0(t_1)), l_0(t_1)) - \sigma(s, R_0(s), l_0(s)) \\
&\quad = \sigma(s, d(s, l_0(t_1)), l_0(t_1)) - \sigma(s, d(s, l_0(s)), l_0(t_1)) \\
&\qquad + \sigma(s, d(s, l_0(s)), l_0(t_1)) - \sigma(s, R_0(s), l_0(s)) > 0,
\end{aligned}$$

where we also used $d(s, l_0(s)) = R_0(s)$. Hence, equality above is not possible. □

As a result, we conclude that if the interfacial curve $l_0(\cdot)$ satisfies $l_0(t) > l_{00}$, then the tangency condition may not hold there, even if this condition is satisfied at $l_{00}$. Indeed, Proposition 3.3 above implies that along the curve there would be a jump in $d$, *i.e.*, the matching condition (3.11) would be violated.

We know from [GR5, Proposition 2.5] that if the tangency condition is satisfied at $l_{00}$ then there is a curve $[0, t_1) \ni t \mapsto l_0^\star(t)$, where the tangency condition holds. Similar curves $l_1^\star(t)$ and $r_i^\star(\cdot)$ exist for $l_{10}$, $r_{i0}$ satisfying the tangency condition. Thus, the matching condition (3.11) defines a new curve $l_0(\cdot)$ (respectively, $l_1(\cdot)$, $r_i(\cdot)$, $i = 0, 1$), for which $l_0(t) < l_0^\star(t)$, for $t > 0$, respectively $l_1(t) > l_1^\star(t)$, for $t > 0$ and the similar properties for $r_i^\star(\cdot)$, $i = 0, 1$. We will call them the *matching curves*. By definition this curve is not a tangency curve. We postpone for a moment the proof of their existence. We would rather concentrate on their properties.

We can deduce further properties of the matching curves. As we mentioned, the interfacial points $l_i$, $i = 0, 1$, are necessary to close the system resulting form re-writing (1.1) in the local coordinates. It is important for us to determine sufficient and necessary conditions on $\sigma$ which guarantee that the functions $l_i$, $i = 0, 1$, are monotone. We begin with the necessary conditions.

**Proposition 3.4** *Let us suppose $(\Gamma, \xi)$ is a variational solution on $(0, T)$ and $\sigma_{x_1}$, $\sigma_{x_2}$, $\sigma_t$ are continuous on $[0, T) \times \mathbb{R}^2$. We also assume that $d(t, \cdot)$ (we suppress the superscript $\Lambda$) is of class $C^{1,1}$ in the complement of the interior of the faceted regions, $l_0(\cdot)$ is a matching curve and $l_0$ is strictly monotone. Moreover, the tangency condition is satisfied at $l_{00} = l_0(0)$ and $l_{00} < l_0(t)$. Finally, we set,*

$$\begin{aligned}
\Sigma_0^\Lambda &= \fint_0^{l_{00}} \sigma_t(0, R_{00}, y)\, dy - \sigma_t(0, R_{00}, l_{00}) \\
&\quad + \sigma(0, R_{00}, l_{00}) \left( \fint_0^{l_{00}} \sigma_{x_1}(0, R_{00}, y)\, dy - \sigma_{x_1}(0, R_{00}, l_{00}) \right).
\end{aligned} \tag{3.14}$$

*Then,*
*(a) If, $d_x^+(t, l_0(t)) > 0$, the right derivative of $d(t, \cdot)$ at $l_0(t)$ is positive, then $l_0(\cdot)$ is differentiable for $t \in (0, T)$ and*

$$\dot{l}_0(t) = \frac{1}{d_x^+(t, l_0(t))} (\dot{R}_0(t) - \sigma(t, R_0(t), l_0(t))),$$

*moreover $\dot{l}_0(0) = 0$.*
*(b) If $d_{0,x}^+(l_{00}) = 0$ and the right derivative second derivative of $d_0$ vanishes at $x = l_{00}$,*

*i.e.,* $d^+_{0,xx}(l_{00}) = 0$*, then* $\dot{l}_0(0) = \frac{1}{2}\Sigma^\Lambda_0/\sigma_{x_2}(0, R_{00}, l_{00})$*. In particular the derivative of* $l_0$ *at* $t = 0$ *is positive if* $\Sigma^\Lambda_0 > 0$*.*
*(c) If* $d^+_{0,x}(l_{00}) = 0$ *and* $d^+_{0,xx}(l_{00}) > 0$*, then*

$$\dot{l}_0(0) = -\frac{\sigma_{x_2}(0, R_{00}, l_{00})}{d_{0,xx}(l_{00})} \left(1 - \sqrt{1 + \frac{\Sigma^\Lambda_0 d^+_{0,xx}(l_{00})}{(\sigma_{x_2}(0, R_{00}, l_{00}))^2}}\right).$$

*In particular the derivative of* $l_0$ *at* $t = 0$ *is positive if* $\Sigma^\Lambda_0 > 0$*.*
*(d) If* $\Sigma^\Lambda_0 = 0$*, then* $\dot{l}_0(0) = 0$*.*

**Remarks.** Some comments on the structure of these formulas and their content are in order. First of all, by Proposition 3.3 the matching curve must be different from the tangency curve.

Moreover, if the tangency condition is satisfied at $t = 0$, then by (a) the condition $d^+_{0,x}(l_{00}) > 0$ implies that $\dot{l}_0(0) = 0$, and no information on $\Sigma^\Lambda_0$ is needed.

It is interesting to note that $\Sigma^\Lambda_0 < 0$ is incompatible with the matching curves. Indeed, if we have a matching curve $l_0(\cdot)$ fulfilling the hypotheses of Proposition 3.4, then by (b) or (c) we conclude that $\dot{l}_0(0) < 0$ which contradicts the assumptions. However, if $\dot{l}_0(0) < 0$ and the tangency condition holds at $l_{00}$, then by [GR5, Proposition 2.5] we have a tangency curve starting at $l_{00}$, which automatically satisfies the matching condition and the tangency condition as well,

$$\int_0^{l_0(t)} \sigma(t, R_0(t), y)\, dy + \gamma(\mathbf{n}_R) = l_0(t)\sigma(t, R_0(t), l_0(t)).$$

Differentiating this with respect to $t$ yields,

$$\begin{aligned} \dot{l}_0(t) &= \frac{\int_0^{l_0(t)} \sigma_t(t, R_0(t), y)\, dy - l_0(t)\sigma_t(t, R_0(t), l_0(t))}{l_0(t)\sigma_{x_2}(t, R_0(t), l_0(t))} \\ &\quad + \sigma(t, R_0(t), l_0(t))\frac{\int_0^{l_0(t)} \sigma_{x_1}(t, R_0(t), y)\, dy - l_0(t)\sigma_{x_1}(t, R_0(t), l_0(t))}{l_0(t)\sigma_{x_2}(t, R_0(t), l_0(t))}. \end{aligned} \tag{3.15}$$

Hence, at $t = 0$,

$$\dot{l}_0(0) = \frac{\Sigma^\Lambda_0}{\sigma_{x_2}(0, R_{00}, l_{00})}.$$

This formula agrees with (b), up to the factor $\frac{1}{2}$. As a result, the sign of $\Sigma^\Lambda_0$ can be used to distinguish the type of curve.

In Proposition 3.4 (a) it is crucial that the function $l_0(\cdot)$ is strictly increasing. Without it we cannot draw any conclusion if $\Sigma^\Lambda_0 = 0$.

We excluded the case of $d$ and $\sigma$ such that $d_{0,x}(l_{00}) = 0$, but the tangency condition fails at $l_{00}$. The situation, when $\Sigma^\Lambda_0 < 0$, *i.e.,* the curved part is faster than the facet does not lead to difficulties (see Theorem 3.7 (c)). On the other hand, the case $\Sigma^\Lambda_0 > 0$ is more involved. The formula in part (a) suggests that $l_0$ is not differentiable at $t = 0$ and the limit of $\dot{l}_0(t)$ blows up when $t$ goes to zero. Our methods do not apply to this case.

The matching condition is defined in an implicit way. This means investigating the difference quotients of $l_0$ is a bit involved. For example, we need the following result.

**Lemma 3.1** *Let us suppose that the hypotheses of Proposition 3.4 are satisfied. We define $\Delta_h l_0 = (l_0(h) - l_{00})/h$, where $h > 0$. Then, there is a constant $M$, which is independent from $h$ and from the value of $d_{0,xx}(l_{00})$, for which we have*

$$0 \le \sup_{h>0} \Delta_h l_0 \le M < \infty.$$

*Proof.* Immediately from the definition of the matching curve (see (3.11) and (3.12)) we obtain the following relation,

$$\int_0^h \dot{R}_0(s)\,ds = \int_0^h \sigma(s, d(s, l_0(h)), l_0(h))\,ds + \int_{l_{00}}^{l_0(h)} d_{0,x}(y)\,dy. \tag{3.16}$$

Due to the definition of $\dot{R}_0(s)$ this equation becomes,

$$\begin{aligned}&\int_0^h [F(s, s, l_0(s)) - \sigma(s, d(s, l_0(h)), l_0(h))]\,ds \\ = \;& (l_0(h) - l_{00}) d_{0,x}(l_{00}) + \int_{l_{00}}^{l_0(h)} \int_{l_{00}}^{y} d_{0,xx}(z)\,dzdy,\end{aligned} \tag{3.17}$$

where we used the following shorthand,

$$F(\tau, s, l) := \fint_0^l \sigma(\tau, R_0(s), y)\,dy + \frac{\gamma(\mathbf{n}_R)}{l}. \tag{3.18}$$

Now, we use a simple rearrangement of the left-hand-side (LHS) of (3.17), where the tangency condition at $t = 0$, *i.e.*, $F(0, 0, l_{00}) = \sigma(0, R_{00}, l_{00})$ plays a key role,

$$\begin{aligned} LHS \;=\; & \int_0^h (F(s, s, l_0(s)) - F(0, s, l_0(s)))\,ds + \int_0^h (F(0, s, l_0(s)) - F(0, s, l_{00}))\,ds \\ & + \int_0^h (F(0, s, l_{00}) - F(0, 0, l_{00}))\,ds \\ & - \int_0^h (\sigma(s, d(s, l_0(h)), l_0(h)) - \sigma(s, d(s, l_0(h)), l_{00}))\,ds \\ & - \int_0^h (\sigma(s, d(s, l_0(h)), l_{00}) - \sigma(0, d(s, l_0(h)), l_{00}))\,ds \\ & - \int_0^h (\sigma(0, d(s, l_0(h)), l_{00}) - \sigma(0, d_0(l_{00}), l_{00}))\,ds \\ =\; & (I_1 + I_2 + I_3) - (J_1 + J_2 + J_3), \end{aligned} \tag{3.19}$$

with the obvious definitions of $I_i$ and $J_i$, $i = 1, 2, 3$. It is convenient to rewrite this equation in the following form,

$$I_1 - J_2 = \int_0^h \int_0^s \left( \fint_0^{l_0(s)} \sigma_t(\tau, R_0(s), y)\,dy - \sigma_t(\tau, d(s, l_0(h)), l_{00}) \right) dsd\tau,$$

$$I_2 = \int_0^h \frac{F(0, s, l_0(s)) - F(0, s, l_{00})}{l_0(s) - l_{00}} (\Delta_s l_0) s\,ds,$$

$$J_1 = (\Delta_h l_0) h \int_0^h \frac{\sigma(s, d(s, l_0(h)), l_0(h)) - \sigma(s, d(s, l_0(h)), l_{00})}{l_0(h) - l_{00}} \, ds$$

and

$$\begin{aligned} J_3 &= \int_0^h \int_0^1 \frac{\partial \sigma}{\partial x_1}(0, p(\tau), l_{00}) \left[ \int_0^s \sigma(\rho, d(\rho, l_0(h)), l_0(h)) \, d\rho + d_0(l_0(h)) - d_0(l_{00}) \right] d\tau ds \\ &= J_{31} + h \Delta_h l_0 J_{32}, \end{aligned}$$

where

$$p(\tau) = \tau(d(s, l_0(h)) - d_0(l_{00})) + d_0(l_{00}).$$

After dividing both sides of (3.17) by $h$ and rearranging terms we shall see that,

$$\begin{aligned} &\Delta_h l_0 \left[ d_{0,x}(l_{00}) + J_{32} + \frac{J_1}{l_0(h) - l_{00}} + \frac{1}{l_0(h) - l_{00}} \int_{l_{00}}^{l_0(h)} \int_{l_{00}}^{x} d_{0,xx}(y) \, dy dx \right] - \frac{1}{h} I_2 \\ &= \frac{1}{h}(I_1 - J_2) - \frac{1}{h} J_{31} + \frac{1}{h} I_3. \end{aligned} \tag{3.20}$$

Let us notice that

$$\lim_{h \to 0^+} \frac{(I_1 - J_2)}{h} = \lim_{h \to 0^+} \frac{1}{h} I_2 = \lim_{h \to 0^+} \frac{1}{h} I_3 = \lim_{h \to 0^+} \frac{J_1}{l_0(h) - l_{00}} = \lim_{h \to 0^+} \frac{1}{h} J_{31} = \lim_{h \to 0^+} J_{32} = 0.$$

Now, we consider several cases. If $d_{0,x}(l_{00}) > 0$, then we see that the coefficient in front of $\Delta_h l_0$ is positive for sufficiently small $h$ and the right-hand-side (RHS) of (3.20) is bounded, actually we shall see that it behaves like $O(h)$. Hence, our first claim follows, in particular $\dot{l}_0(0) = 0$.

