Some Fredholm Integration Operators on A Hilbert Space of Holomorphic Functions on The Unit Disc

By

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Abstract. In this paper, we study when $M_\phi$, $I_\phi$ or $J_\phi$ is a Fredholm operator on a Hilbert space which satisfies few natural axioms.
§ 1. Introduction
Let $D$ be the open unit disc in the complex plane $\mathbb{C}$ and $H(D)$ be the set of all analytic functions on $D$. $H(\bar{D})$ denotes the set of all analytic functions on $\bar{D}$. In this paper, $\mathcal{H}$ is a Hilbert space in $H(D)$ which satisfies the following:

1. $z\mathcal{H} \subset \mathcal{H}$.
2. If $a \in D$ then $(z-a)\mathcal{H} \oplus \mathbb{C} = \mathcal{H}$.
3. $\mathcal{H} \supseteq H(\bar{D})$.

In this paper, we study the following three operators. If $\phi$ is a function in $H(D)$, put for $z \in D$,

$$(M_{\phi}f)(z) = \phi(z)f(z),$$
$$(I_{\phi}f)(z) = \int_0^z f'(\zeta)\phi(\zeta)d\zeta,$$
$$(J_{\phi}f)(z) = \int_0^z f(\zeta)\phi'(\zeta)d\zeta \quad (f \in \mathcal{H}).$$

Then $(M_{\phi}f)(z) = (I_{\phi}f)(z) + (J_{\phi}f)(z) + \phi(0)f(0)$. It is clear that $I_{\phi}$ and $J_{\phi}$ are never invertible.

Put $\mathcal{M}(\mathcal{H}) = \{\phi \in H(D) : M_{\phi}\mathcal{H} \subseteq \mathcal{H}\}$, $\mathcal{I}(\mathcal{H}) = \{\phi \in H(D) : I_{\phi}\mathcal{H} \subseteq \mathcal{H}\}$ and $\mathcal{J}(\mathcal{H}) = \{\phi \in H(D) : J_{\phi}\mathcal{H} \subseteq \mathcal{H}\}$. In this paper, we assume that $H(\bar{D}) \subset \mathcal{M}(\mathcal{H})$, $z \in \mathcal{I}(\mathcal{H})$ and $z \in \mathcal{J}(\mathcal{H})$.

§ 2. Multiplication operator $M_{\phi}$
When $\mathcal{M}(\mathcal{H}) = H^{\infty}(D)$, A. Aleman [1] shows a more general result than Corollary 1 without the condition that $(z-a)\mathcal{H}$ is dense.

**Lemma 1.** If $p$ is a polynomial with no zeros on $\partial D$ then $\dim \mathcal{H}/p\mathcal{H} < \infty$.

**Proof.** If $|a| > 1$ then $(z-a)^{-1} \in H(\bar{D})$ and so $(z-a)^{-1}$ belongs to $M(\mathcal{H})$. Hence we may assume that the zeros of $p$ are contained in $D$. By hypothesis on $\mathcal{H}$, $\dim \mathcal{H}/(z-a)\mathcal{H} = 1$ and so $\dim \mathcal{H}/p\mathcal{H} < \infty$.

**Lemma 2.** If $M$ is a closed invariant subspace of $M_z$ in $\mathcal{H}$ such that $\dim \mathcal{H}/M < \infty$, then there exists a polynomial $p$ such that $p\mathcal{H} \subseteq M$.

**Proof.** Let $N = \mathcal{H} \ominus M$ and $S_z = P_NM_z|N$, then $S_z$ is of finite rank because $\dim N < \infty$. Hence there exists a polynomial $p$ such that $S_{p(z)} = p(S_z) = 0$. Therefore $pN \subseteq M$ and so $p\mathcal{H} \subseteq M$.  

3
Theorem 1.

1. If \( \phi = Bg \) where \( B \) is a finite Blaschke product, and both \( g \) and \( g^{-1} \) are in \( \mathcal{M}(\mathcal{H}) \) then \( M_\phi \) is a Fredholm operator.

