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Some Fredholm Integration Operators on A Hilbert Space of Holomorphic Functions on The Unit Disc

By

Takahiko Nakazi*

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Abstract. In this paper, we study when $M_\phi$, $I_\phi$ or $J_\phi$ is a Fredholm operator on a Hilbert space which satisfies few natural axioms.
§ 1. Introduction

Let $D$ be the open unit disc in the complex plane $\mathbb{C}$ and $H(D)$ be the set of all analytic functions on $D$. $H(\bar{D})$ denotes the set of all analytic functions on $\bar{D}$. In this paper, $H$ is a Hilbert space in $H(D)$ which satisfies the following:

1. $zH \subset H$.
2. If $a \in D$ then $(z-a)H \oplus \mathbb{C} = H$.
3. $H \supseteq H(\bar{D})$.

In this paper, we study the following three operators. If $\phi$ is a function in $H(D)$, put for $z \in D$,

\[
\begin{align*}
(M_\phi f)(z) &= \phi(z)f(z), \\
(I_\phi f)(z) &= \int_0^z f'(\zeta)\phi(\zeta)d\zeta, \\
(J_\phi f)(z) &= \int_0^z f(\zeta)\phi'(\zeta)d\zeta \quad (f \in H).
\end{align*}
\]

Then $(M_\phi f)(z) = (I_\phi f)(z) + (J_\phi f)(z) + \phi(0)f(0)$. It is clear that $I_\phi$ and $J_\phi$ are never invertible.

Put $\mathcal{M}(H) = \{\phi \in H(D) : M_\phi H \subseteq H\}$, $\mathcal{I}(H) = \{\phi \in H(D) : I_\phi H \subseteq H\}$ and $\mathcal{J}(H) = \{\phi \in H(D) : J_\phi H \subseteq H\}$. In this paper, we assume that $H(\bar{D}) \subset \mathcal{M}(H)$, $z \in \mathcal{I}(H)$ and $z \in \mathcal{J}(H)$.

§ 2. Multiplication operator $M_\phi$

When $\mathcal{M}(H) = H^\infty(D)$, A. Aleman [1] shows a more general result than Corollary 1 without the condition that $(z-a)H$ is dense.

**Lemma 1.** If $p$ is a polynomial with no zeros on $\partial D$ then $\dim H/pH < \infty$.

**Proof** If $|a| > 1$ then $(z-a)^{-1} \in H(\bar{D})$ and so $(z-a)^{-1}$ belongs to $M(H)$. Hence we may assume that the zeros of $p$ are contained in $D$. By hypothesis on $H$, $\dim H/(z-a)H = 1$ and so $\dim H/pH < \infty$.

**Lemma 2.** If $M$ is a closed invariant subspace of $M_z$ in $H$ such that $\dim H/M < \infty$, then there exists a polynomial $p$ such that $pH \subseteq M$.

**Proof** Let $N = H \ominus M$ and $S_z = P_NM_z|N$, then $S_z$ is of finite rank because $\dim N < \infty$. Hence there exists a polynomial $p$ such that $S_{p(z)} = p(S_z) = 0$. Therefore $pN \subset M$ and so $pH \subset M$.
Lemma 2. There exists a polynomial $p$ and $\ker\tau$.

We will prove that there exists a function $k$. Since $\ker\tau$ factorized as $\phi H$ and $M$, then $\ker\tau$ is invertible in $\mathcal{M}(H)$ and $g^{-1}$ is in $\mathcal{H}$.

M is invertible in $\mathcal{M}(\mathcal{H})$. Then $I_g$ is a Fredholm operator on $\mathcal{H}$ with index $M_\phi \leq 0$ and there exists a polynomial $q$ such that $q\mathcal{H} \subseteq g\mathcal{H}$ and the zeros are in $\mathbb{C}\setminus D$.