If, however $d_{0,x}(l_{00}) = 0$, then we divide (3.20) one more time by $h$. Then, we obtain,

$$\begin{aligned} &\Delta_h l_0 \left[ \frac{J_{32}}{h} + \frac{J_1}{h(l_0(h) - l_{00})} \right] - \frac{1}{h^2} I_2 + \frac{(\Delta_h l_0)^2}{(l_0(h) - l_{00})^2} \int_{l_{00}}^{l_0(h)} \int_{l_{00}}^{y} d_{0,xx}(y) \, dy \\ = \quad &\frac{1}{h^2}(I_1 - J_2) - \frac{1}{h^2} J_{31} + \frac{1}{h^2} I_3. \end{aligned} \tag{3.21}$$

Let us denote the absolute value RHS of (3.21) by $A$, we shall see that it can be estimated independently from $h$. For this purpose, we recall the following formula, which holds for any continuous function $f$,

$$\lim_{h \to 0^+} \frac{1}{h^2} \int_0^h \int_0^s f(\tau) \, d\tau ds = \frac{1}{2} f(0). \tag{3.22}$$

This fact implies that

$$\lim_{h \to 0^+} \frac{1}{h^2}(I_1 - J_2) = \frac{1}{2} \fint_0^{l_{00}} \sigma_t(0, R_{00}, y) \, dy - \frac{1}{2} \sigma_t(0, R_{00}, l_{00}).$$

By the definition of $J_{31}$ and (3.22) we deduce that,

$$\lim_{h \to 0^+} \frac{1}{h^2} J_{31} = \frac{1}{2} \sigma(0, R_{00}, l_{00}) \sigma_{x_1}(0, R_{00}, l_{00}).$$

Moreover, it is easy to see that (3.22) yields

$$\lim_{h \to 0^+} \frac{1}{h^2} I_3 = \frac{1}{2} \frac{d}{ds} F(0, s, l_{00})|_{s=0} = \frac{1}{2} \fint_0^{l_0} \sigma_{x_1}(0, R_{00}, y) \, dy \sigma(0, R_{00}, l_{00}).$$

where the last equality is the consequence of the tangency condition at $t = 0$. Thus, indeed $A$ is bounded independently from $h$.

Let us now set

$$\sup_{0<s\leq h} |\Delta_s l_0| =: D(h),$$

and

$$C = \frac{1}{(l_0(h) - l_{00})^2} \int_{l_{00}}^{l_0(h)} \int_{l_{00}}^{y} d_{0,xx}(y)\, dy.$$

By (3.22), we immediately conclude that

$$\lim_{h\to 0^+} C(h) = \frac{1}{2} d_{0,xx}(l_{00}) \geq 0. \tag{3.23}$$

Now, we will identify and estimate the coefficient in front of $D(h)$. We notice that

$$\lim_{h\to 0^+} \frac{J_{32}}{h} = \sigma_{x_1}(0, R_{00}, l_{00}) d_{0,x}(l_{00}) = 0,$$

because $d_{0,x}(l_{00}) = 0$. Subsequently,

$$\frac{1}{h^2}|I_2| \leq \left| \frac{F(0, s, l_0(s)) - F(0, s, l_{00})}{l_0(s) - l_{00}} \right| ds$$

and the bound on the right-hand-side tends to zero as $h \to 0^+$, because of the tangency condition at $t = 0$. Finally,

$$\lim_{h\to 0^+} \frac{J_1}{h(l_0(h) - l_{00})} = \sigma_{x_2}(0, R_{00}, l_{00}) > 0.$$

We now set

$$B = \frac{J_1}{h(l_0(h) - l_{00})} - \frac{1}{h} \int_0^h \left| \frac{F(0, s, l_0(s)) - F(0, s, l_{00})}{l_0(s) - l_{00}} \right| ds + \frac{J_{32}}{h}.$$

Then, combining the above estimates, we arrive at the following inequality

$$D(h)B + D^2(h)C \leq A.$$

We have already noticed that $C$ and $B$ are non-negative.

If $d_{0,xx}(l_{00}) > 0$, then inequality (3.23) implies a bound on $\Delta_h l_0$, which is independent from $h$. On the other hand, if $d_{0,xx}(l_{00}) = 0$, then we notice that $C(h)$ tends to zero, but $B(h) \geq B_0 > 0$ and $A \geq 0$. This is sufficient to deduce that the positive numbers $D$ satisfying

$$D^2C + DB - A \leq 0 \tag{3.24}$$

belong to the interval $[0, A/B]$. This is so, because the the only positive numbers $D$ satisfying $DB - A \leq 0$ belong to this interval, and the set of solutions to (3.24) may only be smaller. $\square$

*Proof of Proposition 3.4.* We will first show cases (d), (b) and (c). Once we established boundedness of $\Delta_h l_0$, we may reuse formula (3.21) to calculate $\dot{l}_0(0)$ in the case the right derivative of $d_0$ vanishes at $l_{00}$, *i.e.*, $d^+_{0,x}(l_{00}) = 0$. For this purpose, we use (3.22). Hence, after passing with $h$ to zero in (3.21), we conclude that $\dot{l}_0(0)$ must satisfy

$$\begin{aligned} &2\dot{l}_0(0)\sigma_{x_2}(0, R_{00}, l_{00}) + (\dot{l}_0(0))^2 d_{0,xx}(l_{00}) = \\ &\quad \fint_0^{l_{00}} \sigma_t(0, R_{00}, y)\, dy - \sigma_t(0, R_{00}, l_{00}) \\ &\quad +\sigma(0, R_{00}, l_{00}) \left( \fint_0^{l_{00}} \sigma_{x_1}(0, R_{00}, y)\, dy - \sigma_{x_1}(0, R_{00}, l_{00}) \right) \\ &= \Sigma^\Lambda_0. \end{aligned} \tag{3.25}$$

The formula above was derived for monotone increasing $l_0(\cdot)$, hence $\dot{l}_0(0)$ must be a non-negative solution to this equation. Thus, we conclude from (3.25):
(i) $\Sigma^\Lambda_0 = 0$ is equivalent to $\dot{l}_0(0) = 0$, *i.e.*, part (d) is shown;
(ii) if $d_{0,xx}(l_{00}) = 0$, then $\dot{l}_0(0) = \frac{1}{2}\Sigma_0/\sigma_{x_2}(0, R_{00}, l_{00})$, in particular $\Sigma^\Lambda_0 > 0$ is equivalent to $\dot{l}_0(0) > 0$, *i.e.*, (b) follows;
(iii) if $d_{0,xx}(l_{00}) > 0$, then $\dot{l}_0(0) > 0$ if and only if $\Sigma^\Lambda_0 > 0$ and

$$\dot{l}_0(0) = -\frac{\sigma_{x_2}(0, R_{00}, l_{00})}{d_{0,xx}(l_{00})} \left( 1 - \sqrt{1 + \frac{\Sigma^\Lambda_0 d_{0,xx}(l_{00})}{\sigma_{x_2}(0, R_{00}, l_{00})^2}} \right).$$

Hence, (c) is proven.

The calculations for $t > 0$ are much simpler, because we can differentiate

$$R_0(t) = d(t, l_0(t)).$$

As a result, we obtain,

$$\dot{l}_0(t) = \frac{1}{d^+_x(t, l_0(t))} \left( \fint_0^{l_0(t)} \sigma(t, R_0(t), y)\, dy + \frac{\gamma(\mathbf{n}_R)}{l_0(t)} - \sigma(t, R_0(t), l_0(t)) \right).$$

This formula is valid also at $t = 0$ provided that $d^+_{0,x}(l_{00}) > 0$, thus (a) follows.

Once we established (a), then by the tangency condition, we deduce that $\dot{l}_0(0) = 0$ what is in accordance with our previous calculations.

Our claims follow. □

**Remarks.** We saw from formula (3.16) that the position of the matching curve is determined at time $t$ as the point where the facet catches up with the curved part. This is possible only if the position of interfacial point $l_0$ is an increasing function of time. As a result, this is another argument for impossibility of $\Sigma^\Lambda_0 < 0$ for $t \geq 0$ on matching curves.

Let us also notice that if $x$ is on a matching curve, then for $t > 0$ we have

$$d^+_x(t, x) > 0.$$

Indeed,

$$d^+_x(t, x) = d^+_{0,x}(x) + \int_0^t \sigma_x(s, d(s, x), x)\, ds \geq \int_0^t \sigma_x(s, d(s, x), x)\, ds > 0.$$

An inspection of the proof of Proposition 3.4 reveals that in fact the time regularity of $\sigma$ may be relaxed. Indeed, we have.

**Corollary 3.1** *Let us suppose that the assumptions of Proposition 3.4 are valid except that on $\sigma$. Namely, we assume that $\sigma_{x_1}$ and $\sigma_{x_2}$ are continuous on $[0,T)\times\mathbb{R}^2$, $\sigma$ belongs to $W^{1,1}_{loc}([0,T)\times\mathbb{R}^2)$ and the right derivative $\sigma_t^+$ exists everywhere for $t\geq 0$. In addition $\sigma_t^+(t,R_0(t),\cdot)$ is integrable with the $L^1$-norm independent from time. Then, the conclusions of Proposition 3.4 hold with $\Sigma_0^\Lambda$ replaced by*

$$\begin{aligned}\Sigma_0^\Lambda &= \fint_0^{l_{00}} \sigma_t^+(0,R_{00},y)\,dy - \sigma_t^+(0,R_{00},l_{00})\\ &\quad+\sigma(0,R_{00},l_{00})\left(\fint_0^{l_{00}}\sigma_{x_1}(0,R_{00},y)\,dy-\sigma_{x_1}(0,R_{00},l_{00})\right).\end{aligned}$$

*Proof.* The element of the proof which requires adjustment is the passage to the limit with $h\to 0^+$ in (3.21). Under our assumptions formula (3.22) becomes,

$$\begin{aligned}&\lim_{h\to 0^+}\frac{1}{h^2}\int_0^h\int_0^s \sigma_t(\tau,R_0(s),l_{00})\,d\tau\,ds\\ &=\lim_{h\to 0^+}\frac{1}{h^2}\int_0^1 \tau\frac{\sigma(h\tau,R_0(h\tau),l_{00})-\sigma(0,R_0(h\tau),l_{00})}{h\tau}\,d\tau=\frac{1}{2}\sigma_t^+(0,R_{00},l_{00}).\end{aligned}$$

In a similar manner we conclude that

$$\lim_{h\to 0^+}\frac{1}{h^2}\int_0^h\int_0^s\fint_0^{l_0(s)}\sigma_t(\tau,R_{00},y)\,d\tau dy ds=\frac{1}{2}\fint_0^{l_0(0)}\sigma_t^+(0,R_{00},y)\,dy.$$

Our claim follows. $\square$

Continuing our inspection of the proof of Proposition 3.4 we notice that the argument with minor adjustments is valid also in the case of the interfacial curve $l_1(\cdot)$. In fact, we have to take into account the dependence of $l_1$ upon $L_1$, however, this does not influence significantly our conclusion. Hence, we can state a version of Proposition 3.4 and its corollary for curve $l_1(\cdot)$.

**Proposition 3.5** *Let us suppose $(\Gamma,\xi)$ is a variational solution, $\sigma_t$, $\sigma_{x_1}$ and $\sigma_{x_2}$ are continuous on $[0,T)\times\mathbb{R}^2$, $\sigma$. We also assume that $d(t,\cdot)$ (we suppress the superscript $\Lambda$) is of class $C^{1,1}$ in the complement of the interior of the faceted regions, $l_1(\cdot)$ is a matching curve and $l_1$ is strictly monotone. Moreover, the tangency condition is satisfied at $l_{10}=l_1(0)$ and $l_{10}>l_1(t)$. Finally, we set,*

$$\begin{aligned}\Sigma_1^\Lambda &= \fint_{l_{10}}^{L_1(0)}\sigma_t(0,R_{10},y)\,dy-\sigma_t(0,R_{10},l_{10})\\ &\quad+\sigma(0,R_{10},l_{10})\left(\fint_{l_{10}}^{L_1(0)}\sigma_{x_1}(0,R_{10},y)\,dy-\sigma_{x_1}(0,R_{10},l_{10})\right)\\ &\quad+\frac{\dot L_1(0)}{(L_1(0)-l_{10})}(\sigma(t,R_{10},L_1(0))-\dot R_1(0)),\end{aligned}\tag{3.26}$$

*where*

$$\dot L_1(0)=\fint_{r_{10}}^{R_{10}}\sigma(0,y,L_{10})\,dy-\frac{2\gamma(\mathbf{n}_\Lambda)}{R_{10}-r_{10}},\quad \dot R_1(0)=\fint_{l_{10}}^{L_{10}}\sigma(0,R_{10},y)\,dy-\frac{2\gamma(\mathbf{n}_R)}{L_{10}-l_{10}}\tag{3.27}$$

*(see also (3.28)). Then,*
*(a) If $d_x^-(t,l_1(t))$, the left derivative of $d(t,\cdot)$ at $l_1(t)$ is positive, then $l_1(\cdot)$ is differentiable for $t>0$ and $\dot l_1(t) = \frac{1}{d_x^-(t,l_1(t))}(\dot R_1(t) - \sigma(t,R_1(t),l_1(t)))$, moreover $\dot l_1(0)=0$.*
*(b) If $d^-_{0,x}(l_{10})=0$, and $d^-_{0,xx}(l_{10})$, the second left derivative of $d_0$, vanishes at $l_{10}$, then $\dot l_1(0) = \frac12 \Sigma_1^\Lambda/\sigma_{x_2}(0,R_{10},l_{10})$. In particular, the derivative of $l_1$ at $t=0$ is negative provided that $\Sigma_1^\Lambda<0$.*
*(c) If $d^-_{0,x}(l_{10})=0$ and $d^-_{0,xx}(l_{10})>0$, then*

$$\dot l_1(0) = -\frac{\sigma_{x_2}(0,R_{10},l_{10})}{d^-_{0,xx}(l_{10})}\left(1-\sqrt{1+\frac{\Sigma_1^\Lambda d^-_{0,xx}(l_{10})}{(\sigma_{x_2}(0,R_{10},l_{10}))^2}}\right).$$

