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Some Fredholm Integration Operators on A Hilbert Space of Holomorphic Functions on The Unit Disc

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Abstract. In this paper, we study when $M_\phi$, $I_\phi$ or $J_\phi$ is a Fredholm operator on a Hilbert space which satisfies few natural axioms.
§ 1. Introduction

Let $D$ be the open unit disc in the complex plane $\mathbb{C}$ and $H(D)$ be the set of all analytic functions on $D$. $H(\bar{D})$ denotes the set of all analytic functions on $\bar{D}$. In this paper, $\mathcal{H}$ is a Hilbert space in $H(D)$ which satisfies the following:

(1) $z\mathcal{H} \subset \mathcal{H}$.
(2) If $a \in D$ then $(z-a)\mathcal{H} \oplus \mathbb{C} = \mathcal{H}$.
(3) $\mathcal{H} \supset H(\bar{D})$.

In this paper, we study the following three operators. If $\phi$ is a function in $H(D)$, put for $z \in D$,

$$(M_\phi f)(z) = \phi(z)f(z),$$

$$(I_\phi f)(z) = \int_0^z f'(\zeta)\phi(\zeta) d\zeta,$$

$$(J_\phi f)(z) = \int_0^z f(\zeta)\phi'(\zeta) d\zeta \quad (f \in \mathcal{H}).$$

Then $(M_\phi f)(z) = (I_\phi f)(z) + (J_\phi f)(z) + \phi(0)f(0)$. It is clear that $I_\phi$ and $J_\phi$ are never invertible.

Put $\mathcal{M}(\mathcal{H}) = \{\phi \in H(D) : M_\phi \mathcal{H} \subset \mathcal{H}\}$, $\mathcal{I}(\mathcal{H}) = \{\phi \in H(D) : I_\phi \mathcal{H} \subset \mathcal{H}\}$ and $\mathcal{J}(\mathcal{H}) = \{\phi \in H(D) : J_\phi \mathcal{H} \subset \mathcal{H}\}$. In this paper, we assume that $H(\bar{D}) \subset \mathcal{M}(\mathcal{H})$, $z \in \mathcal{I}(\mathcal{H})$ and $z \in \mathcal{J}(\mathcal{H})$.

§ 2. Multiplication operator $M_\phi$

When $\mathcal{M}(\mathcal{H}) = H^\infty(D)$, A. Aleman [1] shows a more general result than Corollary 1 without the condition that $(z-a)\mathcal{H}$ is dense.

Lemma 1. If $p$ is a polynomial with no zeros on $\partial D$ then $\dim \mathcal{H}/p\mathcal{H} < \infty$.

Proof If $|a| > 1$ then $(z-a)^{-1} \in H(\bar{D})$ and so $(z-a)^{-1}$ belongs to $M(\mathcal{H})$. Hence we may assume that the zeros of $p$ are contained in $D$. By hypothesis on $\mathcal{H}$, $\dim \mathcal{H}/(z-a)\mathcal{H} = 1$ and so $\dim \mathcal{H}/p\mathcal{H} < \infty$.

Lemma 2. If $M$ is a closed invariant subspace of $M_z$ in $\mathcal{H}$ such that $\dim \mathcal{H}/M < \infty$, then there exists a polynomial $p$ such that $p\mathcal{H} \subseteq M$.

Proof Let $N = \mathcal{H} \ominus M$ and $S_z = P_N M_z|N$, then $S_z$ is of finite rank because $\dim N < \infty$. Hence there exists a polynomial $p$ such that $S_{p(z)} = p(S_z) = 0$. Therefore $pN \subset M$ and so $p\mathcal{H} \subset M$.  

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Theorem 1.

(1) If $φ = Bg$ where $B$ is a finite Blaschke product, and both $g$ and $g^{-1}$ are in $\mathcal{M}(\mathcal{H})$ then $M_φ$ is a Fredholm operator.

(2) If $M_φ$ is a Fredholm operator on $\mathcal{H}$ then $φ = Bg$ when $B$ is a finite Blaschke product, $g$ is in $\mathcal{M}(\mathcal{H})$ and $g^{-1}$ is in $\mathcal{H}$.

