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Singularities of timelike Anti de Sitter Gauss images

LIANG CHEN*

Department of Mathematics, Faculty of Science, Hokkaido University,

Sapporo 060-0810, Japan (chenliang@math.sci.hokudai.ac.jp)

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Abstract We study the differential geometry of spacelike surfaces in Anti de Sitter 3-space from the view point of Legendrian singularity theory. We define the timelike Anti de Sitter Gauss image on spacelike surface and investigate the geometric meanings of singularities.

Keywords: Anti de Sitter 3-space; TAdS-Gauss image; AdS-G-K curvature; Legendrain singularities.

2000 *Mathematics Subject classification:* Primary 53A35; 58C27

1 Introduction

Recently, there appeared several articles of differential geometry on submanifolds in Lorentzian space forms as applications of singularity theory [5, 7, 8, 9, 10, 11, 12, 13, 14]. Minkowski space is a flat Lorentzian space form and de Sitter space is the Lorentzian space form with positive constant curvature. The Lorentzian space form with the negative constant curvature is called Anti de Sitter space which is a vacuum solution of the Einstein equation. However, there are very few researches on differential geometry of submanifolds in Anti de Sitter space as applications of singularity theory so far as we know. In this paper we study the differential geometry on spacelike surfaces in Anti de Sitter 3-space from the view point of the theory of Legendrian singularities.

On the other hand, hypersurfaces in hyperbolic space have been studied in [6]. The basic notions and tools for the study of the differential geometry of hypersurfaces in hyperbolic space have been established. Especially, the hyperbolic Gauss indicatrix of a hypersurface in hyperbolic space has been explicity described and the contact of hypersurfaces with model hypersurfaces has been systematically studied as an application of singulary theory to the hyperbolic Gauss indicatrix. Our aim in this paper is to develop the analogous study for spacelike surfaces in Anti de Sitter 3-space. In §2 we first show the basic notions on semi-Euclidean 4-space with index 2 and contact geometry. Especially we have proved the Legendrian duality theorem (Theorem 2.1) between Anti de Sitter 3-spaces, which is the key to see the view of the whole. In §3 we develop the local differential geometry of spacelike surfaces in Anti de Sitter 3-space and introduce the notion of timelike Anti de Sitter Gauss image of a spacelike surface in Anti de Sitter 3-space. Corresponding to this notion we define the Anti de

**On leave from School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, People's Republic of China.*(chenl234@nenu.edu.cn)

Sitter Gauss Kronecker(briefly, AdS-G-K) curvature and consider the geometry meaning of this curvature. One of our conclusions asserts that the AdS-G-K curvature describes the contact of spacelike surfaces with some model srufaces (i.e., AdS-great hyperboloids). We introduce the notion of timelike height function in §4, named AdS-height function, which is useful to show that the TAdS-Gauss image has a singular point if and only if the AdS-G-K curvature vanished at such point. In §5, 6, we apply mainly the theory of Legendrian singularities for the study of TAdS-Gauss image and interpret the TAdS-Gauss image as a Legendrian map in a nature Legendrian fibration whose generating family is the AdS-height function on spacelike surface. We also study the contact of spacelike surfaces with AdS-great-hyperboloids. In §7 we study generic properties. In §8, we give a classification of singularities of TAdS-Gauss image. In the last part, §9 we introduce the notion of the AdS-Monge form of a spacelike surface in Anti de Sitter 3-space and give some examples.

We shall assume throughout the whole paper that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

2 The basic notations and the duality theorem

In this section we prepare basic notions on semi-Euclidean 4-space with index 2 and contact geometry.

Let $\mathbb{R}^4 = \{(x_1, \cdots, x_4) | x_i \in \mathbb{R}\ (i = 1, \cdots, 4)\ \}$ be a 4-dimensional vector space. For any vectors $\boldsymbol{x} = (x_1, \cdots, x_4)$ and $\boldsymbol{y} = (y_1, \cdots, y_4)$ in $\mathbb{R}^4$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined to be $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4$. We call $(\mathbb{R}^4, \langle , \rangle)$ a *smei-Euclidean 4-space with index 2* and write $\mathbb{R}^4_2$ instead of $(\mathbb{R}^4, \langle , \rangle)$.

We say that a non-zero vector $\boldsymbol{x}$ in $\mathbb{R}^4_2$ is *spacelike*, *null* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^4_2$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. For a vector $\boldsymbol{n} \in \mathbb{R}^4_2$ and a real number c, we define the *hyperplane with pseudo-normal* $\boldsymbol{n}$ by

$$HP(\boldsymbol{n}, c) = \{\boldsymbol{x} \in \mathbb{R}^4_2 | \langle \boldsymbol{x}, \boldsymbol{n} \rangle = c\}.$$

We call $HP(\boldsymbol{n}, c)$ a *Lorentz hyperplane*, a *semi-Euclidean hyperplane of index 2* or a *null hyperplane* if $\boldsymbol{n}$ is *timelike, spacelike or null* respectively.

We now define *Anti de Sitter 3-space* (briefly, *AdS 3-space*)by

$$H^3_1 = \{\boldsymbol{x} \in \mathbb{R}^4_2 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\}$$

For any $\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3 \in \mathbb{R}^4_2$. We define a vector $\boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3$ by

$$\boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3 = \begin{vmatrix} -\boldsymbol{e}_1 & -\boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \\ x^1_1 & x^1_2 & x^1_3 & x^1_4 \\ x^2_1 & x^2_2 & x^2_3 & x^2_4 \\ x^3_1 & x^3_2 & x^3_3 & x^3_4 \end{vmatrix},$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4$ is the canonical basis of $\mathbb{R}^4_2$ and $\boldsymbol{X}_i = (x^i_1, x^i_2, x^i_3, x^i_4)$. We can easily check that

$$\langle \boldsymbol{X}, \boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3 \rangle = \det(\boldsymbol{X}, \boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3),$$

so that $\boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3$ is pseudo-orthogonal to any $\boldsymbol{X}_i$ (for $i = 1, 2, 3$).

In this paper We stick to spacelike surfaces in Anti de Sitter 3-space H^3_1. Typical spacelike surfaces in H^3_1 are given by the intersection of H^3_1 with a Lorentz hyperplane in $\mathbb{R}^4_2$:

$$AH(\boldsymbol{n}, c) = H^3_1 \cap HP(\boldsymbol{n}, c),$$

where $\|\boldsymbol{n}\| > |c|$. We say that $AH(\boldsymbol{n}, c)$ is a *AdS-hyperboloid* in the Anti de Sitter 3-space. In particular, we call $AH(\boldsymbol{n}, 0)$ the *AdS-great-hyperboloid*.

On the other hand, we now give a brief review on contact manifolds and Legendrian submanifolds. For some detailed results on contact geometry, please refer to [23]. Let $\pi : PT^*M \longrightarrow M$ be the projective cotangent bundle. This fibration can be considered as a Legendrian fibration with the canonical contact structure K. We now review geometric properties of this space. Consider the tangent bundle $\tau : TPT^*M \longrightarrow PT^*M$ and differential map $d\pi : TPT^*M \longrightarrow TM$ of π. For any $X \in TPT^*M$, there exits an element $\alpha \in T^*M$ such that $\tau(X) = [\alpha]$. For an element $V \in T_xM$, the property $\alpha(V) = 0$ does not depend on the choice of the representative of the class $[\alpha]$. Thus we can define the canonical contact structure on PT^*M by

$$K = \{X \in TPT^*M \mid \tau(X)(d\pi(X)) = 0\}$$

For a local coordinate neighborhood $(U, (x_1, \cdots, x_n))$ on M, we have a trivialization

$$PT^*U \cong U \times P(\mathbb{R}^{n-1})^*$$

and we call $((x_1, \cdots, x_n), [\xi_1 : \cdots : \xi_n])$ homogeneous coordinates, where $[\xi_1 : \cdots : \xi_n]$ are homogeneous coordinates of the dual projective space $P(\mathbb{R}^{n-1})^*$. It is easy to show that $X \in K_{(x,\xi)}$ if and only if $\sum_{i=1}^{n} \mu_i \xi_i$, where $d\pi(X) = \sum_{i=1}^{n} \mu_i \frac{\partial}{\partial x_i}$. An immersion $i : L \longrightarrow PT^*M$ is said to be a *Legendrian immersion* if dim $L = n - 1$ and $di_q(T_qL) \subset K_{i(q)}$ for any $q \in L$. We also call the map $\pi \circ i$ the *Legendrian map* and the set $W(i) =$image $\pi \circ i$ the *wave front* of i. Moreover, i (or, the image of i) is called the *Legendrian lift* of $W(i)$.

We now show the basic theorem in this paper which is the fundamental tool for the study of spacelike surfaces in H_1^3. We consider the following double fibrations:

(1) $H_1^3 \times H_1^3 \supset \Delta = \{(\boldsymbol{v}, \boldsymbol{w}) | \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0\}$,

(2) $\pi_1 : \Delta \longrightarrow H_1^3,\ \pi_2 : \Delta \longrightarrow H_1^3$,

(3) $\theta_1 = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle \mid \Delta,\ \theta_2 = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle \mid \Delta$.

Where

$$\pi_1(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v},\ \pi_2(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{w},$$

$$\langle d\boldsymbol{v}, \boldsymbol{w} \rangle = -w_1 dv_1 - w_2 dv_2 + w_3 dv_3 + w_4 dv_4,$$

$$\langle \boldsymbol{v}, d\boldsymbol{w} \rangle = -v_1 dw_1 - v_2 dw_2 + v_3 dw_3 + v_4 dw_4.$$

The basic theorem in this paper is the following theorem:

Theorem 2.1 *Under the same notations as the above paragraph, each* $(\Delta, \theta_i)(i = 1, 2)$ *is a contact manifold and both of* $\pi_i(i = 1, 2)$ *are Legendrian fibrations.*

Proof. By definition we can easily to show that Δ is a smooth submanifold in $\mathbb{R}_2^4 \times \mathbb{R}_2^4$ and each $\pi_i(i = 1, 2)$ is a smooth fibration.

