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Multipliers For A Quotient Banach Space And The
Nevanlinna-Pick Theorem

By

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Abstract. Let $E$ be a Banach space on a set $X$ and $M(E)$ the space of multipliers of $E$. In this paper, we study the space of multipliers of the quotient space $E/K$ where $K$ is a closed $M(E)$-invariant subspace in $E$. When $E$ is the classical Hilbert Hardy space, the Nevanlinna and Pick theorem shows $M(E/K)$ is a quotient algebra of $M(E)$. 
§1. Introduction

A Banach space $E$ of functions on a set $X$ is a Banach space whose elements are complex-valued functions defined on $X$ with the usual pointwise addition and scalar multiplication. If $\phi$ is a complex-valued function on $X$ and $\phi f$ belongs to $E$ for all $f$ in $E$, then we write that $\phi$ is an element of $M(E)$, the space of multipliers of $E$. We assume that the point evaluations are continuous on $E$, that is, $X$ is embedded in the dual space $E^*$ and that there is no point in $X$ where all the members of $E$ vanish. It is known that $T_\phi : f \to \phi f$ is a bounded operator on $E$ for each $\phi$ in $M(E)$, since by continuity of point evaluation, each such map has closed graph. $M(E)$ is a closed subalgebra of $B(E)$, the set of bounded operators on $E$, indeed $M(E)$ is closed in the weak operator topology. Thus we assume that the space $M(E)$ of multipliers of $E$ is an operator algebra on $E$.

If $K$ is a closed subspace of $E$ then $E/K$ is also a Banach space. We want to define the space of multipliers $M(E/K)$ of $E/K$. For $\phi$ in $M(E)$ put

$$S_\phi(f + K) = \phi f + K \quad (f \in E).$$

In general, $S_\phi$ is not well defined on $E/K$. We need assume that $M(E) K \subset K$. Suppose $M(E/K) = \{ S_\phi; \phi \in M(E) \}$. Then $M(E/K)$ is also an operator algebra on $E/K$. Put

$$K = \{ \phi \in M(E); \phi E \subset K \}$$

then $K$ is a closed ideal in $M(E)$. By the definition, $S_\phi = 0$ if and only if $\phi E \subset K$. Hence $S_\phi = 0$ if and only if $\phi$ belongs to $K$. Therefore $S_{\phi + \psi} = S_\phi$ for any $\psi$ in $K$ and so there exists a one-to-one map from $M(E/K)$ onto $M(E)/K$. Moreover this map is contractive. In fact, for any $g$ in $K$,

$$S_\phi(f + K) = \phi f + K = \phi(f + g) + K$$

and so

$$\|S_\phi(f + K)\| \leq \|\phi(f + g)\| \leq \|\phi\| \|f + g\|.$$  

This implies that $\|S_\phi\| \leq \|\phi\|$. Since $S_{\phi + \psi} = S_\phi$ for any $\psi$ in $K$, $\|S_\phi\| \leq \|\phi + K\|$. Now the following problem is natural.
Problem 1. Is $M(E/K)$ isometrically isomorphic onto $M(E)/K$?

If Problem 1 can be solved positively, then it shows that $M(E)/K$ is an operator algebra on $E/K$. Suppose

$$M(E/K)' = \{ A \in B(E/K); S_\phi A = AS_\phi \text{ for any } \phi \in M(E) \}.$$ 

Then $M(E/K)'$ is a commutative algebra in $B(E/K)$ which contains $M(E/K)$. We are interested in the following problem.

Problem 2. Is $M(E/K)'$ equal to $M(E/K)$?

Problem 2 is related to a problem of commuting dilation, that is, if $A \in B(E/K)$ such that $AS_\phi = S_\phi A$ (for any $\phi \in M(E)$) then does exist $\tilde{A} \in B(E)$ such that $\tilde{A}T_\phi = T_\phi \tilde{A}$ (for any $\phi \in M(E)$) and $A(f + K) = \tilde{A}f + K$ ($f \in E$)? If $M(E)' = \tilde{M}(E)$ then $A = T_\psi$ for some $\psi \in \tilde{M}(E)$ and so $A = S_\psi$.

Let $H^p(1 \leq p \leq \infty)$ be the usual Hardy space of analytic functions on the open unit disc $D$. When $E = H^2$, Sarason [4] solved Problems 1 and 2 positively. Then a theorem of Nevanlinna-Pick and a theorem of Carathéodary follow. When $E = H^p$ (for $1 \leq p \leq \infty$, $p \neq 2$) and $K = BH^p$ for a Blaschke product with simple zeros, Snyder [5] solved Problems 1 and 2. In this paper, we solve them when $E = H^p$ (for $1 \leq p \leq \infty$) and $K$ is arbitrary.

