Multipliers For A Quotient Banach Space And The Nevanlinna-Pick Theorem

By

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Abstract. Let $E$ be a Banach space on a set $X$ and $M(E)$ the space of multipliers of $E$. In this paper, we study the space of multipliers of the quotient space $E/K$ where $K$ is a closed $M(E)$ - invariant subspace in $E$. When $E$ is the classical Hilbert Hardy space, the Nevanlinna and Pick theorem shows $M(E/K)$ is a quotient algebra of $M(E)$. 
§1. Introduction

A Banach space $E$ of functions on a set $X$ is a Banach space whose elements are complex-valued functions defined on $X$ with the usual pointwise addition and scalar multiplication. If $\phi$ is a complex-valued function on $X$ and $\phi f$ belongs to $E$ for all $f$ in $E$, then we write that $\phi$ is an element of $M(E)$, the space of multipliers of $E$. We assume that the point evaluations are continuous on $E$, that is, $X$ is embedded in the dual space $E^*$ and that there is no point in $X$ where all the members of $E$ vanish. It is known that $T_\phi : \ f \rightarrow \phi f$ is a bounded operator on $E$ for each $\phi$ in $M(E)$, since by continuity of point evaluation, each such map has closed graph. $M(E)$ is a closed subalgebra of $B(E)$, the set of bounded operators on $E$, indeed $M(E)$ is closed in the weak operator topology. Thus we assume that the space $M(E)$ of multipliers of $E$ is an operator algebra on $E$.

If $K$ is a closed subspace of $E$ then $E/K$ is also a Banach space. We want to define the space of multipliers $M(E/K)$ of $E/K$. For $\phi$ in $M(E)$ put

$$S_\phi(f + K) = \phi f + K \quad (f \in E).$$

In general, $S_\phi$ is not well defined on $E/K$. We need assume that $M(E)K \subset K$. Suppose $M(E/K) = \{ S_\phi; \phi \in M(E) \}$. Then $M(E/K)$ is also an operator algebra on $E/K$. Put

$$K = \{ \phi \in M(E); \phi E \subset K \}$$

then $K$ is a closed ideal in $M(E)$. By the definition, $S_\phi = 0$ if and only if $\phi E \subset K$. Hence $S_\phi = 0$ if and only if $\phi$ belongs to $K$. Therefore $S_{\phi + \psi} = S_\phi$ for any $\psi$ in $K$ and so there exists a one-to-one map from $M(E/K)$ onto $M(E)/K$. Moreover this map is contractive. In fact, for any $g$ in $K$,

$$S_\phi(f + K) = \phi f + K = \phi(f + g) + K$$

and so

$$\| S_\phi(f + K) \| \leq \| \phi(f + g) \| \leq \| \phi \| \| f + g \|.$$  

This implies that $\| S_\phi \| \leq \| \phi \|$. Since $S_{\phi + \psi} = S_\phi$ for any $\psi$ in $K$,

$$\| S_\phi \| \leq \| \phi + K \|.$$  

Now the following problem is natural.
Problem 1. Is $M(E/K)$ isometrically isomorphic onto $M(E)/K$?

If Problem 1 can be solved positively, then it shows that $M(E/K)$ is an operator algebra on $E/K$. Suppose

$$M(E/K)' = \{ A \in \mathcal{B}(E/K); S_{\phi}A = AS_{\phi} \text{ for any } \phi \in M(E) \}.$$ 

Then $M(E/K)'$ is a commutative algebra in $\mathcal{B}(E/K)$ which contains $M(E/K)$. We are interested in the following problem.

Problem 2. Is $M(E/K)'$ equal to $M(E/K)$?

Problem 2 is related to a problem of commuting dilation, that is, if $A \in \mathcal{B}(E/K)$ such that $AS_{\phi} = T_{\phi}A$ (\(\phi \in M(E)\)) then does exist $\hat{A} \in \mathcal{B}(E)$ such that $\hat{A}S_{\phi} = T_{\phi}\hat{A}$ (\(\phi \in M(E)\)) and $A(f + K) = \hat{A}f + K \ (f \in E)$? If $M(E)' = M(E)$ then $A = T_{\psi}$ for some $\psi \in M(E)$ and so $A = S_{\psi}$.