For any $\boldsymbol{w} = (w_1, w_2, w_3, w_4) \in H_1^3$, we have $w_1 \neq 0$ or $w_2 \neq 0$. Then we can consider a coordinate neighborhood $W_1^+ = \{\boldsymbol{w} = (w_1, w_2, w_3, w_4) \in H_1^3 \mid w_1 > 0\}$ on which we have

$$w_1 = \sqrt{-w_2^2 + w_3^2 + w_4^2 + 1}.$$

Therefore, we regard that (w_2, w_3, w_4) is the local coordinates on W_1^+. We consider a mapping $\Phi : \Delta(W_1^+) \longrightarrow PT^*H_1^3 \mid W_1^+$ defined by

$$\Phi(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{w}, [v_1w_2 - v_2w_1 : -v_1w_3 + v_3w_1 : -v_1w_4 + v_4w_1]).$$

Let $((w_2, w_3, w_4), [\xi_1 : \xi_2 : \xi_3])$ be the homogeneous coordinates of $PT^*H_1^3$ over W_1^+. We have the canonical contact form $\theta = \sum_{i=1}^3 w_{i+1}\xi_i$ on $PT^*H_1^3$ over W_1^+. It follows that

$$\begin{aligned}\Phi^*\theta &= (v_1w_2 - v_2w_1)dw_2 + \textstyle\sum_{i=3}^4(-v_1w_i + v_iw_1)dw_i \\ &= w_1\langle \boldsymbol{v}, d\boldsymbol{w}\rangle \mid \Delta(W_1^+) = w_1\theta_2 \mid W_1^+.\end{aligned}$$

This means that θ_2 is a contact structure such that Φ is a contact morphism. We have the similar calculation as the above on the other coordinate neighborhoods. Thus (Δ, θ_2) is a contact manifold.

On the other hand, from the fact that $\langle \boldsymbol{v}, \boldsymbol{w}\rangle = 0$, we have $\langle d\boldsymbol{v}, \boldsymbol{w}\rangle + \langle \boldsymbol{v}, d\boldsymbol{w}\rangle = 0$. That is

$$\langle d\boldsymbol{v}, \boldsymbol{w}\rangle = 0 \Longleftrightarrow \langle \boldsymbol{v}, d\boldsymbol{w}\rangle = 0.$$

Therefore, both of θ_1 and θ_2 give the common contact structure on Δ. Other assertions are trivial by definition. This completes the proof. □

3 The local differential geometry of spacelike surfaces in Anti de Sitter 3-space

In this section we introduce the local differential geometry of spacelike surfaces in Anti de Sitter 3-space.

Let $\boldsymbol{X} : U \longrightarrow H_1^3$ be a regular surface (i.e., an embedding), where $U \subset \mathbb{R}^2$ is an open subset. We denote $M = \boldsymbol{X}(U)$ and identify M with U through the embedding $\boldsymbol{X}$. The embedding $\boldsymbol{X}$ is said to be spacelike if the induced metric $\mathbf{I}$ of M is Riemannian. Throughout the remain in this paper we assume that M is an spacelike surface in H_1^3. Since $\langle \boldsymbol{X}, \boldsymbol{X}\rangle \equiv -1$, we have

$$\langle \boldsymbol{X}, \boldsymbol{X}_{u_i}\rangle \equiv 0 \ (for\ i = 1, 2),$$

where $u = (u_1, u_2) \in U$. We define a vector $\boldsymbol{e}(u)$ by

$$\boldsymbol{e}(u) = \frac{\boldsymbol{X}(u) \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2}}{\|\boldsymbol{X}(u) \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2}\|}.$$

By definition, we have

$$\langle \boldsymbol{e}, \boldsymbol{X}_{u_i}\rangle \equiv \langle \boldsymbol{e}, \boldsymbol{X}\rangle \equiv 0,$$

Since $\boldsymbol{X}$ is timelike and $\boldsymbol{X}_{u_i}$ $(i = 1, 2)$ are spacelike, $\boldsymbol{e}$ is timelike. Therefore

$$\langle \boldsymbol{e}, \boldsymbol{e}\rangle \equiv -1.$$

We now define a map

$$\mathbb{T} : U \longrightarrow H_1^3$$

by $\mathbb{T}(u) = \boldsymbol{e}(u)$ which is called the *timelike Anti de Sitter Gauss image* (briefly, *TAdS-Gauss image*) of $\boldsymbol{X}$(or M).

We now consider the geometric meanings of the TAdS-Gauss image of a spacelike surface. We have the following proposition.

Proposition 3.1 *Let $\boldsymbol{X} : U \longrightarrow H^3_1$ be a spacelike surface in Anti de Sitter 3-space. If the TAdS-Gauss image $\mathbb{T}$ is constant, then the spacelike surface $\boldsymbol{X}(U) = M$ is a part of a AdS-great-hyperboloid.*

Proof. We consider the set $V = \{\boldsymbol{y} \in \mathbb{R}^4_2 | \langle \boldsymbol{y}, \boldsymbol{e}\rangle = 0\}$. Since $\mathbb{T} = \boldsymbol{e}$ is constant, the set $V = HP(\boldsymbol{e}, 0)$ is a Lorentz hyperplane. We also have $\langle \boldsymbol{X}, \boldsymbol{e}\rangle \equiv 0$, so $\boldsymbol{X}(U) = M \subset V \cap H^3_1$. $\square$

It is easy to show that $\mathbb{T}_{u_i}$ $(i = 1, 2)$ are tangent vectors of M. Therefore we have a linear transformation $W_p = -d\mathbb{T}(u) : T_pM \longrightarrow T_pM$ which is called the *Anti de Sitter shape operator* (briefly, *AdS-shape operator*) of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. We denote the eigenvalue of W_p by $k_i(p)$ $(i = 1, 2)$.

The *Anti de Sitter Gauss-Kronecker curvature* (briefly, *AdS-G-K curvature*) of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$ is defined to be

$$K_{AdS}(u) = \det W_p = k_1(p) \cdot k_2(p).$$

We say that a point $p = \boldsymbol{X}(u)$ is an *Anti de Sitter parabolic point* (or, briefly an *AdS-parabolic point*) of $\boldsymbol{X} : U \longrightarrow H^3_1$ if $K_{AdS}(u) = 0$.

We say that a point $u \in U$ or $p = \boldsymbol{X}(u)$ is an *umbilic point* if $W_p = k(p)id_{T_pM}$. We also say that $M = \boldsymbol{X}(U)$ is *totally umbilic* if all points on M are umbilic. Then we have the following proposition.

Proposition 3.2 *Suppose that $M = \boldsymbol{X}(U)$ is totally umbilic. Then $k(p)$ is constant k. Under this condition, we have the following classification.*

(1) If $k \neq 0$ then M is a part of a AdS-hyperboloid $HP(\boldsymbol{n}, -1) \cap H^3_1$, where $\boldsymbol{n} = \boldsymbol{X} + \frac{1}{k}\boldsymbol{e}$ is a constant timelike vector.

(2) If $k = 0$ then M is a part of a AdS-flat hyperboloid $HP(\boldsymbol{n}, 0) \cap H^3_1$, where $\boldsymbol{n} = \boldsymbol{e}$ is a constant timelike vector.

Proof. By definition, we have $-\mathbb{T}_{u_i} = k\boldsymbol{X}_{u_i}$ for $i = 1, 2$. Therefore we have

$$-\mathbb{T}_{u_iu_j} = k_{u_j}\boldsymbol{X}_{u_i} + k\boldsymbol{X}_{u_iu_j}$$

Since $-\mathbb{T}_{u_iu_j} = -\mathbb{T}_{u_ju_i}$ and $k\boldsymbol{X}_{u_iu_j} = k\boldsymbol{X}_{u_ju_i}$, we have $k_{u_j}\boldsymbol{X}_{u_i} = k_{u_i}\boldsymbol{X}_{u_j}$. From the fact $\{\boldsymbol{X}_{u_1}, \boldsymbol{X}_{u_2}\}$ is linearly independent, so that k is a constant.

We now assume that $k \neq 0$. Since $-\mathbb{T}_{u_i} = -\boldsymbol{e}_{u_i} = k\boldsymbol{X}_{u_i}$, there exists a constant vector $\mathbf{n}$ such that $\boldsymbol{X} = \mathbf{n} - \frac{\boldsymbol{e}}{k}$. We can calculate that

$$\langle \mathbf{n}, \mathbf{n}\rangle = \langle \boldsymbol{X} + \frac{\boldsymbol{e}}{k}, \boldsymbol{X} + \frac{\boldsymbol{e}}{k}\rangle = -1 - \frac{1}{k^2} < 0,$$

and

$$\langle \boldsymbol{X}, \mathbf{n}\rangle = \langle \boldsymbol{X}, \boldsymbol{X} + \frac{\boldsymbol{e}}{k}\rangle = -1.$$

This means that $M = \boldsymbol{X}(U) \subset HP(\mathbf{n}, -1) \cap H^3_1$, so in this case the assertion follows.

If $k = 0$. Then $\mathbb{T} = \boldsymbol{e} = \mathbf{n}$. in this case the assertion follow from the Proposition 3.1. This completes the proof. $\square$

Since $\boldsymbol{X}_{u_1}$ and $\boldsymbol{X}_{u_2}$ are spacelike vectors, we first introduce the Riemannian metric $ds^2 = \sum_{i,j=1}^2 g_{ij}du_idu_j$ on $M = \boldsymbol{X}(U)$, where $g_{ij}(u) = \langle \boldsymbol{X}_{u_i}(u), \boldsymbol{X}_{u_j}(u)\rangle$ for any $u \in U$. We also define the *Anti de Sitter second fundamental invariant* by $h_{ij}(u) = \langle -\mathbb{T}_{u_i}(u), \boldsymbol{X}_{u_j}(u)\rangle$ for any $u \in U$. We have the following results similar to the results of [6].

Proposition 3.3 *With the above notation, we have the following Anti de Sitter Wein-*

garten formula:

$$\mathbb{T}_{u_i} = -\sum_{j=1}^{2} h_i^j \boldsymbol{X}_{u_j},$$

where $(h_i^j) = (h_{ik})(g^{kj})$ *and* $(g^{kj}) = (g_{kj})^{-1}$.

