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<td>Author(s)</td>
<td>Nakazi, Takahiko</td>
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<tr>
<td>Citation</td>
<td>Hokkaido University Preprint Series in Mathematics, 891, 1-9</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2008</td>
</tr>
<tr>
<td>DOI</td>
<td>10.14943/84041</td>
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<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/69700">http://hdl.handle.net/2115/69700</a></td>
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<td>Type</td>
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<td>File Information</td>
<td>pre891.pdf</td>
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<td>Hokkaido University Collection of Scholarly and Academic Papers: HUSCAP</td>
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Some Fredholm Integration Operators on A Hilbert Space of Holomorphic Functions on The Unit Disc

By

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2000 Mathematics Subject Classification. Primary 47 B 38, 47 G 10, Secondary 47 A 53

Keywords and phrases : Fredholm operator, Hilbert space, analytic function, integration operator, multiplication operator

∗ This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education
Abstract. In this paper, we study when $M_\phi$, $I_\phi$ or $J_\phi$ is a Fredholm operator on a Hilbert space which satisfies few natural axioms.
1. Introduction

Let $D$ be the open unit disc in the complex plane $\mathbb{C}$ and $H(D)$ be the set of all analytic functions on $D$. $H(\overline{D})$ denotes the set of all analytic functions on $\overline{D}$. In this paper, $\mathcal{H}$ is a Hilbert space in $H(D)$ which satisfies the following:

1. $z\mathcal{H} \subset \mathcal{H}$.
2. If $a \in D$ then $(z-a)\mathcal{H} \oplus \mathbb{C} = \mathcal{H}$.
3. $\mathcal{H} \supset H(\overline{D})$.

In this paper, we study the following three operators. If $\phi$ is a function in $H(D)$, put for $z \in D$,

\[
(M_\phi f)(z) = \phi(z)f(z),
\]

\[
(I_\phi f)(z) = \int_0^z f'(\zeta)\phi(\zeta)d\zeta,
\]

\[
(J_\phi f)(z) = \int_0^z f(\zeta)\phi'(\zeta)d\zeta \quad (f \in \mathcal{H}).
\]

Then $(M_\phi f)(z) = (I_\phi f)(z) + (J_\phi f)(z) + \phi(0)f(0)$. It is clear that $I_\phi$ and $J_\phi$ are never invertible.

Put $\mathcal{M}(\mathcal{H}) = \{ \phi \in H(D) : M_\phi \mathcal{H} \subseteq \mathcal{H} \}$, $\mathcal{I}(\mathcal{H}) = \{ \phi \in H(D) : I_\phi \mathcal{H} \subseteq \mathcal{H} \}$ and $\mathcal{J}(\mathcal{H}) = \{ \phi \in H(D) : J_\phi \mathcal{H} \subseteq \mathcal{H} \}$. In this paper, we assume that $H(\overline{D}) \subset \mathcal{M}(\mathcal{H})$, $z \in \mathcal{I}(\mathcal{H})$ and $z \in \mathcal{J}(\mathcal{H})$.

2. Multiplication operator $M_\phi$

When $\mathcal{M}(\mathcal{H}) = H^\infty(D)$, A. Aleman [1] shows a more general result than Corollary 1 without the condition that $(z-a)\mathcal{H}$ is dense.

Lemma 1. If $p$ is a polynomial with no zeros on $\partial D$ then $\dim \mathcal{H}/p\mathcal{H} < \infty$.

Proof If $|a| > 1$ then $(z-a)^{-1} \in H(\overline{D})$ and so $(z-a)^{-1}$ belongs to $M(\mathcal{H})$. Hence we may assume that the zeros of $p$ are contained in $D$. By hypothesis on $\mathcal{H}$, $\dim \mathcal{H}/(z-a)\mathcal{H} = 1$ and so $\dim \mathcal{H}/p\mathcal{H} < \infty$.

Lemma 2. If $M$ is a closed invariant subspace of $M_z$ in $\mathcal{H}$ such that $\dim \mathcal{H}/M < \infty$, then there exists a polynomial $p$ such that $p\mathcal{H} \subseteq M$.

Proof Let $N = \mathcal{H} \ominus M$ and $S_z = P_NM_z|N$, then $S_z$ is of finite rank because $\dim N < \infty$. Hence there exists a polynomial $p$ such that $S_{p(z)} = p(S_z) = 0$. Therefore $pN \subset M$ and so $p\mathcal{H} \subset M$. 