Proof (1) Suppose $\phi = Bg, B = \prod_{j=1}^{n}(z - a_j)/(1 - \bar{a}_jz), \{a_j\} \subset D$, and both $g$ and $g^{-1}$ are in $\mathcal{M}(\mathcal{H})$. Since $\mathcal{M}(\mathcal{H}) \supseteq H(D), \prod_{j=1}^{n}(1 - \bar{a}_jz)$ is invertible in $\mathcal{M}(\mathcal{H})$ and so $M_\phi(\mathcal{H}) = p\mathcal{H}$ where $p = \prod_{j=1}^{n}(z - a_j)$.

(2) If $M_\phi$ is a Fredholm operator then $\dim \mathcal{H}/M_\phi(\mathcal{H}) < \infty$ and so by Lemma 2 there exists a polynomial $p$ such that $\phi f = p$. Therefore $\phi$ can be factorized as $\phi = Bg$ where $B$ is a finite Blaschke product and $g \in \mathcal{H}$. For $\phi \in \mathcal{H}$ and $\prod_{j=1}^{n}(1 - \bar{a}_jz)\phi = \prod_{j=1}^{n}(z - a_j)g \in \mathcal{H}$ where $B = \prod_{j=1}^{n}(z - a_j)/(1 - \bar{a}_jz)$. Since $\ker\tau_{a_j} = (z - a_j)\mathcal{H}, g$ belongs to $\mathcal{H}$. By the similar argument, there exists a function $k$ in $\mathcal{H}$ and $gk = 1$ because $Bgf = p$. Thus $g^{-1}$ belongs to $\mathcal{H}$.

We will prove that $g$ belongs to $\mathcal{M}(\mathcal{H})$. Since $B$ is a finite Blaschke product and $\ker\tau_{a_j} = (z - a_j)\mathcal{H}$ for $a \in D, \mathcal{H} = K + B\mathcal{H}$ where $K$ is a finite dimensional subspace such that each function in $K$ is a rational function whose poles are in $\mathbb{C}\setminus D$. Since $g \in \mathcal{H}$ and $\mathcal{M}(\mathcal{H}) \supseteq H(D), gK \subseteq \mathcal{H}$ and so $g\mathcal{H} \subseteq \mathcal{H}$ because $gB\mathcal{H} \subseteq \mathcal{H}$.

(3) By the proof of (2), $p\mathcal{H} \subseteq Bg\mathcal{H} \subseteq g\mathcal{H}$ and so the first statement is clear. Again by the proof of (2), the zeros of $p$ in $D$ is just the zeros of $B$. This implies that there exists a polynomial $q$ such that $q\mathcal{H} \subseteq g\mathcal{H}$ and $q$ does not have any zeros in $D$.

Corollary 1. Suppose that $(z - a)\mathcal{H}$ is dense in $\mathcal{H}$ whenever $a \in \partial D$. Then $M_\phi$ is a Fredholm operator on $\mathcal{H}$ if and only if $\phi = Bg$ where $B$ is a finite Blaschke product, and both $g$ and $g^{-1}$ are in $\mathcal{M}(\mathcal{H})$.

§ 3. Integral operator $I_\phi$

It seems to have not been studied yet in this general setting as Theorem 2.

Lemma 3. If $\phi$ is a function in $I(\mathcal{H})$ then $I_\phi(\mathcal{H}) = I_\phi(z\mathcal{H}) \subseteq z\mathcal{H}$. Therefore $I_\phi(\mathcal{H}) = z\mathcal{H}$ if and only if $\phi$ and $\phi^{-1}$ belongs to $I(\mathcal{H})$.

Proof By the definition of $I_\phi$ the first statement is clear. We will
show the second one. If both $\phi$ and $\phi^{-1}$ belong to $\mathcal{I}(\mathcal{H})$, then

$$z\mathcal{H} = I_1(\mathcal{H}) = I_\phi I_{\phi^{-1}}(\mathcal{H}) \subseteq I_\phi(z\mathcal{H}) \subseteq z\mathcal{H}$$

because $I_\phi$ and $I_{\phi^{-1}}$ are bounded on $\mathcal{H}$. Conversely if $I_\phi(\mathcal{H}) = z\mathcal{H}$ then there exists a function $g$ in $\mathcal{H}$ such that

$$\int_0^z g'(\zeta)\phi(\zeta)d\zeta = z$$

and so $g'(z)\phi(z) = 1$.