(3) For the $g$ in (2), $M_g$ is a Fredholm operator on $\mathcal{H}$ with index $M_φ$ ≤ index $M_g ≤ 0$ and there exists a polynomial $q$ such that $q\mathcal{H} ⊆ g\mathcal{H}$ and the zeros are in $\mathbb{C}\setminus D$.

Proof (1) Suppose $φ = Bg$, $B = \prod_{j=1}^{n}(z - a_j)/(1 - \bar{a}_j z)$, $\{a_j\} \subset D$, and both $g$ and $g^{-1}$ are in $\mathcal{M}(\mathcal{H})$. Since $\mathcal{M}(\mathcal{H}) \supseteq H(\bar{D})$, $\prod_{j=1}^{n}(1 - \bar{a}_j z)$ is invertible in $\mathcal{M}(\mathcal{H})$ and so $M_φ(\mathcal{H}) = p\mathcal{H}$ where $p = \prod_{j=1}^{n}(z - a_j)$.

(2) If $M_φ$ is a Fredholm operator then $\dim \mathcal{H}/M_φ(\mathcal{H}) < \infty$ and so by Lemma 2 there exists a polynomial $p$ such that $φf = p$. Therefore $φ$ can be factorized as $φ = Bg$ where $B$ is a finite Blaschke product and $g \in \mathcal{H}$. For $φ \in \mathcal{H}$ and $\prod_{j=1}^{n}(1 - \bar{a}_j z)φ = \prod_{j=1}^{n}(z - a_j)g \in \mathcal{H}$ where $B = \prod_{j=1}^{n}(z - a_j)/(1 - \bar{a}_j z)$. Since $\ker τ_{a_j} = (z - a_j)\mathcal{H}$, $g$ belongs to $\mathcal{H}$. By the similar argument, there exists a function $k$ in $\mathcal{H}$ and $gk = 1$ because $Bgf = p$. Thus $g^{-1}$ belongs to $\mathcal{H}$. We will prove that $g$ belongs to $\mathcal{M}(\mathcal{H})$. Since $B$ is a finite Blaschke product and $\ker τ_a = (z - a)\mathcal{H}$ for $a \in D$, $\mathcal{H} = K + B\mathcal{H}$ where $K$ is a finite dimensional subspace such that each function in $K$ is a rational function whose poles are in $\mathbb{C}\setminus \bar{D}$. Since $g \in \mathcal{H}$ and $\mathcal{M}(\mathcal{H}) \supseteq H(\bar{D})$, $gK \subseteq \mathcal{H}$ and so $g\mathcal{H} \subseteq \mathcal{H}$ because $gB\mathcal{H} \subseteq \mathcal{H}$.

(3) By the proof of (2), $p\mathcal{H} \subseteq Bg\mathcal{H} \subseteq g\mathcal{H}$ and so the first statement is clear. Again by the proof of (2), the zeros of $p$ in $D$ is just the zeros of $B$. This implies that there exists a polynomial $q$ such that $q\mathcal{H} \subseteq g\mathcal{H}$ and $q$ does not have any zeros in $D$.

Corollary 1. Suppose that $(z - a)\mathcal{H}$ is dense in $\mathcal{H}$ whenever $a \in \partial D$. Then $M_φ$ is a Fredholm operator on $\mathcal{H}$ if and only if $φ = Bg$ where $B$ is a finite Blaschke product, and both $g$ and $g^{-1}$ are in $\mathcal{M}(\mathcal{H})$.

§ 3. Integral operator $I_φ$

It seems to have not been studied yet in this general setting as Theorem 2.

Lemma 3. If $φ$ is a function in $\mathcal{I}(\mathcal{H})$ then $I_φ(\mathcal{H}) = I_φ(z\mathcal{H}) \subseteq z\mathcal{H}$. $I_φ(\mathcal{H}) = z\mathcal{H}$ if and only if $φ$ and $φ^{-1}$ belongs to $\mathcal{I}(\mathcal{H})$.

