Some Fredholm Integration Operators on A Hilbert Space of Holomorphic Functions on The Unit Disc

By

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Abstract. In this paper, we study when $M_\phi$, $I_\phi$ or $J_\phi$ is a Fredholm operator on a Hilbert space which satisfies few natural axioms.
§ 1. Introduction
Let $D$ be the open unit disc in the complex plane $\mathbb{C}$ and $H(D)$ be the set of all analytic functions on $D$. $H(\bar{D})$ denotes the set of all analytic functions on $\bar{D}$. In this paper, $\mathcal{H}$ is a Hilbert space in $H(D)$ which satisfies the following:

1. $z \mathcal{H} \subset \mathcal{H}$.
2. If $a \in D$ then $(z - a)\mathcal{H} \oplus \mathbb{C} = \mathcal{H}$.
3. $\mathcal{H} \supseteq H(\bar{D})$.

In this paper, we study the following three operators. If $\phi$ is a function in $H(D)$, put for $z \in D$,

\begin{align*}
(M_\phi f)(z) &= \phi(z)f(z), \\
(I_\phi f)(z) &= \int_0^z f'(\zeta)\phi(\zeta)d\zeta, \\
(J_\phi f)(z) &= \int_0^z f(\zeta)\phi'(\zeta)d\zeta \quad (f \in \mathcal{H}).
\end{align*}

Then $(M_\phi f)(z) = (I_\phi f)(z) + (J_\phi f)(z) + \phi(0)f(0)$. It is clear that $I_\phi$ and $J_\phi$ are never invertible.

Put $\mathcal{M}(\mathcal{H}) = \{\phi \in H(D) : M_\phi \mathcal{H} \subseteq \mathcal{H}\}$, $\mathcal{I}(\mathcal{H}) = \{\phi \in H(D) : I_\phi \mathcal{H} \subseteq \mathcal{H}\}$ and $\mathcal{J}(\mathcal{H}) = \{\phi \in H(D) : J_\phi \mathcal{H} \subseteq \mathcal{H}\}$. In this paper, we assume that $H(\bar{D}) \subset \mathcal{M}(\mathcal{H})$, $z \in \mathcal{I}(\mathcal{H})$ and $z \in \mathcal{J}(\mathcal{H})$.

§ 2. Multiplication operator $M_\phi$
When $\mathcal{M}(\mathcal{H}) = H^\infty(D)$, A. Aleman [1] shows a more general result than Corollary 1 without the condition that $(z - a)\mathcal{H}$ is dense.

**Lemma 1.** If $p$ is a polynomial with no zeros on $\partial D$ then $\dim \mathcal{H}/p\mathcal{H} < \infty$.

**Proof** If $|a| > 1$ then $(z - a)^{-1} \in H(\bar{D})$ and so $(z - a)^{-1}$ belongs to $\mathcal{M}(\mathcal{H})$. Hence we may assume that the zeros of $p$ are contained in $D$. By hypothesis on $\mathcal{H}$, $\dim \mathcal{H}/(z - a)\mathcal{H} = 1$ and so $\dim \mathcal{H}/p\mathcal{H} < \infty$.

**Lemma 2.** If $M$ is a closed invariant subspace of $M_z$ in $\mathcal{H}$ such that $\dim \mathcal{H}/M < \infty$, then there exists a polynomial $p$ such that $p\mathcal{H} \subseteq M$.

**Proof** Let $N = \mathcal{H} \ominus M$ and $S_z = P_NM_z|N$, then $S_z$ is of finite rank because $\dim N < \infty$. Hence there exists a polynomial $p$ such that $S_{p(z)} = p(S_z) = 0$. Therefore $pN \subset M$ and so $p\mathcal{H} \subset M$. 

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Lemma 2. There exists a polynomial $p$ and $Ker\tau_g$ such that each function in $\mathcal{M}(\mathcal{H})$ and so by Lemma 2 there exists a polynomial $p$ such that $\phi f = p$. Therefore $\phi$ can be factorized as $\phi = Bg$ where $B$ is a finite Blaschke product and $g \in \mathcal{H}$. For $\phi \in \mathcal{H}$ and $\prod_{j=1}^{n}(1 - a_j)\phi = \prod_{j=1}^{n}(z - a_j)g \in \mathcal{H}$ where $B = \prod_{j=1}^{n}(z - a_j)/(1 - \bar{a}_jz)$.