Hence $\phi^{-1} \in H(D)$ and

$$z\mathcal{H} = I_1(\mathcal{H}) = I_{\phi^{-1}} I_\phi(\mathcal{H}) = I_{\phi^{-1}}(z\mathcal{H})$$

and so both $\phi$ and $\phi^{-1}$ belong to $\mathcal{I}(\mathcal{H})$.

**Lemma 4.** If $p$ is a polynomial then $I_p(\mathcal{H}) + \mathbb{C} \supset p^2\mathcal{H}$.

**Proof** Suppose $g \in \mathcal{H}$. Since $z \in \mathcal{I}(\mathcal{H})$ by the hypothesis, $p$ belongs to $\mathcal{I}(\mathcal{H})$ and so

$$\int_0^z g(\zeta)p(\zeta)d\zeta \in \mathcal{H}.$$  

Since $p' \in \mathcal{M}(\mathcal{H})$ and $z \in J(\mathcal{H})$,

$$\int_0^z g(\zeta)p'(\zeta)d\zeta \in \mathcal{H}.$$  

Hence $f(z) = \int_0^z (2p'(\zeta)g(\zeta) + p(\zeta)g'(\zeta))d\zeta$ belongs to $\mathcal{H}$. Now the lemma follows because

$$\int_0^z f'(\zeta)p(\zeta)d\zeta = \int_0^z (p^2(\zeta)g(\zeta))'d\zeta = p^2(z)g(z) + p^2(0)g(0).$$

**Lemma 5.** Suppose that $B$ is a finite Blaschke product, and both $g$ and $g^{-1}$ are in $\mathcal{I}(\mathcal{H})$. If $\phi = Bg$ then $\phi \in \mathcal{I}(\mathcal{H})$ and $\dim \mathcal{H}/I_\phi(\mathcal{H}) < \infty$.

**Proof** By the hypothesis, $I_B(\mathcal{H}) = I_B(z\mathcal{H}) = I_B(I_\phi(\mathcal{H})) = I_\phi(\mathcal{H})$ by Lemma 3. We may assume that

$$B = \prod_{j=1}^n \frac{z-a_j}{1-\bar{a}_j z}$$

and $\{a_j\} \subset D$.

Since $\Pi_{j=1}^n (1-a_jz)$ is invertible in $\mathcal{I}(\mathcal{H})$, by Lemma 3 $I_\phi(\mathcal{H}) = I_p(\mathcal{H})$ where $p = \Pi_{j=1}^n (z-a_j)$. Lemmas 1 and 4 imply that $\dim \mathcal{H}/I_\phi(\mathcal{H}) < \infty$.

**Lemma 6.** If $p$ is a polynomial then $p(S_z) = S_{p(z)}$.

**Proof** By hypothesis, $P^N I_z (I - P^N) = 0$. Hence

$$S_{z^2} = P^N I_{z^2} P^N = P^N I_z I_z P^N = P^N I_z (I - P^N) I_z P^N + P^N I_z P^N I_z P^N = P^N I_z P^N I_z P^N = S_z S_{z^2}.$$
Now it is easy to see that \( p(S_z) = S_{p(z)} \) for a polynomial \( p \).

**Lemma 7.** If \( M \) is a closed invariant subspace of \( I_z \) and \( \dim \mathcal{H}/M = n < \infty \) then there exists a polynomial \( p \) such that the degree of \( p \leq n \) and \( I_p(\mathcal{H}) \subseteq M \).

**Proof** If we put \( N = \mathcal{H} \oplus M \), then \( \dim N < \infty \) and so there exists a polynomial \( p \) such that \( p(S_z) = 0 \) and the degree of \( p \leq n \). By Lemma 6, \( S_p(\mathcal{H}) = 0 \) and so \( I_p(N) \subseteq M \). Since \( I_p(M) \subseteq M \), \( I_p(\mathcal{H}) \subseteq M \).