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Theorem 1.

(1) If $\phi = Bg$ where $B$ is a finite Blaschke product, and both $g$ and $g^{-1}$ are in $\mathcal{M}(\mathcal{H})$ then $M_\phi$ is a Fredholm operator.

(2) If $M_\phi$ is a Fredholm operator on $\mathcal{H}$ then $\phi = Bg$ when $B$ is a finite Blaschke product, $g$ is in $\mathcal{M}(\mathcal{H})$ and $g^{-1}$ is in $\mathcal{H}$.

(3) For the $g$ in (2), $M_g$ is a Fredholm operator on $\mathcal{H}$ with index $M_\phi \leq$ index $M_g \leq 0$ and there exists a polynomial $q$ such that $q\mathcal{H} \subseteq g\mathcal{H}$ and the zeros are in $\mathbb{C} \setminus D$.

Proof (1) Suppose $\phi = Bg$, $B = \prod_{j=1}^n(z - a_j)/(1 - \bar{a}_jz)$, $\{a_j\} \subset D$, and both $g$ and $g^{-1}$ are in $\mathcal{M}(\mathcal{H})$. Since $\mathcal{M}(\mathcal{H}) \supseteq H(D)$, $\prod_{j=1}^n(1 - \bar{a}_jz)$ is invertible in $\mathcal{M}(\mathcal{H})$ and so $M_\phi(\mathcal{H}) = p\mathcal{H}$ where $p = \prod_{j=1}^n(z - a_j)$.

(2) If $M_\phi$ is a Fredholm operator then $\dim \mathcal{H}/M_\phi(\mathcal{H}) < \infty$ and so by Lemma 2 there exists a polynomial $p$ such that $\phi f = p$. Therefore $\phi$ can be factorized as $\phi = Bg$ where $B$ is a finite Blaschke product and $g \in \mathcal{H}$. For $\phi \in \mathcal{H}$ and $\prod_{j=1}^n(1 - \bar{a}_jz)\phi = \prod_{j=1}^n(z - a_j)g \in \mathcal{H}$ where $B = \prod_{j=1}^n(z - a_j)/(1 - \bar{a}_jz)$. Since $\text{Ker}(a_j) = (z - a_j)\mathcal{H}$, $g$ belongs to $\mathcal{H}$. By the similar argument, there exists a function $k$ in $\mathcal{H}$ and $gk = 1$ because $Bgf = p$. Thus $g^{-1}$ belongs to $\mathcal{H}$.

We will prove that $g$ belongs to $\mathcal{M}(\mathcal{H})$. Since $B$ is a finite Blaschke product and $\text{Ker}(a_j) = (z - a_j)\mathcal{H}$ for $a \in D$, $\mathcal{H} = K + B\mathcal{H}$ where $K$ is a finite dimensional subspace such that each function in $K$ is a rational function whose poles are in $\mathbb{C} \setminus \bar{D}$. Since $g \in \mathcal{H}$ and $\mathcal{M}(\mathcal{H}) \supseteq H(D)$, $gK \subseteq \mathcal{H}$ and so $g\mathcal{H} \subseteq \mathcal{H}$ because $gB\mathcal{H} \subseteq \mathcal{H}$.

(3) By the proof of (2), $p\mathcal{H} \subseteq Bg\mathcal{H} \subseteq g\mathcal{H}$ and so the first statement is clear. Again by the proof of (2), the zeros of $p$ in $D$ is just the zeros of $B$. This implies that there exists a polynomial $q$ such that $q\mathcal{H} \subseteq g\mathcal{H}$ and $q$ does not have any zeros in $D$.

Corollary 1. Suppose that $(z - a)\mathcal{H}$ is dense in $\mathcal{H}$ whenever $a \in \partial D$. Then $M_\phi$ is a Fredholm operator on $\mathcal{H}$ if and only if $\phi = Bg$ where $B$ is a finite Blaschke product, and both $g$ and $g^{-1}$ are in $\mathcal{M}(\mathcal{H})$.

§ 3. Integral operator $I_\phi$

It seems to have not been studied yet in this general setting as Theorem 2.

