Some Fredholm Integration Operators on A Hilbert Space of Holomorphic Functions on The Unit Disc

By

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Abstract. In this paper, we study when $M_\phi$, $I_\phi$ or $J_\phi$ is a Fredholm operator on a Hilbert space which satisfies few natural axioms.
§ 1. Introduction

Let \( D \) be the open unit disc in the complex plane \( \mathbb{C} \) and \( H(D) \) be the set of all analytic functions on \( D \). \( H(\overline{D}) \) denotes the set of all analytic functions on \( \overline{D} \). In this paper, \( \mathcal{H} \) is a Hilbert space in \( H(D) \) which satisfies the following:

1. \( z\mathcal{H} \subset \mathcal{H} \).
2. If \( a \in D \) then \( (z-a)\mathcal{H} \oplus \mathbb{C} = \mathcal{H} \).
3. \( \mathcal{H} \supseteq \overline{H(D)} \).

In this paper, we study the following three operators. If \( \phi \) is a function in \( H(D) \), put for \( z \in D \),

\[
(M_\phi f)(z) = \phi(z)f(z),
\]

\[
(I_\phi f)(z) = \int_0^z f'(\zeta)\phi(\zeta)d\zeta,
\]

\[
(J_\phi f)(z) = \int_0^z f(\zeta)\phi'(\zeta)d\zeta \quad (f \in \mathcal{H}).
\]

Then \( (M_\phi f)(z) = (I_\phi f)(z) + (J_\phi f)(z) + \phi(0)f(0) \). It is clear that \( I_\phi \) and \( J_\phi \) are never invertible.

Let \( M(\mathcal{H}) = \{ \phi \in H(D) : M_\phi \mathcal{H} \subseteq \mathcal{H} \} \), \( \mathcal{I}(\mathcal{H}) = \{ \phi \in H(D) : I_\phi \mathcal{H} \subseteq \mathcal{H} \} \) and \( \mathcal{J}(\mathcal{H}) = \{ \phi \in H(D) : J_\phi \mathcal{H} \subseteq \mathcal{H} \} \). In this paper, we assume that \( H(\overline{D}) \subset M(\mathcal{H}) \), \( z \in \mathcal{I}(\mathcal{H}) \) and \( z \in \mathcal{J}(\mathcal{H}) \).

§ 2. Multiplication operator \( M_\phi \)

When \( M(\mathcal{H}) = H^\infty(D) \), A. Aleman [1] shows a more general result than Corollary 1 without the condition that \( (z-a)\mathcal{H} \) is dense.

**Lemma 1.** If \( p \) is a polynomial with no zeros on \( \partial D \) then \( \dim \mathcal{H}/p\mathcal{H} < \infty \).

**Proof** If \( |a| > 1 \) then \( (z-a)^{-1} \in H(\overline{D}) \) and so \( (z-a)^{-1} \) belongs to \( M(\mathcal{H}) \). Hence we may assume that the zeros of \( p \) are contained in \( D \). By hypothesis on \( \mathcal{H} \), \( \dim \mathcal{H}/(z-a)\mathcal{H} = 1 \) and so \( \dim \mathcal{H}/p\mathcal{H} < \infty \).

**Lemma 2.** If \( M \) is a closed invariant subspace of \( M_z \) in \( \mathcal{H} \) such that \( \dim \mathcal{H}/M < \infty \), then there exists a polynomial \( p \) such that \( p\mathcal{H} \subseteq M \).

**Proof** Let \( N = \mathcal{H} \ominus M \) and \( S_z = P_NM_z|N \), then \( S_z \) is of finite rank because \( \dim N < \infty \). Hence there exists a polynomial \( p \) such that \( S_p(z) = p(S_z) = 0 \). Therefore \( pN \subset M \) and so \( p\mathcal{H} \subset M \).
Theorem 1.
(1) If $\phi = Bg$ where $B$ is a finite Blaschke product, and both $g$ and $g^{-1}$ are in $M(\mathcal{H})$ then $M_{\phi}$ is a Fredholm operator.