*In particular, the derivative of $l_1$ at $t=0$ is negative provided that $\Sigma_1^\Lambda<0$.*
*(d) If $\Sigma_1^\Lambda=0$, then $\dot l_1(0)=0$.*

*Proof.* We present only the necessary changes in the calculation. An inspection of the proof of Proposition 3.4 suggests a new definition of $F$, which appears in (3.18), namely we set

$$F(\tau,s,l,L) = -\!\!\!\!\!\!\int_l^L \sigma(\tau,R_1(s),y)\,dy - \frac{2\gamma(\mathbf{n}_R)}{L-l}.$$

The subsequent calculation involving $F$ will require one additional change, precisely, a new term $I_4$ will appear at the right-hand-side of (3.19), without any matching $J_4$,

$$I_4 = \int_0^h (F(0,0,l_{10},L_1(h)) - F(0,0,l_{10},L_1(0)))\,dh.$$

We can easily see that

$$\lim_{h\to 0}\frac{I_4}{h} = 0,$$

while

$$\begin{aligned}
\lim_{h\to0}\frac{I_4}{h^2} &= \frac12\frac{\partial F}{\partial L}(0,0,l_{10},L_{10})\dot L_1(0)\\
&= \frac{1}{(L_{10}-l_{10})}\left(--\!\!\!\!\!\!\int_{l_{10}}^{L_{10}}\sigma(0,R_{10},y)\,dy + \frac{2\gamma(\mathbf{n}_R)}{L_{10}-l_{10}} + \sigma(0,R_{10},L_{10})\right)\\
&\quad\times\left(-\!\!\!\!\!\!\int_{r_{10}}^{R_{10}}\sigma(0,y,L_{10})\,dy - \frac{2\gamma(\mathbf{n}_\Lambda)}{R_{10}-r_{10}}\right)\\
&= \frac12\frac{1}{(L_{10}-l_{10})}(\sigma(0,R_{10},L_{10}) - \dot R_1(0))\dot L_1(0). \qquad (3.28)
\end{aligned}$$

Thus, the additional contribution of $\frac{I_4}{h^2}$ will appear at the right-hand-side of (3.22). Hence, right-hand-side of (3.25), which is the definition of $\Sigma_1^\Lambda$ will take the form we claim. $\square$

It is worth noticing, that similarly to (3.15) we have the following formula for the velocity of the tangency curve emanating from $l_{10}$,

$$\begin{aligned}
\dot l_1(t) &= \frac{1}{(L_1(t)-l_1(t))\sigma_{x_2}(t,R_1(t),l_1(t))}\left(\int_{l_1(t)}^{L_1(t)}[\sigma_t(t,R_1(t),y) + \sigma_{x_1}(t,R_1(t),y)\dot R_1(t)]\,dy\right)\\
&\quad-\frac{\sigma_t(t,R_1(t),l_1(t)) + \sigma_{x_1}(t,R_1(t),l_1(t))\dot R_1(t)}{\sigma_{x_2}(t,R_1(t),l_1(t))}\\
&\quad+\frac{\dot L_1(t)(\sigma(t,R_1(t),L_1(t)) - \sigma(t,R_1(t),l_1(t)))}{(L_1(t)-l_1(t))\sigma_{x_2}(t,R_1(t),l_1(t))}
\end{aligned}$$

Thus, at $t = 0$ we have

$$\dot{l}_1(0) = \frac{\Sigma_1^\Lambda}{\sigma_{x_2}(0, R_{10}, l_{10})},$$

what is in accord with Proposition 3.5 (b) up to the factor of $\frac{1}{2}$.

We also state a version of Proposition 3.5 for less time-regular $\sigma$.

**Corollary 3.2** *Let us suppose that the assumptions of Proposition 3.5 are valid except that on $\sigma$. Namely, we assume that $\sigma_{x_1}$ and $\sigma_{x_2}$ are continuous on $[0,T) \times \mathbb{R}^2$, $\sigma$ belongs to $W^{1,1}_{loc}([0,T) \times \mathbb{R}^2)$ and the right derivative $\sigma_t^+$ exists everywhere for $t \geq 0$. In addition $\sigma_t^+(t, R_1(t), \cdot)$ is integrable with the $L^1$-norm independent from time. Then, the conclusions of Proposition 3.5 hold with $\Sigma_1^\Lambda$ replaced by*

$$\begin{aligned}\Sigma_1^\Lambda \quad = \quad & \fint_{l_{10}}^{L_{10}} \sigma_t^+(0, R_{10}, y)\, dy - \sigma_t^+(0, R_{10}, l_{10}) \\ & + \sigma(0, R_{10}, l_{10}) \left( \fint_{l_{10}}^{L_{10}} \sigma_{x_1}(0, R_{10}, y)\, dy - \sigma_{x_1}(0, R_{10}, l_{10}) \right) \\ & + \frac{1}{(L_{10} - l_{10})} (\sigma(0, R_{10}, L_{10}) - \dot{R}_1(0)) \dot{L}_1(0)\end{aligned}$$

Now, we return to the problem of existence of the matching curves. We will state this so that it will be clear that they depend continuously upon $d$.

We saw in Proposition 3.4 that we have a number of possibilities as far as the behavior of $l_0(\cdot)$ near $t = 0$ is concerned. We shall deal first with the simpler case of $d^+_{0,x}(l_{00}) > 0$. The fact below, stated slightly differently, appeared as Theorem 3.3 in [GR5], but without proof. An analogous proposition for $r_0$ is also valid, we will however omit the obvious statement.

**Proposition 3.6** *Let us suppose that we are given a function $d_0 \in C^{1,1}([l_{00}, L_1]) \to \mathbb{R}$ such that $d^+_{0,x}(l_{00}) > 0$. In addition $\sigma : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and $\sigma(t, \cdot, \cdot)$ satisfies the Berg's effect (1.2) and (1.3). Then, there exists a unique matching curve, which is a solution to the following system of equations*

$$\begin{aligned}\dot{l}_0(t) &= \frac{1}{d_x^+(t, l_0(t))} \left( \fint_0^{l_0(t)} \sigma(t, R_0(t), y)\, dy + \frac{\gamma(\mathbf{n}_R)}{l_0(t)} - \sigma(t, R_0(t), l_0(t)) \right), \quad l_0(0) = l_{00}, \\ \dot{R}_0(t) &= \fint_0^{l_0(t)} \sigma(t, R_0(t), y)\, dy + \frac{\gamma(\mathbf{n}_R)}{l_0(t)}, \quad R_0(0) = d_0(l_{00})\end{aligned} \tag{3.29}$$

*provided that one of conditions below holds:*
*(a) the tangency condition fails at $l_{00}$ but $\dot{R}_0(0) - \sigma(0, R_{00}, l_{00}) > 0$; (b) the tangency condition holds at $l_{00}$ and $\Sigma_0^\Lambda > 0$.*

*Here, $d(t,x)$ is a unique solution to*

$$d_t(t,x) = \sigma(t, d(t,x), x), \qquad d(0,x) = d_0(x). \tag{3.30}$$

*In addition, the curve $l_0(\cdot, d_0)$ depends in a Lipschitz continuous manner upon $d_0$.*

*Proof.* We begin with the condition (a). By the assumptions, it is easy to see that the RHS of (3.29) is a continuous function of $l_0$, $R_0$ and time $t$. In order to establish existence of uniqueness of solutions to (3.29), it is sufficient to check that the the RHS of (3.29) is Lipschitz continuous with respect to $l_0$ and $R_0$. This claim becomes obvious, once we write the integral form of $d(\cdot, x)$. As a result, existence and uniqueness of solutions to (3.29) follow.

The condition (a) implies that the solution is an increasing function, thus we found a matching curve.

Since, by the theory of ODE's the function $x \mapsto d_x^+(t,x)$ is Lipschitz continuous as well as $(x,y) \mapsto \sigma(t,x,y)$, then the Lipschitz continuity of $l_0(\cdot, d_0)$ follows.

In case (b) the above argument yielding existence an uniqueness of solutions to (3.29) is still valid, but we have to make sure that the solution is increasing. If (b) holds, then we know that $\dot{l}_0(0) = 0$. We will use Taylor's formula to show that the numerator in the equation (3.30) for $l_0$ is positive. Indeed,

$$
\begin{aligned}
&\fint_0^{l_0(t)} \sigma(t, R_0(t), y)\, dy + \frac{\gamma(\mathbf{n}_R)}{l_0(t)} - \sigma(t, R_0(t), l_0(t)) \\
&= \dot{R}_0(0) - \sigma(0, R_{00}, l_{00}) + t\frac{d}{dt}(\dot{R}_0(0) - \sigma(0, R_{00}, l_{00}))|_{t=0} + o(t).
\end{aligned}
$$

By assume $\dot{R}_0(0) = \sigma(0, R_{00}, l_{00})$ and simple calculations lead us to the conclusion

$$
\frac{d}{dt}(\dot{R}_0(0) - \sigma(0, R_{00}, l_{00}))|_{t=0} = \Sigma_0^\Lambda > 0.
$$

Thus, the solution to (3.30) is a matching curve. □

A statement analogous to Proposition 3.6 is valid also for the matching curve emanating from $l_{10}$.

**Proposition 3.7** *Let us suppose that we are given a function $d_0 \in C^{1,1}([0, L_{10}]) \to \mathbb{R}$ such that $d_{0,x}^-(l_{10}) > 0$. In addition $\sigma_{x_1}$, $\sigma_{x_2}$ are continuous and $\sigma$ is Lipschitz continuous on $[0, T_*) \times \mathbb{R}^2 \to \mathbb{R}$ and for each $t \in [0, T_*)$ function $\sigma(t, \cdot, \cdot)$ satisfies Berg's effect (1.2) and (1.3). Then, there exists a unique matching curve, which is a solution to the following system of equations*

$$
\begin{aligned}
\dot{l}_1(t) &= \frac{1}{d_x^-(t, l_1(t))}\left(\fint_{l_1(t)}^{L_1(t)} \sigma(t, R_1(t), y)\, dy - \frac{2\gamma(\mathbf{n}_R)}{L_1(t) - l_0(t)} - \sigma(t, R_1(t), l_1(t))\right), \\
& \quad l_1(0) = l_{10}, \qquad (3.31)\\
\dot{R}_1(t) &= \fint_{l_1(t)}^{L_1(t)} \sigma(t, R_1(t), y)\, dy - \frac{2\gamma(\mathbf{n}_R)}{L_1(t) - l_0(t)}, \quad R_1(0) = d_0(l_{10}),
\end{aligned}
$$

*provided that one of conditions below holds:*
*(a) the tangency condition fails at $l_{10}$ but $\dot{R}_1(0) - \sigma(0, R_{10}, l_{10}) < 0$ and $L_1$ is a continuous function;*
*(b) the tangency condition holds at $l_{10}$, $\Sigma_0^\Lambda < 0$ and $L_1$ is a $C^1$ function.*

*Here, $d(t,x)$ is a unique solution to*

$$
d_t(t,x) = \sigma(t, d(t,x), x), \qquad d(0,x) = d_0(x). \tag{3.32}
$$

*In addition, the curve $l_1(\cdot, d_0)$ depends in a Lipschitz continuous manner upon $d_0$.*

We skip the proof which is essentially a repetition the proof of Proposition 3.6.

Now, we turn our attention to the construction of the matching curves emanating form a point where $d_{0,x}$ vanishes and the tangency condition holds.

**Theorem 3.1** *(a) Let us suppose that a function $d_0 \in C^1([l_{00}, L_1])$, where $l_{00} > 0$, is given. We assume that $\sigma_t$, $\sigma_{x_1}$ and $\sigma_{x_2}$ are continuous on $[0, T_*) \times \mathbb{R}^2$. Moreover, for each $t \in [0, T_*)$ function $\sigma(t, \cdot, \cdot)$ satisfies the Berg's effect (1.2) and (1.3). We set $R_{00} = d_0(l_{00})$, we assume that $l_{00}$ is a point, where the tangency condition is satisfied, and $d_{0,x}(l_{00}) = 0$, $\Sigma_0^\Lambda > 0$, (see formula (3.14) for the definition of $\Sigma_0^\Lambda$). Then, there exists $T \in (0, T_*]$ such that there exists a unique matching curve $t \mapsto l_0(t)$ for $t \in [0, T)$ which is of class $C^1$ and it is strictly monotone.*

*(b) If the functions $d_0^1$, $d_0^2$ both satisfy (a) and $l_0^1$, $l_0^2$ are corresponding matching curves, then there exists a constant $K$ such that*

$$\|l_0^1 - l_0^2\|_{C[0,T]} \le K \|d_0^1 - d_0^2\|_{C^1[l_{00}, L_1]}.$$

*Proof.* (a) *Step 1.* It is tempting to take derivative of (3.13) with respect to $t$ in the hope to recover an ODE for the matching curves. Once we do this we will discover that there is a problem. Namely, we shall see that the RHS of the resulting equation is not Lipschitz continuous at $(t, a) = (0, l_{00})$. On top of that the equation is not a regular ODE; its explicit form is found in (3.36) where $l_0$ is replaced by $a$. Thus, we have to worry about uniqueness and part (b). This is why we apply a functional approach. If we do so, we will encounter another difficulty related to the fact that we are interested in monotone solutions, but monotone functions do not form linear spaces.

