Multipliers For A Quotient Banach Space And The Nevanlinna-Pick Theorem

By

Takahiko Nakazi

2000 Mathematics Subject Classification. Primary 47 A 20, Secondary 46 J 15

Keywords and phrases : quotient Banach space, multiplier, Nevanlinna-Pick theorem, Hardy space

* This research was partially supported by Grant-in-Aid for Scientific Research, Japan Society for the Promotion of Science.
Abstract. Let $E$ be a Banach space on a set $X$ and $M(E)$ the space of multipliers of $E$. In this paper, we study the space of multipliers of the quotient space $E/K$ where $K$ is a closed $M(E)$ - invariant subspace in $E$. When $E$ is the classical Hilbert Hardy space, the Nevanlinna and Pick theorem shows $M(E/K)$ is a quotient algebra of $M(E)$. 
§1. Introduction

A Banach space $E$ of functions on a set $X$ is a Banach space whose elements are complex-valued functions defined on $X$ with the usual pointwise addition and scalar multiplication. If $\phi$ is a complex-valued function on $X$ and $\phi f$ belongs to $E$ for all $f$ in $E$, then we write that $\phi$ is an element of $M(E)$, the space of multipliers of $E$. We assume that the point evaluations are continuous on $E$, that is, $X$ is embedded in the dual space $E^*$ and that there is no point in $X$ where all the members of $E$ vanish. It is known that $T_\phi : f \to \phi f$ is a bounded operator on $E$ for each $\phi$ in $M(E)$, since by continuity of point evaluation, each such map has closed graph.

$M(E)$ is a closed subalgebra of $B(E)$, the set of bounded operators on $E$, indeed $M(E)$ is closed in the weak operator topology. Thus we assume that the space $M(E)$ of multipliers of $E$ is an operator algebra on $E$.

If $K$ is a closed subspace of $E$ then $E/K$ is also a Banach space. We want to define the space of multipliers $M(E/K)$ of $E/K$. For $\phi$ in $M(E)$ put

$$S_\phi(f + K) = \phi f + K \quad (f \in E).$$

In general, $S_\phi$ is not well defined on $E/K$. We need assume that $M(E)K \subset K$. Suppose $M(E/K) = \{S_\phi; \phi \in M(E)\}$. Then $M(E/K)$ is also an operator algebra on $E/K$. Put

$$\mathcal{K} = \{\phi \in M(E); \phi E \subset K\}$$

then $\mathcal{K}$ is a closed ideal in $M(E)$. By the definition, $S_\phi = 0$ if and only if $\phi E \subset K$. Hence $S_\phi = 0$ if and only if $\phi$ belongs to $\mathcal{K}$. Therefore $S_{\phi + \psi} = S_\phi$ for any $\psi$ in $\mathcal{K}$ and so there exists a one-to-one map from $M(E/K)$ onto $M(E)/\mathcal{K}$. Moreover this map is contractive. In fact, for any $g$ in $K$,

$$S_\phi(f + K) = \phi f + K = \phi(f + g) + K$$

and so

$$\|S_\phi(f + K)\| \leq \|\phi(f + g)\| \leq \|\phi\|\|f + g\|.$$

This implies that $\|S_\phi\| \leq \|\phi\|$. Since $S_{\phi + \psi} = S_\phi$ for any $\psi$ in $\mathcal{K}$, $\|S_\phi\| \leq \|\phi + \mathcal{K}\|$. Now the following problem is natural.
Problem 1. Is $M(E/K)$ isometrically isomorphic onto $M(E)/K$?

If Problem 1 can be solved positively, then it shows that $M(E/K)$ is an operator algebra on $E/K$. Suppose

$$M(E/K)' = \{ A \in B(E/K); S_\phi A = A S_\phi \text{ for any } \phi \in M(E) \}.$$ 

Then $M(E/K)'$ is a commutative algebra in $B(E/K)$ which contains $M(E/K)$. We are interested in the following problem.

Problem 2. Is $M(E/K)'$ equal to $M(E/K)$?

Problem 2 is related to a problem of commuting dilation, that is, if $A \in B(E/K)$ such that $AS_\phi = S_\phi A$ ($\phi \in M(E)$) then does exist $\hat{A} \in B(E)$ such that $\hat{A}T_\phi = T_\phi \hat{A}$ ($\phi \in M(E)$) and $A(f + K) = \hat{A}f + \hat{K}$ ($f \in E$)? If $M(E)' = M(E)$ then $A = T_\psi$ for some $\psi \in M(E)$ and so $A = S_\psi$.