(2) If $M_{\phi}$ is a Fredholm operator on $\mathcal{H}$ then $\phi = Bg$ when $B$ is a finite Blaschke product, $g$ is in $M(\mathcal{H})$ and $g^{-1}$ is in $\mathcal{H}$.

(3) For the $g$ in (2), $M_{g}$ is a Fredholm operator on $\mathcal{H}$ with index $M_{\phi} \leq \text{index } M_{g} \leq 0$ and there exists a polynomial $q$ such that $q\mathcal{H} \subseteq g\mathcal{H}$ and the zeros are in $\mathbb{C}\setminus D$.

Proof (1) Suppose $\phi = Bg$, $B = \prod_{j=1}^{m}(z - a_{j})/(1 - \bar{a}_{j}z)$, $\{a_{j}\} \subset D$, and both $g$ and $g^{-1}$ are in $M(\mathcal{H})$. Since $M(\mathcal{H}) \supseteq H(D)$, $\prod_{j=1}^{m}(1 - \bar{a}_{j}z)$ is invertible in $M(\mathcal{H})$ and so $M_{\phi}(\mathcal{H}) = p\mathcal{H}$ where $p = \prod_{j=1}^{m}(z - a_{j})$. Since $Ker_{\alpha_{j}} = (z - a_{j})\mathcal{H}$, $g$ belongs to $\mathcal{H}$. By the similar argument, there exists a function $k$ in $\mathcal{H}$ and $gk = 1$ because $Bgf = p$. Thus $g^{-1}$ belongs to $\mathcal{H}$. We will prove that $g$ belongs to $M(\mathcal{H})$. Since $B$ is a finite Blaschke product and $Ker_{\alpha} = (z - a)\mathcal{H}$ for $a \in D$, $\mathcal{H} = K + B\mathcal{H}$ where $K$ is a finite dimensional subspace such that each function in $K$ is a rational function whose poles are in $\mathbb{C}\setminus \bar{D}$. Since $g \in \mathcal{H}$ and $M(\mathcal{H}) \supseteq H(D)$, $gK \subseteq \mathcal{H}$ and so $g\mathcal{H} \subseteq \mathcal{H}$ because $gB\mathcal{H} \subseteq \mathcal{H}$.

(3) By the proof of (2), $p\mathcal{H} \subseteq Bg\mathcal{H} \subseteq g\mathcal{H}$ and so the first statement is clear. Again by the proof of (2), the zeros of $p$ in $D$ is just the zeros of $B$. This implies that there exists a polynomial $q$ such that $q\mathcal{H} \subseteq g\mathcal{H}$ and $q$ does not have any zeros in $D$.

Corollary 1. Suppose that $(z - a)\mathcal{H}$ is dense in $\mathcal{H}$ whenever $a \in \partial D$. Then $M_{\phi}$ is a Fredholm operator on $\mathcal{H}$ if and only if $\phi = Bg$ where $B$ is a finite Blaschke product, and both $g$ and $g^{-1}$ are in $M(\mathcal{H})$.

§ 3. Integral operator $I_{\phi}$
It seems to have not been studied yet in this general setting as Theorem 2.

Lemma 3. If $\phi$ is a function in $\mathcal{I}(\mathcal{H})$ then $I_{\phi}(\mathcal{H}) = I_{\phi}(z\mathcal{H}) \subseteq z\mathcal{H}$. $I_{\phi}(\mathcal{H}) = z\mathcal{H}$ if and only if $\phi$ and $\phi^{-1}$ belongs to $\mathcal{I}(\mathcal{H})$.

Proof By the definition of $I_{\phi}$ the first statement is clear. We will
show the second one. If both $\phi$ and $\phi^{-1}$ belong to $\mathcal{I}(\mathcal{H})$, then

$$z\mathcal{H} = I_1(\mathcal{H}) = I_\phi I_{\phi^{-1}}(\mathcal{H}) \subseteq I_\phi(z\mathcal{H}) \subseteq z\mathcal{H}$$

because $I_\phi$ and $I_{\phi^{-1}}$ are bounded on $\mathcal{H}$. Conversely if $I_\phi(\mathcal{H}) = z\mathcal{H}$ then there exists a function $g$ in $\mathcal{H}$ such that

$$\int_0^z g'(\zeta)\phi(\zeta)d\zeta = z$$

and so $g'(z)\phi(z) = 1$.