In general, $M(E)$ may not be a supnorm algebra (see [6]). Even if $E$ is a Hilbert space, $M(E)$ is a supnorm algebra on $X$ and $\dim M(E)/K = 2$, it is known that we can solve negatively Problem 1 for some $E$ and $K$ (see [1]).

In this paper, for a subset $S$ [S] denotes the closed linear span of $S$.

§2. General case

For each $x$ in $X$, put $\tau_x(f) = f(x)$ for a function $f$ on $X$. We assume that $\tau_x$ is bounded on $E$ and $\|\tau_x\|$ denotes the norm of
\( \tau_x \) on \( E \). \( \tau_x \) is also bounded on \( M(E) \) and the norm is just one. For

\[ |\tau_x(\phi) \cdot \tau_x(f)| = |\tau_x(\phi f)| \leq \|\tau_x\| \cdot \|\phi\| \cdot \|f\| \quad (\phi \in M(E), f \in E) \]

and so \( |\tau_x(\phi)| \cdot \|\tau_x\| \leq \|\tau_x\| \cdot \|\phi\| \). Put \( E_x = \ker \tau_x \cap E \) and \( M(E)_x = \ker \tau_x \cap M(E) \). If \( K = \{0\} \) then \( K = \{0\} \) and so Problem 1 can be solved trivially. Moreover the following Proposition 1 solves Problem 2.

**Proposition 1.** If \( M(E)_x E \) is dense in \( E_x \) for any \( x \) in \( X \) then \( M(E)' = M(E) \).

**Proof.** It is clear that \( M(E) \subset M(E)' \). Suppose \( A \in M(E)' \). If \( T_\phi \in M(E) \) then \( T_\phi \tau_x = \phi(x) \tau_x \) for any \( x \in X \) because \( \tau_x \in E^* \). Hence \( T_{\phi-\phi(x)}^*(A^* \tau_x) = A^*(T_{\phi-\phi(x)}^* \tau_x) = 0 \) because \( AT_{\phi-\phi(x)} = T_{\phi-\phi(x)} A \). Therefore for any \( f \in E \), \( \langle T_{\phi-\phi(x)} f, A^* \tau_x \rangle = 0 \) and so \( A^* \tau_x = 0 \) on \( E_x \) because \( M(E)_x E \) is dense in \( E_x \).

Thus \( A^* \tau_x = \psi(x) \tau_x \) \( (x \in X) \) and so for any \( f \in E \), \( \psi(x)f(x) = \langle f, A^* \tau_x \rangle = \langle Af, \tau_x \rangle = \langle Af \rangle(x) \). Hence \( Af = \psi f = T_\psi f \) \( (f \in E) \). This implies \( A \) belongs to \( M(E) \).

**Proposition 2.** If \( M(E) + K = E \) then \( M(E/K)' = M(E/K) \).

**Proof.** For any \( f \in E \), put \( \tilde{f} = f + K \). Then we may assume that \( f \in M(E) \) by hypothesis \( M(E) + K = E \). For any \( g \in M(E) \), if \( A \in M(E/K)' \) then for any \( x \) in \( (E/K)^* \)

\[
\langle A\tilde{g}, x \rangle = \langle \tilde{g}, A^* x \rangle = \langle \tilde{g} \cdot 1, A^* x \rangle = \langle AS_g \tilde{1}, x \rangle = \langle S_g A\tilde{1}, x \rangle = \langle S_g \tilde{\phi}, x \rangle = \langle \tilde{\phi}\tilde{g}, x \rangle
\]

where \( \tilde{\phi} = A\tilde{1} \). Since \( M(E) + K = E \), we may assume that \( \phi \in M(E) \) and so \( \tilde{A} = S_g \)

For a subset \( S \) of \( X \), let \( E|S \) be the restriction of \( E \) to \( S \) and put \( K = \{f \in E; f = 0 \text{ on } S\} \). Then, \( E|S \) becomes a Banach space of functions on \( S \) under the quotient norm of \( E/K \). We may assume that \( E|S \cong E/K \). Put \( K = \{\phi \in M(E); \phi = 0 \text{ on } S\} \), then \( M(E)|S \cong M(E)/K \). Even if \( K \) is such a special case, Problems 1 and 2 cannot be solved in general. Snyder [5] studied Problem 1,
that is, whether \( M(E)|S = M(E|S) \). In this special case, Problem 1 is just an interpolation problem. That is, if \( f \) is a function on \( S \subset X \) and \( f(E|S) \subset E|S \) then does there exist a function \( F \) on \( X \) such that \( FE \subset E \) and \( F|S = f \) and \( \|F\| = \|f\| \)? Therefore the research of Snyder [5] is contained in our one.

