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Some Fredholm Integration Operators on A Hilbert Space of Holomorphic  
Functions on The Unit Disc

By

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**Abstract.** In this paper, we study when  $M_\phi$ ,  $I_\phi$  or  $J_\phi$  is a Fredholm operator on a Hilbert space which satisfies few natural axioms.

## § 1. Introduction

Let  $D$  be the open unit disc in the complex plane  $\mathbb{C}$  and  $H(D)$  be the set of all analytic functions on  $D$ .  $H(\bar{D})$  denotes the set of all analytic functions on  $\bar{D}$ . In this paper,  $\mathcal{H}$  is a Hilbert space in  $H(D)$  which satisfies the following :

- (1)  $z\mathcal{H} \subset \mathcal{H}$ .
- (2) If  $a \in D$  then  $(z - a)\mathcal{H} \oplus \mathbb{C} = \mathcal{H}$ .
- (3)  $\mathcal{H} \supseteq H(\bar{D})$ .

In this paper, we study the following three operators. If  $\phi$  is a function in  $H(D)$ , put for  $z \in D$ ,

$$\begin{aligned} (M_\phi f)(z) &= \phi(z)f(z), \\ (I_\phi f)(z) &= \int_0^z f'(\zeta)\phi(\zeta)d\zeta, \\ (J_\phi f)(z) &= \int_0^z f(\zeta)\phi'(\zeta)d\zeta \quad (f \in \mathcal{H}). \end{aligned}$$

Then  $(M_\phi f)(z) = (I_\phi f)(z) + (J_\phi f)(z) + \phi(0)f(0)$ . It is clear that  $I_\phi$  and  $J_\phi$  are never invertible.

Put  $\mathcal{M}(\mathcal{H}) = \{\phi \in H(D) : M_\phi \mathcal{H} \subseteq \mathcal{H}\}$ ,  $\mathcal{I}(\mathcal{H}) = \{\phi \in H(D) : I_\phi \mathcal{H} \subseteq \mathcal{H}\}$  and  $\mathcal{J}(\mathcal{H}) = \{\phi \in H(D) : J_\phi \mathcal{H} \subseteq \mathcal{H}\}$ . In this paper, we assume that  $H(\bar{D}) \subset \mathcal{M}(\mathcal{H})$ ,  $z \in \mathcal{I}(\mathcal{H})$  and  $z \in \mathcal{J}(\mathcal{H})$ .

## § 2. Multiplication operator $M_\phi$

When  $\mathcal{M}(\mathcal{H}) = H^\infty(D)$ , A. Aleman [1] shows a more general result than Corollary 1 without the condition that  $(z - a)\mathcal{H}$  is dense.

**Lemma 1.** If  $p$  is a polynomial with no zeros on  $\partial D$  then  $\dim \mathcal{H}/p\mathcal{H} < \infty$ .

**Proof** If  $|a| > 1$  then  $(z - a)^{-1} \in H(\bar{D})$  and so  $(z - a)^{-1}$  belongs to  $M(\mathcal{H})$ . Hence we may assume that the zeros of  $p$  are contained in  $D$ . By hypothesis on  $\mathcal{H}$ ,  $\dim \mathcal{H}/(z - a)\mathcal{H} = 1$  and so  $\dim \mathcal{H}/p\mathcal{H} < \infty$ .

**Lemma 2.** If  $M$  is a closed invariant subspace of  $M_z$  in  $\mathcal{H}$  such that  $\dim \mathcal{H}/M < \infty$ , then there exists a polynomial  $p$  such that  $p\mathcal{H} \subseteq M$ .

**Proof** Let  $N = \mathcal{H} \ominus M$  and  $S_z = P_N M_z|_N$ , then  $S_z$  is of finite rank because  $\dim N < \infty$ . Hence there exists a polynomial  $p$  such that  $S_{p(z)} = p(S_z) = 0$ . Therefore  $pN \subset M$  and so  $p\mathcal{H} \subset M$ .