Proof (1) Suppose $\phi = Bg$, $B = \prod_{j=1}^{n}(z - a_j)/(1 - \bar{a}_jz)$, $\{a_j\} \subset D$, and both $g$ and $g^{-1}$ are in $\mathcal{M}(\mathcal{H})$. Since $\mathcal{M}(\mathcal{H}) \supseteq H(D)$, $\prod_{j=1}^{n}(1 - \bar{a}_jz)$ is invertible in $\mathcal{M}(\mathcal{H})$ and so $M_{{\phi}}(\mathcal{H}) = p\mathcal{H}$ where $p = \prod_{j=1}^{n}(z - a_j)$. Since $Ker\tau_{a_j} = (z - a_j)\mathcal{H}$, $g$ belongs to $\mathcal{H}$. By the similar argument, there exists a function $k$ in $\mathcal{H}$ and $gk = 1$ because $Bgf = p$. Thus $g^{-1}$ belongs to $\mathcal{H}$. We will prove that $g$ belongs to $\mathcal{M}(\mathcal{H})$. Since $B$ is a finite Blaschke product and $Ker\tau_a = (z - a)\mathcal{H}$ for $a \in D$, $\mathcal{H} = K + B\mathcal{H}$ where $K$ is a finite dimensional subspace such that each function in $K$ is a rational function whose poles are in $\mathbb{C} \setminus \bar{D}$. Since $g \in \mathcal{H}$ and $\mathcal{M}(\mathcal{H}) \supseteq H(D)$, $gK \subseteq \mathcal{H}$ and so $g\mathcal{H} \subseteq \mathcal{H}$ because $gB\mathcal{H} \subseteq \mathcal{H}$.

(3) By the proof of (2), $p\mathcal{H} \subseteq Bg\mathcal{H} \subseteq g\mathcal{H}$ and so the first statement is clear. Again by the proof of (2), the zeros of $p$ in $D$ is just the zeros of $B$. This implies that there exists a polynomial $q$ such that $q\mathcal{H} \subseteq g\mathcal{H}$ and $q$ does not have any zeros in $D$.

Corollary 1. Suppose that $(z - a)\mathcal{H}$ is dense in $\mathcal{H}$ whenever $a \in \partial D$. Then $M_{{\phi}}$ is a Fredholm operator on $\mathcal{H}$ if and only if $\phi = Bg$ where $B$ is a finite Blaschke product, and both $g$ and $g^{-1}$ are in $\mathcal{M}(\mathcal{H})$.

§ 3. Integral operator $I_{{\phi}}$

It seems to have not been studied yet in this general setting as Theorem 2.

Lemma 3. If $\phi$ is a function in $\mathcal{I}(\mathcal{H})$ then $I_{{\phi}}(\mathcal{H}) = I_{{\phi}}(z\mathcal{H}) \subseteq z\mathcal{H}$. $I_{{\phi}}(\mathcal{H}) = z\mathcal{H}$ if and only if $\phi$ and $\phi^{-1}$ belongs to $\mathcal{I}(\mathcal{H})$.

Proof By the definition of $I_{{\phi}}$ the first statement is clear. We will
show the second one. If both \( \phi \) and \( \phi^{-1} \) belong to \( \mathcal{I}(\mathcal{H}) \), then

\[
z\mathcal{H} = I_1(\mathcal{H}) = I_\phi I_{\phi^{-1}}(\mathcal{H}) \subseteq I_\phi(z\mathcal{H}) \subseteq z\mathcal{H}
\]

because \( I_\phi \) and \( I_{\phi^{-1}} \) are bounded on \( \mathcal{H} \). Conversely if \( I_\phi(\mathcal{H}) = z\mathcal{H} \) then there exists a function \( g \) in \( \mathcal{H} \) such that

\[
\int_0^z g'(\zeta)\phi(\zeta)d\zeta = z \quad \text{and so} \quad g'(z)\phi(z) = 1.
\]

Hence \( \phi^{-1} \in H(D) \) and

\[
z\mathcal{H} = I_1(\mathcal{H}) = I_{\phi^{-1}}I_\phi(\mathcal{H}) = I_{\phi^{-1}}(z\mathcal{H})
\]

and so both \( \phi \) and \( \phi^{-1} \) belong to \( \mathcal{I}(\mathcal{H}) \).