**Corollary 1.** If \( E \) is a commutative Banach algebra with unit then \( M(E/K)' = M(E/K) \).

Proof. If \( E \) is a commutative Banach algebra with unit then \( M(E) = E \) and \( K = K \). Hence \( M(E) + K = E \). Proposition 2 implies that \( M(E/K)' = M(E/K) \).

**Proposition 3.** If \( E \) is a commutative Banach algebra with unit then \( M(E)/K = M(E/K) \) where \( K = K \).

Proof. By the proof of Corollary 1, \( M(E) = E \) and it is easy to see that \( M(E) \) is isometrically isomorphic to \( E \). Similarly \( M(E/K) \) is isometrically isomorphic to \( E/K \). This implies the proposition.

§3. Two dimensional case

In this section we assume that \( M(E) \subset E \). (1) of Theorem 1 is due to Snyder [5] and (2) of Theorem 1 is new.

\( d_x \) is called the derivation at \( x \) if \( d_x(fg) = d_x(f)\tau_x(g) + \tau_x(f)d_x(g) \) \((f,g \in M(E))\).

**Proposition 4.** Suppose \( E/K \) and \( M(E/K) \) are of finite dimension 2. Then \( (E/K)^* = [\tau_x, \tau_y] \) for \( x, y \in X \) with \( x \neq y \) or \( (E/K)^* = [\tau_x, d_x] \) for \( x \in X \) where \( d_x \) is a point derivation at \( x \).

Proof. By hypothesis, \( M(E/K) = E/K \) as a set. Since \( M(E/K) \) is a commutative Banach algebra and \( \dim M(E/K) = 2 \), by [2, Proposition 1] it is easy to see that \( M(E/K)^* = [\tau_x, \tau_y] \) for \( x, y \in X \) with \( x \neq y \) or \( M(E/K)^* = [\tau_x, d_x] \) for \( x \in X \).

**Lemma 1.** Suppose \( M(E)+K \) is dense in \( E \). If \( \phi \in M(E) \) then \( S_\phi d_x = \overline{d_x(\phi)\tau_x + \tau_x(\phi)d_x} \).
Proof. For $f \in M(E)$
\[
\langle f + K, S^*_d \rangle = \langle \phi + K, d \rangle = \langle \phi, d \rangle
\]
\[
= \langle f + K, \tau_x(\phi) + d \rangle
\]
\[
= \langle f + K, d \rangle \tau_x(\phi) + \tau_x(\phi) d
\]

**Theorem 1.** Suppose that $M(E) + K = E$, $E/K$ and $M(E/K)$ are of two dimension. If $M(E/K)$ is isometrically isomorphic to $M(E)/K$ then the following (1) and (2) are valid.

(1) When $(E/K)^* = [\tau_x, \tau_y]$ for $x, y \in X$ with $x \neq y$, for given $u, v \in \mathbb{C}$, there exists $\phi \in M(E)$ such that $\tau_x(\phi) = u$, $\tau_y(\phi) = v$ and $\|\phi + K\| \leq 1$ if and only if

\[
\|\alpha \bar{u} \tau_x + \beta \bar{v} \tau_y\|_* \leq \|\alpha \tau_x + \beta \tau_y\|_*(\alpha, \beta \in \mathbb{C})
\]

(2) When $(E/K)^* = [\tau_x, d_x]$ for $x \in X$, for given $u, v \in \mathbb{C}$, there exists $\phi \in M(E)$ such that $\tau_x(\phi) = u, d_x(\phi) = v$ and $\|\phi + K\| \leq 1$ if and only if

\[
\|(\alpha \bar{u} + \beta \bar{v}) \tau_x + \beta \bar{u} d_x\|_* \leq \|\alpha \tau_x + \beta d_x\|_*(\alpha, \beta \in \mathbb{C})
\]

Proof. (1) If there exists $\phi \in M(E)$ such that $\tau_x(\phi) = u, \tau_y(\phi) = v$ with $\|\phi + K\| \leq 1$ then $\|S^*_x\| \leq 1$ by hypothesis. This implies that

\[
\|\alpha \bar{u} \tau_x + \beta \bar{v} \tau_y\|_* \leq \|\alpha \tau_x + \beta \tau_y\|_*(\alpha, \beta \in \mathbb{C})
\]

because $S^*_x \tau_x = \tau_x(\phi) \tau_x = \bar{u} \tau_x$ and $S^*_y \tau_y = \bar{v} \tau_y$. For the converse, put $A \in \mathcal{B}(H/K)$, $A^* \tau_x = \bar{u} \tau_x$ and $A^* \tau_y = \bar{v} \tau_y$, then $\|A^*\| \leq 1$ and $A$ belongs to $M(E/K)'$. Since $M(E) + K = E$, by Proposition 2 $A = S^*_\phi$ for some $\phi \in M(E)$. By hypothesis, $\|\phi + K\| \leq 1$ and $\tau_x(\phi) = u$ and $\tau_y(\phi) = v$.