Lemma 3. If $\phi$ is a function in $\mathcal{I}(\mathcal{H})$ then $I_\phi(\mathcal{H}) = I_\phi(z\mathcal{H}) \subseteq z\mathcal{H}$. $I_\phi(\mathcal{H}) = z\mathcal{H}$ if and only if $\phi$ and $\phi^{-1}$ belong to $\mathcal{I}(\mathcal{H})$.

Proof By the definition of $I_\phi$ the first statement is clear. We will
show the second one. If both $\phi$ and $\phi^{-1}$ belong to $I(H)$, then

$$zH = I_1(H) = I_{\phi}I_{\phi^{-1}}(H) \subseteq I_{\phi}(zH) \subseteq zH$$

because $I_{\phi}$ and $I_{\phi^{-1}}$ are bounded on $H$. Conversely if $I_{\phi}(H) = zH$ then there exists a function $g$ in $H$ such that

$$\int_0^z g'(\zeta)\phi(\zeta)d\zeta = z$$

and so $g'(z)\phi(z) = 1$.

Hence $\phi^{-1} \in H(D)$ and

$$zH = I_1(H) = I_{\phi^{-1}}I_{\phi}(H) = I_{\phi^{-1}}(zH)$$

and so both $\phi$ and $\phi^{-1}$ belong to $I(H)$.

**Lemma 4.** If $p$ is a polynomial then $I_p(H) + \mathbb{C} \supset p^2H$.

**Proof** Suppose $g \in H$. Since $z \in I(H)$ by the hypothesis, $p$ belongs to $I(H)$ and so $\int_0^z g(\zeta)p(\zeta)d\zeta \in H$. Since $p' \in M(H)$ and $z \in J(H)$,

$$\int_0^z g(\zeta)p'(\zeta)d\zeta \in H.$$ Hence $f(z) = \int_0^z (2p'(\zeta)g(\zeta) + p(\zeta)g'(\zeta))d\zeta \in H.$

Now the lemma follows because

$$\int_0^z f'(\zeta)p(\zeta)d\zeta = \int_0^z (p^2(\zeta)g(\zeta))'d\zeta = p^2(z)g(z) + p^2(0)g(0).$$

**Lemma 5.** Suppose that $B$ is a finite Blaschke product, and both $g$ and $g^{-1}$ are in $I(H)$. If $\phi = Bg$ then $\phi \in I(H)$ and dim $H/I_{\phi}(H) < \infty$.

**Proof** By the hypothesis, $I_B(H) = I_B(zH) = I_B(I_{\phi}(H)) = I_{\phi}(H)$ by Lemma 3. We may assume that

$$B = \prod_{j=1}^n \frac{z - a_j}{1 - \bar{a}_j z} \text{ and } \{a_j\} \subset D.$$ Since $\prod_{j=1}^n (1 - \bar{a}_j z)$ is invertible in $I(H)$, by Lemma 3 $I_{\phi}(H) = I_p(H)$ where $p = \prod_{j=1}^n (z - a_j)$. Lemmas 1 and 4 imply that dim $H/I_{\phi}(H) < \infty$.

**Lemma 6.** If $p$ is a polynomial then $p(S_z) = S_{p(z)}$.

**Proof** By hypothesis, $P^NI_z(I - P^N) = 0$. Hence

$$S_{z^2} = P^NI_{z^2}P^N = P^NI_zP^N$$

$$= P^NI_z(I - P^N)I_zP^N + P^NI_zP^NP^NI_zP^N$$

$$= P^NI_zP^NI_zP^N = S_zS_{z^2}.$$
Now it is easy to see that \( p(S_z) = S_{p(z)} \) for a polynomial \( p \).

**Lemma 7.** If \( M \) is a closed invariant subspace of \( I_z \) and \( \dim \mathcal{H}/M = n < \infty \) then there exists a polynomial \( p \) such that the degree of \( p \leq n \) and \( I_p(\mathcal{H}) \subseteq M \).

**Proof** If we put \( N = \mathcal{H} \ominus M \), then \( \dim N = n < \infty \) and so there exists a polynomial \( p \) such that \( p(S_z) = 0 \) and the degree of \( p \leq n \). By Lemma 6, \( S_{p(z)} = 0 \) and so \( I_p(N) \subseteq M \). Since \( I_p(M) \subseteq M \), \( I_p(\mathcal{H}) \subseteq M \).