**Lemma 4.** If \( p \) is a polynomial then \( I_p(\mathcal{H}) + \mathbb{C} \supseteq p^2\mathcal{H} \).

**Proof** Suppose \( g \in \mathcal{H} \). Since \( z \in \mathcal{I}(\mathcal{H}) \) by the hypothesis, \( p \) belongs to \( \mathcal{I}(\mathcal{H}) \) and so \( \int_0^z g(t)\phi(t)d\zeta \in \mathcal{H} \). Since \( p'(\zeta) \in \mathcal{M}(\mathcal{H}) \) and \( z \in J(\mathcal{H}) \), \( \int_0^z g(\zeta)p'(\zeta)d\zeta \in \mathcal{H} \). Hence \( f(z) = \int_0^z (2p'(\zeta)g(\zeta) + p(\zeta)g'(\zeta))d\zeta \in \mathcal{H} \). Now the lemma follows because

\[
\int_0^z f'(\zeta)p(\zeta)d\zeta = \int_0^z (p^2(\zeta)g(\zeta))'d\zeta = p^2(z)g(z) + p^2(0)g(0).
\]

**Lemma 5.** Suppose that \( B \) is a finite Blaschke product, and both \( g \) and \( g^{-1} \) are in \( \mathcal{I}(\mathcal{H}) \). If \( \phi = Bg \) then \( \phi \in \mathcal{I}(\mathcal{H}) \) and \( \dim \mathcal{H}/I_\phi(\mathcal{H}) < \infty \).

**Proof** By the hypothesis, \( I_B(\mathcal{H}) = I_B(z\mathcal{H}) = I_B(I_\phi(\mathcal{H})) = I_\phi(\mathcal{H}) \) by Lemma 3. We may assume that

\[
B = \prod_{j=1}^n \frac{z - a_j}{1 - \bar{a}_j z} \quad \text{and} \quad \{a_j\} \subset D.
\]

Since \( \prod_{j=1}^n (1 - a_j z) \) is invertible in \( \mathcal{H} \), by Lemma 3 \( I_\phi(\mathcal{H}) = I_p(\mathcal{H}) \) where \( p = \prod_{j=1}^n (z - a_j) \). Lemmas 1 and 4 imply that \( \dim \mathcal{H}/I_\phi(\mathcal{H}) < \infty \).

**Lemma 6.** If \( p \) is a polynomial then \( p(S_z) = S_{p(z)} \).

**Proof** By hypothesis, \( P^N I_z(I - P^N) = 0 \). Hence

\[
S_{z^2} = P^N I_{z^2}P^N = P^N I_z I_z P^N = P^N I_z(I - P^N)P^N + P^N I_z P^N I_z P^N = P^N I_z P^N I_z P^N = S_z S_z.
\]
Now it is easy to see that $p(S_z) = S_{p(z)}$ for a polynomial $p$.

**Lemma 7.** If $M$ is a closed invariant subspace of $I_z$ and $\dim \mathcal{H}/M = n < \infty$ then there exists a polynomial $p$ such that the degree of $p \leq n$ and $I_p(\mathcal{H}) \subseteq M$.

**Proof** If we put $N = \mathcal{H} \ominus M$, then $\dim N = n < \infty$ and so there exists a polynomial $p$ such that $p(S_z) = 0$ and the degree of $p \leq n$. By Lemma 6, $S_{p(z)} = 0$ and so $I_p(N) \subseteq M$. Since $I_p(M) \subseteq M$, $I_p(\mathcal{H}) \subseteq M$.