We first introduce a function space and re-write equation (3.13) in a functional form. For a fixed $T \in (0, T_*)$ we consider a Banach space $X_T = C([0, T])$ and its subset

$$Y = \{f \in C([0, T]) : \sup_{h \in (0,T)} |\Delta_h f| < \infty, \ f(t) \ge f(0) = l_{00}\},$$

where $\Delta_h f$ was defined in Lemma 3.1. It is an easy exercise to check that $Y$ is closed.

We shall re-use $D(T)$ with $T$ replacing $h$ and explicit usage of the functional argument,

$$D(a, T) = \sup_{0 < s \le T} |\Delta_s a|.$$

We define three continuous operators

$$\mathcal{K} : B(l_{00}, \delta) \times C^1([0, T]) \to X_T, \quad \mathcal{L} : B(l_{00}, \delta) \times C^1([0, T] \times [l_{00}, L_1]) \to X_T,$$

$$\mathcal{M} : Y \times C^1([l_{00}, L_1]) \to X_T,$$

where $B(l_{00}, \delta) \subset X_T$ is the open ball, centered at a constant function $l_{00}$ with radius $\delta = \frac{1}{2} \min\{l_{00}, L_1 - l_{00}\}$. These operators are given by formulas

$$\mathcal{K}(a, R_0)(t) = \frac{1}{t} \int_0^t \left( \fint_0^{a(s)} \sigma(s, R_0(s), y)\, dy + \frac{\gamma(\mathbf{n}_\Lambda)}{a(s)} \right) ds,$$

$$\mathcal{L}(a, d)(t) = \frac{1}{t} \int_0^t \sigma(s, d(s, a(t)), a(t))\, ds, \qquad \mathcal{M}(a, d_0)(t) = \frac{1}{t}(d_0(a(t)) - R_{00}).$$

In the above formulas $R_0 \in C^1([0,T])$ and $d \in C^1([0,T] \times [l_{00}, L_1])$ are not arbitrary. They satisfy the relations: $R_0(0) = R_{00}$ and $d(0,\cdot) = d_0(\cdot)$.

Literally taken, the above definition of $\mathcal{L}$ is correct for any element $a \in X_T$, however, the left-hand-side of (3.12) makes sense only for monotone increasing $l_0(\cdot)$.

The operators $\mathcal{L}$ and $\mathcal{K}$ are not only continuous, which is easy to check, but also differentiable with respect to $a$. Moreover, they are also locally Lipschitz continuous with respect to $R_0$ and $d$. However, in order to make $\mathcal{M}(a, d_0)$ well-defined, we need that $d_{0,x}(l_{00}) = 0$ and $a \in Y$. It is subsequently easy to see that for a fixed $d$ the mapping $\mathcal{M}(\cdot, d_0) : Y \to C[0,T]$ is locally Lipschitz continuous. Namely, we have for $a_1, a_2 \in Y$,

$$\|M(a_1,d_0) - M(a_2,d_0)\|_{C^0} \le \frac{1}{2}\mathrm{Lip}(d_{0,x}(\cdot))(D(a_1,T) + D(a_2,T))\|a_1 - a_2\|_{C^0}.$$

Taking into account the operators defined above, equation (3.13) takes the form

$$\mathcal{K}(a, R_0) = \mathcal{L}(a, d) + \mathcal{M}(a, d_0). \tag{3.33}$$

*Step 2.* We have to specify $R_0$ and $d$. We define $d$ as a unique solution to (3.30) with initial data $d_0$. Moreover, the solution is continuously differentiable with respect to $x$.

Formula (3.33) implies that we should take $R_0$ which is a solution to

$$\dot{R}_0(t) = \fint_0^{a(t)} \sigma(t, R_0(t), y)\, dy + \frac{\gamma(\mathbf{n}_\Lambda)}{a(t)}, \qquad R_0(0) = R_{00}. \tag{3.34}$$

Moreover, the mapping $C([0,T]) \ni a \mapsto R_0 \in C([0,T])$ is Lipschitz continuous. Indeed, if we take $a_1$, $a_2$ and the corresponding $R_0^1$, $R_0^2$, then one can easily see that for $t \le T$,

$$|R_0^1(t) - R_0^2(t)| \qquad \le \int_0^t C(\sigma_{x_1}, l_{00})(\|R_0^1 - R_0^2\|_{C[0,T]} + \|a_1 - a_2\|_{C[0,T]})\, ds.$$

Thus, if we take sufficiently small $T < T_*$, then

$$\frac{1}{2}\|R_0^1 - R_0^2\|_{C[0,T]} \le TC\|a_1 - a_2\|_{C[0,T]}, \tag{3.35}$$

where $CT$ may be made smaller than $\frac{1}{2}$.

If we stick to the above definitions of $R_0$ and $d$, then after differentiating (3.13) we obtain the following equation for a matching curve emanating from $l_{00}$,

$$\dot{a} = \frac{\displaystyle\fint_0^{a(t)} \sigma(t, R_0(t), y)\, dy + \frac{\gamma(\mathbf{n}_\Lambda)}{a(t)} - \sigma(t, d(t, a(t)), a(t))}{\displaystyle\int_0^t (\sigma_{x_1}(s, d(s,a(t)), a(t)) d_x(s, a(t)) + \sigma_{x_2}(s, d(s, a(t)), a(t)))\, ds - d_{0,x}(a(t))},$$
$$a(0) = l_{00}. \tag{3.36}$$

Let us denote the RHS of (3.36) by $H(t, a(t), R_0(t))$ for $t > 0$. We also set

$$H(0, l_{00}, R_{00}) = \lim_{h \to 0+} H(h, a(h), R_0(h)),$$

where $a$ is a matching curve. By Proposition 3.4 (b–d), we see that $H(0, l_{00}, R_{00})$ is finite. We also note that $H(0, a, R_{00})$ must blow up for $a \ne l_{00}$. Thus, the Lipschitz continuity of $H$ does not make much sense. Thus, we will apply the method used in the proof of the Picard Theorem. But first of all, we notice that this equation is equivalent to the following one

$$a(t) = l_{00} + \int_0^t H(s, a(s), R_0(s))\, ds, \tag{3.37}$$

which is coupled to (3.34). For each $\epsilon > 0$ we set,

$$\begin{aligned} a^\epsilon(t) \qquad &= \begin{cases} l_{00} & \text{for } t \le \epsilon, \\ l_{00} + \int_0^{t-\epsilon} H(s, a^\epsilon(s), R_0^\epsilon(s))\, ds & \text{for } t > \epsilon, \end{cases} \\ R_0^\epsilon(t) \qquad &= R_{00} + \int_0^t \left( \fint_0^{a^\epsilon(s)} \sigma(s, R_0^\epsilon(s), y)\, dy + \frac{\gamma(\mathbf{n}_\Lambda)}{a^\epsilon(s)} \right) ds. \end{aligned} \tag{3.38}$$

Due to positivity of $H$ functions $a^\epsilon$ are strictly increasing, the same is true about $R_0^\epsilon$. Moreover, since $H$ is bounded, we deduce that for a fixed $T > 0$ the family of functions $a^\epsilon$ on $[0, T]$ is equibounded and equicontinuous. In addition, monotonicity of boundedness of $\sigma$ and the fact that $a(t) \ge l_{00}$ for all $t \in [0, T]$ imply a uniform bound on $\dot{R}_0^\epsilon$. Hence, we deduce that $R_0^\epsilon$ are bounded and equicontinuous. Thus, by Arzela-Ascoli Theorem we deduce existence of a sequence $\epsilon_k \to 0$ and two continuous functions $a$, $R_0$ such that $a^{\epsilon_k}$ converges uniformly to $a$ on $[0, T]$ as well as $R_0^{\epsilon_k}$ converges uniformly to $R_0$ on $[0, T]$. Thus, we may pass to the limit in (3.38), because on both sides of the equality we have the same function $a^{\epsilon_k}$ and $R_0^{\epsilon_k}$, this yields (3.33).

Finally, $a$ as a limit of increasing functions must be increasing.

*Step 3.* We will show that there is no more than one solution to (3.33). Since, due to (3.34), $R_0$ is defined uniquely once we specify $a$, it is sufficient to prove that there is no more than one $a$ satisfying (3.33).

Let us suppose that $a_i$, $i = 1, 2$, satisfy (3.33), in particular $R_0^i = R_0^i(a_i)$. Hence,

$$\mathcal{K}(a_1, R_0^1) - \mathcal{K}(a_2, R_0^2) = (\mathcal{L}(a_1, d) + \mathcal{M}(a_1, d)) - (\mathcal{L}(a_2, d_0) + \mathcal{M}(a_2, d_0)).$$

Derivatives $a_i'$, $i = 1, 2$, have a common bound and $(D_a\mathcal{K}(a, R_0))(0)$, which is the following expression

$$\frac{1}{l_{00}} \left( \fint_0^{l_{00}} \sigma(0, R_{00}, y)\, dy + \frac{\gamma(\mathbf{n}_\Lambda)}{l_{00}} - \sigma(0, R_{00}, l_{00}) \right),$$

vanishes. Thus, we deduce that for any $\epsilon > 0$ there is $T > 0$ and $\delta > 0$ such that if $\|a_1 - a_2\|_{C^0[0,T]} \le \delta$ and $\|R_0 - R_{00}\|_{C^0[0,T]} \le \delta$, then $\|D_a\mathcal{K}(a, R_0)\| \le \epsilon$. We shall estimate $\|\mathcal{K}(a, R_0^1) - \mathcal{K}(a, R_0^2)\|$ in $C([0, T])$. It is not difficult to see that Lipschitz continuity of $\sigma$ implies that

$$\|\mathcal{K}(a_1, R_0^1) - \mathcal{K}(a_2, R_0^2)\| \le \epsilon\|a_1 - a_2\|_{C^0[0,T]} + M\|R_0^1 - R_0^2\|_{C^0[0,T]}$$

Thus, due to (3.35) we conclude that for sufficiently small $T$ we have

$$\|\mathcal{K}(a_1, R_0^1) - \mathcal{K}(a_2, R_0^2)\|_{C^0[0,T]} \le 2\epsilon\|a_1 - a_2\|_{C^0[0,T]}. \tag{3.39}$$

We shall check that there exists $m_0$ positive, such that

$$\|(\mathcal{L}(a_1, d) + \mathcal{M}(a_1, d_0)) - (\mathcal{L}(a_2, d) + \mathcal{M}(a_2, d_0))\| \ge m_0. \tag{3.40}$$

Indeed, after we set $a(\tau) = (a_1(t) - a_2(t))\tau + a_2(t)$, we see that

$$
\begin{aligned}
&(\mathcal{L}(a_1, d) + \mathcal{M}(a_1, d_0)) - (\mathcal{L}(a_2, d) + \mathcal{M}(a_2, d_0))(t) = \\
&\quad (a_1(t) - a_2(t))\frac{h(t)}{t}\int_0^t\int_0^1 (\sigma_{x_1}(s, d(s, a(\tau)), a(t))d_x(s, a(\tau)) + \sigma_{x_2}(s, d(s, a(t)), a(\tau))\, d\tau ds \\
&\quad +(a_1(t) - a_2(t))\int_0^1 d_{0,x}(a(\tau)))\, d\tau
\end{aligned}
$$

After taking the absolute value we notice

$$
|(\mathcal{L}(a_1, d) + \mathcal{M}(a_1, d_0)) - (\mathcal{L}(a_2, d) + \mathcal{M}(a_2, d_0))(t)| \geq m_0|a_1(t) - a_2(t)|,
$$

where $m_0 > 0$ and

$$
\min_{t\in[0,T]} \frac{1}{t}\int_0^t (\sigma_{x_1}(s, d(s, a(t)), a(t))d_x(s, a(t)) + \sigma_{x_2}(s, d(s, a(t)), a(t)))\, ds + d_{0,x}(a(t)) = m_0 > 0.
$$

Hence,

$$
2\epsilon\|a_1 - a_2\|_{X_T} \geq m_0\|a_1 - a_2\|_{X_T},
$$

where we take $2\epsilon < m_0$, as a result $\|a_1 - a_2\|_{X_T} = 0$. The uniqueness follows.

*Step 4.* Part (b). This in fact is an easy consequence of Lipschitz continuity of the operators $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ with respect to $d$ and $R_0$. Namely, from (3.33) we have

$$
\begin{aligned}
&\mathcal{K}(a_1, R_0(a_1)) - \mathcal{K}(a_2, R_0(a_2)) = (\mathcal{L}(a_1, d^1) + \mathcal{M}(a_1, d_0^1)) - (\mathcal{L}(a_2, d^2) + \mathcal{M}(a_2, d_0^2)) \\
&= (\mathcal{L}(a_1, d^1) - \mathcal{L}(a_1, d^2)) + (\mathcal{L}(a_1, d^2) - \mathcal{L}(a_2, d^2)) \\
&\qquad +(\mathcal{M}(a_1, d_0^1) - \mathcal{M}(a_1, d_0^2)) + (\mathcal{M}(a_1, d_0^2) - \mathcal{M}(a_2, d_0^2)).
\end{aligned}
$$

Combining this with Lipschitz continuity of $\mathcal{L}$ and $\mathcal{M}$, with respect to $d$, and (3.39), (3.40) which are valid for sufficiently small $T$, we come to the conclusion that

$$
2\epsilon\|a_1 - a_2\|_{X_T} + C\|d^1 - d^2\|_{C^0([0,T]\times[l_{00},l_{10}])} + D(a_1, T)\|d_0^1 - d_0^2\|_{C^1[l_{00},l_{10}]} \geq m_0\|a_1 - a_2\|_{X_T}.
$$

We are permitted to take $2\epsilon < m_0$, hence our claim follows from (3.30). □

We have to treat the case of $l_1$ separately because of the additional dependence on $L_1$.