Let $H^p(1 \leq p \leq \infty)$ be the usual Hardy space of analytic functions on the open unit disc $D$. When $E = H^2$, Sarason [4] solved Problems 1 and 2 positively Then a theorem of Nevanlinna-Pick and a theorem of Carathéodary follow. When $E = H^p$ (1 \leq p \leq \infty, p \neq 2) and $K = BH^p$ for a Blaschke product with simple zeros, Snyder [5] solved Problems 1 and 2. In this paper, we solve them when $E = H^p$ (1 \leq p \leq \infty) and $K$ is arbitrary.

In general, $M(E)$ may not be a supnorm algebra (see [6]). Even if $E$ is a Hilbert space, $M(E)$ is a supnorm algebra on $X$ and dim $M(E)/K = 2$, it is known that we can solve negatively Problem 1 for some $E$ and $K$(see [1]).

In this paper, for a subset $S$ [S] denotes the closed linear span of $S$.

§2. General case

For each $x$ in $X$, put $\tau_x(f) = f(x)$ for a function $f$ on $X$. We assume that $\tau_x$ is bounded on $E$ and $\|\tau_x\|$ denotes the norm of
Thus $A = 0$ and so $M$. This implies $X = \ker\tau$ and so $|\tau_x| \leq \|\tau_x\| \cdot \|\phi\| \cdot \|f\|$ (\(\phi \in M(E), f \in E\)) and so $|\tau_x(\phi)| \cdot \|\tau_x\| \leq \|\tau_x\| \cdot \|\phi\|$. Put $E_x = \ker\tau_x \cap E$ and $M(E)_x = \ker\tau_x \cap M(E)$. If $K = \{0\}$ then $K = \{0\}$ and so Problem 1 can be solved trivially. Moreover the following Proposition 1 solves Problem 2.

**Proposition 1.** If $M(E)_xE$ is dense in $E_x$ for any $x$ in $X$ then $M(E)' = M(E)$.

Proof. It is clear that $M(E) \subseteq M(E)'$. Suppose $A \in M(E)'$. If $T_\phi \in M(E)$ then $T_\phi \tau_x = \phi(x)\tau_x$ for any $x \in X$ because $\tau_x \in E^\ast$. Hence $T_{\phi-\phi(x)}(A\ast \tau_x) = A\ast(T_{\phi-\phi(x)}\tau_x) = 0$ because $AT_{\phi-\phi(x)} = 0$ and so $A\ast \tau_x = 0$ on $E_x$ because $M(E)_xE$ is dense in $E_x$. Thus $A\ast \tau_x = \phi(x)\tau_x$ (\(x \in X\)) and so for any $f \in E$, $\langle T_{\phi-\phi(x)}f, A\ast \tau_x \rangle = 0$ and so $A\ast \tau_x = 0$ on $E_x$ because $M(E)_xE$ is dense in $E_x$. Hence $\tau_x = \phi(x)\tau_x$ (\(x \in X\)) and so for any $f \in E$, $\langle f, A\ast \tau_x \rangle = \langle Af, \tau_x \rangle = \langle Af \rangle (x)$. Hence $Af = \psi f = T\psi f$ (\(f \in E\)). This implies $A$ belongs to $M(E)$.

**Proposition 2.** If $M(E) + K = E$ then $M(E/K)' = M(E/K)$.

Proof. For any $f \in E$, put $\tilde{f} = f + K$. Then we may assume that $f \in M(E)$ by hypothesis $M(E) + K = E$. For any $g \in M(E)$, if $A \in M(E/K)'$ then for any $x$ in $(E/K)^\ast$

\[
\langle A\tilde{g}, x \rangle = \langle \tilde{g}, A\ast x \rangle = \langle \tilde{g} \cdot \tilde{1}, A\ast x \rangle = \langle A\tilde{S}_g\tilde{1}, x \rangle = \langle S_gA\tilde{1}, x \rangle = \langle S_g\tilde{\phi}, x \rangle = \langle \tilde{\phi}\tilde{g}, x \rangle
\]

where $\tilde{\phi} = A\tilde{1}$. Since $M(E) + K = E$, we may assume that $\phi \in M(E)$ and so $A = S_\tilde{\phi}$.

For a subset $S$ of $X$, let $E|S$ be the restriction of $E$ to $S$ and put $K = \{f \in E; f = 0$ on $S\}$. Then, $E|S$ becomes a Banach space of functions on $S$ under the quotient norm of $E/K$. We may assume that $E|S \cong E/K$. Put $\mathcal{K} = \{\phi \in M(E); \phi = 0$ on $S\}$, then $M(E)|S \cong M(E)/\mathcal{K}$. Even if $K$ is such a special case, Probrems 1 and 2 cannot be solved in general. Snyder [5] studied Problem 1,
that is, whether $M(E)|S = M(E|S)$. In this special case, Problem 1 is just an interpolation problem. That is, if $f$ is a function on $S \subset X$ and $f(E|S) \subset E|S$ then does there exist a function $F$ on $X$ such that $FE \subset E$ and $F|S = f$ and $\|F\| = \|f\|$? Therefore the research of Snyder [5] is contained in our one.