Let $H^p(1 \leq p \leq \infty)$ be the usual Hardy space of analytic functions on the open unit disc $D$. When $E = H^2$, Sarason [4] solved Problems 1 and 2 positively Then a theorem of Nevanlinna-Pick and a theorem of Carathéodary follow. When $E = H^p \ (1 \leq p \leq \infty, p \neq 2)$ and $K = BH^p$ for a Blaschke product with simple zeros, Snyder [5] solved Problems 1 and 2. In this paper, we solve them when $E = H^p \ (1 \leq p \leq \infty)$ and $K$ is arbitrary.

In general, $M(E)$ may not be a supnorm algebra (see [6]). Even if $E$ is a Hilbert space, $M(E)$ is a supnorm algebra on $X$ and $\dim M(E)/K = 2$, it is known that we can solve negatively Problem 1 for some $E$ and $K$(see [1]).

In this paper, for a subset $S$ $[S]$ denotes the closed linear span of $S$.

§2. General case

For each $x$ in $X$, put $\tau_x(f) = f(x)$ for a function $f$ on $X$. We assume that $\tau_x$ is bounded on $E$ and $\|\tau_x\|$ denotes the norm of
Thus $= 0$ and so $A$.

This implies $A$.

X then $\tau_x \cap E$ and $M(E)_x = \ker x$. If $K = \{0\}$ then $\mathcal{K} = \{0\}$ and so Problem 1 can be solved trivially. Moreover the following Proposition 1 solves Problem 2.

**Proposition 1.** If $M(E)_x E$ is dense in $E_x$ for any $x$ in $X$ then $M(E)' = M(E)$.

Proof. It is clear that $M(E) \subseteq M(E)'$. Suppose $A \in M(E)'$. If $T_0 \in M(E)$ then $T_0 \tau_x = \phi(x)\tau_x$ for any $x \in X$ because $\tau_x \in E^*$. Hence $T_0^*\phi(x)\tau_x = A^*(T_0^*\phi(x)\tau_x) = 0$ because $AT_0\phi(x) = T_0\phi(x)A$. Therefore for any $f \in E$, $\langle T_0\phi(x)\tau_x, A^*\tau_x \rangle = 0$ and so $A^*\tau_x = 0$ on $E_x$ because $M(E)_x E$ is dense in $E_x$. Thus $A^*\tau_x = \psi(x)\tau_x$ ($x \in X$) and so for any $f \in E$, $\psi(x)f(x) = \langle f, A^*\tau_x \rangle = \langle Af, \tau_x \rangle = \langle (Af)(x), x \rangle$. Hence $Af = \psi f = T_x f \quad (f \in E)$. This implies $A$ belongs to $M(E)$.

**Proposition 2.** If $M(E) + K = E$ then $M(E/K)' = M(E/K)$.

Proof. For any $f \in E$, put $\tilde{f} = f + K$. Then we may assume that $f \in M(E)$ by hypothesis $M(E) + K = E$. For any $g \in M(E)$, if $A \in M(E/K)'$ then for any $x$ in $(E/K)^*$

\[
\langle A\tilde{g}, x \rangle = \langle \tilde{g}, A^*x \rangle = \langle \tilde{g} \cdot \tilde{1}, A^*x \rangle = \langle ASg\tilde{1}, x \rangle = \langle SgA\tilde{1}, x \rangle = \langle Sg\tilde{\phi}, x \rangle = \langle \tilde{\phi}g, x \rangle
\]

where $\tilde{\phi} = A\tilde{1}$. Since $M(E) + K = E$, we may assume that $\phi \in M(E)$ and so $A = Sg$.

For a subset $S$ of $X$, let $E|S$ be the restriction of $E$ to $S$ and put $K = \{ f \in E; f = 0 \text{ on } S \}$. Then, $E|S$ becomes a Banach space of functions on $S$ under the quotient norm of $E/K$. We may assume that $E|S \cong E/K$. Put $K = \{ \phi \in M(E); \phi = 0 \text{ on } S \}$, then $M(E)|S \cong M(E)/K \cong M(E)$.

Even if $K$ is such a special case, Problems 1 and 2 cannot be solved in general. Snyder [5] studied Problem 1,
that is, whether $M(E)|S = M(E|S)$. In this special case, Problem 1 is just an interpolation problem. That is, if $f$ is a function on $S \subset X$ and $f(E|S) \subset E|S$ then does there exist a function $F$ on $X$ such that $FE \subset E$ and $F|S = f$ and $\|F\| = \|f\|$? Therefore the research of Snyder [5] is contained in our one.