**Theorem 1.**

- (1) If  $\phi = Bg$  where  $B$  is a finite Blaschke product, and both  $g$  and  $g^{-1}$  are in  $\mathcal{M}(\mathcal{H})$  then  $M_\phi$  is a Fredholm operator.
- (2) If  $M_\phi$  is a Fredholm operator on  $\mathcal{H}$  then  $\phi = Bg$  when  $B$  is a finite Blaschke product,  $g$  is in  $\mathcal{M}(\mathcal{H})$  and  $g^{-1}$  is in  $\mathcal{H}$ .
- (3) For the  $g$  in (2),  $M_g$  is a Fredholm operator on  $\mathcal{H}$  with index  $M_\phi \leq$  index  $M_g \leq 0$  and there exists a polynomial  $q$  such that  $q\mathcal{H} \subseteq g\mathcal{H}$  and the zeros are in  $\mathbb{C} \setminus D$ .

**Proof** (1) Suppose  $\phi = Bg$ ,  $B = \prod_{j=1}^n (z - a_j)/(1 - \bar{a}_j z)$ ,  $\{a_j\} \subset D$ , and both  $g$  and  $g^{-1}$  are in  $\mathcal{M}(\mathcal{H})$ . Since  $\mathcal{M}(\mathcal{H}) \supseteq H(\bar{D})$ ,  $\prod_{j=1}^n (1 - \bar{a}_j z)$  is invertible in  $\mathcal{M}(\mathcal{H})$  and so  $M_\phi(\mathcal{H}) = p\mathcal{H}$  where  $p = \prod_{j=1}^n (z - a_j)$ .

(2) If  $M_\phi$  is a Fredholm operator then  $\dim \mathcal{H}/M_\phi(\mathcal{H}) < \infty$  and so by Lemma 2 there exists a polynomial  $p$  such that  $\phi f = p$ . Therefore  $\phi$  can be factorized as  $\phi = Bg$  where  $B$  is a finite Blaschke product and  $g \in \mathcal{H}$ . For  $\phi \in \mathcal{H}$  and  $\prod_{j=1}^n (1 - \bar{a}_j z)\phi = \prod_{j=1}^n (z - a_j)g \in \mathcal{H}$  where  $B = \prod_{j=1}^n (z - a_j)/(1 - \bar{a}_j z)$ . Since  $\text{Ker}\tau_{a_j} = (z - a_j)\mathcal{H}$ ,  $g$  belongs to  $\mathcal{H}$ . By the similar argument, there exists a function  $k$  in  $\mathcal{H}$  and  $gk = 1$  because  $Bgf = p$ . Thus  $g^{-1}$  belongs to  $\mathcal{H}$ . We will prove that  $g$  belongs to  $\mathcal{M}(\mathcal{H})$ . Since  $B$  is a finite Blaschke product and  $\text{Ker}\tau_a = (z - a)\mathcal{H}$  for  $a \in D$ ,  $\mathcal{H} = K + B\mathcal{H}$  where  $K$  is a finite dimensional subspace such that each function in  $K$  is a rational function whose poles are in  $\mathbb{C} \setminus \bar{D}$ . Since  $g \in \mathcal{H}$  and  $\mathcal{M}(\mathcal{H}) \supseteq H(\bar{D})$ ,  $gK \subseteq \mathcal{H}$  and so  $g\mathcal{H} \subseteq \mathcal{H}$  because  $gB\mathcal{H} \subseteq \mathcal{H}$ .

(3) By the proof of (2),  $p\mathcal{H} \subseteq Bg\mathcal{H} \subseteq g\mathcal{H}$  and so the first statement is clear. Again by the proof of (2), the zeros of  $p$  in  $D$  is just the zeros of  $B$ . This implies that there exists a polynomial  $q$  such that  $q\mathcal{H} \subseteq g\mathcal{H}$  and  $q$  does not have any zeros in  $D$ .