Hence $\phi^{-1} \in H(D)$ and

$$z\mathcal{H} = I_1(\mathcal{H}) = I_{\phi^{-1}} I_\phi(\mathcal{H}) = I_{\phi^{-1}}(z\mathcal{H})$$

and so both $\phi$ and $\phi^{-1}$ belong to $\mathcal{I}(\mathcal{H})$.

**Lemma 4.** If $p$ is a polynomial then $I_p(\mathcal{H}) + \mathbb{C} \supset p^2\mathcal{H}$.

**Proof** Suppose $g \in \mathcal{H}$. Since $z \in \mathcal{I}(\mathcal{H})$ by the hypothesis, $p$ belongs to $\mathcal{I}(\mathcal{H})$ and so $\int_0^z g'(\zeta)p(\zeta)d\zeta \in \mathcal{H}$. Since $p' \in \mathcal{M}(\mathcal{H})$ and $z \in J(\mathcal{H})$,

$$\int_0^z g(\zeta)p'(\zeta)d\zeta \in \mathcal{H}.$$ 

Hence $f(z) = \int_0^z (2p'(\zeta)g(\zeta) + p(\zeta)g'(\zeta))d\zeta$ belongs to $\mathcal{H}$. Now the lemma follows because

$$\int_0^z f'(\zeta)p(\zeta)d\zeta = \int_0^z (p^2(\zeta)g(\zeta))'d\zeta = p^2(z)g(z) + p^2(0)g(0).$$

**Lemma 5.** Suppose that $B$ is a finite Blaschke product, and both $g$ and $g^{-1}$ are in $\mathcal{I}(\mathcal{H})$. If $\phi =Bg$ then $\phi \in \mathcal{I}(\mathcal{H})$ and $\dim \mathcal{H}/I_\phi(\mathcal{H}) < \infty$.

**Proof** By the hypothesis, $I_B(\mathcal{H}) = I_B(z\mathcal{H}) = I_B(I_\phi(\mathcal{H})) = I_\phi(\mathcal{H})$ by Lemma 3. We may assume that

$$B = \prod_{j=1}^n \frac{z - a_j}{1 - \bar{a}_j z} \text{ and } \{a_j\} \subset D.$$ 

Since $\prod_{j=1}^n (1 - \bar{a}_j z)$ is invertible in $\mathcal{I}(\mathcal{H})$, by Lemma 3 $I_\phi(\mathcal{H}) = I_p(\mathcal{H})$ where $p = \prod_{j=1}^n (z - a_j)$. Lemmas 1 and 4 imply that $\dim \mathcal{H}/I_\phi(\mathcal{H}) < \infty$.

**Lemma 6.** If $p$ is a polynomial then $p(S_z) = S_{p(z)}$.

**Proof** By hypothesis, $P^N I_z(I - P^N) = 0$. Hence

$$S_z = P^N I_z P^N = P^N I_z P^N$$

$$= P^N I_z (I - P^N) I_z P^N + P^N I_z P^N I_z P^N$$

$$= P^N I_z P^N I_z P^N = S_z S_z.$$
Now it is easy to see that \( p(S_z) = S_{p(z)} \) for a polynomial \( p \).

**Lemma 7.** If \( M \) is a closed invariant subspace of \( I_z \) and \( \dim \mathcal{H}/M = n < \infty \) then there exists a polynomial \( p \) such that the degree of \( p \leq n \) and \( I_p(\mathcal{H}) \subseteq M \).

**Proof** If we put \( N = \mathcal{H} \cap M \), then \( \dim N = n < \infty \) and so there exists a polynomial \( p \) such that \( p(S_z) = 0 \) and the degree of \( p \leq n \). By Lemma 6, \( S_{p(z)} = 0 \) and so \( I_p(N) \subseteq M \). Since \( I_p(M) \subseteq M \), \( I_p(\mathcal{H}) \subseteq M \).