2. If \( M_\phi \) is a Fredholm operator on \( \mathcal{H} \) then \( \phi = Bg \) when \( B \) is a finite Blaschke product, \( g \) is in \( \mathcal{M}(\mathcal{H}) \) and \( g^{-1} \) is in \( \mathcal{H} \).

3. For the \( g \) in (2), \( M_g \) is a Fredholm operator on \( \mathcal{H} \) with index \( M_g \leq 0 \) and there exists a polynomial \( q \) such that \( q\mathcal{H} \subseteq g\mathcal{H} \) and the zeros are in \( \mathbb{C} \setminus D \).

Proof (1) Suppose \( \phi = Bg, B = \prod_{j=1}^{n}(z - a_j)/(1 - \bar{a}_j z), \{a_j\} \subset D \), and both \( g \) and \( g^{-1} \) are in \( \mathcal{M}(\mathcal{H}) \). Since \( \mathcal{M}(\mathcal{H}) \supseteq H(D), \prod_{j=1}^{n}(1 - \bar{a}_j z) \) is invertible in \( \mathcal{M}(\mathcal{H}) \) and so \( M_\phi(\mathcal{H}) = p\mathcal{H} \) where \( p = \prod_{j=1}^{n}(z - a_j) \).

2) If \( M_\phi \) is a Fredholm operator then \( \dim \mathcal{H}/M_\phi(\mathcal{H}) < \infty \) and so by Lemma 2 there exists a polynomial \( p \) such that \( \phi f = p \). Therefore \( \phi \) can be factorized as \( \phi = Bg \) where \( B \) is a finite Blaschke product and \( g \in \mathcal{H} \). For \( \phi \in \mathcal{H} \) and \( \prod_{j=1}^{n}(1 - \bar{a}_j z)\phi = \prod_{j=1}^{n}(z - a_j)g \in \mathcal{H} \) where \( B = \prod_{j=1}^{n}(z - a_j)/(1 - \bar{a}_j z) \). Since \( \text{Ker} \tau_{a_j} = (z - a_j)\mathcal{H}, g \) belongs to \( \mathcal{H} \). By the similar argument, there exists a function \( k \) in \( \mathcal{H} \) and \( gk = 1 \) because \( Bgf = p \). Thus \( g^{-1} \) belongs to \( \mathcal{H} \).

We will prove that \( g \) belongs to \( \mathcal{M}(\mathcal{H}) \). Since \( B \) is a finite Blaschke product and \( \text{Ker} \tau = (z - a)\mathcal{H} \) for \( a \in D, \mathcal{H} = K + B\mathcal{H} \) where \( K \) is a finite dimensional subspace such that each function in \( K \) is a rational function whose poles are in \( \mathbb{C} \setminus D \). Since \( g \in \mathcal{H} \) and \( \mathcal{M}(\mathcal{H}) \supseteq H(D) \), \( gK \subseteq \mathcal{H} \) and so \( g\mathcal{H} \subseteq \mathcal{H} \) because \( gB\mathcal{H} \subseteq \mathcal{H} \).

3) By the proof of (2), \( p\mathcal{H} \subseteq Bg\mathcal{H} \subseteq g\mathcal{H} \) and so the first statement is clear. Again by the proof of (2), the zeros of \( p \) in \( D \) is just the zeros of \( B \). This implies that there exists a polynomial \( q \) such that \( q\mathcal{H} \subseteq g\mathcal{H} \) and \( q \) does not have any zeros in \( D \).

Corollary 1. Suppose that \((z - a)\mathcal{H}\) is dense in \( \mathcal{H} \) whenever \( a \in \partial D \). Then \( M_\phi \) is a Fredholm operator on \( \mathcal{H} \) if and only if \( \phi = Bg \) where \( B \) is a finite Blaschke product, and both \( g \) and \( g^{-1} \) are in \( \mathcal{M}(\mathcal{H}) \).

§ 3. Integral operator \( I_\phi \)

It seems to have not been studied yet in this general setting as Theorem 2.

Lemma 3. If \( \phi \) is a function in \( \mathcal{I}(\mathcal{H}) \) then \( I_\phi(\mathcal{H}) = I_\phi(z\mathcal{H}) \subseteq z\mathcal{H} \). \( I_\phi(\mathcal{H}) = z\mathcal{H} \) if and only if \( \phi \) and \( \phi^{-1} \) belongs to \( \mathcal{I}(\mathcal{H}) \).