**Corollary 1.** Suppose that  $(z - a)\mathcal{H}$  is dense in  $\mathcal{H}$  whenever  $a \in \partial D$ . Then  $M_\phi$  is a Fredholm operator on  $\mathcal{H}$  if and only if  $\phi = Bg$  where  $B$  is a finite Blaschke product, and both  $g$  and  $g^{-1}$  are in  $\mathcal{M}(\mathcal{H})$ .

**§ 3. Integral operator  $I_\phi$** 

It seems to have not been studied yet in this general setting as Theorem 2.

**Lemma 3.** If  $\phi$  is a function in  $\mathcal{I}(\mathcal{H})$  then  $I_\phi(\mathcal{H}) = I_\phi(z\mathcal{H}) \subseteq z\mathcal{H}$ .  $I_\phi(\mathcal{H}) = z\mathcal{H}$  if and only if  $\phi$  and  $\phi^{-1}$  belongs to  $\mathcal{I}(\mathcal{H})$ .

**Proof** By the definition of  $I_\phi$  the first statement is clear. We will

show the second one. If both  $\phi$  and  $\phi^{-1}$  belong to  $\mathcal{I}(\mathcal{H})$ , then

$$z\mathcal{H} = I_1(\mathcal{H}) = I_\phi I_{\phi^{-1}}(\mathcal{H}) \subseteq I_\phi(z\mathcal{H}) \subseteq z\mathcal{H}$$

because  $I_\phi$  and  $I_{\phi^{-1}}$  are bounded on  $\mathcal{H}$ . Conversely if  $I_\phi(\mathcal{H}) = z\mathcal{H}$  then there exists a function  $g$  in  $\mathcal{H}$  such that

$$\int_0^z g'(\zeta)\phi(\zeta)d\zeta = z \text{ and so } g'(z)\phi(z) = 1.$$

Hence  $\phi^{-1} \in H(D)$  and

$$z\mathcal{H} = I_1(\mathcal{H}) = I_{\phi^{-1}}I_\phi(\mathcal{H}) = I_{\phi^{-1}}(z\mathcal{H})$$

and so both  $\phi$  and  $\phi^{-1}$  belong to  $\mathcal{I}(\mathcal{H})$ .

**Lemma 4.** If  $p$  is a polynomial then  $I_p(\mathcal{H}) + \mathbb{C} \supset p^2\mathcal{H}$ .

**Proof** Suppose  $g \in \mathcal{H}$ . Since  $z \in \mathcal{I}(\mathcal{H})$  by the hypothesis,  $p$  belongs to  $\mathcal{I}(\mathcal{H})$  and so  $\int_0^z g'(\zeta)p(\zeta)d\zeta \in \mathcal{H}$ . Since  $p' \in \mathcal{M}(\mathcal{H})$  and  $z \in J(\mathcal{H})$ ,  $\int_0^z g(\zeta)p'(\zeta)d\zeta$  belongs to  $\mathcal{H}$ . Hence  $f(z) = \int_0^z (2p'(\zeta)g(\zeta) + p(\zeta)g'(\zeta))d\zeta$  belongs to  $\mathcal{H}$ . Now the lemma follows because

$$\int_0^z f'(\zeta)p(\zeta)d\zeta = \int_0^z (p^2(\zeta)g(\zeta))'d\zeta = p^2(z)g(z) + p^2(0)g(0).$$

**Lemma 5.** Suppose that  $B$  is a finite Blaschke product, and both  $g$  and  $g^{-1}$  are in  $\mathcal{I}(\mathcal{H})$ . If  $\phi = Bg$  then  $\phi \in \mathcal{I}(\mathcal{H})$  and  $\dim \mathcal{H}/I_\phi(\mathcal{H}) < \infty$ .