**Corollary 1.** If $E$ is a commutative Banach algebra with unit then $M(E/K)' = M(E/K)$.

Proof. If $E$ is a commutative Banach algebra with unit then $M(E) = E$ and $\mathcal{K} = \mathcal{K}$. Hence $M(E) + K = E$. Proposition 2 implies that $M(E/K)' = M(E/K)$.

**Proposition 3.** If $E$ is a commutative Banach algebra with unit then $M(E)/\mathcal{K} = M(E/K)$ where $\mathcal{K} = \mathcal{K}$.

Proof. By the proof of Corollary 1, $M(E) = E$ and it is easy to see that $M(E)$ is isometrically isomorphic to $E$. Similarly $M(E/K)$ is isometrically isomorphic to $E/K$. This implies the proposition.

§3. Two dimensional case

In this section we assume that $M(E) \subset E$. (1) of Theorem 1 is due to Snyder [5] and (2) of Theorem 1 is new.

$d_x$ is called the derivation at $x$ if $d_x(fg) = d_x(f)\tau_x(g) + \tau_x(f)d_x(g)$ ($f, g \in M(E)$).

**Proposition 4.** Suppose $E/K$ and $M(E/K)$ are of finite dimension 2. Then $(E/K)^* = [\tau_x, \tau_y]$ for $x, y \in X$ with $x \neq y$ or $(E/K)^* = [\tau_x, d_x]$ for $x \in X$ where $d_x$ is a point derivation at $x$.

Proof. By hypothesis, $M(E/K) = E/K$ as a set. Since $M(E/K)$ is a commutative Banach algebra and $\dim M(E/K) = 2$, by [2, Proposition 1] it is easy to see that $M(E/K)^* = [\tau_x, \tau_y]$ for $x, y \in X$ with $x \neq y$ or $M(E/K)^* = [\tau_x, d_x]$ for $x \in X$.

**Lemma 1.** Suppose $M(E)+K$ is dense in $E$. If $\bar{\phi} \in M(E)$ then $S_\phi d_x = \bar{d}_x(\bar{\phi})\tau_x + \tau_x(\bar{\phi})d_x$. 
Proof. For \( f \in M(E) \)

\[
\langle f + K, S_\phi^* d_x \rangle = \langle \phi f + K, d_x \rangle = \langle \phi f, d_x \rangle
\]

\[
= d_x(\phi) \tau_x(f) + \tau_x(\phi) d_x(f)
\]

\[
= \langle f + K, d_x(\phi) \tau_x + \tau_x(\phi) d_x \rangle
\]

**Theorem 1.** Suppose that \( M(E) + K = E \), \( E/K \) and \( M(E/K) \) are of two dimension. If \( M(E/K) \) is isometrically isomorphic to \( M(E)/K \) then the following (1) and (2) are valid.

(1) When \((E/K)^* = [\tau_x, \tau_y] \) for \( x, y \in X \) with \( x \neq y \), for given \( u, v \in \mathbb{C} \), there exists \( \phi \in M(E) \) such that \( \tau_x(\phi) = u, \tau_y(\phi) = v \) and \( \|\phi + K\| \leq 1 \) if and only if

\[
\|\alpha \bar{u} \tau_x + \beta \bar{v} \tau_y\|_* \leq \|\alpha \tau_x + \beta \tau_y\|_* \quad (\alpha, \beta \in \mathbb{C}).
\]

(2) When \((E/K)^* = [\tau_x, d_x] \) for \( x \in X \), for given \( u, v \in \mathbb{C} \), there exists \( \phi \in M(E) \) such that \( \tau_x(\phi) = u, d_x(\phi) = v \) and \( \|\phi + K\| \leq 1 \) if and only if

\[
\|\alpha \bar{u} \tau_x + \beta \bar{v} d_x\|_* \leq \|\alpha \tau_x + \beta d_x\|_* \quad (\alpha, \beta \in \mathbb{C}).
\]

Proof. (1) If there exists \( \phi \in M(E) \) such that \( \tau_x(\phi) = u, \tau_y(\phi) = v \) with \( \|\phi + K\| \leq 1 \) then \( \|S_\phi^*\| \leq 1 \) by hypothesis. This implies that

\[
\|\alpha \bar{u} \tau_x + \beta \bar{v} \tau_y\|_* \leq \|\alpha \tau_x + \beta \tau_y\|_* \quad (\alpha, \beta \in \mathbb{C})
\]

because \( S_{\phi}^* \tau_x = \tau_x(\phi) \tau_x = \bar{u} \tau_x \) and \( S_{\phi}^* \tau_y = \bar{v} \tau_y \). For the converse, put \( A \in B(H/K) \), \( A^* \tau_x = \bar{u} \tau_x \) and \( A^* \tau_y = \bar{v} \tau_y \), then \( \|A^*\| \leq 1 \) and \( A \) belongs to \( M(E/K)' \). Since \( M(E) + K = E \), by Proposition 2 \( A = S_{\phi} \) for some \( \phi \in M(E) \). By hypothesis, \( \|\phi + K\| \leq 1 \) and \( \tau_x(\phi) = u \) and \( \tau_y(\phi) = v \).