Proof By the definition of $I_φ$ the first statement is clear. We will
show the second one. If both $\phi$ and $\phi^{-1}$ belong to $\mathcal{I}(\mathcal{H})$, then

$$z\mathcal{H} = I_1(\mathcal{H}) = I_\phi I_{\phi^{-1}}(\mathcal{H}) \subseteq I_\phi(z\mathcal{H}) \subseteq z\mathcal{H}$$

because $I_\phi$ and $I_{\phi^{-1}}$ are bounded on $\mathcal{H}$. Conversely if $I_\phi(\mathcal{H}) = z\mathcal{H}$ then there exists a function $g$ in $\mathcal{H}$ such that

$$\int_0^z g(\zeta)\phi(\zeta)d\zeta = z$$

and so $g'(z)\phi(z) = 1$.

Hence $\phi^{-1} \in H(D)$ and

$$z\mathcal{H} = I_1(\mathcal{H}) = I_{\phi^{-1}} I_\phi(\mathcal{H}) = I_{\phi^{-1}}(z\mathcal{H})$$

and so both $\phi$ and $\phi^{-1}$ belong to $\mathcal{I}(\mathcal{H})$.

**Lemma 4.** If $p$ is a polynomial then $I_p(\mathcal{H}) + \mathbb{C} \supseteq p^2\mathcal{H}$.

**Proof** Suppose $g \in \mathcal{H}$. Since $z \in \mathcal{I}(\mathcal{H})$ by the hypothesis, $p$ belongs to $\mathcal{I}(\mathcal{H})$ and so $\int_0^z g(\zeta)p(\zeta)d\zeta \in \mathcal{H}$. Since $p' \in \mathcal{M}(\mathcal{H})$ and $z \in J(\mathcal{H})$, $\int_0^z g(\zeta)p'(\zeta)d\zeta$ belongs to $\mathcal{H}$. Hence $f(z) = \int_0^z (2p'(\zeta)g(\zeta) + p(\zeta)g'(\zeta))d\zeta$ belongs to $\mathcal{H}$. Now the lemma follows because

$$\int_0^z f'(\zeta)p(\zeta)d\zeta = \int_0^z (p^2(\zeta)g(\zeta))'d\zeta = p^2(z)g(z) + p^2(0)g(0).$$

**Lemma 5.** Suppose that $B$ is a finite Blaschke product, and both $g$ and $g^{-1}$ are in $\mathcal{I}(\mathcal{H})$. If $\phi = Bg$ then $\phi \in \mathcal{I}(\mathcal{H})$ and $\dim \mathcal{H}/I_\phi(\mathcal{H}) < \infty$.

**Proof** By the hypothesis, $I_B(\mathcal{H}) = I_B(z\mathcal{H}) = I_B(I_\phi(\mathcal{H})) = I_\phi(\mathcal{H})$ by Lemma 3. We may assume that

$$B = \prod_{j=1}^n \frac{z - a_j}{1 - \bar{a}_j z} \text{ and } \{a_j\} \subset D.$$ 

Since $\prod_{j=1}^n (1 - \bar{a}_j z)$ is invertible in $\mathcal{I}(\mathcal{H})$, by Lemma 3 $I_\phi(\mathcal{H}) = I_p(\mathcal{H})$ where $p = \prod_{j=1}^n (z - a_j)$. Lemmas 1 and 4 imply that $\dim \mathcal{H}/I_\phi(\mathcal{H}) < \infty$.

**Lemma 6.** If $p$ is a polynomial then $p(S_z) = S_p(z)$.

**Proof** By hypothesis, $P^N I_z (I - P^N) = 0$. Hence

$$S_{z^2} = P^N I_{z^2} P^N = P^N I_z I_z P^N$$

$$= P^N I_z (I - P^N) I_z P^N + P^N I_z P^N I_z P^N$$

$$= P^N I_z P^N I_z P^N = S_z S_z.$$
Now it is easy to see that \( p(S_\zeta) = S_{p(z)} \) for a polynomial \( p \).

**Lemma 7.** If \( M \) is a closed invariant subspace of \( I_\zeta \) and \( \dim H/M = n < \infty \) then there exists a polynomial \( p \) such that the degree of \( p \leq n \) and \( I_p(H) \subseteq M \).