**Theorem 2.** Suppose \( \mathcal{I}(\mathcal{H}) \) contains \( H(D) \) and if \( f \in \mathcal{I}(\mathcal{H}) \) and \( f(a) = 0 \) for some \( a \in D \) then \( f/(z-a) \) belongs to \( \mathcal{I}(\mathcal{H}) \). \( I_\phi \) is a Fredholm operator on \( \mathcal{H} \) if and only if \( \phi = Bg \) where \( B \) is a finite Blaschke product, and \( g \) and \( g^{-1} \) are in \( \mathcal{I}(\mathcal{H}) \).

**Proof** If \( \phi = Bg \), \( B \) is a finite Blaschke product, \( g \in \mathcal{I}(\mathcal{H}) \) and \( g^{-1} \in \mathcal{I}(\mathcal{H}) \) then by Lemma 5 \( I_\phi(\mathcal{H}) \) is closed and \( \dim \ker I_\phi < \infty \). Since \( \ker I_\phi = \mathbb{C} \), index \( I_\phi = 1 - \dim \ker I_\phi^* \) and so \( I_\phi \) is Fredholm. Conversely if \( I_\phi \) is Fredholm then \( I_\phi(\mathcal{H}) \) is closed and \( \dim \mathcal{H}/I_\phi(\mathcal{H}) < \infty \). Since \( I_z I_\phi(\mathcal{H}) \subseteq I_\phi(\mathcal{H}) \), by Lemma 7 there exists a polynomial such that \( I_p(\mathcal{H}) \subseteq I_\phi(\mathcal{H}) \). By Lemma 4 \( I_p(\mathcal{H}) + \mathbb{C} \supseteq p^2 \mathcal{H} \). Hence there exists a function \( F \) in \( \mathcal{H} \) and \( c \in \mathbb{C} \) such that \( I_\phi(F) + c = p^2 \). Therefore \( F'(z)\phi(z) = 2p(z)p'(z) \) and so the Blaschke part of \( \phi \) is a finite one \( B \). Thus \( \phi \) can be factorized as \( \phi = Bg \) where \( g \in \mathcal{I}(\mathcal{H}) \) and \( g \) has no zeros on \( D \) because \( \mathcal{I}(\mathcal{H}) \) is a subalgebra in \( \mathcal{B}(\mathcal{H}) \) and both \( B \) and \( B^{-1} \) are in \( \mathcal{I}(\mathcal{H}) \). Hence

\[
I_{g^{-1}p}(\mathcal{H}) \subseteq I_{g^{-1}I_\phi}(\mathcal{H}) = I_B(\mathcal{H}) \subseteq \mathcal{H}
\]

and so \( g^{-1}p \) belongs to \( \mathcal{I}(\mathcal{H}) \). By hypothesis on \( \mathcal{I}(\mathcal{H}) \), \( g^{-1} \) belongs to \( \mathcal{I}(\mathcal{H}) \).

§ 4. Integral operator \( J_\phi \)

A Fredholm integral operator \( J_\phi \) have not studied. But if \( J_\phi \) is compact then it is not Fredholm. In some special Hilbert space \( \mathcal{H} \), the compactness of \( J_\phi \) have studied.

**Lemma 8.** If \( \phi \) and \( \psi \) are in \( H(D) \) then \( I_\psi J_\phi = J_\phi M_\psi \).

**Proof** For \( f \in \mathcal{H} \)

\[
(I_\psi J_\phi f)(z) = \int_0^z (J_\phi f)'(\zeta)\psi(\zeta)d\zeta = \int_0^z f(\zeta)\phi'(\zeta)\psi(\zeta)d\zeta = (J_\phi M_\psi f)(z)
\]
Lemma 9. If $J_\phi$ is a Fredholm operator on $\mathcal{H}$ then $J_\phi \mathcal{H}$ is a closed invariant subspace of $I_z$ and $\dim \mathcal{H}/J_\phi \mathcal{H} < \infty$. Hence there exists a polynomial $p$ such that $J_\phi \mathcal{H} \supseteq I_p \mathcal{H}$ and so $J_\phi \mathcal{H} + \mathbb{C} \supseteq p^2 \mathcal{H}$.