**Theorem 3.2** *(a) Let us suppose that $L_1 \in C^1([0, T_*])$ and $d_0 \in C^1([l_{10}, M])$, where $l_{10} > 0$, $L_1(t) < M$ are given, we set $R_{10} = d_0(l_{10})$. We assume that $\sigma_t$, $\sigma_{x_1}$ and $\sigma_{x_2}$ are continuous on $[0, T_*) \times \mathbb{R}^2$. Moreover, for each $t \in [0, T_*)$ function $\sigma(t, \cdot, \cdot)$ satisfies the Berg's effect (1.2) and (1.3). We assume that $l_{10}$ is a point where the tangency condition is satisfied, $d_{0,x}(l_{10}) = 0$, and $\Sigma_1^\Lambda > 0$ (see (3.26) for the definition). Then, there exists $T \in (0, T_*]$ such that there exists a unique matching curve $t \mapsto l_1(t)$ for $t \in [0, T)$ which is of class $C^1$.*

*(b) If the couples $(L_1^1, d^1)$, $(L_1^2, d^2)$ both satisfy (a) and $l_1^1$, $l_1^2$ are the corresponding matching curves, then there exists a constant $K$ such that*

$$
\|l_1^1 - l_1^2\|_{C[0,T]} \leq K(\|d_0^1 - d_0^2\|_{C^1[l_{00},L_1]} + \|L_1^1 - L_1^2\|_{C[0,T]}).
$$

*Proof.* The line of reasoning is exactly as in Theorem 3.1. In particular, we define three continuous operators

$$\mathcal{K} : B(l_{00}, \delta) \times C^1([0,T])^2 \to X_T, \quad \mathcal{L} : B(l_{00}, \delta) \times C^1([0,T] \times [l_{00}, L_1]) \to X_T,$$
$$\mathcal{M} : Y \times C^1([l_{00}, L_1]) \to X_T,$$

$\mathcal{K}, \mathcal{L} : B(l_{00}, \delta) \to X_T$ and $\mathcal{M} : Y \to X_T$, given by formulas

$$\mathcal{K}(a, R_1, L_1)(t) = \frac{1}{t} \int_0^t \left( \fint_{a(s)}^{L_1} \sigma(s, R_1(s), y)\, dy - \frac{2\gamma(\mathbf{n}_R)}{L_1 - a(s)} \right) ds,$$
$$\mathcal{L}(a, d)(t) = \frac{1}{t} \int_0^t \sigma(s, d(s, a(t)), a(t))\, ds, \qquad \mathcal{M}(a, d_0)(t) = \frac{1}{t}(R_{10} - d_0(a(t))).$$

The desired matching curve $l_1(\cdot)$ is a solution to the equation, (where the dependence upon $d_0$ and $L_1$ has been suppressed)

$$\mathcal{K}(a, R_1) = \mathcal{L}(a, d) + \mathcal{M}(a, d_0).$$

The only difference with the proof of Theorem 3.1 is that $\mathcal{K}$ depends in the $C^1$–manner upon $L_1$, hence the additional dependence of $l_1$, a solution of the above equation, upon $L_1$. $\square$

**Remark 3.1** We need the same results on $r_0$, $r_1$ which we proved about $l_0$, $l_1$. They are obtained by obvious change of notation, in particular we have to define $\Sigma_0^R$, $\Sigma_1^R$:

$$\begin{aligned}
\Sigma_0^R \;&=\; \fint_0^{r_{00}} \sigma_t(0, y, L_{00})\, dy - \sigma_t(0, r_{00}, L_{00}) + \\
&\quad \sigma(0, r_{00}, L_{00}) \left( \fint_0^{r_{00}} \sigma_{x_1}(0, y, L_{00})\, dy - \sigma_{x_1}(0, r_{00}, L_{00}) \right), \\
\Sigma_1^R \;&=\; \fint_{r_{10}}^{R_{10}} \sigma_t(0, y, L_{10})\, dy - \sigma_t(0, r_{10}, L_{10}) \\
&\quad + \sigma(0, r_{10}, L_{10}) \left( \fint_{r_{10}}^{R_{10}} \sigma_{x_1}(0, y, L_{10})\, dy - \sigma_{x_1}(0, r_{10}, L_{10}) \right) \\
&\quad + \frac{1}{R_{10} - r_{10}} (\sigma(0, R_{10}, L_{10}) - \dot{L}_1(0)) \dot{R}_1(0),
\end{aligned} \tag{3.41}$$

where $\dot{L}_1(0)$ and $\dot{R}_1(0)$ are given by (3.27). We will just state those results. All these theorems permit us to close system (3.7).

**Theorem 3.3** *(a) Let us suppose that $d_0 \in C^1([r_{00}, R_1])$, where $r_{00} > 0$, is given, we set $L_{00} = d_0(r_{00})$. We assume that $\sigma_t$, $\sigma_{x_1}$ and $\sigma_{x_2}$ are continuous on $[0, T_*) \times \mathbb{R}^2$. Moreover, for each $t \in [0, T_*)$ function $\sigma(t, \cdot, \cdot)$ satisfies the Berg's effect (1.2) and (1.3). We assume that $r_{00}$ is a point, where the tangency condition is satisfied and $d_{0,x}(r_{00}) = 0$, $\Sigma_0^R > 0$. Then, there exists $T \in (0, T_*]$ such that there exists a unique matching curve $t \mapsto r_0(t)$ for $t \in [0, T)$ which is of class $C^1$.*

*(b) If the couples $d_0^1$, $d_0^2$ both satisfy (a) and $r_0^1$, $r_0^2$ are the corresponding matching curves, then there exists a constant $K$ such that*

$$\|r_0^1 - r_0^2\|_{C[0,T]} \le K \|d_0^1 - d_0^2\|_{C^1[r_{00}, R_1]}.$$

By the similar token we have.

**Theorem 3.4** *(a) Let us suppose that $R_1 \in C^1([0,T_*])$, and $d_0 \in C^1([r_{10}, M])$, where $r_{10} > 0$, $R_1(t) < M$ are given. We assume that $\sigma_t$, $\sigma_{x_1}$ and $\sigma_{x_2}$ are continuous on $[0,T_*) \times \mathbb{R}^2$. Moreover, for each $t \in [0,T_*)$ function $\sigma(t,\cdot,\cdot)$ satisfies the Berg's effect (1.2) and (1.3). We set $L_{10} = d_0(r_{10})$. We assume that $r_{10}$ is a point where the tangency condition is satisfied, $d_{0,x}(r_{10}) = 0$ (the superscript $R$ is suppressed) and $\Sigma_1^R > 0$. Then, there exists $T \in (0,T_*]$ such that there exists a unique matching curve $t \mapsto r_1(t)$ for $t \in [0,T)$ which is of class $C^1$.*

*(b) If the couples $(R_1^1, d_0^1)$, $(R_1^2, d_0^2)$ both satisfy (a) and $r_1^1$, $r_1^2$ are the corresponding matching curves, then there exists a constant $K$ such that*

$$\|r_1^1 - r_1^2\|_{C[0,T]} \le K(\|d_0^1 - d_0^2\|_{C^1[r_{00},R_1]} + \|R_1^1 - R_1^2\|_{C[0,T]}).$$

Once we closed system (1.2) by supplying the interfacial curves $l_i$, $r_i$, $i = 0, 1$, we may show existence of solutions. We shall show Theorem 1.1 for $l_{00} < l_{10}$ and $r_{00} < r_{10}$, the first step toward this goal is to consider data leading to a matching curve emanating from $r_{00}$, $r_{10}$, $l_{00}$ or $l_{10}$. Without loss of generality we may assume that a matching curve starts at $l_{00}$.

**Theorem 3.5** *We adopt the following hypotheses. Function $\sigma$ is of $C^1$–class on $[0,T_*)\times \mathbb{R}^2$ and for each $t \in [0,T_*)$ function $\sigma(t,\cdot,\cdot)$ satisfies Berg's effect (1.2) and (1.3). In addition, $\beta$ fulfills (2.7). Two Lipschitz continuous function $d_0^\Lambda$, $d_0^R$ are given. They define a bent rectangle $\Gamma_0$ through (BR). Moreover, $d_0^\Lambda$, $d_0^R$ are of class $C^{1,1}$ in the complement of the interior of faceted regions. At the point $l_{00}$ the tangency condition is satisfied and $d_{0,x}^\Lambda(l_{00}) = 0$. The quantity $\Sigma_0^\Lambda$ is defined by (3.14) and $\Sigma_0^\Lambda > 0$. The other interfacial point $l_{10}$ is given and $l_{00} < l_{10}$. Furthermore, the interfacial curves $l_1(\cdot)$, $r_0(\cdot)$ and $r_1(\cdot)$ are well-defined and they are Lipschitz continuous with respect to $d^\Lambda$, $d^R$. Then, there exist $T \in (0,T_*]$ and a variational solution to (1.1) on $[0,T)$ and $d^\Lambda(t,\cdot)$, $d^R(t,\cdot)$, defining the bent rectangle $\Gamma(t)$, are of class $C^1$ in the complement of the interior of faceted regions at each time $t > 0$. Finally, the right derivative of $d^\Lambda(t,\cdot)$ is positive at $x = l_0(t)$ for $t > 0$, thus we witness the phenomenon of loss of regularity.*

**Remark.** We stated the above Theorem in such a way to separate behavior of $l_0(\cdot)$ from the character of the other interfacial points. The assumption that the curves $l_1(\cdot)$, $r_0(\cdot)$ and $r_1(\cdot)$ depend in a Lipschitz continuous manner upon the data holds by Theorems 3.1, 3.2, 3.3, 3.4 and [GR5, Proposition 2.5]. We will conduct the proof in such a way that it carries over to the case of $l_1$, only after minor changes.

*Proof of Theorem 3.5.* We will write equation (3.7) as an integral equation. We note that if $\vec{d} = (d^\Lambda, d^R)$ is a solution to (3.7), then due to Theorem 3.1 the interfacial curve $l_0$ is uniquely determined. The other interfacial curves $l_1$, $r_i$, $i = 0, 1$ are uniquely determined in virtue of one of the Theorems 3.3, 3.4 or Proposition 3.7, its counterpart for $r_1$ or the counterpart of Proposition 3.6 for $r_0$.

Integrating (3.7) with respect to time yields,

$$\vec{d} = \vec{d}_0 + \int_0^t \vec{V}(s, \vec{d})\, ds, \tag{3.42}$$

where $\vec{d}_0 = (d_0^\Lambda, d_0^R)$, $\vec{V} = (V^\Lambda, V^R)$ and $V^\Lambda$ is given by RHS of $(3.7)_{1,2,3}$ while $V^R$ is given by RHS of $(3.7)_{4,5,6}$. We stress that this definition takes into account the changing in time domain of definition of $R_i$, $L_j$, $i,j = 0,1$.

On the other hand, if $\vec{d}$ is a solution to (3.42), then the curve $l_0$ is uniquely determined by Theorem 3.1, as well as the curves $l_1$, $r_i$, $i = 0,1$. Subsequently, taking $\frac{\partial \vec{d}}{\partial t}$ yields a solution to (3.7). Thus, it is sufficient to show that operator $\mathcal{H}(\vec{d})$, defined as the RHS of (3.42), has a unique fixed point in a properly chosen Banach space. Here we take $X = C([0,T];(C[0,M])^2)$ for a suitable $T \in (0,T_*)$ and in accordance with part (b) of the definition of solution we set $M = \max\{L_1, R_1\} + 1$. We also have to define (3.7) for $x_1, x_2 \in [-M, M]$. Namely, we set

$$\begin{aligned} \dot{R}_1 &= \fint_{l_1}^{L_1} \sigma(t, R_1, s)\, ds - \frac{2\gamma(\mathbf{n}_R)}{L_1 - l_1} \qquad \text{on} \quad [l_1, M], \\ \dot{L}_1 &= \fint_{r_1}^{R_1} \sigma(t, s, L_1)\, ds - \frac{2\gamma(\mathbf{n}_\Lambda)}{R_1 - r_1} \qquad \text{on} \quad [r_1, M]. \end{aligned} \tag{3.43}$$

We notice that the above definition of $M$ is a restriction on $L_1(t)$, $R_1(t)$ and time $T$.

We consider here a closed set

$$\begin{aligned} \mathcal{F} \ = \{(f,g) \in X : \ \ & f(t,\cdot),\ g(t,\cdot) \text{ are increasing and Lipschitz continuous and} \\ & f(0,\cdot) = d_0^\Lambda(\cdot), g(0,\cdot) = d_0^R(\cdot)\}. \end{aligned}$$

If $\vec{d} \in \mathcal{F}$, then we know from Theorem 3.1 that the curve $l_0$ exists and it is unique. The other interfacial curves $r_0$, $l_1$ and $r_1$ are also well-defined. The assumptions on the data guarantee that they are monotone and in particular $l_0$ is increasing. Existence and Lipschitz dependence upon the data of the interfacial curves $l_0$, $l_1$, $r_0$ and $r_1$ permits us to define $\vec{d}$ and consequently $\mathcal{H}(\vec{d})$, which belongs to $\mathcal{F}$, due to the signs of the velocities $V^R$ and $V^\Lambda$. Moreover, since $\vec{V}$ and $l_i$, $r_i$, $i = 0,1$ are Lipschitz continuous functions of its arguments, we conclude existence of a small $T > 0$ such that operator $\vec{V}$ is a contraction in $\mathcal{F}$. Hence, existence of a unique fixed point follows.