Proof. There exist real numbers $\alpha, \beta, \lambda_i^j$ such that

$$\mathbb{T}_{u_i} = \alpha \boldsymbol{X} + \beta \boldsymbol{e} + \sum_{j=1}^{2} \lambda_i^j \boldsymbol{X}_{u_j}$$

Since $\langle \mathbb{T}, \boldsymbol{X} \rangle = 0,\ \langle \mathbb{T}, \boldsymbol{X}_{u_i} \rangle = 0,\ \langle \mathbb{T}, \mathbb{T} \rangle = -1$, we have

$$0 = \langle \mathbb{T}_{u_i}, \boldsymbol{X} \rangle = -\alpha,\ \ 0 = \langle \mathbb{T}_{u_i}, \mathbb{T} \rangle = -\beta.$$

Therefore, we have

$$\mathbb{T}_{u_i} = \sum_{j=1}^{2} \lambda_i^j \boldsymbol{X}_{u_j}.$$

By definition, we have

$$-h_{ik} = \langle \mathbb{T}_{u_i}, \boldsymbol{X}_{u_k} \rangle = \sum_{l=1}^{2} \lambda_i^l g_{lk}.$$

Hence, we have

$$-h_i^j = -\sum_{l=1}^{2} h_{ik} g^{kj} = \lambda_i^j.$$

This completes the proof of the AdS-weingarten formula. □

As a corollary of the above proposition, we have an explicit expression for the AdS-G-K curvature by Riemannian metric and the Anti de Sitter second fundamental invariant.

Corollary 3.4 *With the same notation as in the above Proposition, we can give the AdS-G-K curvature as follows:*

$$K_{AdS} = \frac{\det(h_{ij})}{\det(g_{\alpha\beta})}.$$ □

Since ds^2 is a Riemannian metric, we have the section curvature K_I of M, which we call an intrinsic Gaussian curvature. By B. O'Neil [22] (Page 107 Corollary 20), we remark that $K_{AdS} = -1 - K_I$.

4 The timelike Anti de Sitter height function

In this section we define a family of functions on a spacelike surface in Anti de Sitter 3-space which is useful for the study of singularities of TAdS-Gauss image.

Let $\boldsymbol{X} : U \longrightarrow H_1^3$ be a spacelike surface. We define a family of functions

$$H : U \times H_1^3 \longrightarrow \mathbb{R}$$

by $H(u, \boldsymbol{v}) = \langle \boldsymbol{X}(u), \boldsymbol{v} \rangle$. We call H *a timelike Anti de Sitter height function* (or, a *AdS-height function*) on $M = \boldsymbol{X}(U)$. We denote the *Hessian matrix* of the AdS-height function

$h_{v_0}(u) = H(u, \boldsymbol{v}_0)$ at u_0 by $\mathrm{Hess}(h_{v_0})(u_0)$. Then we have the following proposition.

Proposition 4.1 *Let $M = \boldsymbol{X}(U)$ be a spacelike surface in H_1^3 and $H : U \times H_1^3 \longrightarrow \mathbb{R}$ be a AdS-height function. Then we have the following assertions:*

(1) $H(u, \boldsymbol{v}) = \frac{\partial H}{\partial u_i}(u, \boldsymbol{v}) = 0$ *(for $i = 1, 2$) if and only if $\boldsymbol{v} = \pm \boldsymbol{e}(u) = \pm \mathbb{T}(u)$;*

(2) *Let $\boldsymbol{v}_0 = \boldsymbol{e}(u_0)$, then* $\det\mathrm{Hess}(h_{v_0})(u_0) = 0$ *if and only if $K_{AdS}(u_0) = 0$.*

Proof. (1) Since $\{\boldsymbol{X}, \boldsymbol{e}, \boldsymbol{X}_{u_1}, \boldsymbol{X}_{u_2}\}$ is a basis of the vector space $T_p\mathbb{R}_2^4$ where $p = \boldsymbol{X}(u)$, there exist real numbers $\lambda, \eta, \alpha_1, \alpha_2$ such that $\boldsymbol{v} = \lambda \boldsymbol{X} + \eta \boldsymbol{e} + \alpha_1 \boldsymbol{X}_{u_1} + \alpha_2 \boldsymbol{X}_{u_2}$. Therefore $H(u, \boldsymbol{v}) = 0$ if and only if $\lambda = -\langle \boldsymbol{X}(u), \boldsymbol{v} \rangle = 0$. Since $0 = \frac{\partial H}{\partial u_i}(u, \boldsymbol{v}) = \langle \boldsymbol{X}_{u_i}, \boldsymbol{v} \rangle = \sum_{j=1}^{2} g_{ij}\alpha_i$. Since (g_{ij}) is non-degenerate, we have $\alpha_i = 0$ (for $i = 1, 2$). Therefore we have $\boldsymbol{v} = \eta \boldsymbol{e}$. Then from a straight forward calculation, we have $\eta = \pm 1$.

(2) By definition, we have

$$\mathrm{Hess}(h_{v_0})(u_0) = (\langle \boldsymbol{X}_{u_i u_j}(u_0), \mathbb{T}(u_0) \rangle) = (-\langle \boldsymbol{X}_{u_i}(u_0), \mathbb{T}_{u_j}(u_0) \rangle).$$

By the AdS-Weingarten formula, we have

$$-\langle \boldsymbol{X}_{u_i}, \mathbb{T}_{u_j} \rangle = \sum_{\alpha=1}^{2} h_i^\alpha \langle \boldsymbol{X}_{u_\alpha}, \boldsymbol{X}_{u_j} \rangle = \sum_{\alpha=1}^{2} h_i^\alpha g_{\alpha j} = h_{ij}.$$

Therefore we have

$$K_{AdS} = \frac{\det(h_{i,j})}{\det(g_{\alpha\beta})} = \frac{\det\mathrm{Hess}(h_{v_0})(u_0)}{\det(g_{\alpha\beta}(u_0))}.$$

Then we complete the proof. □

As an application of the above proposition, we have the following.

Corollary 4.2 quad *Let $H : U \times H_1^3 \longrightarrow \mathbb{R}$, with $H(u, \boldsymbol{v}) = h_v(u)$ be a AdS-height function on spacelike surface $M = \boldsymbol{X}(U)$ and $\mathbb{T}$ be the TAdS-Gauss image, $p = \boldsymbol{X}(u)$. Then the following conditions are equivalent:*

(1) $\exists\ \boldsymbol{v} \in H_1^3$, *such that $p \in M$ is a degenerate singular point of AdS-height function h_v ;*

(2) $\exists\ \boldsymbol{v} \in H_1^3$, *such that $p \in M$ is a singular point of TAdS-Gauss image $\mathbb{T}$;*

(3) $K_{AdS}(u) = 0$.

Proof. By definition, (2) and (3) are equivalent. By the assertion (2) of above proposition, we have (1) and (3) are also equivalent. □

5 TAdS-Gauss images as Legendrian maps

In this section we naturally interpret the TAdS-Gauss image $\mathbb{T}$ of M as a Legendrian map in the framework of Legendrian singularity theory. We give a brief review on Legendrian singularity theory mainly due to Arnold [1]. The main tool of Legendrian singularities theory is the notion of generating families. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, \boldsymbol{0}) \longrightarrow (\mathbb{R}, 0)$ be a function germ. We say that F is a *Morse family* if the mapping

$$\Delta^* F = (F, \frac{\partial F}{\partial q_1}, \cdots, \frac{\partial F}{\partial q_k}) : (\mathbb{R}^k \times \mathbb{R}^n, \boldsymbol{0}) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, \boldsymbol{0})$$

is non-singular, where $(q,x)=(q_1,\cdots,q_k,x_1,\cdots,x_n)\in(\mathbb{R}^k\times\mathbb{R}^n,\mathbf{0})$. In this case we have a smooth $(n-1)-$dimensional submanifold,

$$\Sigma_*(F)=\{(q,x)\in(\mathbb{R}^k\times\mathbb{R}^n,\mathbf{0})|F(q,x)=\frac{\partial F}{\partial q_1}(q,x)=\cdots=\frac{\partial F}{\partial q_k}(q,x)=0\}$$

and the map germ $\Phi_F:(\Sigma_*(F),\mathbf{0})\longrightarrow PT^*\mathbb{R}^n$ defined by

$$\Phi_F(q,x)=(x,[\frac{\partial F}{\partial x_1}(q,x):\cdots:\frac{\partial F}{\partial x_n}(q,x)])$$

is a Legendrian immersion germ. Then we have the following fundamental theorem of Arnold [1] and Zakalyukin [20].

Proposition 5.1 *All Legendrian submanifold germs in $PT^*\mathbb{R}^n$ are constructed by the above method.*

We call F a *generating family* of $\Phi_F(\Sigma_*(F))$. Therefore the corresponding wave front is

$$W(\Phi_F)=\{x\in\mathbb{R}^n|\exists\ q\in\mathbb{R}^k \text{ such that } F(q,x)=\frac{\partial F}{\partial q_1}(q,x)=\cdots=\frac{\partial F}{\partial q_k}(q,x)=0\}.$$

We sometimes denote $\mathcal{D}_F=W(\Phi_F)$ and call it the *discriminant set* of F.

Now we can apply the above arguments to our situation. Let $\boldsymbol{X}:U\longrightarrow H^3_1$ be a spacelike surface in H^3_1 and $\mathbb{T}$ be the TAdS-Gauss image on $M=\boldsymbol{X}(U)$. We define a mapping

$$\mathcal{L}:U\longrightarrow\Delta$$

by $\mathcal{L}(u)=(\boldsymbol{X}(u),\mathbb{T}(u))$. Since $\langle\boldsymbol{X}(u),\mathbb{T}(u)\rangle=\langle d\boldsymbol{X}(u),\mathbb{T}(u)\rangle=0$, the mapping $\mathcal{L}$ is a Legendrian embedding. We denote $\boldsymbol{X}(u)=(x_1,x_2,x_3,x_4)$ and $\mathbb{T}(u)=(v_1,v_2,v_3,v_4)$ as coordinate representations. We define a smooth mapping

$$\mathcal{T}:U\longrightarrow PT^*(H^3_1)$$

by $\mathcal{T}(u)=(\mathbb{T}(u),[(x_1v_2-x_2v_1):(-x_1v_3+x_3v_1):(-x_1v_4+x_4v_1)])$.