**Theorem 2.** Suppose $\mathcal{I}(\mathcal{H})$ contains $H(\overline{D})$ and if $f \in \mathcal{I}(\mathcal{H})$ and $f(a) = 0$ for some $a \in D$ then $f/(z - a)$ belongs to $\mathcal{I}(\mathcal{H})$. $I_\phi$ is a Fredholm operator on $\mathcal{H}$ if and only if $\phi = Bg$ where $B$ is a finite Blaschke product, and $g$ and $g^{-1}$ are in $\mathcal{I}(\mathcal{H})$.

**Proof** If $\phi = Bg$, $B$ is a finite Blaschke product, $g \in \mathcal{I}(\mathcal{H})$ and $g^{-1} \in \mathcal{I}(\mathcal{H})$ then by Lemma 5 $I_\phi(\mathcal{H})$ is closed and dim $\text{Ker} I_\phi < \infty$. Since $\text{Ker} I_\phi = \mathbb{C}$, index $I_\phi = 1 - \text{dim Ker} I_\phi$ and so $I_\phi$ is Fredholm. Conversely if $I_\phi$ is Fredholm then $I_\phi(\mathcal{H})$ is closed and dim $\mathcal{H}/I_\phi(\mathcal{H}) < \infty$. Since $I_z I_\phi(\mathcal{H}) \subseteq I_\phi(\mathcal{H})$, by Lemma 7 there exists a polynomial such that $I_p(\mathcal{H}) \subseteq I_\phi(\mathcal{H})$. By Lemma 4 $I_p(\mathcal{H}) + \mathbb{C} = p^2 \mathcal{H}$. Hence there exists a function $F$ in $\mathcal{H}$ and $c \in \mathbb{C}$ such that $I_\phi(F) + c = p^2$. Therefore $F'(z)\phi(z) = 2p(z)p'(z)$ and so the Blaschke part of $\phi$ is a finite one $B$. Thus $\phi$ can be factorized as $\phi = Bg$ where $g \in \mathcal{I}(\mathcal{H})$ and $g$ has no zeros on $D$ because $\mathcal{I}(\mathcal{H})$ is a subalgebra in $\mathcal{B}(\mathcal{H})$ and both $B$ and $B^{-1}$ are in $\mathcal{I}(\mathcal{H})$. Hence

$$I_{g^{-1}p}(\mathcal{H}) \subseteq I_{g^{-1}I_\phi(\mathcal{H})} = I_B(\mathcal{H}) \subseteq \mathcal{H}$$

and so $g^{-1}p$ belongs to $\mathcal{I}(\mathcal{H})$. By hypothesis on $\mathcal{I}(\mathcal{H})$, $g^{-1}$ belongs to $\mathcal{I}(\mathcal{H})$.

§ 4. Integral operator $J_\phi$

A Fredholm integral operator $J_\phi$ have not studied. But if $J_\phi$ is compact then it is not Fredholm. In some special Hilbert space $\mathcal{H}$, the compactness of $J_\phi$ have studied.

**Lemma 8.** If $\phi$ and $\psi$ are in $H(D)$ then $I_\psi J_\phi = J_\phi M_\psi$.

**Proof** For $f \in \mathcal{H}$

$$(I_\psi J_\phi f)(z) = \int_0^z (J_\phi f)'(\zeta)\psi(\zeta)d\zeta = \int_0^z f'(\zeta)\phi'(\zeta)\psi(\zeta)d\zeta = (J_\phi M_\psi f)(z)$$
Lemma 9. If $J_\phi$ is a Fredholm operator on $\mathcal{H}$ then $J_\phi \mathcal{H}$ is a closed invariant subspace of $I_z$ and $\dim \mathcal{H} / J_\phi \mathcal{H} < \infty$. Hence there exists a polynomial $p$ such that $J_\phi \mathcal{H} \supseteq I_p \mathcal{H}$ and so $J_\phi \mathcal{H} + \mathbb{C} \supseteq p^2 \mathcal{H}$.

Proof If $J_\phi$ is Fredholm on $\mathcal{H}$ then $J_\phi \mathcal{H}$ is a closed subspace and by Lemma 8 $I_z(J_\phi \mathcal{H}) \subseteq J_\phi \mathcal{H}$. By Lemma 7 there exists a polynomial $q$ such that $I_q \mathcal{H} \subseteq J_\phi \mathcal{H}$. Lemma 4 implies this lemma.