(2) If there exists $\phi \in M(E)$ with $\tau_x(\phi) = u, d_x(\phi) = v$ with $\|\phi + K\| \leq 1$ then $\|S^*_x\| \leq 1$ by hypothesis. This and Lemma 1 imply

\[
\|(\alpha \bar{u} + \beta \bar{v}) \tau_x + \beta \bar{u} d_x\|_* \leq \|\alpha \tau_x + \beta d_x\|_*(\alpha, \beta \in \mathbb{C})
\]
For the converse, put $A \in B(H/K)$, $A^*\tau_x = \bar{u}\tau_x$ and $A^*d_x = \bar{v}\tau_x + \bar{u}d_x$, then $\|A^*\| \leq 1$ and $A$ belongs to $M(E/K)'$ by Lemma 1. By Proposition 2 $A = S_\phi$ for some $\phi \in M(E)$. By hypothesis, $\|\phi + \mathcal{K}\| \leq 1$ and $\tau_x(\phi) = u$ and $\tau_y(\phi) = v$.

In Theorem 1, if (1) or (2) is valid then $M(E/K)$ is isometrically isomorphic to $M(E)/\mathcal{K}$.

**Corollary 2.** In Theorem 1, if $E$ is a Hilbert space then there exist $k_x$ and $h_x$ in $E$ such that

$$\tau_x(f) = (f, k_x) \quad (f \in E)$$

and

$$d_x(f) = (f, h_x) \quad (f \in E)$$

and the following (1) and (2) are valid.

1. When $(E/K)^* = [\tau_x, \tau_y]$ for $x, y \in X$ with $x \neq y$, for given $u, v \in \mathbb{C}$, there exists $\phi \in M(E)$ such that $\tau_x(\phi) = u, \tau_y(\phi) = v$ and $\|\phi + \mathcal{K}\| \leq 1$ if and only if

$$|\alpha|^2(1 - |u|^2)(k_x, k_x) + \alpha\bar{\beta}(1 - \bar{u}v)(k_x, k_y) + \bar{\alpha}\beta(1 - u\bar{v})(k_y, k_x) + |\beta|^2(1 - |v|^2)(k_y, k_y) \geq 0$$

for any $\alpha, \beta \in \mathbb{C}$.

2. When $(E/K)^* = [\tau_x, d_x]$ for $x \in X$, for given $u, v \in \mathbb{C}$, there exists $\phi \in M(E)$ such that $\tau_x(\phi) = u, d_x(\phi) = v$ and $\|\phi + \mathcal{K}\| \leq 1$ if and only if

$$\begin{align*}
(|\alpha|^2 - |\alpha\bar{u} + \beta\bar{v}|^2)(k_x, k_x) + (\alpha\bar{\beta} - (\alpha\bar{\beta}|u|^2 + |\beta|^2u\bar{v}))(k_x, k_y) \\
+ (\bar{\alpha}\beta - (\bar{\alpha}\beta|u|^2 + |\beta|^2\bar{u}v))(h_x, k_x) + |\beta|^2(1 - |u|^2)(h_x, h_x) \geq 0
\end{align*}$$

for any $\alpha, \beta \in \mathbb{C}$.

The condition of (1) in Corollary 2 shows that the $2 \times 2$ matrix $\{(1 - |u|^2)(k_x, k_x), (1 - \bar{u}v)(k_x, k_y), (1 - u\bar{v})(k_y, k_x), (1 - |v|^2)(k_y, k_y)\}$ is nonnegative. When $(k_x, h_x) = 0$, the condition of (2) in Corollary 2 shows that the $2 \times 2$ matrix $\{(1 - |u|^2)(k_x, k_x), \bar{u}v(h_x, k_x), u\bar{v}(k_x, h_x), (1 - |u|^2 - |v|^2)(h_x, h_x)\}$ is nonnegative.
When \( \dim E/K \geq 3 \), even if \( \dim E/K \) is finite, it is difficult to describe \((E/K)^*\) except \( K = \{ f \in E : f(x) = 0 \ 1 \leq j \leq \dim E/K \} \) and \( x_i \neq x_j(i \neq j) \). Therefore we could not generalize (2) of Theorem 1.