Corollary 1. If $E$ is a commutative Banach algebra with unit then $M(E/K)' = M(E/K)$.
Proof. If $E$ is a commutative Banach algebra with unit then $M(E) = E$ and $K = K$. Hence $M(E) + K = E$. Proposition 2 implies that $M(E/K)' = M(E/K)$.

Proposition 3. If $E$ is a commutative Banach algebra with unit then $M(E)/K = M(E/K)$ where $K = K$.
Proof. By the proof of Corollary 1, $M(E) = E$ and it is easy to see that $M(E)$ is isometrically isomorphic to $E$. Similarly $M(E/K)$ is isometrically isomorphic to $E/K$. This implies the proposition.

§3. Two dimensional case

In this section we assume that $M(E) \subset E$. (1) of Theorem 1 is due to Snyder [5] and (2) of Theorem 1 is new.

d$_x$ is called the derivation at $x$ if $d_x(fg) = d_x(f)\tau_x(g) + \tau_x(f)d_x(g)$ $(f, g \in M(E))$.

Proposition 4. Suppose $E/K$ and $M(E/K)$ are of finite dimension 2. Then $(E/K)^* = [\tau_x, \tau_y]$ for $x, y \in X$ with $x \neq y$ or $(E/K)^* = [\tau_x, d_x]$ for $x \in X$ where $d_x$ is a point derivation at $x$.
Proof. By hypothesis, $M(E/K) = E/K$ as a set. Since $M(E/K)$ is a commutative Banach algebra and $\dim M(E/K) = 2$, by [2, Proposition 1] it is easy to see that $M(E/K)^* = [\tau_x, \tau_y]$ for $x, y \in X$ with $x \neq y$ or $M(E/K)^* = [\tau_x, d_x]$ for $x \in X$.

Lemma 1. Suppose $M(E)+K$ is dense in $E$. If $\phi \in M(E)$ then $S'\phi d_x = \overline{d_x(\phi)}\tau_x + \tau_x(\phi)d_x$. 
Proof. For \( f \in M(E) \)
\[
\langle f + K, S_\phi^*d_x \rangle = \langle \phi f + K, d_x \rangle = \langle \phi f, d_x \rangle = d_x(\phi) \tau_x(f) + \tau_x(\phi)d_x(f) = (f + K, d_x(\phi) + \tau_x(\phi)d_x)
\]

**Theorem 1.** Suppose that \( M(E) + K = E \), \( E/K \) and \( M(E/K) \) are of two dimension. If \( M(E/K) \) is isometrically isomorphic to \( M(E)/K \) then the following (1) and (2) are valid.

(1) When \( (E/K)^* = [\tau_x, \tau_y] \) for \( x, y \in X \) with \( x \neq y \), for given \( u, v \in \mathbb{C}, \) there exists \( \phi \in M(E) \) such that \( \tau_x(\phi) = u, \tau_y(\phi) = v \) and \( \|\phi + K\| \leq 1 \) if and only if

\[
\|\alpha \bar{u} \tau_x + \beta \bar{v} \tau_y\|_* \leq \|\alpha \tau_x + \beta \tau_y\|_* \quad (\alpha, \beta \in \mathbb{C}).
\]

(2) When \( (E/K)^* = [\tau_x, d_x] \) for \( x \in X \), for given \( u, v \in \mathbb{C}, \) there exists \( \phi \in M(E) \) such that \( \tau_x(\phi) = u, d_x(\phi) = v \) and \( \|\phi + K\| \leq 1 \) if and only if

\[
\|\alpha \bar{u} \tau_x + \beta \bar{v} d_x\|_* \leq \|\alpha \tau_x + \beta d_x\|_* \quad (\alpha, \beta \in \mathbb{C}).
\]