(2) If there exists \( \phi \in M(E) \) with \( \tau_x(\phi) = u, d_x(\phi) = v \) with \( \|\phi + K\| \leq 1 \) then \( \|S_{\phi}^*\| \leq 1 \) by hypothesis. This and Lemma 1 imply

\[
\|\alpha \bar{u} \tau_x + \beta \bar{v} \tau_y\|_* \leq \|\alpha \tau_x + \beta \tau_y\|_* \quad (\alpha, \beta \in \mathbb{C}).
\]
For the converse, put $A \in \mathcal{B}(H/K)$, $A^* \tau_x = \bar{u} \tau_x$ and $A^* d_x = \bar{v} \tau_x + \bar{u} d_x$, then $\|A^*\| \leq 1$ and $A$ belongs to $M(E/K)'$ by Lemma 1. By Proposition 2 $A = S_\phi$ for some $\phi \in M(E)$. By hypothesis, $\|\phi + K\| \leq 1$ and $\tau_x(\phi) = u$ and $\tau_y(\phi) = v$.

In Theorem 1, if (1) or (2) is valid then $M(E/K)$ is isometrically isomorphic to $M(E)/K$.

**Corollary 2.** In Theorem 1, if $E$ is a Hilbert space then there exist $k_x$ and $h_x$ in $E$ such that

$$\tau_x(f) = (f, k_x) \quad (f \in E)$$

and

$$d_x(f) = (f, h_x) \quad (f \in E)$$

and the following (1) and (2) are valid.

(1) When $(E/K)^* = [\tau_x, \tau_y]$ for $x, y \in X$ with $x \neq y$, for given $u, v \in \mathbb{C}$, there exists $\phi \in M(E)$ such that $\tau_x(\phi) = u, \tau_y(\phi) = v$ and $\|\phi + K\| \leq 1$ if and only if

$$|\alpha|^2(1 - |u|^2)(k_x, k_x) + \alpha \bar{\beta}(1 - \bar{u}v)(k_x, k_y) + \bar{\alpha} \beta(1 - u\bar{v})(k_y, k_x) + |\beta|^2(1 - |v|^2)(k_y, k_y) \geq 0$$

for any $\alpha, \beta \in \mathbb{C}$.

(2) When $(E/K)^* = [\tau_x, d_x]$ for $x \in X$, for given $u, v \in \mathbb{C}$, there exists $\phi \in M(E)$ such that $\tau_x(\phi) = u, d_x(\phi) = v$ and $\|\phi + K\| \leq 1$ if and only if

$$((|\alpha|^2 - |\alpha \bar{u} + \beta \bar{v}|^2)(k_x, k_x) + (\alpha \bar{\beta} - (\alpha \bar{\beta}|u|^2 + |\beta|^2 u\bar{v}))(k_x, h_x) + (\bar{\alpha} \beta - (\bar{\alpha} \beta|u|^2 + |\beta|^2 \bar{u}v))(h_x, k_x) + |\beta|^2(1 - |v|^2)(h_x, h_x) \geq 0$$

for any $\alpha, \beta \in \mathbb{C}$.

The condition of (1) in Corollary 2 shows that the $2 \times 2$ matrix $\{((1 - |u|^2)(k_x, k_x), (1 - \bar{u}v)(k_x, k_y), (1 - u\bar{v})(k_y, k_x), (1 - |v|^2)(k_y, k_y))\}$ is nonnegative. When $(k_x, h_x) = 0$, the condition of (2) in Corollary 2 shows that the $2 \times 2$ matrix $\{(1 - |u|^2)(k_x, k_x), \bar{u}v(h_x, k_x), u\bar{v}(k_x, h_x), (1 - |u|^2 - |v|^2)(h_x, h_x)\}$ is nonnegative.
When dim $E/K \geq 3$, even if dim $E/K$ is finite, it is difficult to describe $(E/K)^*$ except $K = \{ f \in E : f(x_j) = 0 \; 1 \leq j \leq \dim E/K \}$ and $x_i \neq x_j (i \neq j)$. Therefore we could not generalize (2) of Theorem 1.