Theorem 3. Suppose that there exists a function $g$ in $\mathcal{H}$ such that $g'$ does not belong to $H^2$. Suppose that any function in $\mathcal{H}$ has radial limits almost everywhere. Then there does not exist $J_\phi$ which is a Fredholm operator on $\mathcal{H}$.

Proof If $J_\phi$ is Fredholm on $\mathcal{H}$ then $J_\phi \mathcal{H} + \mathbb{C} \supseteq p^2 \mathcal{H}$ for some polynomial by Lemma 9. For any $G$ in $p^2 \mathcal{H}$ there exists a function $f$ in $\mathcal{H}$ such that $f(z)\phi'(z) = G'(z) \quad (z \in D)$.

By hypothesis, there exists $G$ in $p^2 \mathcal{H}$ such that $G' \notin H^2$ and so $G'$ does not have radial limits on a set of positive measure on $\partial D$ (see [2, Appendix A]). On the other hand, if $G = p^2$ then $G$ has radial limits almost everywhere on $\partial D$. By hypothesis, $f$ has radial limits almost everywhere. This contradiction implies that $J_\phi$ is not Fredholm.

§ 5. Relation between $M_\phi$ and $I_\phi$

Put $Df(z) = f'(z)$ and $J = J_z$, that is, $Jf(z) = \int_0^z f(\zeta) d\zeta$. Then $DJf = f$ and $JDf = f - f(0)$.

It is easy to see that $I_\phi J = JM_\phi$ and $DI_\phi = M_\phi D$. Put $\mathcal{H}^D = \{ f \in H(D) : Df \in \mathcal{H} \}$

Suppose that $D$ and $J$ are bounded on $\mathcal{H}$ and for $f$ in $\mathcal{H}^D$ put $\|f\|^2_D = \|Df\|^2 + |f(0)|^2$. Then $\mathcal{H}^D$ is a Hilbert space. Put $\mathcal{H}^I = \{ f \in H(D) : Jf \in \mathcal{H} \}$

and for $f$ in $\mathcal{H}^I$ $\|f\|_I = \|Jf\|$. Then $\mathcal{H}^I$ is a Hilbert space.

$D$ is isometric from $\mathcal{H}_0^D = \{ f \in \mathcal{H}^D : f(0) = 0 \}$ onto $\mathcal{H}$. $J$ is isometric from $\mathcal{H}^I$ onto $\mathcal{H}_0 = \{ f \in \mathcal{H} : f(0) = 0 \}$. Since $DI_\phi = M_\phi D$, $I_\phi$ is bounded on $\mathcal{H}^D$ if and only if $M_\phi$ is bounded on $\mathcal{H}$. Hence $I(\mathcal{H}^D) = \mathcal{M}(\mathcal{H})$. Moreover $I_\phi$ is Fredholm on $\mathcal{H}^D$ if and only if $M_\phi$ is Fredholm on $\mathcal{H}$. Since $JM_\phi = I_\phi J$, 

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\( \mathcal{I}(\mathcal{H}^J) = \mathcal{M}(\mathcal{H}), \) and \( I_\phi \) is Fredholm on \( \mathcal{H}^J \) if and only if \( M_\phi \) is Fredholm on \( \mathcal{H} \). Moreover \( (\mathcal{H}^J)^D = (\mathcal{H}^D)^J = \mathcal{H} \). Hence \( \mathcal{I}(\mathcal{H}) = \mathcal{M}(\mathcal{H}^D) = \mathcal{M}(\mathcal{H}^J) \), and \( I_\phi \) is Fredholm on \( \mathcal{H} \) if and only if \( M_\phi \) is Fredholm on \( \mathcal{H}^D \) and \( \mathcal{H}^J \).