By theory of ODE we deduce that $\vec{d}(t,\cdot)$ is $C^1$ in the complement of the faceted regions. Moreover, one can see

$$(d^\Lambda)_x^+(t, l_0(t)) = d_0^\Lambda(l_0(t)) + \int_0^t \sigma_{x_1}(s, d^\Lambda(s, l_0(t)), l_0(t)) d_x^\Lambda(s, l_0(t))\, ds.$$

Due to positivity of $\sigma_{x_1}$ we conclude that $(d^\Lambda)_x^+(t, l_0(t))$ is always positive for $t > 0$. Thus, we witness the loss of differentiability of solutions.

We have to exhibit $\xi$ and to show that the pair $(\Gamma, \xi)$ is a variational solution. In fact, $\xi$ is given by formula (3.2). In order to show that this $\xi$ is a minimizer we will adapt the methods of [GR5, Lemma 2.1]. We shall compare $\mathcal{E}_j(\xi + h)$ and $\mathcal{E}_j(\xi)$, $j = \Lambda, R$. We have to examine the assumption that $\xi + h$ is in $\mathcal{D}_j$, $j = \Lambda, R$. Let us write $h(\tilde{d}(t,x)) = (h_1(x), h_2(x))$. First, since $\partial\gamma(\mathbf{n}_\Lambda) \cap \partial\gamma(\mathbf{n}_R) = \{p\}$, then $h(\pm R_1, \pm L_1) = (0,0)$. As a result we may consider each side $S_j^\pm$, $j = \Lambda$, $R$ separately. Thus, we will present the argument only for $\mathcal{E}_\Lambda$, because the other functional is handled in the same way.

Moreover, the structure of $\mathcal{D}_\Lambda$ implies that $h_1(x) = 0$ for all $x \in [-L_1, L_1]$. In addition, due to $\xi_2(t,x) = -\gamma(\mathbf{n}_R)$ at $x = l_i$, $i = 0,1$, we have $h_2(l_i) \geq 0$. By a similar argument $h_2(-l_i) \leq 0$.

In general $\mathcal{E}_\Lambda(\xi)$ is a curvilinear integral over $S^+_\Lambda$. It can be written as

$$\mathcal{E}_\Lambda(\xi) = \int_{-L_1}^{L_1} \frac{1}{2}|\sigma(t, d^\Lambda(t,x),x) - \tau \cdot \frac{\partial \xi}{\partial \tau}(t, d^\Lambda(t,x),x)|^2 \sqrt{1 + (d^\Lambda_x(t,x))^2}\, dx.$$

On inverse images of faceted regions this integral simplifies because $\tau \cdot \frac{\partial \xi}{\partial \tau} = \frac{\partial \xi_2}{\partial x}$ and $d^\Lambda_x = 0$.

On $(l_0, l_1)$ the following formula for solutions $d^\Lambda$

$$d^\Lambda(t,x) = d^\Lambda_0(x) + \int_0^t \sigma(s, d^\Lambda(s,x),x)\, ds$$

is valid. Hence, $d^\Lambda$ is strictly increasing and due to $\frac{\partial \sigma}{\partial x_2}(t, d^\Lambda(x), x) > 0$ for $x > 0$ the derivative $\frac{\partial d^\Lambda}{\partial x}(t,x)$ never vanishes on $(l_0, l_1)$. As a result, for each $t > 0$ the exceptional set $E^\Lambda_Z$ contains at most four point, hence Proposition 3.1 yields $\mathrm{div}_S \xi = 0$ on $(-l_1, -l_0) \cup (l_0, l_1)$. In addition, we deduce that $\mathbf{n} \neq \mathbf{n}_\Lambda, \mathbf{n}_R$ there, hence $\partial\gamma(\mathbf{n})$ is a singleton. Thus $h = 0$ on $(-l_1, -l_0) \cup (l_0, l_1)$.

Our calculations require the knowledge on the behavior of the difference $\sigma - \mathrm{div}_S \xi$ on $[-L, -l_1]$, $[-l_1, -l_0]$, $[-l_0, l_0]$, $[l_0, l_1]$, $[l_1, L]$. Due to symmetries involved, we may consider only positive arguments. First, we take $x \in [0, l_0]$, thus $\mathrm{div}_S \xi = \frac{\partial \xi_2}{\partial x_2}$. We can immediately see that

$$\sigma(t, R_0, l_0) - \frac{\partial \xi_2}{\partial x_2}(t, R_0, l_0) = \fint_0^{l_0} \sigma(t, R_0, s)\, ds + \frac{\gamma(\mathbf{n}_R)}{l_0} = \dot{R}_0 \geq \sigma(t, R_0, l_0) \quad (3.44)$$

holds on $[0, l_0]$, hence on $[-l_0, l_0]$.

If $R_0 = \sigma(t, R_0, l_0)$, then by [GR5, Lemma 2.1], (see also the proof of [GR5, Theorem 3.1]) the $\xi$ we constructed (see (3.2)) is a minimizers. Below, we shall deal with the case $\dot{R}_0 > \sigma(t, R_0, l_0)$.

Next, we consider $[-l_1, -l_0] \cup [l_0, l_1]$. Here, we obviously have

$$\sigma(t, d^\Lambda(t,x), x) - \frac{\partial \xi_2}{\partial x_2}(d^\Lambda(t,x), x) \equiv \sigma(t, d^\Lambda(t,x), x).$$

On interval $[l_1, L_1]$ it holds

$$\sigma(t, R_1, x) - \frac{\partial \xi_2}{\partial x_2}(t, R_1, x) = \fint_{l_1}^{L_1} \sigma(t, R_1, s)\, ds - \frac{2\gamma(\mathbf{n}_R)}{L_1 - l_1}.$$

Let us set

$$\delta = \frac{1}{2}(\dot{R}_0 - \sigma(t, R_0, l_0)) > 0$$

and consider $h_2 \geq 0$ such that

$$\|\frac{\partial h_2}{\partial x_2}\|_{L^\infty} \leq \delta. \quad (3.45)$$

Hence,

$$|\sigma - \frac{\partial \xi_2}{\partial x_2} - \frac{\partial h_2}{\partial x_2}| = |\dot{R}_0 - \frac{\partial h_2}{\partial x_2}| \geq |\sigma(t, R_0, l_0) - \frac{\partial h_2}{\partial x_2}|.$$

After collecting the above information we can see that,

$$
\begin{aligned}
\mathcal{E}_\Lambda(\xi+h) \geq \quad & \frac{1}{2}\int_{-l_0}^{l_0} |\sigma(t,R_0,l_0) - \frac{\partial h_2}{\partial x_2}(x)|^2\,dx \\
+ \quad & \frac{1}{2}\left(\int_{-l_1}^{-l_0} + \int_{l_0}^{l_1}\right) |\sigma(t,d^\Lambda(t,x),x)|^2 \sqrt{1+(d_x^\Lambda)^2(t,x)}\,dx \\
+ \quad & \frac{1}{2}\left(\int_{-L_1}^{-l_1} + \int_{l_1}^{L_1}\right) |\sigma(t,R_1,l_1) - \frac{\partial h_2}{\partial x_2}(x)|^2\,dx. \qquad (3.46)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\mathcal{E}_\Lambda(\xi+h) - \mathcal{E}_\Lambda(\xi) = \\
&\int_{-l_0}^{l_0} [\frac{1}{2}(\frac{\partial h_2}{\partial x_2})^2 - \sigma(t,R_0,l_0)\frac{\partial h_2}{\partial x_2}]\,dx + \left(\int_{-L_1}^{-l_1} + \int_{l_1}^{L_1}\right)[\frac{1}{2}(\frac{\partial h_2}{\partial x_2})^2 - \sigma(t,R_1,l_1)\frac{\partial h_2}{\partial x_2}]\,dx.
\end{aligned}
$$

Integration yields

$$
\mathcal{E}_\Lambda(\xi+h) - \mathcal{E}_\Lambda(\xi) \geq -\sigma(t,R_0,l_0)h_2|_{-l_0}^{l_0} - \sigma(t,R_1,l_1)\left(h_2|_{l_1}^{L} + h|_{-L}^{-l_1}\right).
$$

The last conclusion follows from $\dfrac{\partial\sigma}{\partial x_2}(t,d^\Lambda(t,x),x) > 0$ for $x > 0$, $h_2(l_i) \geq 0$ and $\frac{\partial\sigma}{\partial x_2}(t,d^\Lambda(t,x),x) < 0$ for $x < 0$, $h_2(-l_i) \leq 0$. □

**Remarks.** The question of uniqueness of solutions will be treated separately.

In the above proof we refer to Lipschitz continuity of $l_0$ with respect to $L_1$, $R_1$. However, according to Theorem 3.1 this dependence is trivial. On the other hand, in case of $l_1$ this dependence is not trivial and the above argument is substantial.

**Theorem 3.6** *Let us suppose that $\sigma_t$, $\sigma_{x_1}$ and $\sigma_{x_2}$ are continuous on $[0,T_*)\times\mathbb{R}^2$. For each $t\in[0,T_*)$ function $\sigma(t,\cdot,\cdot)$ satisfies the Berg's effect (1.2) and (1.3). In addition $\beta$ satisfies (2.7). We assume that $d_0^\Lambda$, $d_0^R$ are given of class $C^{1,1}$ in the complement of the interior of the faceted regions. Moreover, $\Sigma_0^\Lambda$ is defined by (3.14), $\Sigma_0^\Lambda > 0$ and $(d_0^\Lambda)_x^+(l_{00}) > 0$. In addition, we assume that $l_{00} < l_{10}$ and that the interfacial curves $l_1(\cdot)$, $r_0(\cdot)$ and $r_1(\cdot)$ are well-defined and they are Lipschitz continuous with respect to $d^\Lambda$, $d^R$. Then, there exist $T\in(0,T_*]$ and a variational solution to (1.1) on $[0,T)$.*

*Proof.* We essentially repeat the argument of Theorem 3.5. Here, instead of Theorem 3.1 we rely on Proposition 3.6 for existence of the matching curve emanating form $l_{00}$. The details are omitted. □

The point of Theorem 3.6 is that it shows existence of solutions for data violating the tangency condition and such that we have a jump discontinuity of $d_x^\Lambda$ at $l_0$.

Finally, we would like to recall the statement of Theorem 1.1, yielding the summary of the existence.

**Theorem 1.1** *Let us suppose that $\sigma$ is $C^1$ on $[0,T_*)\times\mathbb{R}^2$. It satisfies (1.2) and (1.3), $\beta$ is given by (2.7) and $\gamma$ is defined by the formula (1.4). If the initial curve $\Gamma_0$ is a bent rectangle, $l_{00} < l_{10}$ and none of the quantities $\Sigma_0^\Lambda$, $\Sigma_1^\Lambda$, $\Sigma_0^R$, $\Sigma_1^R$ is zero, then there exist $T\in(0,T_*]$ and a unique local-in-time variational solution to (1.1) on $[0,T)$.*

In order to prove it, we have to make its detailed content explicit. This is done in the Theorem below, where we restrict the statement just to a single side $S_\Lambda$ and a single interfacial point which does not lead to any loss of generality as we have already remarked in the proof of Theorem 3.5. During the course of constructing the interfacial curves we have seen that we have to take into account the following factors: (1) the sign of $\Sigma_0^\Lambda$; (2) the tangency condition at $l_{00}$; (3) the vanishing of $d^+_{0,x}(l_{00})$. The theorem below is a report of a book keeper about behavior of the interfacial curve whose formula for the derivative is $\dot{l}_0(t) = (\dot{R}_0(t) - \sigma(t, R_0(t), l_0(t)))/d_x(t, l_0(t))$. Obviously, the case $\dot{l}_0(0) = 0/0$ corresponding to the tangency condition being satisfied at $l_{00}$ and to vanishing of $d^+_{0,x}(l_{00})$ is involved. It is also transparent that the case corresponding to $d^+_{0,x}(l_{00}) = 0$ which the tangency condition fails and $\Sigma_0^\Lambda > 0$ is left out. It is so because it requires different methods, it will be dealt with elsewhere.

**Theorem 3.7** *Let us suppose that $\sigma$ satisfies the Berg's effect and (1.3), it is $C^1$ on $[0, T_*) \times \mathbb{R}^2$, $\beta$ is defined by (2.7) and $\gamma$ is defined by the formula (1.4). We assume that the initial curve $\Gamma_0$ is a bent rectangle (but not a rectangle) and $l_{00} < l_{10}$. Then, there exists $T \in (0, T_*]$ such that:*

*(a) If $\Sigma_0^\Lambda < 0$ and the tangency condition holds at $l_{00}$, then there exists a unique local-in-time variational solution to (1.1) on $[0, T)$ and $l_0(\cdot)$ is a tangency curve.*

*(b) If $\Sigma_0^\Lambda > 0$ and the tangency condition holds at $l_{00}$, then there exists a unique local-in-time variational solution to (1.1) on $[0, T)$ and $l_0(\cdot)$ is a matching curve.*

*(c) If $\Sigma_0^\Lambda < 0$ and the tangency condition does not hold at $l_{00}$, then there exists $\lambda_{00} < l_{00}$ where the tangency condition is satisfied at $\lambda_{00}$. If the assumptions either (a) or (b) are satisfied at $\lambda_{00}$, then there exists a variational solution on $[0, T)$.*

*(d) If $\Sigma_0^\Lambda > 0$, the tangency condition does not hold at $l_{00}$ and $d^+_{0,x}(l_{00}) > 0$, then there exists a unique local-in-time variational solution to (1.1) on $[0, T)$ and $l_0(\cdot)$ is a matching curve.*

*Proof.* (a) This has been proved in [GR5, Theorem 3.1].