**Proof** By the hypothesis,  $I_B(\mathcal{H}) = I_B(z\mathcal{H}) = I_B(I_g(\mathcal{H})) = I_\phi(\mathcal{H})$  by Lemma 3. We may assume that

$$B = \prod_{j=1}^n \frac{z - a_j}{1 - \bar{a}_j z} \text{ and } \{a_j\} \subset D.$$

Since  $\prod_{j=1}^n (1 - \bar{a}_j z)$  is invertible in  $\mathcal{I}(\mathcal{H})$ , by Lemma 3  $I_\phi(\mathcal{H}) = I_p(\mathcal{H})$  where  $p = \prod_{j=1}^n (z - a_j)$ . Lemmas 1 and 4 imply that  $\dim \mathcal{H}/I_\phi(\mathcal{H}) < \infty$ .

**Lemma 6.** If  $p$  is a polynomial then  $p(S_z) = S_{p(z)}$ .

**Proof** By hypothesis,  $P^N I_z (I - P^N) = 0$ . Hence

$$\begin{aligned} S_{z^2} &= P^N I_{z^2} P^N = P^N I_z I_z P^N \\ &= P^N I_z (I - P^N) I_z P^N + P^N I_z P^N I_z P^N \\ &= P^N I_z P^N I_z P^N = S_z S_z. \end{aligned}$$

Now it is easy to see that  $p(S_z) = S_{p(z)}$  for a polynomial  $p$ .

**Lemma 7.** If  $M$  is a closed invariant subspace of  $I_z$  and  $\dim \mathcal{H}/M = n < \infty$  then there exists a polynomial  $p$  such that the degree of  $p \leq n$  and  $I_p(\mathcal{H}) \subseteq M$ .

**Proof** If we put  $N = \mathcal{H} \ominus M$ , then  $\dim N = n < \infty$  and so there exists a polynomial  $p$  such that  $p(S_z) = 0$  and the degree of  $p \leq n$ . By Lemma 6,  $S_{p(z)} = 0$  and so  $I_p(N) \subseteq M$ . Since  $I_p(M) \subseteq M$ ,  $I_p(\mathcal{H}) \subseteq M$ .

**Theorem 2.** Suppose  $\mathcal{I}(\mathcal{H})$  contains  $H(\bar{D})$  and if  $f \in \mathcal{I}(\mathcal{H})$  and  $f(a) = 0$  for some  $a \in D$  then  $f/(z - a)$  belongs to  $\mathcal{I}(\mathcal{H})$ .  $I_\phi$  is a Fredholm operator on  $\mathcal{H}$  if and only if  $\phi = Bg$  where  $B$  is a finite Blaschke product, and  $g$  and  $g^{-1}$  are in  $\mathcal{I}(\mathcal{H})$ .

**Proof** If  $\phi = Bg$ ,  $B$  is a finite Blaschke product,  $g \in \mathcal{I}(\mathcal{H})$  and  $g^{-1} \in \mathcal{I}(\mathcal{H})$  then by Lemma 5  $I_\phi(\mathcal{H})$  is closed and  $\dim \text{Ker} I_\phi^* < \infty$ . Since  $\text{Ker} I_\phi = \mathbb{C}$ ,  $\text{index } I_\phi = 1 - \dim \text{Ker} I_\phi^*$  and so  $I_\phi$  is Fredholm. Conversely if  $I_\phi$  is Fredholm then  $I_\phi(\mathcal{H})$  is closed and  $\dim \mathcal{H}/I_\phi(\mathcal{H}) < \infty$ . Since  $I_z I_\phi(\mathcal{H}) \subseteq I_\phi(\mathcal{H})$ , by Lemma 7 there exists a polynomial such that  $I_p(\mathcal{H}) \subseteq I_\phi(\mathcal{H})$ . By Lemma 4  $I_p(\mathcal{H}) + \mathbb{C} \supset p^2 \mathcal{H}$ . Hence there exists a function  $F$  in  $\mathcal{H}$  and  $c \in \mathbb{C}$  such that  $I_\phi(F) + c = p^2$ . Therefore  $F'(z)\phi(z) = 2p(z)p'(z)$  and so the Blaschke part of  $\phi$  is a finite one  $B$ . Thus  $\phi$  can be factorized as  $\phi = Bg$  where  $g \in \mathcal{I}(\mathcal{H})$  and  $g$  has no zeros on  $D$  because  $\mathcal{I}(\mathcal{H})$  is a subalgebra in  $\mathcal{B}(\mathcal{H})$  and both  $B$  and  $B^{-1}$  are in  $\mathcal{I}(\mathcal{H})$ . Hence