§ 6. Examples

Let \( dA \) denote the normalized Lebesgue area measure on \( D \) and \( \omega \) a positive function on \( D \) which is summable with respect to \( dA \). Put

\[
\mathcal{D}^2(\omega) = \{ f \in H(D) : \| f \|_D^2 = | f(0) |^2 + \int_{D} | f'(z) |^2 \omega(z) dA(z) < \infty \}
\]

and

\[
L_a^2(\omega) = \{ f \in H(D) : \| f \|_{L_a^2}^2 = \int_{D} | f(z) |^2 \omega(z) dA(z) < \infty \}.
\]

Then \( \mathcal{D}^2(\omega) \) is called a weighted Dirichlet space and \( L_a^2(\omega) \) is called a weighted Bergman space when \( \mathcal{D}^2(\omega) \) and \( L_a^2(\omega) \) are nontrivial Hilbert spaces. It is easy to see that \((\mathcal{D}^2(\omega))^J = L_a^2(\omega) \) and \( (L_a^2(\omega))^D = \mathcal{D}^2(\omega) \).

If \( \omega(z) = (1 - |z|^2)^\alpha \) and \( \alpha > -1 \), we will write \( \mathcal{D}^2(\omega) = \mathcal{D}^2_\alpha \) and \( L_a^2(\omega) = L_a^2_\alpha \). It is known that \( \mathcal{D}^2_\alpha \) and \( L_a^2_\alpha \) are nontrivial Hilbert spaces. \( \mathcal{D}_1 \) is the Hardy space \( H^2 \), \( \mathcal{D}_2 \) is the Bergman space \( L_a^2 \) and \( \mathcal{D}_0 \) is the Dirichlet space. If \( \mathcal{H} = \mathcal{D}_\alpha \) or \( L_a^2_\alpha \) then \( \mathcal{H} \) satisfies the condition (1), (2) and (3) in Introduction. It is known that \( H(\bar{\mathcal{D}}) \subset \mathcal{M}(\mathcal{D}_\alpha) \subset H^\infty(D) \) and \( \mathcal{M}(L_a^2_\alpha) = H^\infty(D) \). Hence Theorem 1 can apply to \( \mathcal{D}_\alpha \) for any \( \alpha > -1 \). If \( \alpha \geq 1 \) then \( (z - a) \mathcal{D}_\alpha \) is dense in \( \mathcal{D}_\alpha \) whenever \( a \in \partial D \). Hence Corollary 1 can apply to \( \mathcal{D}_\alpha \) for \( \alpha \geq 1 \).

\( \mathcal{I}(L_a^2_\alpha) = \mathcal{M}(L_a^2_\alpha)^D = \mathcal{M}(\mathcal{D}_\alpha) \) and \( H(\bar{\mathcal{D}}) \subset \mathcal{M}(\mathcal{D}_\alpha) \subset H^\infty(D) \). Since \( \mathcal{I}(\mathcal{D}_\alpha) = \mathcal{M}(L_a^2_\alpha) = H^\infty(D) \), Theorem 2 can apply to \( \mathcal{D}_\alpha \) for \( \alpha > -1 \). It is known [3] that \( \mathcal{M}(\mathcal{D}_\alpha) = H^\infty(D) \) for \( \alpha > 1 \) and \( \mathcal{M}(\mathcal{D}_\alpha) = \mathcal{D}_\alpha \) for \( -1 < \alpha < 0 \). Hence \( \mathcal{I}(L_a^2_\alpha) = H^\infty(D) \) for \( \alpha > 1 \) and \( \mathcal{I}(L_a^2_\alpha) = \mathcal{D}_\alpha \) for \( -1 < \alpha < 0 \). Hence Theorem 2 can apply to \( L_a^2_\alpha \) for \( \alpha > 1 \) and \( -1 < \alpha < 0 \). By a theorem in [3], it is easy to see that \( \mathcal{I}(L_a^2_\alpha) = \mathcal{M}(\mathcal{D}_\alpha) \) (0 \( \leq \alpha \leq 1 \)) satisfies the conditions in Theorem 2. Hence Theorem 2 can apply to \( L_a^2_\alpha \).

When \( \mathcal{D}^2(\omega) \) or \( L_a^2(\omega) \) is a Hilbert space \( \mathcal{H} \), it is important in order to study composition operator that \( \mathcal{H} \) satisfies three conditions in Introduction. It will be interesting to determine such a weight \( \omega \).

References


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