Proof. (1) If there exists \( \phi \in M(E) \) such that \( \tau_x(\phi) = u, \tau_y(\phi) = v \) with \( \|\phi + K\| \leq 1 \) then \( S_\phi^* \leq 1 \) by hypothesis. This implies that

\[
\|\alpha \bar{u} \tau_x + \beta \bar{v} \tau_y\|_* \leq \|\alpha \tau_x + \beta \tau_y\|_* \quad (\alpha, \beta \in \mathbb{C})
\]

because \( S_\phi^* \tau_x = \tau_x(\phi)\tau_x = \bar{u} \tau_x \) and \( S_\phi^* \tau_y = \bar{v} \tau_y \). For the converse, put \( A \in \mathcal{B}(H/K) \), \( A^* \tau_x = \bar{u} \tau_x \) and \( A^* \tau_y = \bar{v} \tau_y \), then \( \|A^*\| \leq 1 \) and \( A \) belongs to \( M(E/K)^* \). Since \( M(E) + K = E \), by Proposition 2 \( A = S_\phi \) for some \( \phi \in M(E) \). By hypothesis, \( \|\phi + K\| \leq 1 \) and \( \tau_x(\phi) = u \) and \( \tau_y(\phi) = v \).

(2) If there exists \( \phi \in M(E) \) with \( \tau_x(\phi) = u, d_x(\phi) = v \) with \( \|\phi + K\| \leq 1 \) then \( S_\phi^* \leq 1 \) by hypothesis. This and Lemma 1 imply

\[
\|\alpha \bar{u} \tau_x + \beta \bar{v} d_x\|_* \leq \|\alpha \tau_x + \beta d_x\|_* \quad (\alpha, \beta \in \mathbb{C}).
\]
For the converse, put \( A \in B(H/K) \), \( A^* \tau_x = \bar{u} \tau_x \) and \( A^* d_x = \bar{v} \tau_x + \bar{u} d_x \), then \( \|A^*\| \leq 1 \) and \( A \) belongs to \( M(E/K) \) by Lemma 1. By Proposition 2 \( A = S_\phi \) for some \( \phi \in M(E) \). By hypothesis, \( \|\phi + K\| \leq 1 \) and \( \tau_x(\phi) = u \) and \( \tau_y(\phi) = v \).

In Theorem 1, if (1) or (2) is valid then \( M(E/K) \) is isometrically isomorphic to \( M(E)/K \).

**Corollary 2.** In Theorem 1, if \( E \) is a Hilbert space then there exist \( k_x \) and \( h_x \) in \( E \) such that

\[
\tau_x(f) = (f, k_x) \quad (f \in E)
\]

and

\[
d_x(f) = (f, h_x) \quad (f \in E)
\]

and the following (1) and (2) are valid.

1. When \( (E/K)^* = [\tau_x, \tau_y] \) for \( x, y \in X \) with \( x \neq y \), for given \( u, v \in \mathbb{C} \), there exists \( \phi \in M(E) \) such that \( \tau_x(\phi) = u, \tau_y(\phi) = v \) and \( \|\phi + K\| \leq 1 \) if and only if

\[
|\alpha|^2(1 - |u|^2)(k_x, k_x) + \alpha \bar{\beta}(1 - \bar{u}v)(k_x, k_y) + \bar{\alpha} \beta(1 - u\bar{v})(k_y, k_x) + |\beta|^2(1 - |v|^2)(k_y, k_y) \geq 0
\]

for any \( \alpha, \beta \in \mathbb{C} \).

2. When \( (E/K)^* = [\tau_x, d_x] \) for \( x \in X \), for given \( u, v \in \mathbb{C} \), there exists \( \phi \in M(E) \) such that \( \tau_x(\phi) = u, d_x(\phi) = v \) and \( \|\phi + K\| \leq 1 \) if and only if

\[
(|\alpha|^2 - |\alpha u + \beta \bar{u}|^2)(k_x, k_x) + (\alpha \bar{\beta} - (\alpha \bar{\beta}|u|^2 + |\beta|^2 u\bar{v}))(k_x, h_x) + (\bar{\alpha} \beta - (\bar{\alpha} \beta|u|^2 + |\beta|^2 \bar{u}v))(h_x, k_x) + |\beta|^2(1 - |u|^2)(h_x, h_x) \geq 0
\]

for any \( \alpha, \beta \in \mathbb{C} \).

The condition of (1) in Corollary 2 shows that the \( 2 \times 2 \) matrix \( \{(1 - |u|^2)(k_x, k_x), (1 - \bar{u}v)(k_x, k_y), (1 - u\bar{v})(k_y, k_x), (1 - |v|^2)(k_y, k_y)\} \) is nonnegative. When \( (k_x, h_x) = 0 \), the condition of (2) in Corollary 2 shows that the \( 2 \times 2 \) matrix \( \{(1 - |u|^2)(k_x, k_x), \bar{u}v(h_x, k_x), u\bar{v}(k_x, h_x), (1 - |u|^2 - |v|^2)(h_x, h_x)\} \) is nonnegative.
When \( \dim E/K \geq 3 \), even if \( \dim E/K \) is finite, it is difficult to describe \( (E/K)^* \) except \( K = \{ f \in E : f(x_j) = 0, 1 \leq j \leq \dim E/K \} \) and \( x_i \neq x_j (i \neq j) \). Therefore we could not generalize (2) of Theorem 1.