§4. Hardy space $H^p$ ($1 \leq p \leq \infty$)

In this section, we solve Problems 1 and 2 when $E = H^p$ for $1 \leq p \leq \infty$. When $E = H^\infty$, we can solve trivially Problems 1 and 2 by Corollary 2 and Proposition 4. If dim $H^p/K < \infty$ then $M(H^p) + K = H^p$ and so $M(H^p/K)' = M(H^p/K)$ by Proposition 2. However we have to work more in order to prove $M(H^p/K) = M(H^p)/K$.

Let $W$ be a nonnegative function in $L^1$ with $\log W$ in $L^1 = L^1(d\theta/2\pi)$. Then there exists an outer function $h$ in $H^1$ with $W = |h|$. For $1 \leq p < \infty$, $H^p(W)$ denotes the closure of analytic polynomials in $L^p(W) = L^1(Wd\theta/2\pi)$. Then $H^p(W) = h^{-1/p}H^p$ and so we may assume that $H^p(W)$ is a Banach space of analytic functions on $D$. It is known that the point evaluations of points in $D$ are continuous on $H^p(W)$. It is well known that $M(H^p(W)) = H^\infty$.

**Theorem 2.** For $1 \leq p \leq \infty$, let $K$ be a closed subspace of $H^p(W)$ with $M(H^p(W))/K \subseteq K$. Then $M(H^p(W)/K)' = M(H^p(W)/K)$ and $M(H^p(W)/K) = M(H^p(W))/K$ where $K = \{ \phi \in M(H^p(W)) : \phi H^p(W) \subseteq K \}$.

Proof. Since $M(H^p(W)) = H^\infty$, $K = QH^\infty$ for some inner function and $K = QH^p(W)$. Since $M(H^p(W)/K) \subseteq M(H^p(W)/K)'$, we will show that $M(H^p(W)/K)' \subseteq M(H^p(W)/K)$. If $A \in M(H^p(W)/K)'$ then there exists $\psi$ in $H^p(W)$ such that $A(1 + K) = \psi + K$. For any polynomial $f = h^{-1/p}F$ in $H^p(W)$, $\|\psi f + K\|_W \leq \|A\|\|f + K\|_W$.

Since $K = QH^p(W) = h^{-1/p}QH^p$, if $1/p + 1/q = 1$

\[
\|\psi f + QH^p(W)\|_W
= \sup \{ |\psi f, g|_W : g \in \{ QH^p(W) \}^\perp \text{ and } \|g\|_W \leq 1 \}
= \sup \left\{ |\int \psi h^{-1/p}F \bar{Q} h^{-1/q}G |h|dm| : G \in H^q_0 \text{ and } \|G\|_q \leq 1 \right\}
\]
\[
\sup \left\{ \left| \int \psi \bar{Q} F G dm \right| : G \in H_0^q \text{ and } \|G\|_q \leq 1 \right\} \\
\leq \|A\| \|F\|_p = \|A\| \|F\|_p
\]
because \( \{Q H^p(W)\}^\perp = Q \tilde{h}^{1/p}[h]^{-1} \tilde{H}_0^q \). Thus
\[
\sup \left\{ \left| \int \psi \bar{Q} G d\theta / 2\pi \right| : F \in H^p, G \in H_0^q, \|F\|_p \leq 1 \text{ and } \|G\|_q \leq 1 \right\} \leq \|A\|
\]
By the factorization theorem of \( H^1 \),
\[
\sup \left\{ \left| \int \psi \bar{Q} K d\theta / 2\pi \right| : K \in H_0^1 \text{ and } \|K\|_1 \leq 1 \right\} \leq \|A\|
\]
Since \((\bar{Q} H_0^1)^* = L^\infty / Q H^\infty\), \( \|\psi + Q H^\infty\| \leq \|A\| \). Hence there exists a function \( \phi \) in \( H^\infty \) such that \( S_\phi = A \) and \( \|\phi + K\| = \|S_\phi\| \). Thus \( A \) belongs to \( M(H^p(W)/K) \). Therefore \( M(H^p(W)/K)' = M(H^p(W)/K) \) and \( M(H^p(W)/K) = M(H^p(W)/K) \).

**Corollary 3.** For \( 1 \leq p \leq \infty \), \( M(H^p(W)/Q H^p(W)) = H^\infty / Q H^\infty \) for any inner function \( Q \).
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