Proof By the definition of \( I_\phi \) the first statement is clear. We will
show the second one. If both $\phi$ and $\phi^{-1}$ belong to $\mathcal{I}(\mathcal{H})$, then

$$z\mathcal{H} = I_1(\mathcal{H}) = I_\phi I_{\phi^{-1}}(\mathcal{H}) \subseteq I_{\phi}(z\mathcal{H}) \subseteq z\mathcal{H}$$

because $I_\phi$ and $I_{\phi^{-1}}$ are bounded on $\mathcal{H}$. Conversely if $I_{\phi}(\mathcal{H}) = z\mathcal{H}$ then there exists a function $g$ in $\mathcal{H}$ such that

$$\int_0^z g'(\zeta)\phi(\zeta)d\zeta = z$$

and so $g'(z)\phi(z) = 1$. Hence $\phi^{-1} \in H(D)$ and

$$z\mathcal{H} = I_1(\mathcal{H}) = I_{\phi^{-1}}(\mathcal{H}) = I_{\phi}(z\mathcal{H})$$

and so both $\phi$ and $\phi^{-1}$ belong to $\mathcal{I}(\mathcal{H})$.

**Lemma 4.** If $p$ is a polynomial then $I_p(\mathcal{H}) + \mathbb{C} \supset p^2\mathcal{H}$. 

**Proof** Suppose $g \in \mathcal{H}$. Since $z \in \mathcal{I}(\mathcal{H})$ by the hypothesis, $p$ belongs to $\mathcal{I}(\mathcal{H})$ and so $\int_0^z g(\zeta)p(\zeta)d\zeta \in \mathcal{H}$. Since $p' \in \mathcal{M}(\mathcal{H})$ and $z \in J(\mathcal{H})$, $\int_0^z g(\zeta)p'(\zeta)d\zeta \in \mathcal{H}$. Hence $f(z) = \int_0^z (2p'(\zeta)g(\zeta) + p(\zeta)g'(\zeta))d\zeta \in \mathcal{H}$. Now the lemma follows because

$$\int_0^z f'(\zeta)p(\zeta)d\zeta = \int_0^z (p^2(\zeta)g(\zeta))'d\zeta = p^2(z)g(z) + p^2(0)g(0).$$

**Lemma 5.** Suppose that $B$ is a finite Blaschke product, and both $g$ and $g^{-1}$ are in $\mathcal{I}(\mathcal{H})$. If $\phi = Bg$ then $\phi \in \mathcal{I}(\mathcal{H})$ and dim $\mathcal{H}/I_\phi(\mathcal{H}) < \infty$.

**Proof** By the hypothesis, $I_B(\mathcal{H}) = I_B(z\mathcal{H}) = I_B(I_\phi(\mathcal{H})) = I_\phi(\mathcal{H})$ by Lemma 3. We may assume that

$$B = \prod_{j=1}^n \frac{z - a_j}{1 - \overline{a_j} z}$$

and $\{a_j\} \subset D$. Since $\prod_{j=1}^n (1 - a_j z)$ is invertible in $\mathcal{I}(\mathcal{H})$, by Lemma 3 $I_\phi(\mathcal{H}) = I_p(\mathcal{H})$ where $p = \prod_{j=1}^n (z - a_j)$. Lemmas 1 and 4 imply that dim $\mathcal{H}/I_\phi(\mathcal{H}) < \infty$.

**Lemma 6.** If $p$ is a polynomial then $p(S_z) = S_{p(z)}$.

**Proof** By hypothesis, $P^N I_z(I - P^N) = 0$. Hence

$$S_z = P^N I_z P^N = P^N I_z I_z P^N = P^N I_z(I - P^N) I_z P^N + P^N I_z P^N I_z P^N = P^N I_z P^N I_z P^N = S_z S_z.$$
Now it is easy to see that \( p(S_z) = S_{p(z)} \) for a polynomial \( p \).