$$I_{g^{-1}p}(\mathcal{H}) \subseteq I_{g^{-1}} I_\phi(\mathcal{H}) = I_B(\mathcal{H}) \subseteq \mathcal{H}$$

and so  $g^{-1}p$  belongs to  $\mathcal{I}(\mathcal{H})$ . By hypothesis on  $\mathcal{I}(\mathcal{H})$ ,  $g^{-1}$  belongs to  $\mathcal{I}(\mathcal{H})$ .

#### § 4. Integral operator $J_\phi$

A Fredholm integral operator  $J_\phi$  have not studied. But if  $J_\phi$  is compact then it is not Fredholm. In some special Hilbert space  $\mathcal{H}$ , the compactness of  $J_\phi$  have studied.

**Lemma 8.** If  $\phi$  and  $\psi$  are in  $H(D)$  then  $I_\psi J_\phi = J_\phi M_\psi$ .

**Proof** For  $f \in \mathcal{H}$

$$\begin{aligned} (I_\psi J_\phi f)(z) &= \int_0^z (J_\phi f)'(\zeta) \psi(\zeta) d\zeta = \int_0^z f(\zeta) \phi'(\zeta) \psi(\zeta) d\zeta \\ &= (J_\phi M_\psi f)(z) \end{aligned}$$

**Lemma 9.** If  $J_\phi$  is a Fredholm operator on  $\mathcal{H}$  then  $J_\phi\mathcal{H}$  is a closed invariant subspace of  $I_z$  and  $\dim \mathcal{H}/J_\phi\mathcal{H} < \infty$ . Hence there exists a polynomial  $p$  such that  $J_\phi\mathcal{H} \supseteq I_p\mathcal{H}$  and so  $J_\phi\mathcal{H} + \mathbb{C} \supseteq p^2\mathcal{H}$ .

**Proof** If  $J_\phi$  is Fredholm on  $\mathcal{H}$  then  $J_\phi\mathcal{H}$  is a closed subspace and by Lemma 8  $I_z(J_\phi\mathcal{H}) \subseteq J_\phi\mathcal{H}$ . By Lemma 7 there exists a polynomial  $q$  such that  $I_q\mathcal{H} \subseteq J_\phi\mathcal{H}$ . Lemma 4 implies this lemma.

**Theorem 3.** Suppose that there exists a function  $g$  in  $\mathcal{H}$  such that  $g'$  does not belong to  $H^2$ . Suppose that any function in  $\mathcal{H}$  has radial limits almost everywhere. Then there does not exist  $J_\phi$  which is a Fredholm operator on  $\mathcal{H}$ .

**Proof** If  $J_\phi$  is Fredholm on  $\mathcal{H}$  then  $J_\phi\mathcal{H} + \mathbb{C} \supseteq p^2\mathcal{H}$  for some polynomial by Lemma 9. For any  $G$  in  $p^2\mathcal{H}$  there exists a function  $f$  in  $\mathcal{H}$  such that

$$f(z)\phi'(z) = G'(z) \quad (z \in D).$$

By hypothesis, there exists  $G$  in  $p^2\mathcal{H}$  such that  $G' \notin H^2$  and so  $G'$  does not have radial limits on a set of positive measure on  $\partial D$  (see [2, Appendix A]). On the other hand, if  $G = p^2$  then  $G$  has radial limits almost everywhere on  $\partial D$ . By hypothesis,  $f$  has radial limits almost everywhere. This contradiction implies that  $J_\phi$  is not Fredholm.