Proof If $J_\phi$ is Fredholm on $\mathcal{H}$ then $J_\phi \mathcal{H}$ is a closed subspace and by Lemma 8 $I_z(J_\phi \mathcal{H}) \subseteq J_\phi \mathcal{H}$. By Lemma 7 there exists a polynomial $q$ such that $I_q \mathcal{H} \subseteq J_\phi \mathcal{H}$. Lemma 4 implies this lemma.

Theorem 3. Suppose that there exists a function $g$ in $\mathcal{H}$ such that $g'$ does not belong to $H^2$. Suppose that any function in $\mathcal{H}$ has radial limits almost everywhere. Then there does not exist $J_\phi$ which is a Fredholm operator on $\mathcal{H}$.

Proof If $J_\phi$ is Fredholm on $\mathcal{H}$ then $J_\phi \mathcal{H} + \mathbb{C} \supseteq p^2 \mathcal{H}$ for some polynomial by Lemma 9. For any $G$ in $p^2 \mathcal{H}$ there exists a function $f$ in $\mathcal{H}$ such that $f(z)\phi'(z) = G'(z)$ ($z \in D$). By hypothesis, there exists $G$ in $p^2 \mathcal{H}$ such that $G' \notin H^2$ and so $G'$ does not have radial limits on a set of positive measure on $\partial D$ (see [2, Appendix A]). On the other hand, if $G = p^2$ then $G$ has radial limits almost everywhere on $\partial D$. By hypothesis, $f$ has radial limits almost everywhere. This contradiction implies that $J_\phi$ is not Fredholm.

§ 5. Relation between $M_\phi$ and $I_\phi$

Put $Df(z) = f'(z)$ and $J = J_z$, that is, $Jf(z) = \int_0^z f(\zeta)d\zeta$. Then

$$DJf = f \quad \text{and} \quad JDf = f - f(0).$$

It is easy to see that $I_\phi J = JM_\phi$ and $DI_\phi = M_\phi D$. Put

$$\mathcal{H}^D = \{f \in H(D) : Df \in \mathcal{H}\}$$

Suppose that $D$ and $J$ are bounded on $\mathcal{H}$ and for $f$ in $\mathcal{H}^D$ put $\|f\|^2_D = \|Df\|^2 + |f(0)|^2$. Then $\mathcal{H}^D$ is a Hilbert space. Put

$$\mathcal{H}^J = \{f \in H(D) : Jf \in \mathcal{H}\}$$

and for $f$ in $\mathcal{H}^J$ $\|f\|_J = \|Jf\|$. Then $\mathcal{H}^J$ is a Hilbert space.

$D$ is isometric from $\mathcal{H}^D_0 = \{f \in \mathcal{H}^D : f(0) = 0\}$ onto $\mathcal{H}$. $J$ is isometric from $\mathcal{H}^J$ onto $\mathcal{H}_0 = \{f \in \mathcal{H} : f(0) = 0\}$. Since $DI_\phi = M_\phi D$, $I_\phi$ is bounded on $\mathcal{H}^D$ if and only if $M_\phi$ is bounded on $\mathcal{H}$. Hence $I(\mathcal{H}^D) = \mathcal{M}(\mathcal{H})$. Moreover $I_\phi$ is Fredholm on $\mathcal{H}^D$ if and only if $M_\phi$ is Fredholm on $\mathcal{H}$. Since $JM_\phi = I_\phi J$,
\(I(\mathcal{H}^d) = \mathcal{M}(\mathcal{H})\), and \(I_\phi\) is Fredholm on \(\mathcal{H}^d\) if and only if \(M_\phi\) is Fredholm on \(\mathcal{H}\). Moreover \((\mathcal{H}^d)^D = (\mathcal{H}^D)^I = \mathcal{H}\). Hence \(I(\mathcal{H}) = M(\mathcal{H}^D) = \mathcal{M}(\mathcal{H}^d)\), and \(I_\phi\) is Fredholm on \(\mathcal{H}\) if and only if \(M_\phi\) is Fredholm on \(\mathcal{H}^D\) and \(\mathcal{H}^d\).