§4. Hardy space \( H^p \) \( (1 \leq p \leq \infty) \)

In this section, we solve Problems 1 and 2 when \( E = H^p \) for \( 1 \leq p \leq \infty \). When \( E = H^\infty \), we can solve trivially Problems 1 and 2 by Corollary 2 and Proposition 4. If \( \dim H^p/K < \infty \) then \( M(H^p) + K = H^p \) and so \( M(H^p/K)' = M(H^p/K) \) by Proposition 2. However we have to work more in order to prove \( M(H^p/K) = M(H^p)/K \).

Let \( W \) be a nonnegative function in \( L^1 \) with \( \log W \) in \( L^1(d\theta/2\pi) \). Then there exists an outer function \( h \) in \( H^1 \) with \( W = |h| \). For \( 1 \leq p < \infty \), \( H^p(W) \) denotes the closure of analytic polynomials in \( L^p(W) = L^1(Wd\theta/2\pi) \). Then \( H^p(W) = h^{-1/p}H^p \) and so we may assume that \( H^p(W) \) is a Banach space of analytic functions on \( D \). It is known that the point evaluations of points in \( D \) are continuous on \( H^p(W) \). It is well known that \( M(H^p(W)) = H^\infty \).

**Theorem 2.** For \( 1 \leq p \leq \infty \), let \( K \) be a closed subspace of \( H^p(W) \) with \( M(H^p(W))/K \subseteq K \). Then \( M(H^p(W))/K' = M(H^p(W)/K) \) and \( M(H^p(W)/K) = M(H^p(W))/K \) where \( K = \{ \phi \in M(H^p(W)) : \phi H^p(W) \subseteq K \} \).

**Proof.** Since \( M(H^p(W)) = H^\infty \), \( K = QH^\infty \) for some inner function and \( K = QH^p(W) \). Since \( M(H^p(W)/K) \subseteq M(H^p(W)/K)' \), we will show that \( M(H^p(W)/K)' \subseteq M(H^p(W)/K) \). If \( A \in M(H^p(W)/K)' \), then there exists \( \psi \in H^p(W) \) such that \( A(1+K) = \psi + K \). For any polynomial \( f = h^{-1/p}F \) in \( H^p(W) \), \( \|\psi f + K\|_W \leq \|A\|\|f + K\|_W \). Since \( K = QH^p(W) = h^{-1/p}QH^p \), if \( 1/p + 1/q = 1 \)

\[
\|\psi f + QH^p(W)\|_W \\
= \sup \{ |\langle \psi f, g \rangle_W | : g \in \{ QH^p(W) \}^\perp \text{ and } \|g\|_W \leq 1 \} \\
= \sup \left\{ \int \psi h^{-1/p}F \bar{Q} h^{-1/q} G |h|dm : G \in H^q_0 \text{ and } \|G\|_q \leq 1 \right\}
\]
\[
\begin{align*}
= \sup \Big\{ \left| \int \psi \bar{Q}FGd\theta \right| : & \ G \in H_0^q \text{ and } \|G\|_q \leq 1 \Big\} \\
\leq & \ |A|F\|_w = |A|F\|_p
\end{align*}
\]

because \( \{QH^p(W)\}^\perp = Q\bar{h}^{1/p}|h|^{-1}H_0^q \). Thus
\[
\sup \Big\{ \left| \int \psi \bar{Q}FGd\theta/2\pi \right| : \ F \in H^p, G \in H_0^q, \|F\|_p \leq 1 \text{ and } \|G\|_q \leq 1 \Big\} \leq \|A\|
\]

By the factorization theorem of \( H^1 \),
\[
\sup\{ \left| \int \psi \bar{Q}Kd\theta/2\pi \right| : K \in H_0^1 \text{ and } \|K\|_1 \leq 1 \} \leq \|A\|
\]

Since \( (\bar{Q}H_0^1)^* = L^\infty/QH^\infty \), \( \|\psi + QH^\infty\| \leq \|A\|. \) Hence there exists a function \( \phi \) in \( H^\infty \) such that \( S_\phi = A \) and \( \|\phi + K\| = \|S_\phi\|. \) Thus \( A \) belongs to \( M(H^p(W)/K) \). Therefore \( M(H^p(W)/K)' = M(H^p(W)/K) \) and \( M(H^p(W)/K) = M(H^p(W)/K). \)

**Corollary 3.** For \( 1 \leq p \leq \infty \), \( M(H^p(W)/QH^p(W)) = H^\infty/QH^\infty \) for any inner function \( Q \).
References


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