**Theorem 2.** Suppose \( \mathcal{I}(\mathcal{H}) \) contains \( H(\bar{D}) \) and if \( f \in \mathcal{I}(\mathcal{H}) \) and \( f(a) = 0 \) for some \( a \in D \) then \( f/(z - a) \) belongs to \( \mathcal{I}(\mathcal{H}) \). \( I_\phi \) is a Fredholm operator on \( \mathcal{H} \) if and only if \( \phi = Bg \) where \( B \) is a finite Blaschke product, and \( g \) and \( g^{-1} \) are in \( \mathcal{I}(\mathcal{H}) \).

**Proof** If \( \phi = Bg \), \( B \) is a finite Blaschke product, \( g \in \mathcal{I}(\mathcal{H}) \) and \( g^{-1} \in \mathcal{I}(\mathcal{H}) \) then by Lemma 5 \( I_\phi(\mathcal{H}) \) is closed and \( \dim \text{Ker} I_\phi < \infty \). Since \( \text{Ker} I_\phi = \mathbb{C} \), index \( I_\phi = 1 - \dim \text{Ker} I_\phi^* \) and so \( I_\phi \) is Fredholm. Conversely if \( I_\phi \) is Fredholm then \( I_\phi(\mathcal{H}) \) is closed and \( \dim \mathcal{H}/I_\phi(\mathcal{H}) < \infty \). Since \( I_z I_\phi(\mathcal{H}) \subseteq I_\phi(\mathcal{H}) \), by Lemma 7 there exists a polynomial such that \( I_p(\mathcal{H}) \subseteq I_\phi(\mathcal{H}) \). By Lemma 4 \( I_p(\mathcal{H}) + \mathbb{C} \supseteq p^2 \mathcal{H} \). Hence there exists a function \( F \) in \( \mathcal{H} \) and \( c \in \mathbb{C} \) such that \( I_\phi(F) + c = p^2 \). Therefore \( F'(z)\phi(z) = 2p(z)p'(z) \) and so the Blaschke part of \( \phi \) is a finite one \( B \). Thus \( \phi \) can be factorized as \( \phi = Bg \) where \( g \in \mathcal{I}(\mathcal{H}) \) and \( g \) has no zeros on \( D \) because \( \mathcal{I}(\mathcal{H}) \) is a subalgebra in \( \mathcal{B}(\mathcal{H}) \) and both \( B \) and \( B^{-1} \) are in \( \mathcal{I}(\mathcal{H}) \). Hence

\[
I_{g^{-1}p}(\mathcal{H}) \subseteq I_{g^{-1}}I_\phi(\mathcal{H}) = I_B(\mathcal{H}) \subseteq \mathcal{H}
\]

and so \( g^{-1}p \) belongs to \( \mathcal{I}(\mathcal{H}) \). By hypothesis on \( \mathcal{I}(\mathcal{H}) \), \( g^{-1} \) belongs to \( \mathcal{I}(\mathcal{H}) \).

§ 4. Integral operator \( J_\phi \)

A Fredholm integral operator \( J_\phi \) have not studied. But if \( J_\phi \) is compact then it is not Fredholm. In some special Hilbert space \( \mathcal{H} \), the compactness of \( J_\phi \) have studied.

**Lemma 8.** If \( \phi \) and \( \psi \) are in \( H(D) \) then \( I_\psi J_\phi = J_\phi M_\psi \).

**Proof** For \( f \in \mathcal{H} \)

\[
(I_\psi J_\phi f)(z) = \int_0^z (J_\phi f)'(\zeta)\psi(\zeta)d\zeta = \int_0^z f(\zeta)\phi'(\zeta)\psi(\zeta)d\zeta = (J_\phi M_\psi f)(z)
\]

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Lemma 9. If $J_\phi$ is a Fredholm operator on $\mathcal{H}$ then $J_\phi \mathcal{H}$ is a closed invariant subspace of $I_z$ and $\dim \mathcal{H}/J_\phi \mathcal{H} < \infty$. Hence there exists a polynomial $p$ such that $J_\phi \mathcal{H} \supseteq I_p \mathcal{H}$ and so $J_\phi \mathcal{H} + \mathbb{C} \supseteq p^2 \mathcal{H}$.