**Lemma 7.** If \( M \) is a closed invariant subspace of \( I_z \) and \( \dim \mathcal{H}/M = n < \infty \) then there exists a polynomial \( p \) such that the degree of \( p \leq n \) and \( I_p(\mathcal{H}) \subseteq M \).

**Proof** If we put \( N = \mathcal{H} \ominus M \), then \( \dim N = n < \infty \) and so there exists a polynomial \( p \) such that \( p(S_z) = 0 \) and the degree of \( p \leq n \). By Lemma 6, \( S_{p(z)} = 0 \) and so \( I_p(N) \subseteq M \). Since \( I_p(M) \subseteq M \), \( I_p(\mathcal{H}) \subseteq M \).

**Theorem 2.** Suppose \( \mathcal{I}(\mathcal{H}) \) contains \( H(D) \) and if \( f \in \mathcal{I}(\mathcal{H}) \) and \( f(a) = 0 \) for some \( a \in D \) then \( f/(z-a) \) belongs to \( \mathcal{I}(\mathcal{H}) \). \( I_\phi \) is a Fredholm operator on \( \mathcal{H} \) if and only if \( \phi = Bg \) where \( B \) is a finite Blaschke product, and \( g \) and \( g^{-1} \) are in \( \mathcal{I}(\mathcal{H}) \).

**Proof** If \( \phi = Bg \), \( B \) is a finite Blaschke product, \( g \in \mathcal{I}(\mathcal{H}) \) and \( g^{-1} \in \mathcal{I}(\mathcal{H}) \) then by Lemma 5 \( I_\phi(\mathcal{H}) \) is closed and \( \dim \text{Ker} I_\phi < \infty \). Since \( \text{Ker} I_\phi = \mathbb{C} \), index \( I_\phi = 1 - \dim \text{Ker} I_\phi^* \) and so \( I_\phi \) is Fredholm. Conversely if \( I_\phi \) is Fredholm then \( I_\phi(\mathcal{H}) \) is closed and \( \dim \mathcal{H}/I_\phi(\mathcal{H}) < \infty \). Since \( I_z I_\phi(\mathcal{H}) \subseteq I_\phi(\mathcal{H}) \), by Lemma 7 there exists a polynomial such that \( I_p(\mathcal{H}) \subseteq I_\phi(\mathcal{H}) \). By Lemma 4 \( I_p(\mathcal{H}) \) is a subalgebra in \( \mathcal{B}(\mathcal{H}) \) and both \( B \) and \( B^{-1} \) are in \( \mathcal{I}(\mathcal{H}) \). Hence

\[
I_{g^{-1}p}(\mathcal{H}) \subseteq I_{g^{-1}I_\phi}(\mathcal{H}) = I_{Bp}(\mathcal{H}) \subseteq \mathcal{H}
\]

and so \( g^{-1}p \) belongs to \( \mathcal{I}(\mathcal{H}) \). By hypothesis on \( \mathcal{I}(\mathcal{H}) \), \( g^{-1} \) belongs to \( \mathcal{I}(\mathcal{H}) \).

§ 4. **Integral operator \( J_\phi \)**

A Fredholm integral operator \( J_\phi \) have not studied. But if \( J_\phi \) is compact then it is not Fredholm. In some special Hilbert space \( \mathcal{H} \), the compactness of \( J_\phi \) have studied.

**Lemma 8.** If \( \phi \) and \( \psi \) are in \( H(D) \) then \( I_\psi J_\phi = J_\phi M_\psi \).

**Proof** For \( f \in \mathcal{H} \)

\[
(I_\psi J_\phi f)(z) = \int_0^z (J_\phi f)'(\zeta)\psi(\zeta)d\zeta = \int_0^z f(\zeta)\phi'(\zeta)\psi(\zeta)d\zeta = (J_\phi M_\psi f)(z)
\]
Lemma 9. If $J_\phi$ is a Fredholm operator on $\mathcal{H}$ then $J_\phi \mathcal{H}$ is a closed invariant subspace of $I_z$ and $\dim \mathcal{H}/J_\phi \mathcal{H} < \infty$. Hence there exists a polynomial $p$ such that $J_\phi \mathcal{H} \supseteq I_p \mathcal{H}$ and so $J_\phi \mathcal{H} + C \supseteq p^2 \mathcal{H}$.