**Theorem 2.** Suppose \( \mathcal{I}(\mathcal{H}) \) contains \( H(\bar{D}) \) and if \( f \in \mathcal{I}(\mathcal{H}) \) and \( f(a) = 0 \) for some \( a \in D \) then \( f/(z-a) \) belongs to \( \mathcal{I}(\mathcal{H}) \). \( I_{\phi} \) is a Fredholm operator on \( \mathcal{H} \) if and only if \( \phi = Bg \) where \( B \) is a finite Blaschke product, and \( g \) and \( g^{-1} \) are in \( \mathcal{I}(\mathcal{H}) \).

**Proof** If \( \phi = Bg \), \( B \) is a finite Blaschke product, \( g \in \mathcal{I}(\mathcal{H}) \) and \( g^{-1} \in \mathcal{I}(\mathcal{H}) \) then by Lemma 5 \( I_{\phi}(\mathcal{H}) \) is closed and \( \dim \ker I_{\phi} < \infty \). Since \( \ker I_{\phi} = \mathbb{C} \), index \( I_{\phi} = 1 - \dim \ker I_{\phi} \) and so \( I_{\phi} \) is Fredholm. Conversely if \( I_{\phi} \) is Fredholm then \( I_{\phi}(\mathcal{H}) \) is closed and \( \dim \mathcal{H}/I_{\phi}(\mathcal{H}) < \infty \). Since \( I_z I_{\phi}(\mathcal{H}) \subseteq I_{\phi}(\mathcal{H}) \), by Lemma 7 there exists a polynomial such that \( I_p(\mathcal{H}) \subseteq I_{\phi}(\mathcal{H}) \). By Lemma 4 \( I_p(\mathcal{H}) + \mathbb{C} \supset p^2 \mathcal{H} \). Hence there exists a function \( F \) in \( \mathcal{H} \) and \( c \in \mathbb{C} \) such that \( I_{\phi}(F) + c = p^2 \). Therefore \( F'(z)\phi(z) = 2p(z)\phi'(z) \) and so the Blaschke part of \( \phi \) is a finite one \( B \). Thus \( \phi \) can be factorized as \( \phi = Bg \) where \( g \in \mathcal{I}(\mathcal{H}) \) and \( g \) has no zeros on \( D \) because \( \mathcal{I}(\mathcal{H}) \) is a subalgebra in \( \mathcal{B}(\mathcal{H}) \) and both \( B \) and \( B^{-1} \) are in \( \mathcal{I}(\mathcal{H}) \). Hence

\[
I_{g^{-1}p}(\mathcal{H}) \subseteq I_{g^{-1}I_{\phi}}(\mathcal{H}) = I_B(\mathcal{H}) \subseteq \mathcal{H}
\]

and so \( g^{-1}p \) belongs to \( \mathcal{I}(\mathcal{H}) \). By hypothesis on \( \mathcal{I}(\mathcal{H}) \), \( g^{-1} \) belongs to \( \mathcal{I}(\mathcal{H}) \).

§ 4. **Integral operator** \( J_{\phi} \)

A Fredholm integral operator \( J_{\phi} \) have not studied. But if \( J_{\phi} \) is compact then it is not Fredholm. In some special Hilbert space \( \mathcal{H} \), the compactness of \( J_{\phi} \) have studied.

**Lemma 8.** If \( \phi \) and \( \psi \) are in \( H(D) \) then \( I_{\psi} J_{\phi} = J_{\phi} M_{\psi} \).

**Proof** For \( f \in \mathcal{H} \)

\[
(I_{\psi} J_{\phi} f)(z) = \int_0^z (J_{\phi} f)'(\zeta) \psi(\zeta) d\zeta = \int_0^z f(\zeta) \phi'(\zeta) \psi(\zeta) d\zeta = (J_{\phi} M_{\psi} f)(z)
\]
Lemma 9. If $J_\phi$ is a Fredholm operator on $\mathcal{H}$ then $J_\phi \mathcal{H}$ is a closed invariant subspace of $I_\phi$ and $\dim \mathcal{H}/J_\phi \mathcal{H} < \infty$. Hence there exists a polynomial $p$ such that $J_\phi \mathcal{H} \supseteq I_p \mathcal{H}$ and so $J_\phi \mathcal{H} + \mathbb{C} \supseteq p^2 \mathcal{H}$.