(b) This is the content of Theorem 3.5 and Theorem 4.1 below.

(c) The interfacial curve wants to be decreasing while the tangency condition is violated. We encountered such a situation in [GR5, §2.4]. In this case $\dot{R} < \sigma(t, R_0, l_0)$, hence the existence of the postulated $\lambda_{00} < l_{00}$ is obvious. If $\Sigma_0^\Lambda \neq 0$ at the new location, then we are back to cases (a) or (b) of the present theorem.

(d) This is the content of Theorem 3.6 and Theorem 4.1 below. □

We left out undecided the case of $\Sigma_0^\Lambda = 0$. However, such $\sigma$'s are in a $C^1$-neighborhood of another $\sigma_1$, which satisfies one of the conditions above. This is why we claim we have solved the existence problem for the case of generic data.

We also note that if (c) in the above Theorem occurs, then the interval $[\lambda_{00}, l_{00}]$ will bend immediately.

## 3.2 Bending the rectangles

In the previous section we excluded the case $l_{00} = l_{10}$. We will treat it now. Let us notice that if $l_{00} = l_{10}$, then we have a flat facet which is partitioned into three pre-images of faceted regions, $(-L_1, -l_{00})$, $(-l_{00}, l_{00})$, $(l_{00}, L_1)$ and at points belonging to these intervals vector field $\xi$ is in the relative interior of $\partial\gamma(\mathbf{n})$. The solution to the minimization problem (2.6) satisfies (3.4). The definition of the faceted region implies that $\xi(t, l_{00})$ must belong to the boundary of $\partial\gamma(\mathbf{n})$. Thus, it follows from Proposition 3.2 and its proof that

$$\frac{\partial \xi}{\partial x_2}(t, R_0, l_{00}) = 0,$$

*i.e.*, the tangency condition holds at $l_{00}$, thus $V_0 = V_1$ in (3.4).

This fact restricts the possible behavior of the interfacial curves $l_i(\cdot)$, $i = 0, 1$, and explains our interest in data satisfying the tangency conditions.

We notice that some of the configurations are not possible. We begin with complications related to tangency curves $l_i^\star(\cdot)$, $i = 0, 1$.

**Proposition 3.8** *Let us suppose that $(\Gamma, \xi)$ is a variational solution and $l_{00} = l_{10}$. If $l_i^\star(\cdot)$, $i = 0, 1$ are tangency curves satisfying*

$$l_0^\star(t) > l_1^\star(t) \qquad \text{for } t > 0. \tag{3.47}$$

*Then, for $t > 0$ interval $(-L_1, L_1)$ is the pre-image of a single faceted region.*

*Proof.* From the geometry of the problem we infer that the inequality $l_0^\star(t) > l_1^\star(t)$ implies that the lines tangent to $G$ at $l_0^\star(t)$ and $l_1^\star(t)$ are below the graph of $G$. The first line connects the point $(0, 0)$ and $(l_0^\star(t), G(l_0^\star(t) + \gamma(\mathbf{n}_R)))$, the second one connects $(l_1^\star(t), G(l_1^\star(t + \gamma(\mathbf{n}_R))))$ and $(L_1, G(L_1))$. Due to convexity of $G$, they are below the graph of $G$. Hence, the line joining the points $(0, 0)$ and $(L_1, G(L_1))$ is below these tangents. Thus, we showed a different solution to the variational problem (2.6) for which the interfacial points disappear. We reached a contradiction, as a result no bending of the facet occurs and our claim follows. □

Furthermore, a situation when both curves $l_0(\cdot)$, $l_1(\cdot)$ are matching curves is not possible. Indeed, this would imply that $l_1(t) < l_0(t)$, which contradict the possibility of defining these matching curves.

Finally, we have the situation when one of $l_0(\cdot)$, $l_1(\cdot)$ is a matching curve, while the other one is a tangency one. For the sake of definiteness, we assume that $l_0$ is matching while $l_1$ is a tangency curve. Thus, $l_1(t) = l_1^\star(t) > l_{10}$ and $l_0(t) > l_{10}$. We notice that by Proposition 3.8 inequality $l_0^\star(t) > l_1^\star(t)$ is excluded. Since the slope of the tangent to $G(t, \cdot)$ continuously depends upon time, we observe that there is a tangent to the graph of $G(t, \cdot)$ for $0 < t < \epsilon$ and passing through $(0, 0)$. Its tangency point $l_0^\star(t)$ must be close to $l_{10}$ and $l_0^\star(t) < l_1^\star(t)$. Since $l_0(t)$ is a matching curve, then $l_0^\star(t) < l_0(t)$, in particular $\frac{dG}{dx}(l_0(t)) > \frac{dG}{dx}(l_0^\star(t))$, hence the line connecting $(0, 0)$ and $(l_0(t), G_0(t) + \gamma(\mathbf{n}_R))$ must intersect $G$. Thus, $\xi$ is not a solution to the variational principle (2.6), a contradiction. Our claim follows.

Now, we can show the existence result, knowing that only tangency curves starting from $l_{00} = l_{10}$ are possible.

**Theorem 3.8** *Let us suppose that $\sigma$ and $\beta$ are as in Theorem 3.7, $l_{00} = l_{10}$ and $\Sigma_0^\Lambda < 0$, $\Sigma_1^\Lambda > 0$. Then, there is $T > 0$ such a variational solution to (1.1) exists for $t \in [0, T)$. Moreover, the interfacial curves $l_i^\star(\cdot)$, $i = 0, 1$, are the tangency curves and $l_0^\star(t) < l_1^\star(t)$ for $t > 0$.*

*Proof.* What we have shown so far implies that $l_0^\star(t) \le l_1^\star(t)$. Since for the tangency curves we have $\dot{R}_0 < \dot{R}_1$, then $l_0^\star(t) < l_1^\star(t)$. Existence of variational solutions follows now from Theorem 3.7 (a) and its proof. □

## 3.3 Examples

In [GR5] we considered a couple of examples of $\sigma$'s. They were

$$\sigma_1 = 2\sigma^\infty - \sigma^\infty\left(\frac{1}{1+x^2} + \frac{1}{1+y^2}\right),$$
$$\sigma_2 = 2\sigma^\infty - \frac{\sigma^\infty}{1+x^2+y^2}.$$

Let us look first at $\sigma_1$ with constant $\sigma^\infty$. It is easy to check that

$$\Sigma_0^i \equiv 0, \quad i = R, \Lambda,$$
$$\Sigma_1^\Lambda = \dot L_1(\sigma_1(R_1, L_1) - \sigma_1(R_1, l_1)) > 0, \qquad \Sigma_1^R = \dot R_1(\sigma_1(R_1, L_1) - \sigma_1(r_1, L_1)) > 0.$$

Thus, if the tangency condition is satisfied at the initial time at $l_{00}$ or $l_{10}$, then these points are the starting points of the tangency curves, moreover, $l_0(t) \equiv l_{00}$. A similar conclusion holds for $r_0$ and $r_1$.

If we assume that the tangency condition is violated at $l_{10}$ (resp. at $r_{10}$), then we can solve equation (1.1) only if the left derivative of $d_0^\Lambda$ (resp. $d_0^R$) is positive at $l_1$, (resp. $r_1$).

Let us turn our attention to the $\sigma_2$, which is more interesting, because we cannot guarantee in general the sign of all $\Sigma_j^i$, $i = R, \Lambda$, $j = 0, 1$. We first consider $\sigma^\infty$ independent from time. It is easy to notice that, if the tangency condition is satisfied at $l_{00}$, then $\dot l_0(0) > 0$, because for a constant $\sigma^\infty$, the quantity $\Sigma_0^\Lambda$ takes the form

$$\tilde\Sigma_0^\Lambda = 2R_{00}\dot R_0(0)\left(\fint_0^{l_{00}} \frac{dy}{(1+R_{00}^2+y^2)^2} - \frac{1}{(1+R_{00}^2+l_{00}^2)^2}\right) < 0.$$

Hence, a tangency curve emanates from $l_{00}$. We also notice that in general, we cannot determine the sign of $\dot l_1(0)$, without the detailed knowledge about $l_{00}$, $l_{10}$, $R_{00}$, $R_{10}$ and $L_{10}$ because $\Sigma_1^\Lambda$ equals to

$$\begin{aligned}\tilde\Sigma_1^\Lambda \;&=\; 2R_{10}\dot R_1(0)\left(\fint_{l_{10}}^{L_{10}} \frac{dy}{(1+R_{10}^2+y^2)^2} - \frac{1}{(1+R_{10}^2+l_{10}^2)^2}\right)\\ &\quad + \frac{\dot L_1(0)}{L_{10}-l_{10}}\left(\frac{1}{1+R_{10}^2+l_{10}^2} - \frac{1}{1+R_{10}^2+L_{10}^2}\right).\end{aligned}$$

Thus, we cannot determine the character of the curve starting at $l_{10}$.

We can change this situation if $\sigma^\infty$ depends upon time. Namely,

$$\Sigma_0^\Lambda = \tilde\Sigma_0^\Lambda + \sigma_t^\infty\left(\frac{1}{1+R_{00}^2+l_{00}^2} - \fint_0^{l_{00}} \frac{dy}{1+R_{00}^2+y^2}\right)$$

and we notice that the term in the parenthesis is negative. Similarly,

$$\Sigma_1^\Lambda = \tilde\Sigma_1^\Lambda + \sigma_t^\infty\left(\frac{1}{1+R_{10}^2+l_{10}^2} - \fint_{l_{10}}^{L_{10}} \frac{dy}{1+R_{10}^2+y^2}\right)$$

and the in the parenthesis is positive. If we choose large negative $\sigma_t^\infty$, then we can obtain $\Sigma_0^\Lambda > 0$ while $\Sigma_1^\Lambda < 0$, *i.e.*, matching curves start at $l_{00}$ and at $l_{10}$. If we reverse the sign of $\sigma_t^\infty$, we will obtain two tangency curves starting from $l_{00}$ and $l_{10}$.

# 4 Uniqueness of solutions

Here, we essentially use the methods of [GR5, Theorem 3.2]. They depend on the monotonicity of the RHS of (3.7) and on regularity of the interfacial curves, $r_i$, $l_i$, $i = 0, 1$. For the sake of completeness we present below the proof, which is valid for both types of curves.

**Theorem 4.1** *Let us suppose that $\beta$ satisfies (2.7), $\sigma$ is of $C^1$-class on $[0, T) \times \mathbb{R}^2$, for each $t \in [0, T)$ function $\sigma(t, \cdot, \cdot)$ fulfills (1.2) and (1.3). We are given $(\Gamma^i, \xi^i)$, $i = 1, 2$, are two variational solutions to (1.1) defined on $[0, T)$ and $\Gamma^1(0, \cdot) = \Gamma^2(0, \cdot)$, $\xi^1(0, \cdot) = \xi^2(0, \cdot)$. Moreover, the initial data satisfy the assumptions of Theorem 3.7 or §3.2. Then, $\Gamma^1(t, \cdot) = \Gamma^2(t, \cdot)$, $\xi^1(t, \cdot) = \xi^2(t, \cdot)$ for all $t \in [0, T)$.*

*Proof.* One of the problems we have to overcome is the time dependence of domains of $d^{R_i}$, $d^{\Lambda_i}$, $i = 1, 2$. We have to extend these to fixed domains. By $M$ we mean the number defined in part (b) of the definition on the notion of a solution. We recall that by assumption $\sigma(t, \cdot, \cdot)$ is defined over $\mathbb{R}^2$, while $\xi^i(t, \cdot, \cdot)$, $i = 1, 2$ are over $[-M, M]^2$, see (2.2). We will extend $d^{R_i}$ and $d^{\Lambda_i}$ to $[-M, M]^2$. However, in order to simplify the notation we will concentrate on $d^{\Lambda_i}$, $i = 1, 2$. The argument for $d^{R_i}$ is the same and thus the details will be omitted. We extend $d^{\Lambda_i}$, $i = 1, 2$ is by the solution to the system

$$\begin{aligned}
\frac{\partial d^{\Lambda_i}}{\partial t} &= \sigma(t, R_1^i, L_1^i) - \frac{\partial^+ \xi_2^i}{\partial x_2}(t, R_1^i, L_1^i), \qquad x \in [L_1^i, M], \qquad (4.1)\\
\frac{\partial d^{\Lambda_i}}{\partial t} &= \sigma(t, R_1^i, -L_1^i) - \frac{\partial^- \xi_2^i}{\partial x_2}(t, R_1^i, -L_1^i), \qquad x \in [-M, -L_1^i],
\end{aligned}$$

where $\frac{\partial^{\pm}\xi_i}{\partial x_2}$ are one-sided derivatives. We stress that the speed of evolution of $d^{\Lambda_i}$ in (4.1) is set to be constant and equal to the horizontal speed of vertex $(R_1^i, L_1^i)$. Since $\sigma(t, R_1^i, x) - \frac{\partial^+ \xi_2^i}{\partial x_2}(t, R_1^i, x)$ is constant for $x \in [l_1^i, L_1^i]$ and equal to $\dot{R}_1^i$. We see that the above definition is compatible with (3.43).