Proposition 5.2 *The AdS-height function $H:U\times H^3_1\longrightarrow\mathbb{R}$ is a Morse family.*

Proof. For any $\boldsymbol{v}=(v_1,v_2,v_3,v_4)\in H^3_1$, we have $v_1\neq 0$ or $v_2\neq 0$. Without loss of the generality, we might assume that $v_1>0$, then $v_1=\sqrt{1+v_3^2+v_4^2-v_2^2}$. So that

$$H(u,\boldsymbol{v})=-x_1(u)\sqrt{1+v_3^2+v_4^2-v_2^2}-x_2(u)v_2+x_3(u)v_3+x_4(u)v_4$$

where $\boldsymbol{X}(u)=(x_1(u),x_2(u),x_3(u),x_4(u))$. We have to prove the mapping

$$\Delta^*H=(H,\frac{\partial H}{\partial u_1},\frac{\partial H}{\partial u_2})$$

is non-singular at any point. The Jacobian matrix of Δ^*H is given as follows:

$$\begin{pmatrix} \langle\boldsymbol{X}_{u_1},\boldsymbol{v}\rangle & \langle\boldsymbol{X}_{u_2},\boldsymbol{v}\rangle & x_1\frac{v_2}{v_1}-x_2 & -x_1\frac{v_3}{v_1}+x_3 & -x_1\frac{v_4}{v_1}+x_4 \\ \langle\boldsymbol{X}_{u_1u_1},\boldsymbol{v}\rangle & \langle\boldsymbol{X}_{u_1u_2},\boldsymbol{v}\rangle & x_{1u_1}\frac{v_2}{v_1}-x_{2u_1} & -x_{1u_1}\frac{v_3}{v_1}+x_{3u_1} & -x_{1u_1}\frac{v_4}{v_1}+x_{4u_1} \\ \langle\boldsymbol{X}_{u_2u_1},\boldsymbol{v}\rangle & \langle\boldsymbol{X}_{u_2u_2},\boldsymbol{v}\rangle & x_{1u_2}\frac{v_2}{v_1}-x_{2u_2} & -x_{1u_2}\frac{v_3}{v_1}+x_{3u_2} & -x_{1u_2}\frac{v_4}{v_1}+x_{4u_2} \end{pmatrix}.$$

We claim that it will suffice to show that the determinant of the matrix

$$A=\begin{pmatrix} x_1\frac{v_2}{v_1}-x_2 & -x_1\frac{v_3}{v_1}+x_3 & -x_1\frac{v_4}{v_1}+x_4 \\ x_{1u_1}\frac{v_2}{v_1}-x_{2u_1} & -x_{1u_1}\frac{v_3}{v_1}+x_{3u_1} & -x_{1u_1}\frac{v_4}{v_1}+x_{4u_1} \\ x_{1u_2}\frac{v_2}{v_1}-x_{2u_2} & -x_{1u_2}\frac{v_3}{v_1}+x_{3u_2} & -x_{1u_2}\frac{v_4}{v_1}+x_{4u_2} \end{pmatrix},$$

does not vanish at $(u, \boldsymbol{v}) \in \Delta^* H^{-1}(\mathbf{0})$. In this case, $\boldsymbol{v} = \mathbb{T}(u)$ and we denote

$$\boldsymbol{b}_1 = \begin{pmatrix} x_1 \\ x_{1u_1} \\ x_{1u_2} \end{pmatrix}, \ \boldsymbol{b}_2 = \begin{pmatrix} x_2 \\ x_{2u_1} \\ x_{2u_2} \end{pmatrix}, \ \boldsymbol{b}_3 = \begin{pmatrix} x_3 \\ x_{3u_1} \\ x_{3u_2} \end{pmatrix}, \ \boldsymbol{b}_4 = \begin{pmatrix} x_4 \\ x_{4u_1} \\ x_{4u_2} \end{pmatrix}.$$

Then we have

$$\det A = -\frac{v_1}{v_1}\det(\boldsymbol{b}_2\ \boldsymbol{b}_3\ \boldsymbol{b}_4) + \frac{v_2}{v_1}\det(\boldsymbol{b}_1\ \boldsymbol{b}_3\ \boldsymbol{b}_4) - \frac{v_3}{v_1}\det(\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_4) + \frac{v_4}{v_1}\det(\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3).$$

On the other hand, we have

$$\boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2} = (-\det(\boldsymbol{b}_2\ \boldsymbol{b}_3\ \boldsymbol{b}_4),\ \det(\boldsymbol{b}_1\ \boldsymbol{b}_3\ \boldsymbol{b}_4),\ \det(\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_4),\ -\det(\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3))$$

Therefore we have

$$\begin{aligned} \det A &= \langle(-\frac{v_1}{v_1}, -\frac{v_2}{v_1}, -\frac{v_3}{v_1}, -\frac{v_4}{v_1}), \boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2}\rangle \\ &= -\frac{1}{v_1}\langle \mathbb{T}, \| \boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2} \| \boldsymbol{e}\rangle \\ &= \frac{\| \boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2} \|}{v_1} \neq 0. \end{aligned}$$

□

We now show that H is a generating family of $\mathcal{L}(U) \subset \Delta$.

Proposition 5.3 *For any spacelike surfaces $\boldsymbol{X} : U \longrightarrow H^3_1$, the AdS-height function $H : U \times H^3_1 \longrightarrow \mathbb{R}$ of $\boldsymbol{X}$ is a generating family of the Legendrian embedding $\mathcal{L}$.*

Proof. We consider a coordinate neighborhood $W_1^+ = \{\boldsymbol{w} = (w_1, w_2, w_3, w_4) \in H^3_1 \mid w_1 > 0\}$. Remember the contact morphism $\Phi : \Delta(W_1^+) \longrightarrow PT^*H^3_1 \mid W_1^+$ defined in the proof of Theorem 2.1. Since AdS-height function H is a Morse family, we have a Legendrian immersion

$$\mathcal{L}_H : \Sigma_*(H) \mid (U \times W_1^+) \longrightarrow PT^*H^3_1 \mid W_1^+$$

defined by

$$\mathcal{L}_H(u, \boldsymbol{w}) = (\boldsymbol{w}, [\frac{\partial H}{\partial w_2} : \frac{\partial H}{\partial w_3} : \frac{\partial H}{\partial w_4}]).$$

By Proposition 4.1, we have

$$\Sigma_*(H) = \{(u, \mathbb{T}(u)) \in U \times H^3_1 \mid u \in U\}.$$

Since $\boldsymbol{w} = \mathbb{T}(u)$ and $w_1 = \sqrt{-w_2^2 + w_3^2 + w_4^2 + 1}$, we have

$$\begin{aligned} \frac{\partial H}{\partial w_2}(u, \mathbb{T}(u)) &= x_1(u)\frac{v_2(u)}{v_1(u)} - x_2(u), \\ \frac{\partial H}{\partial w_3}(u, \mathbb{T}(u)) &= x_3(u) - x_1(u)\frac{v_3(u)}{v_1(u)}, \\ \frac{\partial H}{\partial w_4}(u, \mathbb{T}(u)) &= x_4(u) - x_1(u)\frac{v_4(u)}{v_1(u)}, \end{aligned}$$

where $\boldsymbol{X} = (x_1, x_2, x_3, x_4)$ and $\mathbb{T} = (v_1, v_2, v_3, v_4)$. It follows that

$$\mathcal{L}_H(u, \mathbb{T}(u)) = (\mathbb{T}(u), [x_1v_2 - x_2v_1 : -x_1v_3 + x_3v_1 : -x_1v_4 + x_4v_1]) = \mathcal{T}(u).$$

Therefore we have $\Phi \circ \mathcal{L}(u) = \mathcal{T}(u)$ on W_1^+. We also have the same relation as the above on the other local coordinates. This means that H is a generating family of $\mathcal{L} \subset \Delta$. □

Therefore we conclude that the TAdS-Gauss image $\mathbb{T}$ can be regarded as a Legendrian map.

6 Contact with AdS-great-hyperboloids

In this section we consider the geometric meaning of the singularities of the TAdS-Gauss image of spacelike surface $M = \boldsymbol{X}(U)$ in H^3_1. We consider the contact of spacelike surfaces with AdS-great-hyperboloids. We now briefly review the theory of contact due to Montaldi [16]. Let $X_i, Y_i (i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that the *contact* of X_1 and Y_1 at y_1 is the same type as the *contact* of X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. It is clear that in the definition $\mathbb{R}^n$ could be replaced by any manifold. In his paper [16], Montaldi gives a characterization of the notion of contact by using the terminology of singularity theory.