**Proof** If we put \( N = H \ominus M \), then \( \dim N = n < \infty \) and so there exists a polynomial \( p \) such that \( p(S_\zeta) = 0 \) and the degree of \( p \leq n \). By Lemma 6, \( S_{p(z)} = 0 \) and so \( I_p(N) \subseteq M \). Since \( I_p(M) \subseteq M \), \( I_p(H) \subseteq M \).

**Theorem 2.** Suppose \( I(H) \) contains \( H(\bar{D}) \) and if \( f \in I(H) \) and \( f(a) = 0 \) for some \( a \in D \) then \( f/(z-a) \) belongs to \( I(H) \). \( I_\phi \) is a Fredholm operator on \( H \) if and only if \( \phi = Bg \) where \( B \) is a finite Blaschke product, and \( g \) and \( g^{-1} \) are in \( I(H) \).

**Proof** If \( \phi = Bg \), \( B \) is a finite Blaschke product, \( g \in I(H) \) and \( g^{-1} \in I(H) \) then by Lemma 5 \( I_\phi(H) \) is closed and \( \dim \ker I_\phi < \infty \). Since \( \ker I_\phi = C \), index \( I_\phi = 1 - \dim \ker I_\phi \) and so \( I_\phi \) is Fredholm. Conversely if \( I_\phi \) is Fredholm then \( I_\phi(H) \) is closed and \( \dim H/\ker I_\phi < \infty \). Since \( I_\zeta I_\phi(H) \subseteq I_\phi(H) \), by Lemma 7 there exists a polynomial such that \( I_p(H) \subseteq I_\phi(H) \). By Lemma 4 \( I_p(H) + C \supseteq p^2H \). Hence there exists a function \( F \) in \( H \) and \( c \in C \) such that \( I_\phi(F) + c = p^2 \). Therefore \( F'(z)\phi(z) = 2p(z)p'(z) \) and so the Blaschke part of \( \phi \) is a finite one \( B \). Thus \( \phi \) can be factorized as \( \phi = Bg \) where \( g \in I(H) \) and \( g \) has no zeros on \( D \) because \( I(H) \) is a subalgebra in \( B(H) \) and both \( B \) and \( B^{-1} \) are in \( I(H) \). Hence

\[
I_{g^{-1}p}(H) \subseteq I_{g^{-1}}I_\phi(H) = I_B(H) \subseteq H
\]

and so \( g^{-1}p \) belongs to \( I(H) \). By hypothesis on \( I(H) \), \( g^{-1} \) belongs to \( I(H) \).

§ 4. Integral operator \( J_\phi \)

A Fredholm integral operator \( J_\phi \) have not studied. But if \( J_\phi \) is compact then it is not Fredholm. In some special Hilbert space \( H \), the compactness of \( J_\phi \) have studied.

**Lemma 8.** If \( \phi \) and \( \psi \) are in \( H(D) \) then \( I_\psi J_\phi = J_\phi M_\psi \).

**Proof** For \( f \in H \)

\[
(I_\psi J_\phi f)(z) = \int_0^z (J_\phi f)'(\zeta)\psi(\zeta)d\zeta = \int_0^z f(\zeta)\phi'(\zeta)\psi(\zeta)d\zeta = (J_\phi M_\psi f)(z)
\]
Lemma 9. If \( J_{\phi} \) is a Fredholm operator on \( \mathcal{H} \) then \( J_{\phi} \mathcal{H} \) is a closed invariant subspace of \( I_{z} \) and \( \dim \mathcal{H}/J_{\phi} \mathcal{H} < \infty \). Hence there exists a polynomial \( p \) such that \( J_{\phi} \mathcal{H} \supseteq I_{p} \mathcal{H} \) and so \( J_{\phi} \mathcal{H} + \mathbb{C} \supseteq p^{2} \mathcal{H} \).