### § 5. Relation between $M_\phi$ and $I_\phi$

Put  $Df(z) = f'(z)$  and  $J = J_z$ , that is,  $Jf(z) = \int_0^z f(\zeta)d\zeta$ . Then

$$DJf = f \text{ and } JDf = f - f(0).$$

It is easy to see that  $I_\phi J = JM_\phi$  and  $DI_\phi = M_\phi D$ . Put

$$\mathcal{H}^D = \{f \in H(D) : Df \in \mathcal{H}\}$$

Suppose that  $D$  and  $J$  are bounded on  $\mathcal{H}$  and for  $f$  in  $\mathcal{H}^D$  put  $\|f\|_D^2 = \|Df\|^2 + |f(0)|^2$ . Then  $\mathcal{H}^D$  is a Hilbert space. Put

$$\mathcal{H}^J = \{f \in H(D) : Jf \in \mathcal{H}\}$$

and for  $f$  in  $\mathcal{H}^J$   $\|f\|_J = \|Jf\|$ . Then  $\mathcal{H}^J$  is a Hilbert space.

$D$  is isometric from  $\mathcal{H}_0^D = \{f \in \mathcal{H}^D : f(0) = 0\}$  onto  $\mathcal{H}$ .  $J$  is isometric from  $\mathcal{H}^J$  onto  $\mathcal{H}_0 = \{f \in \mathcal{H} : f(0) = 0\}$ . Since  $DI_\phi = M_\phi D$ ,  $I_\phi$  is bounded on  $\mathcal{H}^D$  if and only if  $M_\phi$  is bounded on  $\mathcal{H}$ . Hence  $\mathcal{I}(\mathcal{H}^D) = \mathcal{M}(\mathcal{H})$ . Moreover  $I_\phi$  is Fredholm on  $\mathcal{H}^D$  if and only if  $M_\phi$  is Fredholm on  $\mathcal{H}$ . Since  $JM_\phi = I_\phi J$ ,



$\mathcal{I}(\mathcal{H}^J) = \mathcal{M}(\mathcal{H})$ , and  $I_\phi$  is Fredholm on  $\mathcal{H}^J$  if and only if  $M_\phi$  is Fredholm on  $\mathcal{H}$ . Moreover  $(\mathcal{H}^J)^D = (\mathcal{H}^D)^J = \mathcal{H}$ . Hence  $\mathcal{I}(\mathcal{H}) = \mathcal{M}(\mathcal{H}^D) = \mathcal{M}(\mathcal{H}^J)$ , and  $I_\phi$  is Fredholm on  $\mathcal{H}$  if and only if  $M_\phi$  is Fredholm on  $\mathcal{H}^D$  and  $\mathcal{H}^J$ .

### § 6. Examples

Let  $dA$  denote the normalized Lebesgue area measure on  $D$  and  $\omega$  a positive function on  $D$  which is summable with respect to  $dA$ . Put

$$\mathcal{D}^2(\omega) = \{f \in H(D) : \|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \int_D |f'(z)|^2 \omega(z) dA(z) < \infty\}$$

and

$$L_a^2(\omega) = \{f \in H(D) : \|f\|_{L_a^2}^2 = \int_D |f(z)|^2 \omega(z) dA(z) < \infty\}.$$

Then  $\mathcal{D}^2(\omega)$  is called a weighted Dirichlet space and  $L_a^2(\omega)$  is called a weighted Bergman space when  $\mathcal{D}^2(\omega)$  and  $L_a^2(\omega)$  are nontrivial Hilbert spaces. It is easy to see that  $(\mathcal{D}^2(\omega))^J = L_a^2(\omega)$  and  $(L_a^2(\omega))^D = \mathcal{D}^2(\omega)$ .