Proof If $J_\phi$ is Fredholm on $\mathcal{H}$ then $J_\phi \mathcal{H}$ is a closed subspace and by Lemma 8 $I_z(J_\phi \mathcal{H}) \subseteq J_\phi \mathcal{H}$. By Lemma 7 there exists a polynomial $q$ such that $I_q \mathcal{H} \subseteq J_\phi \mathcal{H}$. Lemma 4 implies this lemma.

Theorem 3. Suppose that there exists a function $g$ in $\mathcal{H}$ such that $g'$ does not belong to $H^2$. Suppose that any function in $\mathcal{H}$ has radial limits almost everywhere. Then there does not exist $J_\phi$ which is a Fredholm operator on $\mathcal{H}$.

Proof If $J_\phi$ is Fredholm on $\mathcal{H}$ then $J_\phi \mathcal{H} + \mathbb{C} \supseteq p^2 \mathcal{H}$ for some polynomial by Lemma 9. For any $G$ in $p^2 \mathcal{H}$ there exists a function $f$ in $\mathcal{H}$ such that $f(z)\phi'(z) = G'(z) \ (z \in D)$. By hypothesis, there exists $G$ in $p^2 \mathcal{H}$ such that $G' \notin H^2$ and so $G'$ does not have radial limits on a set of positive measure on $\partial D$ (see [2, Appendix A]). On the other hand, if $G = p^2$ then $G$ has radial limits almost everywhere on $\partial D$. By hypothesis, $f$ has radial limits almost everywhere. This contradiction implies that $J_\phi$ is not Fredholm.

§ 5. Relation between $M_\phi$ and $I_\phi$

Put $Df(z) = f'(z)$ and $J = J_z$, that is, $Jf(z) = \int_0^z f(\zeta)d\zeta$. Then

$$DJf = f \quad \text{and} \quad JDf = f - f(0).$$

It is easy to see that $I_\phi J = JM_\phi$ and $DI_\phi = M_\phi D$. Put

$$\mathcal{H}^D = \{f \in H(D) : Df \in \mathcal{H}\}$$

Suppose that $D$ and $J$ are bounded on $\mathcal{H}$ and for $f$ in $\mathcal{H}^D$ put $\|f\|^2_D = \|Df\|^2 + |f(0)|^2$. Then $\mathcal{H}^D$ is a Hilbert space. Put

$$\mathcal{H}^I = \{f \in H(D) : Jf \in \mathcal{H}\}$$

and for $f$ in $\mathcal{H}^I$ $\|f\|_I = \|Jf\|$. Then $\mathcal{H}^I$ is a Hilbert space.

$D$ is isometric from $\mathcal{H}_0^D = \{f \in \mathcal{H}^D : f(0) = 0\}$ onto $\mathcal{H}$. $J$ is isometric from $\mathcal{H}^I$ onto $\mathcal{H}_0 = \{f \in \mathcal{H} : f(0) = 0\}$. Since $DI_\phi = M_\phi D$, $I_\phi$ is bounded on $\mathcal{H}^D$ if and only if $M_\phi$ is bounded on $\mathcal{H}$. Hence $I(\mathcal{H}^D) = \mathcal{M}(\mathcal{H})$. Moreover $I_\phi$ is Fredholm on $\mathcal{H}^D$ if and only if $M_\phi$ is Fredholm on $\mathcal{H}$. Since $JM_\phi = I_\phi J$,
\( \mathcal{I}(\mathcal{H}^J) = \mathcal{M}(\mathcal{H}), \) and \( I_\phi \) is Fredholm on \( \mathcal{H}^J \) if and only if \( M_\phi \) is Fredholm on \( \mathcal{H} \). Moreover \( (\mathcal{H}^J)^D = (\mathcal{H}^D)^J = \mathcal{H} \). Hence \( \mathcal{I}(\mathcal{H}) = \mathcal{M}(\mathcal{H}^D) = \mathcal{M}(\mathcal{H}^J) \), and \( I_\phi \) is Fredholm on \( \mathcal{H} \) if and only if \( M_\phi \) is Fredholm on \( \mathcal{H}^D \) and \( \mathcal{H}^J \).