Theorem 6.1 *Let $X_i, Y_i (i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $dimX_1 = dimX_2$ and $dimY_1 = dimY_2$. Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, \boldsymbol{0})$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(\boldsymbol{0}), y_i)$. Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.*

For the definition of the $\mathcal{K}$-equivalent, See Martinet [15]. We now consider a function $\mathcal{H}$: $H^3_1 \times H^3_1 \longrightarrow \mathbb{R}$ defined by $\mathcal{H}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle$. For any $\boldsymbol{v}_0 \in H^3_1$, we denote $\mathfrak{h}_{v_0}(\boldsymbol{u}) = \mathcal{H}(\boldsymbol{u}, \boldsymbol{v}_0)$ and we have the AdS-great-hyperboloid $\mathfrak{h}^{-1}_{v_0}(0) = H^3_1 \cap HP(\boldsymbol{v}_0, 0)$. We write $AH(\boldsymbol{v}_0, 0) = H^3_1 \cap HP(\boldsymbol{v}_0, 0)$. For any $u_0 \in U$, we consider the timelike vector $\boldsymbol{v}_0 = \mathbb{T}(u_0)$. Then we have

$$\mathfrak{h}_{v_0} \circ \boldsymbol{X}(u_0) = \mathcal{H} \circ (\boldsymbol{X} \times id_{H^3_1})(u_0, \boldsymbol{v}_0) = H(u_0, \mathbb{T}(u_0)) = 0.$$

We also have relations

$$\frac{\partial \mathfrak{h}_{v_0} \circ \boldsymbol{X}}{\partial u_i}(u_0) = \frac{\partial H}{\partial u_i}(u_0, \mathbb{T}(u_0)) = 0,$$

for $i = 1, 2$. This means that the AdS-great-hyperboloid $AH(\boldsymbol{v}_0, 0)$ is tangent to $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u_0)$. In this case, we call $AH(\boldsymbol{v}_0, 0)$ the *tangent AdS-great-hyperboloid* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u_0)$ (or, u_0), which we write $AH(\boldsymbol{X}, u_0)$. Let $\boldsymbol{v}_1, \boldsymbol{v}_2$ be timelike vectors. If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly dependent, then $HP(\boldsymbol{v}_1, 0)$ and $HP(\boldsymbol{v}_2, 0)$ are equal. Therefore, AdS-great-hyperboloids $AH(\boldsymbol{v}_1, 0) = AH(\boldsymbol{v}_2, 0)$. Then we have the following simple lemma.

Lemma 6.2 *Let $\boldsymbol{X} : U \longrightarrow H^3_1$ be a spacelike surface. Consider two points $u_1, u_2 \in U$. Then we have the following assertion:*

$$\mathbb{T}(u_1) = \mathbb{T}(u_2) \text{ if and only if } AH(\boldsymbol{X}, u_1) = AH(\boldsymbol{X}, u_2). \qquad \square$$

We now consider the contact of M with tangent AdS-great-hyperboloid at $p \in M$ as an application of Legendrian singularity theorey. We introduce an equivalence relation among Legendrian immersion germs. Let $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ and $i : (L', p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs. Then we say that i and i' are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, p) \longrightarrow (PT^*\mathbb{R}^n, p')$ such that H preserves fibres of π and that $H(L) = L'$. A Legendrian germ into $PT^*\mathbb{R}^n$ at a point is said to be *Legendrian stable* if for every map with the given germ there are a neighbourhood in the space of Legendrian immersion (in the Whitney $C^\infty-$topology) and a neighbourhood of the original point such that each Legendrian immersion belonging to the first neighbourhood has, in the second neighbourhood, a point at which its germ is Legendrian equivalent to the original germ.

Since the Legendrian lift $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ is uniquely determined on the regular part of the wave front $W(i)$, we have the following simple but significant property of Legendrian immersion germs.

Proposition 6.3 *Let $i : (L,p) \subset (PT^*\mathbb{R}^n, p)$ and $i : (L',p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs such that regular sets of $\pi \circ i$ and $\pi \circ i'$ respectively are dense. Then i and i' are Legemndrian equivalent if and only if wave front sets $W(i)$ and $W(i')$ are diffeomorphic as set germs.*

This result had been firstly pointed out by Zakalyukin [21]. The assumption in the above proposition is a generic condition for i and i'. In particular, if i and i' are Legendrian stable, then these satisfy the assumption.

We can interpret the Legendrian equivalence by using the notion of generating families. We denote $\mathcal{E}_n$ the local ring of function germs $(\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal $\mathcal{M}_n = \{h \in \mathcal{E}_n | h(\mathbf{0}) = 0\}$. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be function germs. We say that F and G are $P\!-\!\mathcal{K}$ *equivalent* if there exists a diffeomorphsim germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ of the form $\Psi(q,x) = (\psi_1(q,x), \psi_2(x))$ for $(q,x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ such that $\Psi^*(\langle F \rangle_{\mathcal{E}_{k+n}}) = \langle G \rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \longrightarrow \mathcal{E}_{k+n}$ is the pull back $\mathbb{R}$-algebra isomorphim defined by $\Psi^*(h) = h \circ \Psi$.

Let $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be a function germ. We say that F is a $\mathcal{K}$*-versal deformation* of $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$ if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \langle \frac{\partial F}{\partial x_1}|\mathbb{R}^k \times \{\mathbf{0}\}, \cdots, \frac{\partial F}{\partial x_n}|\mathbb{R}^k \times \{\mathbf{0}\} \rangle_{\mathbb{R}},$$

where

$$T_e(\mathcal{K})(f) = \langle \frac{\partial f}{\partial q_1}, \cdots, \frac{\partial f}{\partial q_k}, f \rangle_{\mathcal{E}_k}.$$

The main result in the theory of Arnold [1] and Zakalyukin [20] is the following:

Theorem 6.4 *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families. Then*
(1) *Φ_F and Φ_G are Legendrian equivalent if and only if F and G are $P\!-\!\mathcal{K}$ equivalent;*
(2) *Φ_F is Legendrian stable if and only if F is a $\mathcal{K}$-versal deformation of $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$.*

Since F and G are function germs on the common space germ $(\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$, we do not need the notion of stably $P\!-\!\mathcal{K}$ equivalences under this situation (cf., [1]). By the uniqueness result of the $\mathcal{K}$-versal deformation of a function germ, Proposition 6.3 and Theorem 6.4, we have the following classification result of Legendrian stable germs (cf. [3]). For any map germ $f : (\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^p, \mathbf{0})$, we define the *local ring* of f by $Q(f) = \mathcal{E}_n / f^*(\mathcal{M}_p)\mathcal{E}_n$.

Proposition 6.5 *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families. Suppose that Φ_F and Φ_G are Legendrian stable. Then the following conditions are equivalent:*
(1) *$(W(\Phi_F), \mathbf{0})$ and $(W(\Phi_G), \mathbf{0})$ are diffeomorphic as germs;*
(2) *Φ_F and Φ_G are Legendrian equivalent;*
(3) *$Q(f)$ and $Q(g)$ are isomorphic as $\mathbb{R}$-algebras, where $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$ and $g = G|\mathbb{R}^k \times \{\mathbf{0}\}$.*

Proof. See [6] □

We have the tools for study of the contact of spacelike surfaces with AdS-great-hyperboloids. Let $\mathbb{T}_i : (U, u_i) \longrightarrow (H^3_1, \boldsymbol{v}_i)$ (for $i = 1, 2$) be TAdS-Gauss image germs of spacelike surface germs $\boldsymbol{X}_i : (U, u_i) \longrightarrow (H^3_1, \boldsymbol{X}_i(u_i))$. We say that $\mathbb{T}_1$ and $\mathbb{T}_2$ are $\mathcal{A}$*-equivalent* if there exist diffeomorphism germs $\phi : (U, u_1) \longrightarrow (U, u_2)$ and $\Phi : (H^3_1, \boldsymbol{v}_1) \longrightarrow (H^3_1, \boldsymbol{v}_2)$ such that $\Phi \circ \mathbb{T}_1 = \mathbb{T}_2 \circ \phi$.

Suppose the regular set of $\mathbb{T}_i$ is dense in (U, u_i) for each $i = 1, 2$. It follows from Proposition 6.3 that $\mathbb{T}_1$ and $\mathbb{T}_2$ are $\mathcal{A}$-equivalent if and only if the corresponding Legendrian embedding germs $\mathcal{L}^1 : (U, u_1) \longrightarrow (\Delta, \mathbf{z}_1)$ and $\mathcal{L}^2 : (U, u_2) \longrightarrow (\Delta, \mathbf{z}_2)$ are Legendrian equivalent. This condition is also equivalent to the condition that two generating families H_1 and H_2 are $P{-}\mathcal{K}$ equivalent by Theorem 6.4. Here, $H_i : (U \times H^3_1, (u_i, \boldsymbol{v}_i)) \longrightarrow \mathbb{R}$ is the corresponding AdS-height function germ of $\boldsymbol{X}_i$.

On the other hand, we denote $h_{i,v_i} = H_i(u, \boldsymbol{v}_i)$; then we have $h_{i,v_i}(u) = \mathfrak{h}_{v_i} \circ \boldsymbol{X}_i(u)$. By Theorem 6.1,

$$K(\boldsymbol{X}_1(U), AH(\boldsymbol{X}_1, u_1), \boldsymbol{v}_1) = K(\boldsymbol{X}_2(U), AH(\boldsymbol{X}_2, u_2), \boldsymbol{v}_2)$$

if and only if h_{1,v_1} and h_{2,v_2} are $\mathcal{K}$-equivalent. Therefore, we can apply the above arguments to our situation. We denote by $Q(x, u_0)$ the local ring of the function germ $h_{v_0} : (U, u_0) \longrightarrow \mathbb{R}$, where $\boldsymbol{v}_0 = \mathbb{T}(u_0)$. We remark that we can write the local ring explicitly as follows:

$$Q(x, u_0) = \frac{C^\infty_{u_0}(U)}{\langle \langle \boldsymbol{X}(u), \mathbb{T}(u_0) \rangle \rangle_{C^\infty_{u_0}(U)}},$$

where $C^\infty_{u_0}(U)$ is the local ring of function germs at u_0 with the unique maximal ideal $\mathcal{M}_{u_0}(U)$.