If  $\omega(z) = (1 - |z|^2)^\alpha$  and  $\alpha > -1$ , we will write  $\mathcal{D}^2(\omega) = \mathcal{D}_\alpha^2$  and  $L_a^2(\omega) = L_{a,\alpha}^2$ . It is known that  $\mathcal{D}_\alpha^2$  and  $L_{a,\alpha}^2$  are nontrivial Hilbert spaces.  $\mathcal{D}_1$  is the Hardy space  $H^2$ ,  $\mathcal{D}_2$  is the Bergman space  $L_a^2$  and  $\mathcal{D}_0$  is the Dirichlet space. If  $\mathcal{H} = \mathcal{D}_\alpha$  or  $L_{a,\alpha}^2$  then  $\mathcal{H}$  satisfies the condition (1), (2) and (3) in Introduction. It is known that  $H(\bar{D}) \subset \mathcal{M}(\mathcal{D}_\alpha) \subset H^\infty(D)$  and  $\mathcal{M}(L_{a,\alpha}^2) = H^\infty(D)$ . Hence Theorem 1 can apply to  $\mathcal{D}_\alpha$  for any  $\alpha > -1$ . If  $\alpha \geq 1$  then  $(z - a)\mathcal{D}_\alpha$  is dense in  $\mathcal{D}_\alpha$  whenever  $a \in \partial D$ . Hence Corollary 1 can apply to  $\mathcal{D}_\alpha$  for  $\alpha \geq 1$ .  $\mathcal{I}(L_{a,\alpha}^2) = \mathcal{M}((L_{a,\alpha}^2)^D) = \mathcal{M}(\mathcal{D}_\alpha)$  and  $H(\bar{D}) \subset \mathcal{M}(\mathcal{D}_\alpha) \subset H^\infty(D)$ . Since  $\mathcal{I}(\mathcal{D}_\alpha) = \mathcal{M}(L_{a,\alpha}^2) = H^\infty(D)$ , Theorem 2 can apply to  $\mathcal{D}_\alpha$  for  $\alpha > -1$ . It is known [3] that  $\mathcal{M}(\mathcal{D}_\alpha) = H^\infty(D)$  for  $\alpha > 1$  and  $\mathcal{M}(\mathcal{D}_\alpha) = \mathcal{D}_\alpha$  for  $-1 < \alpha < 0$ . Hence  $\mathcal{I}(L_{a,\alpha}^2) = H^\infty(D)$  for  $\alpha > 1$  and  $\mathcal{I}(L_{a,\alpha}^2) = \mathcal{D}_\alpha$  for  $-1 < \alpha < 0$ . Hence Theorem 2 can apply to  $L_{a,\alpha}^2$  for  $\alpha > 1$  and  $-1 < \alpha < 0$ . By a theorem in [3], it is easy to see that  $\mathcal{I}(L_{a,\alpha}^2) = \mathcal{M}(\mathcal{D}_\alpha)$  ( $0 \leq \alpha \leq 1$ ) satisfies the conditions in Theorem 2. Hence Theorem 2 can apply to  $L_{a,\alpha}^2$ .

When  $\mathcal{D}^2(\omega)$  or  $L_a^2(\omega)$  is a Hilbert space  $\mathcal{H}$ , it is important in order to study composition operator that  $\mathcal{H}$  satisfies three conditions in Introduction. It will be interesting to determine such a weight  $\omega$ .

### References

1. A. Aleman, Finite codimensional invariant subspaces in Hilbert spaces of analytic functions, J. Funct. Anal. 119(1994), 1-18.
2. P. Duren, Theory of  $H^p$  Spaces, Academic Press, New York, 1970.

3. D. A. Stegenga, Multipliers of the Dirichlet space, Illinois J. Math. 24(1980), 113-139.

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