Proof If $J_\phi$ is Fredholm on $\mathcal{H}$ then $J_\phi \mathcal{H}$ is a closed subspace and by Lemma 8 $I_z(J_\phi \mathcal{H}) \subseteq J_\phi \mathcal{H}$. By Lemma 7 there exists a polynomial $q$ such that $I_q \mathcal{H} \subseteq J_\phi \mathcal{H}$. Lemma 4 implies this lemma.

Theorem 3. Suppose that there exists a function $g$ in $\mathcal{H}$ such that $g'$ does not belong to $H^2$. Suppose that any function in $\mathcal{H}$ has radial limits almost everywhere. Then there does not exist $J_\phi$ which is a Fredholm operator on $\mathcal{H}$.

Proof If $J_\phi$ is Fredholm on $\mathcal{H}$ then $J_\phi \mathcal{H} + \mathbb{C} \supseteq p^2 \mathcal{H}$ for some polynomial by Lemma 9. For any $G$ in $p^2 \mathcal{H}$ there exists a function $f$ in $\mathcal{H}$ such that $f(z) = G'(z)$ $(z \in D)$.

By hypothesis, there exists $G$ in $p^2 \mathcal{H}$ such that $G' \notin H^2$ and so $G'$ does not have radial limits on a set of positive measure on $\partial D$ (see [2, Appendix A]). On the other hand, if $G = p^2$ then $G$ has radial limits almost everywhere on $\partial D$. By hypothesis, $f$ has radial limits almost everywhere. This contradiction implies that $J_\phi$ is not Fredholm.

§ 5. Relation between $M_\phi$ and $I_\phi$

Put $Df(z) = f'(z)$ and $J = J_z$, that is, $Jf(z) = \int_0^z f(\zeta) d\zeta$. Then

$$DJf = f \quad \text{and} \quad JDf = f - f(0).$$

It is easy to see that $I_\phi J = JM_\phi$ and $DI_\phi = M_\phi D$. Put

$$\mathcal{H}^D = \{ f \in H(D) : Df \in \mathcal{H} \}$$

Suppose that $D$ and $J$ are bounded on $\mathcal{H}$ and for $f$ in $\mathcal{H}^D$ put $\|f\|^2_D = \|Df\|^2 + |f(0)|^2$. Then $\mathcal{H}^D$ is a Hilbert space. Put

$$\mathcal{H}^J = \{ f \in H(D) : Jf \in \mathcal{H} \}$$

and for $f$ in $\mathcal{H}^J$ $\|f\|_J = \|Jf\|$. Then $\mathcal{H}^J$ is a Hilbert space.

$D$ is isometric from $\mathcal{H}^D_0 = \{ f \in \mathcal{H}^D : f(0) = 0 \}$ onto $\mathcal{H}$. $J$ is isometric from $\mathcal{H}^J$ onto $\mathcal{H}_0 = \{ f \in \mathcal{H} : f(0) = 0 \}$. Since $DI_\phi = M_\phi D$, $I_\phi$ is bounded on $\mathcal{H}^D$ if and only if $M_\phi$ is bounded on $\mathcal{H}$. Hence $I(\mathcal{H}^D) = M(\mathcal{H})$. Moreover $I_\phi$ is Fredholm on $\mathcal{H}^D$ if and only if $M_\phi$ is Fredholm on $\mathcal{H}$. Since $JM_\phi = I_\phi J$, 

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\( \mathcal{I}(\mathcal{H}^J) = \mathcal{M}(\mathcal{H}), \) and \( I_\phi \) is Fredholm on \( \mathcal{H}^J \) if and only if \( M_\phi \) is Fredholm on \( \mathcal{H} \). Moreover \( (\mathcal{H}^J)^D = (\mathcal{H}^D)^J = \mathcal{H} \). Hence \( \mathcal{I}(\mathcal{H}) = \mathcal{M}(\mathcal{H}^D) = \mathcal{M}(\mathcal{H}^J) \), and \( I_\phi \) is Fredholm on \( \mathcal{H} \) if and only if \( M_\phi \) is Fredholm on \( \mathcal{H}^D \) and \( \mathcal{H}^J \).