§ 6. Examples

Let \(dA\) denote the normalized Lebesgue area measure on \(D\) and \(\omega\) a positive function on \(D\) which is summable with respect to \(dA\). Put

\[
D^2(\omega) = \{ f \in H(D) : \| f \|_{D^2}^2 = |f(0)|^2 + \int_D |f'(z)|^2 \omega(z) dA(z) < \infty \}
\]

and

\[
L^2_a(\omega) = \{ f \in H(D) : \| f \|_{L^2_a}^2 = \int_D |f(z)|^2 \omega(z) dA(z) < \infty \}.
\]

Then \(D^2(\omega)\) is called a weighted Dirichlet space and \(L^2_a(\omega)\) is called a weighted Bergman space when \(D^2(\omega)\) and \(L^2_a(\omega)\) are nontrivial Hilbert spaces. It is easy to see that \((D^2(\omega))^D = L^2_a(\omega)\) and \((L^2_a(\omega))^D = D^2(\omega)\).

If \(\omega(z) = (1 - \lvert z \rvert^2)^\alpha\) and \(\alpha > -1\), we will write \(D^2(\omega) = D^2_\alpha\) and \(L^2_a(\omega) = L^2_{a,\alpha}\). It is known that \(D^2_\alpha\) and \(L^2_{a,\alpha}\) are nontrivial Hilbert spaces. \(D_1\) is the Hardy space \(H^2\), \(D_2\) is the Bergman space \(L^2_a\) and \(D_0\) is the Dirichlet space. If \(\mathcal{H} = D_\alpha\) or \(L^2_{a,\alpha}\) then \(\mathcal{H}\) satisfies the condition (1), (2) and (3) in Introduction. It is known that \(H(\bar{D}) \subset M(D_\alpha) \subset H^\infty(D)\) and \(M(L^2_{a,\alpha}) = H^\infty(D)\). Hence Theorem 1 can apply to \(D_\alpha\) for any \(\alpha > -1\). If \(\alpha \geq 1\) then \((z - a)D_\alpha\) is dense in \(D_\alpha\) whenever \(a \in \partial D\). Hence Corollary 1 can apply to \(D_\alpha\) for \(\alpha \geq 1\).

\(I(L^2_{a,\alpha}) = \mathcal{M}(L^2_{a,\alpha}) = M(D_\alpha)\) and \(H(\bar{D}) \subset M(D_\alpha) \subset H^\infty(D)\). Since \(I(D_\alpha) = M(L^2_{a,\alpha}) = H^\infty(D)\), Theorem 2 can apply to \(D_\alpha\) for \(\alpha > -1\). It is known [3] that \(M(D_\alpha) = H^\infty(D)\) for \(\alpha > 1\) and \(M(D_\alpha) = D_\alpha\) for \(-1 < \alpha < 0\). Hence \(I(L^2_{a,\alpha}) = H^\infty(D)\) for \(\alpha > 1\) and \(I(L^2_{a,\alpha}) = D_\alpha\) for \(-1 < \alpha < 0\). Hence Theorem 2 can apply to \(L^2_{a,\alpha}\) for \(\alpha > 1\) and \(-1 < \alpha < 0\). By a theorem in [3], it is easy to see that \(I(L^2_{a,\alpha}) = M(D_\alpha)\) \((0 \leq \alpha \leq 1)\) satisfies the conditions in Theorem 2. Hence Theorem 2 can apply to \(L^2_{a,\alpha}\).

When \(D^2(\omega)\) or \(L^2_a(\omega)\) is a Hilbert space \(\mathcal{H}\), it is important in order to study composition operator that \(\mathcal{H}\) satisfies three conditions in Introduction. It will be interesting to determine such a weight \(\omega\).

References


Takahiko Nakazi
Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060-0810, Japan
nakazi@math.sci.hokudai.ac.jp