\section{Examples}

Let \( dA \) denote the normalized Lebesgue area measure on \( D \) and \( \omega \) a positive function on \( D \) which is summable with respect to \( dA \). Put

\[
\mathcal{D}^2(\omega) = \{ f \in H(D) : \| f \|_{2,\omega}^2 = |f(0)|^2 + \int_D |f'(z)|^2 \omega(z)dA(z) < \infty \}
\]

and

\[
L^2_a(\omega) = \{ f \in H(D) : \| f \|_{L^2_a}^2 = \int_D |f(z)|^2 \omega(z)dA(z) < \infty \}.
\]

Then \( \mathcal{D}^2(\omega) \) is called a weighted Dirichlet space and \( L^2_a(\omega) \) is called a weighted Bergman space when \( \mathcal{D}^2(\omega) \) and \( L^2_a(\omega) \) are nontrivial Hilbert spaces. It is easy to see that \( (\mathcal{D}^2(\omega))^J = L^2_a(\omega) \) and \( (L^2_a(\omega))^D = \mathcal{D}^2(\omega) \).

If \( \omega(z) = (1 - |z|^2)^\alpha \) and \( \alpha > -1 \), we will write \( \mathcal{D}^2(\omega) = \mathcal{D}^2_\alpha \) and \( L^2_a(\omega) = \mathcal{L}^2_a,\alpha \). It is known that \( \mathcal{D}^2_\alpha \) and \( \mathcal{L}^2_a,\alpha \) are nontrivial Hilbert spaces. \( \mathcal{D}_1 \) is the Hardy space \( H^2 \), \( \mathcal{D}_2 \) is the Bergman space \( L^2_a \) and \( \mathcal{D}_0 \) is the Dirichlet space. If \( \mathcal{H} = \mathcal{D}_\alpha \) or \( \mathcal{L}^2_a,\alpha \) then \( \mathcal{H} \) satisfies the condition (1), (2) and (3) in Introduction. It is known that \( H(\bar{D}) \subset \mathcal{M}(\mathcal{D}_a) \subset H^\infty(D) \) and \( \mathcal{M}(\mathcal{L}^2_a,\alpha) = H^\infty(D) \). Hence Theorem 1 can apply to \( \mathcal{D}_a \) for any \( \alpha > -1 \). If \( \alpha > 1 \) then \( (z - a)\mathcal{D}_a \) is dense in \( \mathcal{D}_\alpha \) whenever \( a \in \partial D \). Hence Corollary 1 can apply to \( \mathcal{D}_\alpha \) for \( \alpha > 1 \). If \( \alpha = 1 \) then \( \mathcal{D}(\mathcal{D}_a) = \mathcal{M}(\mathcal{D}_a) = H^\infty(D) \), Theorem 2 can apply to \( \mathcal{D}_a \) for \( \alpha > -1 \). It is known [3] that \( \mathcal{M}(\mathcal{D}_a) = H^\infty(D) \) for \( \alpha > 1 \) and \( \mathcal{M}(\mathcal{D}_a) = \mathcal{D}_a \) for \( -1 < \alpha < 0 \). Hence \( \mathcal{I}(\mathcal{L}^2_a,\alpha) = H^\infty(D) \) for \( \alpha > 1 \) and \( \mathcal{I}(\mathcal{L}^2_a,\alpha) = \mathcal{D}_a \) for \( -1 < \alpha < 0 \). Hence Theorem 2 can apply to \( \mathcal{L}^2_a,\alpha \) for \( \alpha > 1 \) and \( -1 < \alpha < 0 \). By a theorem in [3], it is easy to see that \( \mathcal{I}(\mathcal{L}^2_a,\alpha) = \mathcal{M}(\mathcal{D}_\alpha) \) \( (0 \leq \alpha \leq 1 \) satisfies the conditions in Theorem 2. Hence Theorem 2 can apply to \( \mathcal{L}^2_a,\alpha \).

When \( \mathcal{D}^2(\omega) \) or \( L^2_a(\omega) \) is a Hilbert space \( \mathcal{H} \), it is important in order to study composition operator that \( \mathcal{H} \) satisfies three conditions in Introduction. It will be interesting to determine such a weight \( \omega \).

References


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