Theorem 6.6 *Let $\boldsymbol{X}_i : (U, u_i) \longrightarrow (H^3_1, \boldsymbol{X}_i(u_i))$ (for $i = 1, 2$) be spacelike surface germs such that the corresponding Legendrian embedding germs $\mathcal{L}^i : (U, u_i) \longrightarrow (\Delta, \boldsymbol{z}_i)$ are Legendrian stable. Then the following conditions are equivalent:*

(1) *TAdS-Gauss image germs $\mathbb{T}_1$ and $\mathbb{T}_2$ are $\mathcal{A}$-equivalent;*
(2) *H_1 and H_2 are $P{-}\mathcal{K}$-equivalent;*
(3) *h_{1,v_1} and h_{2,v_2} are $\mathcal{K}$-equivalent;*
(4) *$K(\boldsymbol{X}_1(U), AH(\boldsymbol{X}_1, u_1), \boldsymbol{v}_1) = K(\boldsymbol{X}_2(U), AH(\boldsymbol{X}_2, u_2), \boldsymbol{v}_2)$*
(5) *$Q(\boldsymbol{X}_1, u_1)$ and $Q(\boldsymbol{X}_2, u_2)$ are isomorphic as $\mathbb{R}$-algebras.*

Proof. By the previous arguments (mainly from Theorem 6.1), it has already been shown that conditions (3) and (4) are equivalent. Other assersions follow from Proposition 6.5. □

For a spacelike surface germ

$$\boldsymbol{X} : (U, u_0) \longrightarrow (H^3_1, \boldsymbol{X}(u_0)),$$

we call $\boldsymbol{X}^{-1}(AH(\mathbb{T}(u_0), 0), u_0)$ the *tangent AdS-great-hyperboloidic indicatrix germ* of $\boldsymbol{X}$. In general we have the following proposition:

Proposition 6.7 *Let $\boldsymbol{X}_i : (U, u_i) \longrightarrow (H^3_1, \boldsymbol{X}_i(u_i))$ (for $i = 1, 2$) be spacelike surface germs such that their AdS-parabolic sets have no interior points as subspaces of U. If TAdS-Gauss image germs $\mathbb{T}_1$ and $\mathbb{T}_2$ are $\mathcal{A}$-equivalent, then*

$$K(\boldsymbol{X}_1(U), AH(\boldsymbol{X}_1, u_1), \boldsymbol{v}_1) = K(\boldsymbol{X}_2(U), AH(\boldsymbol{X}_2, u_2), \boldsymbol{v}_2).$$

In this case, $\boldsymbol{X}_1^{-1}(AH(\mathbb{T}_1(u_1), 0), u_1)$ and $\boldsymbol{X}_2^{-1}(AH(\mathbb{T}_2(u_2), 0), u_2)$ are diffeomorphic as set germs.

Proof. The AdS-parabolic set is the set of singular points of the TAdS-Gauss image. So the corresponding Legendrian embedding $\mathcal{L}^i$ satisfy the hypothesis of Proposition 6.3. If TAdS-Gauss image germs $\mathbb{T}_1$ and $\mathbb{T}_2$ are $\mathcal{A}$-equivalent, then $\mathcal{L}^1$ and $\mathcal{L}^2$ are Legendrian equivalent, so that H_1 and H_2 are $p{-}\mathcal{K}$-equivalent. Therefore, h_{1,v_1} and h_{2,v_2} are $\mathcal{K}$-equivalent. By

Theorem 5.1, this condition is equivalent to the condition that $K(\boldsymbol{X}_1(U), AH(\boldsymbol{X}_1, u_1), \boldsymbol{v}_1) = K(\boldsymbol{X}_2(U), AH(\boldsymbol{X}_2, u_2), \boldsymbol{v}_2)$.

On the other hand, we have $\boldsymbol{X}_i^{-1}(AH(\mathbb{T}_i(u_0), 0), u_0) = (h_{i,v_i}^{-1}(0), u_0)$. It follows from this fact that $\boldsymbol{X}_1^{-1}(AH(\mathbb{T}_1(u_1), 0), u_1)$ and $\boldsymbol{X}_2^{-1}(AH(\mathbb{T}_2(u_2), 0), u_2)$ are diffeomorphic as set germs because the $\mathcal{K}$-equivalent preserves the zero level sets. □

From the above proposition, the diffeomorphism type of the tangent AdS-great-hyperboloidic indicatrix germ is an invariant of $\mathcal{A}$-classification of the TAdS-Gauss image germ of $\boldsymbol{X}$. Moreover, we can borrow some basic invariants from the singularity theory on function germs. We need $\mathcal{K}$-invariants for a function germ. The local ring of a function is a complete $\mathcal{K}$-invariant for generic function germs. It is, however, not a numerical invariant. The $\mathcal{K}$-codimension of a function germ is a numerical $\mathcal{K}$-invariant of function germs. We denote

$$\text{AdS-ord}(\boldsymbol{X}, u_0) = \dim \frac{C^\infty_{u_0}(U)}{\langle h_{v_0}, \partial h_{v_0}/\partial u_i \rangle_{C^\infty_{u_0}(U)}},$$

where $\boldsymbol{v}_0 = \mathbb{T}(u_0)$. Usually AdS-ord$(\boldsymbol{X}, u_0)$ is called the $\mathcal{K}$-codimension of h_{v_0}. However, We call it the order of contact with tangent AdS-great-hyperboloid at $\boldsymbol{X}(u_0)$. We also have the notion of *corank* of function germs:

$$\text{AdS-corank}(\boldsymbol{X}, u_0) = 2 - \text{rankHess}(h_{v_0})(u_0),$$

where $\boldsymbol{v}_0 = \mathbb{T}(u_0)$.

By Proposition 4.1, $\boldsymbol{X}(u_0)$ is an AdS-parabolic point if and only if AdS-corank $(\boldsymbol{X}, u_0) \geq 1$. On the other hand, a function germ $f : (\mathbb{R}^{n-1}, \boldsymbol{a}) \longrightarrow \mathbb{R}$ has the A_k-type singularity if f is $\mathcal{K}$-equivalent to the germ $\pm u_1^2 \pm \cdots \pm u_{n-2}^2 + u_{n-1}^{k+1}$. If AdS-corank$(\boldsymbol{X}, u_0) = 1$, the AdS-height function h_{v_0} has the A_k-type singularity at u_0 and is generic. In this case we have AdS-ord$(\boldsymbol{X}, u_0) = k$. This number is equal to the order of contact in the classical sense (cf., [2]). This is the reason why we call AdS-ord$(\boldsymbol{X}, u_0)$ the order of contact with the AdS-great-hyperboloid at $\boldsymbol{X}(u_0)$.

7 Generic properties

In this section we consider generic properties of spacelike surfaces in H^3_1. The main tool is a kind of transversality theorem. We consider the space of spacelike embeddings $\text{Emb}_S(U, H^3_1)$ with Whitney $C^\infty-$topology. We also consider the function $\mathcal{H}$: $H^3_1 \times H^3_1 \longrightarrow \mathbb{R}$ which is given in §6. We claim that $\mathcal{H}_u$ is a submersion for any $\boldsymbol{u} \in H^3_1$, where $\mathcal{H}_u(\boldsymbol{v}) = \mathcal{H}(\boldsymbol{u}, \boldsymbol{v})$. For any $\boldsymbol{X} \in \text{Emb}_S(U, H^3_1)$, we have $H = \mathcal{H} \circ (\boldsymbol{X} \times id_{H^3_1})$. We also have the $l-$jet extension

$$j^l_1 H : U \times H^3_1 \longrightarrow J^l(U, \mathbb{R})$$

defined by $j^l_1 H(u, \boldsymbol{v}) = j^l h_v(u)$. We consider the trivialisation

$$J^l(U, \mathbb{R}) \equiv U \times \mathbb{R} \times J^l(2, 1).$$

For any submanifold $Q \subset J^l(2, 1)$, we denote $\widetilde{Q} = U \times \{0\} \times Q$. Then we have the following proposition as a corollary of Lemma 6 of Wassermann [18].(See also Izumiya *et al.*[6] and Montaldi [17]).

Proposition 7.1 *Let Q be a submanifold of $J^l(2, 1)$. Then the set*
$$T_Q = \{\boldsymbol{X} \in \text{Emb}_S(U, H^3_1) | j^l_1 H \text{ is transversal to } \widetilde{Q}\}$$

is a residual subset of $\mathrm{Emb}_S(U, H^3_1)$*. If* Q *is a closed subset, then* T_Q *is open.*

On the other hand, let $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse family and Φ_F is the legendrian immersion with generating family F. By Theorem 6.4, we already have Φ_F is Legendrian stable if and only if F is a $\mathcal{K}$-versal deformation of $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$. We need the following characterization of $\mathcal{K}$-versality of generating family. Let $J^l(\mathbb{R}^k, \mathbb{R})$ be the l-jet bundle of k-variable functions which has the canonical decomposion: $J^l(\mathbb{R}^k, \mathbb{R}) \equiv \mathbb{R}^k \times \mathbb{R} \times J^l(k, 1)$. For any Morse family of hypersurfaces F, we define a map germ

$$j^l_1 F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow J^l(\mathbb{R}^k, \mathbb{R})$$

by $j^l_1 F(q, x) = j^l F_x(q)$, where $F_x(q) = F(q, x)$. We denote $\mathcal{K}^l(z)$ the $\mathcal{K}$-orbit through $z = j^l(0) \in J^l(k, 1)$. (cf.,[15]). If $f(q) = F(q, 0)$ is l-determined relative to $\mathcal{K}$, then F is $\mathcal{K}$-versal deformation of f if and only if $j^l_1 F$ is transversal to $\mathbb{R}^k \times \{0\} \times \mathcal{K}^l(z)$ (cf.,[15]). Therefore we can apply this characterization to the AdS-height function. By the classification of stable Legendrian singularities of $n < 6$ and Proposition 7.1, we have the following theorem.