Proof If $J_\phi$ is Fredholm on $\mathcal{H}$ then $J_\phi \mathcal{H}$ is a closed subspace and by Lemma 8 $I_z(J_\phi \mathcal{H}) \subseteq J_\phi \mathcal{H}$. By Lemma 7 there exists a polynomial $q$ such that $I_q \mathcal{H} \subseteq J_\phi \mathcal{H}$. Lemma 4 implies this lemma.

Theorem 3. Suppose that there exists a function $g$ in $\mathcal{H}$ such that $g'$ does not belong to $H^2$. Suppose that any function in $\mathcal{H}$ has radial limits almost everywhere. Then there does not exist $J_\phi$ which is a Fredholm operator on $\mathcal{H}$.

Proof If $J_\phi$ is Fredholm on $\mathcal{H}$ then $J_\phi \mathcal{H} + C \supseteq p^2 \mathcal{H}$ for some polynomial by Lemma 9. For any $G$ in $p^2 \mathcal{H}$ there exists a function $f$ in $\mathcal{H}$ such that $f(z)\phi'(z) = G'(z)$ $(z \in D)$. By hypothesis, there exists $G$ in $p^2 \mathcal{H}$ such that $G' \notin H^2$ and so $G'$ does not have radial limits on a set of positive measure on $\partial D$ (see [2, Appendix A]). On the other hand, if $G = p^2$ then $G$ has radial limits almost everywhere on $\partial D$. By hypothesis, $f$ has radial limits almost everywhere. This contradiction implies that $J_\phi$ is not Fredholm.

§ 5. Relation between $M_\phi$ and $I_\phi$

Put $Df(z) = f'(z)$ and $J = J_z$, that is, $Jf(z) = \int_0^z f(\zeta) d\zeta$. Then

$$DJf = f \text{ and } JDf = f - f(0).$$

It is easy to see that $I_\phi J = JM_\phi$ and $DI_\phi = M_\phi D$. Put

$$\mathcal{H}^D = \{ f \in H(D) : Df \in \mathcal{H} \}$$

Suppose that $D$ and $J$ are bounded on $\mathcal{H}$ and for $f$ in $\mathcal{H}^D$ put $\|f\|_D^2 = \|Df\|^2 + |f(0)|^2$. Then $\mathcal{H}^D$ is a Hilbert space. Put

$$\mathcal{H}^I = \{ f \in H(D) : Jf \in \mathcal{H} \}$$

and for $f$ in $\mathcal{H}^I$ $\|f\|_I = \|Jf\|$. Then $\mathcal{H}^I$ is a Hilbert space. $D$ is isometric from $\mathcal{H}^D_0 = \{ f \in \mathcal{H}^D : f(0) = 0 \}$ onto $\mathcal{H}$. $J$ is isometric from $\mathcal{H}^I$ onto $\mathcal{H}_0 = \{ f \in \mathcal{H} : f(0) = 0 \}$. Since $DI_\phi = M_\phi D$, $I_\phi$ is bounded on $\mathcal{H}^D$ if and only if $M_\phi$ is bounded on $\mathcal{H}$. Hence $I(\mathcal{H}^D) = M(\mathcal{H})$. Moreover $I_\phi$ is Fredholm on $\mathcal{H}^D$ if and only if $M_\phi$ is Fredholm on $\mathcal{H}$. Since $JM_\phi = I_\phi J$,
\( \mathcal{I}(\mathcal{H}^j) = \mathcal{M}(\mathcal{H}), \) and \( I_\phi \) is Fredholm on \( \mathcal{H}^j \) if and only if \( M_\phi \) is Fredholm on \( \mathcal{H}. \) Moreover \( (\mathcal{H}^j)^D = (\mathcal{H}^D)^j = \mathcal{H}. \) Hence \( \mathcal{I}(\mathcal{H}) = \mathcal{M}(\mathcal{H}^D) = \mathcal{M}(\mathcal{H}^j) \), and \( I_\phi \) is Fredholm on \( \mathcal{H} \) if and only if \( M_\phi \) is Fredholm on \( \mathcal{H}^D \) and \( \mathcal{H}^j \).