Proof If \( J_{\phi} \) is Fredholm on \( \mathcal{H} \) then \( J_{\phi} \mathcal{H} \) is a closed subspace and by Lemma 8 \( I_{z}(J_{\phi} \mathcal{H}) \subseteq J_{\phi} \mathcal{H} \). By Lemma 7 there exists a polynomial \( q \) such that \( I_{q} \mathcal{H} \subseteq J_{\phi} \mathcal{H} \). Lemma 4 implies this lemma.

Theorem 3. Suppose that there exists a function \( g \) in \( \mathcal{H} \) such that \( g' \) does not belong to \( H^{2} \). Suppose that any function in \( \mathcal{H} \) has radial limits almost everywhere. Then there does not exist \( J_{\phi} \) which is a Fredholm operator on \( \mathcal{H} \).

Proof If \( J_{\phi} \) is Fredholm on \( \mathcal{H} \) then \( J_{\phi} \mathcal{H} + \mathbb{C} \supseteq p^{2} \mathcal{H} \) for some polynomial by Lemma 9. For any \( G \) in \( p^{2} \mathcal{H} \) there exists a function \( f \) in \( \mathcal{H} \) such that \( f(z)\phi'(z) = G'(z) \) \( (z \in D) \).

By hypothesis, there exists \( G \) in \( p^{2} \mathcal{H} \) such that \( G' \notin H^{2} \) and so \( G' \) does not have radial limits on a set of positive measure on \( \partial D \) (see [2, Appendix A]). On the other hand, if \( G = p^{2} \) then \( G \) has radial limits almost everywhere on \( \partial D \). By hypothesis, \( f \) has radial limits almost everywhere. This contradiction implies that \( J_{\phi} \) is not Fredholm.

§ 5. Relation between \( M_{\phi} \) and \( I_{\phi} \)

Put \( Df(z) = f'(z) \) and \( J = J_{z} \), that is, \( Jf(z) = \int_{0}^{z} f(\zeta) d\zeta \). Then

\[ DJf = f \quad \text{and} \quad JDf = f - f(0). \]

It is easy to see that \( I_{\phi} J = JM_{\phi} \) and \( DI_{\phi} = M_{\phi} D \). Put

\[ \mathcal{H}^{D} = \{ f \in H(D) : Df \in \mathcal{H} \} \]

Suppose that \( D \) and \( J \) are bounded on \( \mathcal{H} \) and for \( f \) in \( \mathcal{H}^{D} \) put \( \| f \|_{D}^{2} = \| Df \|^{2} + |f(0)|^{2} \). Then \( \mathcal{H}^{D} \) is a Hilbert space. Put

\[ \mathcal{H}^{J} = \{ f \in H(D) : Jf \in \mathcal{H} \} \]

and for \( f \) in \( \mathcal{H}^{J} \) \( \| f \|_{J} = \| Jf \| \). Then \( \mathcal{H}^{J} \) is a Hilbert space.

\( D \) is isometric from \( \mathcal{H}_{0}^{D} = \{ f \in \mathcal{H}^{D} : f(0) = 0 \} \) onto \( \mathcal{H} \). \( J \) is isometric from \( \mathcal{H}^{J} \) onto \( \mathcal{H}_{0} = \{ f \in \mathcal{H} : f(0) = 0 \} \). Since \( DI_{\phi} = M_{\phi} D \), \( I_{\phi} \) is bounded on \( \mathcal{H}^{D} \) if and only if \( M_{\phi} \) is bounded on \( \mathcal{H} \). Hence \( I(\mathcal{H}^{D}) = \mathcal{M}(\mathcal{H}) \). Moreover \( I_{\phi} \) is Fredholm on \( \mathcal{H}^{D} \) if and only if \( M_{\phi} \) is Fredholm on \( \mathcal{H} \). Since \( JM_{\phi} = I_{\phi} J \),
\[ I(\mathcal{H}^J) = \mathcal{M}(\mathcal{H}), \] and \( I_\phi \) is Fredholm on \( \mathcal{H}^J \) if and only if \( M_\phi \) is Fredholm on \( \mathcal{H} \). Moreover \( (\mathcal{H}^J)^D = (\mathcal{H}^D)^J = \mathcal{H} \). Hence \( I(\mathcal{H}) = \mathcal{M}(\mathcal{H}^D) = \mathcal{M}(\mathcal{H}^J) \), and \( I_\phi \) is Fredholm on \( \mathcal{H} \) if and only if \( M_\phi \) is Fredholm on \( \mathcal{H}^D \) and \( \mathcal{H}^J \).