§ 6. Examples

Let \( dA \) denote the normalized Lebesgue area measure on \( D \) and \( \omega \) a positive function on \( D \) which is summable with respect to \( dA \). Put

\[
\mathcal{D}^2(\omega) = \{ f \in H(D) : \| f \|_{2,\omega}^2 = |f(0)|^2 + \int_D |f'(z)|^2 \omega(z) dA(z) < \infty \}
\]

and

\[
L^2_a(\omega) = \{ f \in H(D) : \| f \|_{L^2_a(\omega)}^2 = \int_D |f(z)|^2 \omega(z) dA(z) < \infty \}.
\]

Then \( \mathcal{D}^2(\omega) \) is called a weighted Dirichlet space and \( L^2_a(\omega) \) is called a weighted Bergman space when \( \mathcal{D}^2(\omega) \) and \( L^2_a(\omega) \) are nontrivial Hilbert spaces. It is easy to see that \( (\mathcal{D}^2(\omega))^D = L^2_a(\omega) \) and \( (L^2_a(\omega))^D = \mathcal{D}^2(\omega) \).

If \( \omega(z) = (1 - |z|^2)^\alpha \) and \( \alpha > -1 \), we will write \( \mathcal{D}^2(\omega) = \mathcal{D}^2 \) and \( L^2_a(\omega) = L^2_a \). It is known that \( \mathcal{D}^2 \) and \( L^2_a \) are nontrivial Hilbert spaces. \( \mathcal{D}_1 \) is the Hardy space \( H^2 \), \( \mathcal{D}_2 \) is the Bergman space \( L^2_a \) and \( \mathcal{D}_0 \) is the Dirichlet space.

If \( \mathcal{H} = \mathcal{D}_a \) or \( L^2_a \), then \( \mathcal{H} \) satisfies the condition (1), (2) and (3) in Introduction. It is known that \( H(D) \subset \mathcal{M}(\mathcal{D}_a) \subset H^\infty(D) \) and \( \mathcal{M}(L^2_a) = H^\infty(D) \). Hence Theorem 1 can apply to \( \mathcal{D}_a \) for any \( \alpha > -1 \). If \( \alpha \geq 1 \) then \( (z - a)^\alpha \mathcal{D}_a \) is dense in \( \mathcal{D}_a \) whenever \( a \in \partial D \). Hence Corollary 1 can apply to \( \mathcal{D}_a \) for \( \alpha \geq 1 \).

\( \mathcal{I}(L^2_{a,\alpha}) = \mathcal{M}((L^2_{a,\alpha})^D) = \mathcal{M}(\mathcal{D}_a) \) and \( H(D) \subset \mathcal{M}(\mathcal{D}_a) \subset H^\infty(D) \). Since \( \mathcal{I}(\mathcal{D}_a) = \mathcal{M}(L^2_{a,\alpha}) = H^\infty(D) \), Theorem 2 can apply to \( \mathcal{D}_a \) for \( \alpha > -1 \). It is known [3] that \( \mathcal{M}(\mathcal{D}_a) = H^\infty(D) \) for \( \alpha > 1 \) and \( \mathcal{M}(\mathcal{D}_a) = \mathcal{D}_a \) for \(-1 < \alpha < 0 \). Hence \( \mathcal{I}(L^2_{a,\alpha}) = H^\infty(D) \) for \( \alpha > 1 \) and \( \mathcal{I}(L^2_{a,\alpha}) = \mathcal{D}_a \) for \(-1 < \alpha < 0 \). Hence Theorem 2 can apply to \( L^2_{a,\alpha} \) for \( \alpha > 1 \) and \(-1 < \alpha < 0 \). By a theorem in [3], it is easy to see that \( \mathcal{I}(L^2_{a,\alpha}) = \mathcal{M}(\mathcal{D}_a) \) \( (0 \leq \alpha \leq 1) \) satisfies the conditions in Theorem 2. Hence Theorem 2 can apply to \( L^2_{a,\alpha} \).

When \( \mathcal{D}^2(\omega) \) or \( L^2_a(\omega) \) is a Hilbert space \( \mathcal{H} \), it is important in order to study composition operator that \( \mathcal{H} \) satisfies three conditions in Introduction. It will be interesting to determine such a weight \( \omega \).

References


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