Theorem 7.2 *There exists an open dense subset* $\mathcal{O} \subset \mathrm{Emb}_S(U, H^3_1)$ *such that for any* $\boldsymbol{X} \in \mathcal{O}$*, the germ of the corresponding Legendrian embedding* $\mathcal{L}$ *at each point is Legendrian stable .*

8 Classification of singularities of TAdS-Gauss images

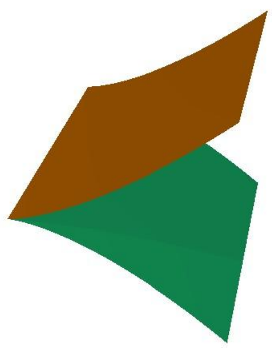
In this section we consider the generic singularities of TAdS-Gauss images. By Theorem 7.2 and the classification of function germs [1], We have the following theorem:

Theorem 8.1 *There exists an open dense subset* $\mathcal{O} \subset \mathrm{Emb}_S(U, H^3_1)$ *such that for any* $\boldsymbol{X} \in \mathcal{O}$ *the following conditions hold.*

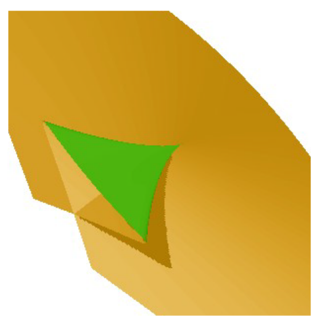
(1) *The AdS-parabolic set* $K^{-1}_{AdS}(0)$ *is a regular curve. We call such a curve the AdS-parabolic curve.*

(2) *The TAdS-Gauss image* $\mathbb{T}$ *along the AdS-parabolic curve is a cuspidal edge except at isolated points. At such the point* $\mathbb{T}$ *is the swallowtail.*

Here, a map germ $f : (\mathbb{R}^2, \boldsymbol{a}) \longrightarrow (\mathbb{R}^3, \boldsymbol{b})$ *is called a cuspidal edge if it is* $\mathcal{A}$*-equivalent to the germ* (u_1, u_2^2, u_2^3) *and a swallowtail if it is* $\mathcal{A}$*-equivalent to the germ* $(3u_1^4 + u_1^2 u_2, 4u_1^3 + 2u_1 u_2, u_2)$*.*



Cuspital edge



Swallowtail

Figure 1

The assertion of Theorem 8.1 can be interpreted as saying that the Legendrian embedding $\mathcal{L}$ of the TAdS-Gauss image $\mathbb{T}$ of $\boldsymbol{X}$ is Legendrian stable at each point. Following the terminology of Whitney [19], we say that a spacelike surface $\boldsymbol{X} : U \longrightarrow H^3_1$ has the *excellent TAdS-Gauss image* $\mathbb{T}$ if $\mathcal{L}$ is a stable Legendrian immersion germ at each point. In this case, the TAdS-Gauss image $\mathbb{T}$ has only cuspidal edges and swallowtails as singularities. Theorem 8.1 assert that a spacelike surface with the excellent TAdS-Gauss image is generic in the space of all spacelike surfaces in H^3_1.

We now consider the geometric meanings of cuspidal edges and swallowtails of the TAdS-Gauss image. We have the following results analogous to the results of Izumiya *et al*. [6].

Theorem 8.2 *Let* $\mathbb{T} : (U, u_0) \longrightarrow (H^3_1, \boldsymbol{v}_0)$ *be the excellent TAdS-Gauss image germ of a spacelike surface* $\boldsymbol{X}$ *and* $h_{v_0} : (U, u_0) \longrightarrow \mathbb{R}$ *be the AdS-height function germ at* $\boldsymbol{v}_0 = \mathbb{T}(u_0)$*. Then we have the following.*

(1) *The point* u_0 *is an AdS-parabolic point of* $\boldsymbol{X}$ *if and only if* AdS-corank $(\boldsymbol{X}, u_0)= 1$.

(2) *If* u_0 *is an AdS-parabolic point of* $\boldsymbol{X}$*, then* h_{v_0} *has the* A_k-*type singularity for* $k = 2, 3$.

(3) *Suppose that* u_0 *is an AdS-parabolic point of* $\boldsymbol{X}$*. Then the following conditions are equivalent:*

(a) $\mathbb{T}$ *has the cuspidal edge at* u_0*;*

(b) h_{v_0} *has the* A_2-*type singularity;*

(c) AdS-order$(\boldsymbol{X}, u_0) = 2$*;*

(d) *the tangent AdS-great-hyperboloidic indicatrix is an ordinary cusp, where a curve* $C \subset \mathbb{R}^2$ *is called an ordinary cusp if it is diffeomorphic to the curve given by* $\{(u_1, u_2)|u_1^2 - u_2^3 = 0\}$*.*

(4) *Suppose that* u_0 *is an AdS-parabolic point of* $\boldsymbol{X}$*. Then the following conditions are equivalent:*

(a) $\mathbb{T}$ *has the swallowtail at* u_0*;*

(b) h_{v_0} *has the* A_3-*type singularity;*

(c) AdS-order$(\boldsymbol{X}, u_0) = 3$*;*

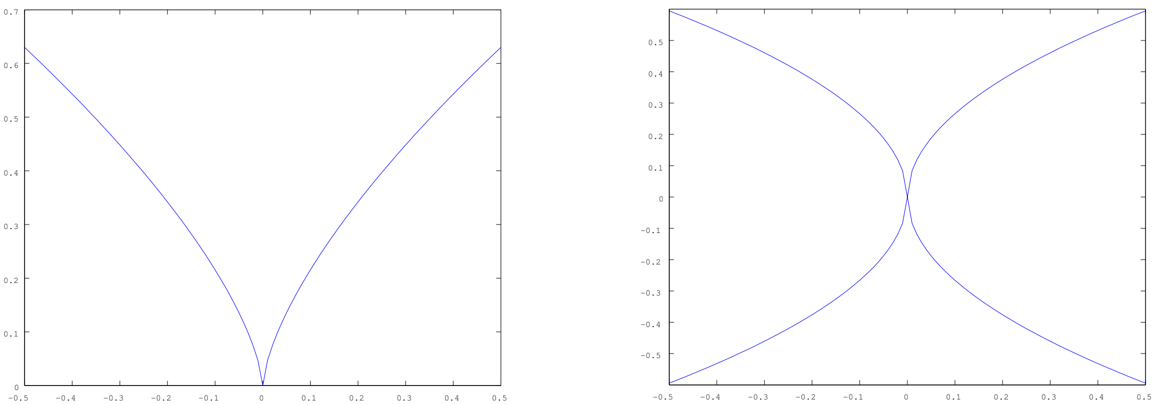
(d) *the tangent AdS-great-hyperboloidic indicatrix is an point or a tachnodal, where a curve* $C \subset \mathbb{R}^2$ *is called a tachnodal if it is diffeomorphic to the curve given by* $\{(u_1, u_2)|u_1^2 - u_2^4 = 0\}$*.*

(e) *for each* $\varepsilon > 0$*, there exit two points* $u_1, u_2 \in U$ *such that* $|u_0 - u_i| < \varepsilon$ *for* $i = 1, 2$*, neither of* u_1 *nor* u_2 *is an AdS-parabolic point and the tangent AdS-great-hyperboloids to* $M = \boldsymbol{X}(U)$ *at* u_1 *and* u_2 *are equal.*

Proof. By the Proposition 4.1, we have shown that u_0 is an AdS-parabolic point if and only if AdS-corank$(\boldsymbol{X}, u_0) \geq 1$. Since $n = 3$, we have AdS-corank$(\boldsymbol{X}, u_0) \leq 2$. Since AdS-height function germ $H : (U \times H^3_1, (u_0, \boldsymbol{v}_0)) \longrightarrow \mathbb{R}$ can be considered as a generating family of the Legendrian embedding germ $\mathcal{L}$, h_{v_0} has only the A_k-type singularities $(k = 1, 2, 3)$. This means that the corank of the Hessian matrix of the h_{v_0} at an AdS-parabolic point is 1. The assertion (2) also follows. For the same reason, the conditions $(3)\{(a), (b), (c)\}$(respectively, $(4)\{(a), (b), (c)\}$) are equivalent.

On the other hand, if the AdS-height function germ h_{v_0} has the A_2-type singularity, it is $\mathcal{K}$-equivalent to the germ $\pm u_1^2 + u_2^3$. Since the $\mathcal{K}$-equivalence preserves the zero level sets, the tangent AdS-great-hyperboloidic indicatrix is diffeomorphic to the curve given by $\pm u_1^2 + u_2^3 = 0$. This is the ordinary cusp. The normal form for the A_3-type singularity is given by $\pm u_1^2 + u_2^4$, so the tangent AdS-great-hyperboloidic indicatrix is diffeomorphic to the curve given by $\pm u_1^2 + u_2^4 = 0$. This means that the condition $(3)\{(d)\}$(respectively, $(4)\{(d)\}$) is also equivalent to the other conditions.

For the swallowtail point u_0, there is a self-intersection curve approaching u_0. On this curve, there are two distinct points u_1 and u_2 such that $\mathbb{T}(u_1) = \mathbb{T}(u_2)$. By Lemma 6.2, this means that the tangent AdS-great-hyperboloids to $M = \boldsymbol{X}(U)$ at u_1 and u_2 are equal. Since there are no other singularities in this case, the condition (4){(e)} characterizes a swallowtail point of $\mathbb{T}$. This completes the proof. $\square$



Ordinary cusp Tachnodal

Figure 2

9 AdS-Monge form

The notion of the Monge form of a surface in Euclidean 3-space is one of the powerful tools for the study of local properties of the surface from the view point of differential geometry.In this section we consider the analogous notion for a spacelike surface in H^3_1.

We now consider a function $f(u_1, u_2)$ with $f(0) = f_{u_i}(0) = 0$. Then we have a spacelike surface in H^3_1 defined by

$$\boldsymbol{X}_f(u_1, u_2) = (\sqrt{1 + u_1^2 + u_2^2 - f^2(u_1, u_2)}, f(u_1, u_2), u_1, u_2).$$

We can easily calculate $\boldsymbol{e}(0) = (0, 1, 0, 0)$; therefore $\mathbb{T}(0) = (0, 1, 0, 0)$. We call $\boldsymbol{X}_f$ a *Anti de Sitter Monge form* (briefly, *AdS-Monge form*). Then we have the following proposition.