§ 6. Examples

Let \( dA \) denote the normalized Lebesgue area measure on \( D \) and \( \omega \) a positive function on \( D \) which is summable with respect to \( dA \). Put

\[ D^2(\omega) = \{ f \in H(D) : \| f \|^2_D = |f(0)|^2 + \int_D |f'(z)|^2 \omega(z) dA(z) < \infty \} \]

and

\[ L^2_\alpha(\omega) = \{ f \in H(D) : \| f \|^2_D = \int_D |f(z)|^2 \omega(z) dA(z) < \infty \}. \]

Then \( D^2(\omega) \) is called a weighted Dirichlet space and \( L^2_\alpha(\omega) \) is called a weighted Bergman space when \( D^2(\omega) \) and \( L^2_\alpha(\omega) \) are nontrivial Hilbert spaces. It is easy to see that \( (D^2(\omega))^J = L^2_\alpha(\omega) \) and \( (L^2_\alpha(\omega))^D = D^2(\omega) \).

If \( \omega(z) = (1 - |z|^2)^\alpha \) and \( \alpha > -1 \), we will write \( D^2(\omega) = D^2_\alpha \) and \( L^2_\alpha(\omega) = L^2_\alpha \). It is known that \( D^2_\alpha \) and \( L^2_\alpha \) are nontrivial Hilbert spaces. \( D_1 \) is the Hardy space \( H^2 \), \( D_2 \) is the Bergman space \( L^2_0 \) and \( D_0 \) is the Dirichlet space. If \( \mathcal{H} = D_\alpha \) or \( L^2_\alpha \) then \( \mathcal{H} \) satisfies the condition (1), (2) and (3) in Introduction. It is known that \( H(D) \subset \mathcal{M}(D_\alpha) \subset H^\infty(D) \) and \( \mathcal{M}(L^2_\alpha) = H^\infty(D) \). Hence Theorem 1 can apply to \( D_\alpha \) for any \( \alpha > -1 \). If \( \alpha \geq 1 \) then \( (z - a)^\alpha \) is dense in \( D_\alpha \) whenever \( a \in \partial D \). Hence Corollary 1 can apply to \( D_\alpha \) for \( \alpha \geq 1 \). \( I(L^2_\alpha) = \mathcal{M}(L^2_\alpha)^D = \mathcal{M}(D_\alpha) \) and \( (H(D) \subset \mathcal{M}(D_\alpha) \subset H^\infty(D) \). Since \( I(D_\alpha) = \mathcal{M}(D_\alpha)^J = H^\infty(D) \), Theorem 2 can apply to \( D_\alpha \) for \( \alpha > -1 \). It is known [3] that \( \mathcal{M}(D_\alpha) = H^\infty(D) \) for \( \alpha > 1 \) and \( \mathcal{M}(D_\alpha) = D_\alpha \) for \( -1 < \alpha < 0 \). Hence \( I(L^2_\alpha) = H^\infty(D) \) for \( \alpha > 1 \) and \( I(L^2_\alpha) = D_\alpha \) for \( -1 < \alpha < 0 \). Hence Theorem 2 can apply to \( L^2_\alpha \) for \( \alpha > 1 \) and \( -1 < \alpha < 0 \). By a theorem in [3], it is easy to see that \( I(L^2_\alpha) = \mathcal{M}(D_\alpha) \) \( (0 \leq \alpha \leq 1) \) satisfies the conditions in Theorem 2. Hence Theorem 2 can apply to \( L^2_\alpha \).

When \( D^2(\omega) \) or \( L^2_\alpha(\omega) \) is a Hilbert space \( \mathcal{H} \), it is important in order to study composition operator that \( \mathcal{H} \) satisfies three conditions in Introduction. It will be interesting to determine such a weight \( \omega \).

References


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