Subsequently, we proceed as in the proof of [GR5, Theorem 3.2], *i.e.* we take the difference of (4.1) for $i = 1, 2$, multiply by $d^{\Lambda_2} - d^{\Lambda_1}$ and integrate over $[-M, M]$. Hence, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{-M}^{M} |d^{\Lambda_2}(t, x) - d^{\Lambda_1}(t, x)|^2\, dx \\
= &\int_{-M}^{M} \left(-\frac{\partial \xi_2^2}{\partial x_2}(t, x) + \frac{\partial \xi_2^1}{\partial x_2}(t, x)\right)(d^{\Lambda_2}(t, x) - d^{\Lambda_1(t,x)})\, dx \\
&\quad + \int_{-M}^{M} [\sigma(t, d^{\Lambda_2}(t, x), x) - \sigma(t, d^{\Lambda_1}(t, x), x)](d^{\Lambda_2}(t, x) - d^{\Lambda_1}(t, x))\, dx \\
= &\ J + I.
\end{aligned}$$

The second term is easily handled due to Lipschitz continuity of $\sigma$, we obtain

$$I \le C \int_{-M}^{M} |d^{\Lambda_2} - d^{\Lambda_1}|^2\, dx.$$

In order to proceed, we have to examine $\xi^i$ and to introduce some notation for that purpose. Namely, we shall write $\xi(\cdot) = \xi(d^{\Lambda}, L, \cdot)$ to denote the unique solution to (2.6) for $d = d^{\Lambda}$

defined over $[-L,L]$. In fact, as we have seen in the course of the proof of [GR5, Theorem 3.2], it is a unique solution to the corresponding Euler–Lagrange equation. Hence, in our case $\xi^i(\cdot)=\xi^i(d^{\Lambda_i},L^i,\cdot)$, $i=1,2$. Similar notation should be used for the Cahn-Hoffman vectors defined over the other pair of sides $S_R$. However, for the sake of simplicity of notation we shall not do this.

Using the new notation, we will rewrite the term $J$, namely

$$\begin{aligned} J &= \int_{-M}^{M}\left(-\frac{\partial\xi_2}{\partial x_2}(d^{\Lambda_2},L^2,x)+\frac{\partial\xi_2}{\partial x_2}(d^{\Lambda_1},L^2,x)\right)(d^{\Lambda_2}-d^{\Lambda_1})(t,x)\,dx \\ &\quad +\int_{-M}^{M}\left(-\frac{\partial\xi_2}{\partial x_2}(d^{\Lambda_1},L^2,x)+\frac{\partial\xi_2}{\partial x_2}(d^{\Lambda_1},L^1,x)\right)(d^{\Lambda_2}-d^{\Lambda_1})(t,x)\,dx \\ &= J_1+J_2. \end{aligned}$$

An argument based on monotonicity of the operator $\partial\gamma$, as in the proof of [GR5, Theorem 2.2], yields that $J_1\le 0$. Namely, by the definition of $\xi$'s

$$\begin{aligned} J_1 &= \int_{-\min\{L_1^1,L_1^2\}}^{\min\{L_1^1,L_1^2\}}\left(-\frac{\partial\xi_2}{\partial x_2}(d^{\Lambda_2},L_2,x)+\frac{\partial\xi_2}{\partial x_2}(d^{\Lambda_1},L_2,x)\right)(d^{\Lambda_2}-d^{\Lambda_1})(t,x)\,dx+ \\ &\quad \left(\int_{-\max\{L_1^1,L_1^2\}}^{-\min\{L_1^1,L_1^2\}}+\int_{\min\{L_1^1,L_1^2\}}^{-\max\{L_1^1,L_1^2\}}\right)\left(\frac{\partial\xi_2}{\partial x_2}(d^{\Lambda_1},L_2,x)-\frac{\partial\xi_2}{\partial x_2}(d^{\Lambda_2},L_2,x)\right)|R_1^1-R_1^2|\,dx \\ &= J_{11}+J_{12}. \end{aligned}$$

The integration by parts and the boundary conditions (2.5) on $\xi^i$ at $x=\pm L_1^i$, $i=1,2$, lead us to

$$J_{11}=\int_{-\min\{L_1^1,L_1^2\}}^{\min\{L_1^1,L_1^2\}}(\xi_2(d^{\Lambda_2},L_2,x)-\xi_2(d^{\Lambda_1},L_2,x))(d_x^{\Lambda_2}-d_x^{\Lambda_1})(t,x)\,dx.$$

Now, we notice that the integrand equals to the following inner product

$$-(\xi(d^{\Lambda_2},L_2,x)-\xi(d^{\Lambda_1},L_2,x))\cdot(d_x^{\Lambda_2}(t,x)-d_x^{\Lambda_1}(t,x),0)=:I_1.$$

Since $\xi^i(x)\in\partial\gamma((-d_x^{\Lambda_i},1))$, because $\mathbf{n}(x)=(-d_x^{\Lambda_i},1)/\sqrt{1+(d_x^{\Lambda_i})^2}$, we conclude by monotonicity of the subdifferential that $I_1\le 0$. Hence

$$J_{11}\le 0.$$

Later we will deal with $J_{12}$. It requires treatment similar to that applied to $J_{21}$ below.

Now, we turn our attention to $J_2$. Vector fields $\xi(d^{\Lambda_1},L^i)$, $i=1,2$ are obtained as solutions to the Euler–Lagrange equation for the same $d^{\Lambda_1}$ but different $L_1^i$.

Let us notice that the data uniquely imply whether the interfacial curve $l_0^i(\cdot)$, $i=1,2$, is a tangency curve or a matching curve. If $l_0^i(\cdot)$, $i=1,2$, are tangency curves then, by [GR5, Proposition 2.5] they are uniquely defined by $\sigma$ and $R_0(\cdot)=R_0^1(\cdot)=R_0^2(\cdot)$. Moreover, they are of class $C^1$. On the other hand, if $l_0^i(\cdot)$, $i=1,2$, are matching curves, then by Theorem 3.1 they are also uniquely defined by $\sigma$ and $R_0(\cdot)$, in addition $l_0^i\in C^1[0,T]$.

As a result, by the formula for $\xi(d^{\Lambda_1},L^i,x)$, on $(-l_0,l_0)$, we deduce

$$\frac{\partial\xi_2}{\partial x_1}(d^{\Lambda_1},L^2,x)=\frac{\partial\xi_2}{\partial x_1}(d^{\Lambda_1},L^1,x)\qquad\text{for } x\in(-l_0,l_0).$$

However, the above argument is no longer valid for $l_1^i$, $i=1,2$, hence we obtain

$$J_2 = \left(\int_{-M}^{-l_0} + \int_{l_0}^{M}\right)\left(-\frac{\partial\xi_1}{\partial x_1}(d^{\Lambda_1}, L^2, x) + \frac{\partial\xi_1}{\partial x_1}(d^{\Lambda_1}, L^1, x)\right)(d^{\Lambda_2} - d^{\Lambda_1})(x)\,dx.$$

Let us introduce further notation

$$l_1^k = \min\{l_1^2, l_1^1\}, \quad l_1^m = \max\{l_1^2, l_1^1\} \qquad L_1^i = \min\{L_1^2, L_1^1\}, \quad L_1^j = \max\{L_1^2, L_1^1\}.$$

Let us notice that

$$\begin{aligned} &\tfrac{\partial\xi_2}{\partial x_1}(d^{\Lambda_1}, L^p, x) = 0, &&\text{for } x_1 \in (l_0, l_1^k), \quad p = 1,2,\\ &\tfrac{\partial\xi_2}{\partial x_1}(d^{\Lambda_1}, L^m, x) = 0, &&\text{for } x_1 \in (l_1^k, l_1^m), \end{aligned}$$

where the superscript $m$ in $L^m$ means the index $m$ in $l_1^m$, and

$$\frac{\partial\xi_2}{\partial x_1}(d^{\Lambda_1}, L^i, x) = 0, \quad \text{for } x_1 \in (L_1^i, L_1^j).$$

Hence,

$$\begin{aligned} J_2 \quad &\le 2\int_{l_1^k}^{l_1^m}\left|\frac{\partial\xi_2}{\partial x_2}(d^{\Lambda_2}, L^k, x)\right| |d^{\Lambda_2} - d^{\Lambda_1}|(t,x)\,dx\\ &+2\int_{l_1^m}^{L_1^i}\left|\frac{\partial\xi_2}{\partial x_2}(d^{\Lambda_2}, L^2, x) - \frac{\partial\xi_2}{\partial x_2}(d^{\Lambda_2}, L^1, x)\right| |d^{\Lambda_2} - d^{\Lambda_1}|(t,x)\,dx\\ &+2\int_{L_1^i}^{L_1^j}\left|\frac{\partial\xi_2}{\partial x_2}(d^{\Lambda_2}, L^j, x)\right| |d^{\Lambda_2} - d^{\Lambda_1}|(t,x)\,dx. \end{aligned}$$

The formulas for $\frac{\partial\xi_1}{\partial x_1}$, see (3.2), permit us to write

$$J_2 \le K\left(\int_{l_1^k}^{l_1^m} |d^{\Lambda_2} - d^{\Lambda_1}|\,dx_1 + |L_1^2 - L_1^1|\|d^{\Lambda_2} - d^{\Lambda_1}\|_{L^2} + |L_1^2 - L_1^1||R_1^2 - R_1^1|\right).$$

In order to reach the desired bound, we have to show the following "reverse Hölder inequalities",

$$|L_1^2 - L_1^1| \le C\|d^{R_2} - d^{R_1}\|_{L^2}, \qquad |R_1^2 - R_1^1| \le C\|d^{\Lambda_2} - d^{\Lambda_1}\|_{L^2}. \tag{4.2}$$

We will show them, possibly on after restricting the time intervals by the condition

$$l_1^i(t) \le l_{10} + a, \quad t \le T_1, \qquad r_1^i(t) \le r_{10} + a, \quad t \le T_1,$$

for some $a > 0$. Such $T_1$ exists because of differentiability of $l_1^i$ and $r_1^i$ (see [GR5, Proposition 2.5] for the tangency curves and Theorem 3.6 for the matching curves). Thus, after recalling that $L_1^i(t) \ge L_{10}$ for $\sigma$ satisfying (1.2),

$$\|d^{\Lambda_2} - d^{\Lambda_1}\|_{L^2}^2 = \int_{-M}^{M} |d^{\Lambda_2} - d^{\Lambda_1}|^2\,dx_1 \ge \int_{l_{10}+a}^{L_{10}} |d^{\Lambda_2} - d^{\Lambda_1}|^2\,dx_1 = |L_{10} - (l_{10}+a)||R_1^2 - R_1^1|^2.$$

Hence, (4.2) follows. The other inequality follows by the same token.

In order to estimate that the remaining term $\int_{l_1^k}^{l_1^m} |d^{\Lambda_2} - d^{\Lambda_1}|\, dx_1$, we have to work a little bit more. We note that equations $(3.7)_2$ and $(3.7)_5$ imply that $d^{\Lambda_i}$, $d^{\Lambda_i}$, $i = 1, 2$ are bounded on $[0, T_1] \times [-M, M]$ as long as

$$\|d^j_{0,x}\|_{L^\infty} \le K < \infty, \qquad j = R, \Lambda.$$

Using this information we arrive at

$$J_{21} := \int_{l_1^k}^{l_1^m} \left|d^{\Lambda_2} - d^{\Lambda_1}\right|\, dx_1 = \int_{l_1^k}^{l_1^m} \left|R_1^2 - R_1^1 + \int_{x_1}^{l_1^m} (d_x^{\Lambda_2} - d_x^{\Lambda_1})\, ds\right|\, dx_1.$$

Hence,

$$J_{21} \le |l_1^2 - l_1^1||R_1^2 - R_1^1| + K|l_1^2 - l_1^1|^2.$$

We recall that $l_1^i$, no matter what is the character of these curves, they are of class $C^1$ and if $L_1^2 = L_1^1$, then $l_1^1 = l_1^2$. Thus,

$$|l_1^2 - l_1^1| \le K|L_1^2 - L_1^1|.$$

Taking into account of above bounds we arrive at the estimate

$$\frac{d}{dt}\|d^{\Lambda_2} - d^{\Lambda_1}\|^2_{L^2} \le K(\|d^{\Lambda_2} - d^{\Lambda_1}\|^2_{L^2} + \|d^{\Lambda_2} - d^{\Lambda_1}\|^2_{L^2}).$$

A similar estimate is valid for the difference $d^{R_2} - d^{R_1}$, after adding them up we reach

$$\frac{d}{dt}(\|d^{R_2} - d^{R_1}\|^2_{L^2} + \|d^{\Lambda_2} - d^{\Lambda_1}\|^2_{L^2}) \le K(\|d^{R_2} - d^{R_1}\|^2_{L^2} + \|d^{\Lambda_2} - d^{\Lambda_1}\|^2_{L^2}).$$

Using Gronwall inequality we deduce that

$$\|d^{\Lambda_2} - d^{\Lambda_1}\|^2_{L^2} + \|d^{R_2} - d^{R_1}\|^2_{L^2} = 0$$

for $t \in [0, T_1]$. Once we show that $\Gamma_1 = \Gamma_2$, then $\xi^1 = \xi^2$ follows from the strict convexity of the integrand in $\mathcal{E}_i$, $i = R, \Lambda$ as in the proof of Theorem 2.2. □

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