Proposition 9.1 *Any spacelike surface in H^3_1 is locally given by the AdS-Monge form.*

Proof. Let $\boldsymbol{X} : U \longrightarrow H^3_1$ be a spacelike surface. We consider Lorentzian motion of H^3_1 which is a transitive action. Therefore, without loss of the generality, we assume that $p = \boldsymbol{X}(0) = (1, 0, 0, 0)$. We denote $M = \boldsymbol{X}(U)$, we have a basis $\{\boldsymbol{X}(0), \boldsymbol{e}(0), \boldsymbol{X}_{u_1}(0), \boldsymbol{X}_{u_2}(0)\}$ of $T_p\mathbb{R}^4_2$ such that $T_pM = \langle \boldsymbol{X}_{u_1}(0), \boldsymbol{X}_{u_2}(0)\rangle_{\mathbb{R}}$. Applying the Gram-Schmidt procedure we have a pseudo-orthonormal basis $\{\boldsymbol{X}(0), \boldsymbol{e}(0), \boldsymbol{e}_1, \boldsymbol{e}_2\}$ of $T_p\mathbb{R}^4_2$ such that $T_pM = \langle \boldsymbol{e}_1, \boldsymbol{e}_2\rangle_{\mathbb{R}}$. In particular, $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is an orthonormal basis of T_pM. Since $p = (1, 0, 0, 0)$, T_pM is considered to be a subspace of ${}_0\mathbb{R}^3_1 = \{(0, x_1, x_2, x_3) | x_i \in \mathbb{R}\}$. By a rotation of the space ${}_0\mathbb{R}^3_1$, we might assume that $T_pM = \{(0, 0, u_1, u_2) | u_i \in \mathbb{R}\} \subset \mathbb{R}^4_2$. Then the germ (M, p) might be written in the form

$$(f_0(u_1, u_2), f(u_1, u_2), u_1, u_2)$$

with function germs $f_0(u_1, u_2), f(u_1, u_2)$. Since $M \subset H^3_1$, we have the relation

$$f_0(u_1, u_2) = \sqrt{1 + u_1^2 + u_2^2 - f^2(u_1, u_2)}$$

Since we have $T_pM = \{(0,0,u_1,u_2)|u_i \in \mathbb{R}\}$, the condition $f(0)=0,\ f_{u_i}(0)=0$ are automatically satisfied. $\square$

For the timelike vector $\boldsymbol{v}_0 = (0,1,0,0)$, we consider the AdS-great-hyperboloid AH$(\boldsymbol{v}_0, 0)$. Then we have the AdS-Monge form of AH$(\boldsymbol{v}_0, 0)$:

$$\boldsymbol{a}(u_1,u_2) = (\sqrt{1+u_1^2+u_2^2}, 0, u_1, u_2).$$

Here, we can easily check the relation $\langle \boldsymbol{a}(u), \boldsymbol{v}_0 \rangle = 0$.

On the other hand, $\boldsymbol{a}(0) = (1,0,0,0) = p$ and $\boldsymbol{a}_{u_i}(0)$ is equal to the x_{i+2}-axis for $i=1,2$. This means that $T_pM = T_p(\boldsymbol{a}(u))$. Therefore $\boldsymbol{a}(u)$ =AH$(\boldsymbol{v}_0, 0)$ is the tangent AdS-great-hyperboloid of $M = \boldsymbol{X}_f(U)$ at $p = \boldsymbol{X}_f(0)$. It follows from this fact that the tangent AdS-great-hyperboloidic indicatrix of the AdS-Monge form germ $(\boldsymbol{X}_f, 0)$ is given as follows:

$$\boldsymbol{X}_f^{-1}(AH(\boldsymbol{v}_0,0)) = \{(u_1,u_2)|f(u_1,u_2)=0\}.$$

Since the height function of $\boldsymbol{X}_f$ at $\boldsymbol{v}_0$ is

$$h_{v_0}(u) = \langle \boldsymbol{X}_f(u), \boldsymbol{v}_0 \rangle = f(u_1,u_2),$$

we can calculate the Hessian matrix; then we have Hess$(h_{v_0})(0) =$ Hess$(f)(0)$. Thus we conclude that AdS-corank$(\boldsymbol{X}_f, 0) = 2 -$ rankHess$(f)(0)$.

On the other hand, since $f(0) = f_{u_i}(0) = 0$, we may write

$$f(u_1,u_2) = \frac{1}{2}\bar{k}_1 u_1^2 + \frac{1}{2}\bar{k}_2 u_2^2 + g(u_1,u_2)$$

where $g \in \mathcal{M}_2^3$ and $\bar{k}_1, \bar{k}_2$ are eigenvalues of $(f_{u_1u_2}(0))$. Under this representation, we can easily calculate $\boldsymbol{X}_{f,u_1u_2}(0) = (\delta_{ij}, f_{u_1u_2}(0), 0, 0)$. It follows from this fact that

$$h_{ij}(0) = \langle \boldsymbol{e}(0), \boldsymbol{X}_{f,u_1u_2}(0) \rangle = f_{u_1u_2}(0) = \delta_{ij}\bar{k}_i,$$

and

$$g_{ij}(0) = \langle \boldsymbol{X}_{f,u_1}(0), \boldsymbol{X}_{f,u_2}(0) \rangle = \delta_{ij}.$$

Therefore, we have $k_i(0) = \bar{k}_i$ and

$$K_{AdS}(0) = k_1(0)k_2(0) = \bar{k}_1\bar{k}_2.$$

The tangent AdS-great-hyperboloidic indicatrix is given by

$$\begin{aligned}\boldsymbol{X}_f^{-1}(AH(\boldsymbol{v}_0,0)) &= \{(u_1,u_2)| \pm \frac{1}{2}\bar{k}_1 u_1^2 \pm \frac{1}{2}\bar{k}_2 u_2^2 \pm g(u_1,u_2) = 0\} \\ &= \{(u_1,u_2)| \pm k_1(0)u_1^2 \pm k_2(0)u_2^2 \pm 2g(u_1,u_2) = 0\}.\end{aligned}$$

If we try to draw picture of the TAdS-Gauss image, it might be very hard to give a parameterization. However, by the AdS-Monge form of the tangent AdS-great-hyperboloidic indicatrix germ, we can easy to detect the type of singularities of the TAdS-Gauss image $\mathbb{T}$.

Example 9.1 *Consider the function given by*

$$f(u_1,u_2) = 2u_1^2 - 3u_2^3.$$

Then $\bar{k_1} = 4,\ \bar{k_2} = 0$. *We have* $k_1 = 4,\ k_2 = 0$, *so that the origin is an AdS- parabolic point. The tangent AdS-great-hyperboloidic indicatrix germ at the origin is the ordinary cusp. By Theorem 8.2,* $\mathbb{T}(0)$ *is the cuspidal edge.*

Example 9.2 *Consider the function given by*

$$f(u_1, u_2) = 2u_1^2 - 4u_2^4.$$

Then $\bar{k_1} = 4,\ \bar{k_2} = 0$. *We have* $k_1 = 4,\ k_2 = 0$, *so that the origin is an AdS- parabolic point. The tangent AdS-great-hyperboloidic indicatrix germ at the origin is the tachnodal. By Theorem 8.2,* $\mathbb{T}(0)$ *is the swallowtail.*

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References

[1] V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, Singularities of Differentiable Maps vol. I, *Birkhäuser* (1986).

[2] J. W. Bruce and P. J. Giblin, Curves and singularities (second edition), Cambridge Univ. Press, (1992).

[3] S. Izumiya, On Legendrian singularities, *Proc. Amer. Math. Soc.***101** (1987), 748–752.

[4] S. Izumiya, Topological properties of Legendrian singularities, *Canad. Math. Bull.***34** (1991), 357–363.

[5] S. Izumiya, D-H.Pei and T.Sano, The lightcone Gauss map and the lightcone developable of a spacelike curve in Minkowski 3-space, *Glasgow. Math. J.* **42** (2000), 75–89.

[6] S. Izumiya, D-H.Pei and T.Sano, Singularities of hyperbolic Gauss maps, *Proc. London. Math. Soc.* (3) **86** (2003), 485–512.

[7] S. Izumiya, D. Pei and M. C. Romero Fuster, Umbilicity of spacelike surfaces in Minkowski space, *Proc. Royal Society of Edinburgh.* 134A, (2004), 375–387.

[8] S. Izumiya, D. Pei and M. C. Romero Fuster, Singularities of Lightcone Gauss maps of space like surfaces in Minkowski 4-space, *Asian Journal of Mathematics.* **8**, No.3, (2004), 511–530.

[9] S. Izumiya, M. Kossowski, D. Pei and M. C. Romero Fuster, Singularities of lightlike hypersurfaces in Minkowski four-space, *Tohoku Mathematical Journal.* **58** (2006), 71–88.

[10] S. Izumiya, M. Kikuchi and M. Takahashi, Global properties of spacelike curves in Minkowski 3-space, *Journal of Knot theory and its Ramifications.* **15** (2006), 869–881.

[11] S. Izumiya and M. Takahashi, Spacelike parallels and evolutes in Minkowski pseudospheres, *J. Geometry and Physics.* **57** (2007), 1569–1600.

[12] S. Izumiya and T. Fusho, Lightlike surfaces of spacelike curves in de Sitter 3-space, *to appear in Journal of Geometry.*

[13] S. Izumiya, Timelike hypersurfaces in de Sitter space and Legendrian singularities, *Journal of Mathematical Sciences.* **144**, No.1, (2007), 3789-3803.

[14] S. Izumiya and M. C. Romero Fuster, The Lightlike flat geometry on spacelike submanifolds of codimension two in Minkowski space, *Selecta Math.* (N.S.) **13** (2007), 23–55.

[15] J. Martinet, Singularities of Smooth Functions and Maps, *London Math. Soc. Lecture Note Series.* Cambridge Univ. Press. **58** (1982).

[16] J. A. Montaldi, On contact between submanifolds, *Michigan Math. J.* **33** (1986), 81–85.

[17] J. A. Montaldi, On generic composites of maps, *Bull. London Math. Soc.* **23** (1991), 81–85.

[18] G. Wassermann, Stability of Caustics, *Math. Ann.* **2210** (1975), 443–50.

[19] H. Whitney, On singularities of mappings of Euclidean spaces I, *Ann. of Math.* **62** (1955), 374–410.

[20] V. M. Zakalyukin, Lagrangian and Legendrian singularities, *Funct. Anal. Appl.* **10** (1976), 23–31.

[21] V. M. Zakalyukin, Reconstructions of fronts and caustics depending one parameter and versality of mappings, *J. Sov. Math.* **27** (1984), 2713-2735.

[22] B. O'neil, Semi-Riemannian Geometry, Academic Press, New York, (1983).

[23] D. E. Blair, Contact Manifolds in Riemannian Geometry, *Lecture Notes in Mathematics.* **509** (Springer 1983).