§ 6. Examples
Let \( dA \) denote the normalized Lebesgue area measure on \( D \) and \( \omega \) a positive function on \( D \) which is summable with respect to \( dA \). Put

\[
\mathcal{D}^2(\omega) = \{ f \in H(D) : \| f \|^2_D = |f(0)|^2 + \int_D |f'(z)|^2 \omega(z) dA(z) < \infty \}
\]
and

\[
L^2_a(\omega) = \{ f \in H(D) : \| f \|^2_{L^2_a} = \int_D |f(z)|^2 \omega(z) dA(z) < \infty \}.
\]

Then \( \mathcal{D}^2(\omega) \) is called a weighted Dirichlet space and \( L^2_a(\omega) \) is called a weighted Bergman space when \( \mathcal{D}^2(\omega) \) and \( L^2_a(\omega) \) are nontrivial Hilbert spaces. It is easy to see that \( (\mathcal{D}^2(\omega))^j = L^2_a(\omega) \) and \( (L^2_a(\omega))^D = \mathcal{D}^2(\omega) \).

If \( \omega(z) = (1 - |z|^2)^\alpha \) and \( \alpha > -1 \), we will write \( \mathcal{D}^2(\omega) = \mathcal{D}^2_\alpha \) and \( L^2_a(\omega) = L^2_{a,\alpha} \). It is known that \( \mathcal{D}^2_\alpha \) and \( L^2_{a,\alpha} \) are nontrivial Hilbert spaces. \( \mathcal{D}_1 \) is the Hardy space \( H^2 \), \( \mathcal{D}_2 \) is the Bergman space \( L^2_a \) and \( \mathcal{D}_0 \) is the Dirichlet space.

If \( \mathcal{H} = \mathcal{D}_\alpha \) or \( L^2_{a,\alpha} \) then \( \mathcal{H} \) satisfies the condition (1), (2) and (3) in Introduction. It is known that \( \mathcal{H}(\bar{D}) \subset \mathcal{M}(\mathcal{D}_\alpha) \subset H^\infty(D) \) and \( \mathcal{M}(L^2_{a,\alpha}) = H^\infty(D) \). Hence Theorem 1 can apply to \( \mathcal{D}_\alpha \) for any \( \alpha > -1 \). If \( \alpha \geq 1 \) then \( (z-a)\mathcal{D}_\alpha \) is dense in \( \mathcal{D}_\alpha \) whenever \( a \in \partial D \). Hence Corollary 1 can apply to \( \mathcal{D}_\alpha \) for \( \alpha \geq 1 \).

\( \mathcal{I}(L^2_{a,\alpha}) = \mathcal{M}(L^2_{a,\alpha}) \subset H^\infty(D), \) Theorem 2 can apply to \( \mathcal{D}_\alpha \) for \( \alpha > -1 \). It is known [3] that \( \mathcal{M}(\mathcal{D}_\alpha) = H^\infty(D) \) for \( \alpha > 1 \) and \( \mathcal{M}(\mathcal{D}_\alpha) = \mathcal{D}_\alpha \) for \( -1 < \alpha < 0 \). Hence \( \mathcal{I}(L^2_{a,\alpha}) = H^\infty(D) \) for \( \alpha > 1 \). Since \( \mathcal{I}(\mathcal{D}_\alpha) = \mathcal{M}(L^2_{a,\alpha}) = H^\infty(D) \), Theorem 2 can apply to \( \mathcal{D}_\alpha \) for \( \alpha > -1 \). By a theorem in [3], it is easy to see that \( \mathcal{I}(L^2_{a,\alpha}) = \mathcal{M}(\mathcal{D}_\alpha) \) (0 \( \leq \alpha \leq 1 \)) satisfies the conditions in Theorem 2. Hence Theorem 2 can apply to \( L^2_{a,\alpha} \).

When \( \mathcal{D}^2(\omega) \) or \( L^2_a(\omega) \) is a Hilbert space \( \mathcal{H} \), it is important in order to study composition operator that \( \mathcal{H} \) satisfies three conditions in Introduction. It will be interesting to determine such a weight \( \omega \).

References


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