\[ \text{§4. Hardy space } H^p \quad (1 \leq p \leq \infty) \]

In this section, we solve Problems 1 and 2 when \( E = H^p \) for \( 1 \leq p \leq \infty \). When \( E = H^\infty \), we can solve trivially Problems 1 and 2 by Corollary 2 and Proposition 4. If \( \dim H^p/K < \infty \) then \( M(H^p) + K = H^p \) and so \( M(H^p/K)' = M(H^p/K) \) by Proposition 2. However we have to work more in order to prove \( M(H^p/K) = M(H^p)/K \).

Let \( W \) be a nonnegative function in \( L^1 \) with \( \log W \) in \( L^1 = L^1(d\theta/2\pi) \). Then there exists an outer function \( h \) in \( H^1 \) with \( W = |h| \). For \( 1 \leq p < \infty \), \( H^p(W) \) denotes the closure of analytic polynomials in \( L^p(W) = L^1(Wd\theta/2\pi) \). Then \( H^p(W) = h^{-1/p}H^p \) and so we may assume that \( H^p(W) \) is a Banach space of analytic functions on \( D \). It is known that the point evaluations of points in \( D \) are continuous on \( H^p(W) \). It is well known that \( M(H^p(W)) = H^\infty \).

**Theorem 2.** For \( 1 \leq p \leq \infty \), let \( K \) be a closed subspace of \( H^p(W) \) with \( M(H^p(W))K \subseteq K \). Then \( M(H^p(W)/K)' = M(H^p(W)/K) \) and \( M(H^p(W)/K) = M(H^p(W))/K \) where \( K = \{ \phi \in M(H^p(W)) : \phi H^p(W) \subseteq K \} \).

Proof. Since \( M(H^p(W)) = H^\infty \), \( K = QH^\infty \) for some inner function and \( K = QH^p(W) \). Since \( M(H^p(W)/K) \subseteq M(H^p(W)/K)' \), we will show that \( M(H^p(W)/K)' \subseteq M(H^p(W)/K) \). If \( A \in M(H^p(W)/K)' \) then there exists \( \psi \) in \( H^p(W) \) such that \( A(1+K) = \psi + K \). For any polynomial \( f = h^{-1/p}F \) in \( H^p(W) \), \( \| \psi f + K \|_W \leq \| A \| \| f + K \|_W \).

Since \( K = QH^p(W) = h^{-1/p}QH^p \), if \( 1/p + 1/q = 1 \)

\[ \| \psi f + QH^p(W) \|_W \]
\[ = \sup \{ | \langle \psi f, g \rangle_W | : g \in (QH^p(W))^\perp \text{ and } \| g \|_W \leq 1 \} \]
\[ = \sup \left\{ \left| \int \psi h^{-1/p}FQh^{-1/q}G \right| dm : G \in H^q_0 \text{ and } \| G \|_q \leq 1 \right\} \]
\[
\begin{align*}
= \sup \left\{ \left| \int \psi \bar{Q} F G d\theta \right| : G \in H^0_0 \text{ and } \|G\|_q \leq 1 \right\} \\
\leq \|A\| \|f\|_W = \|A\| \|F\|_p
\end{align*}
\]

because \( \{QH^p(W)\}^\perp = Q\bar{h}^{1/p}|h|^{-1} \bar{H}^q_0 \). Thus

\[
\sup \left\{ \left| \int \psi \bar{Q} F G d\theta /2\pi \right| : F \in H^p, G \in H^0_0, \|F\|_p \leq 1 \text{ and } \|G\|_q \leq 1 \right\} \leq \|A\|
\]

By the factorization theorem of \( H^1 \),

\[
\sup \left\{ \left| \int \psi \bar{Q} K d\theta /2\pi \right| : K \in H^1_0 \text{ and } \|K\|_1 \leq 1 \right\} \leq \|A\|
\]

Since \((\bar{Q}H^1_0)^* = L^\infty /QH^\infty\), \( \|\psi + QH^\infty\| \leq \|A\|\). Hence there exists a function \( \phi \) in \( H^\infty \) such that \( S_{\phi} = A \) and \( \|\phi + K\| = \|S_{\phi}\|\). Thus \( A \) belongs to \( M(H^p(W)/K) \). Therefore \( M(H^p(W)/K)' = M(H^p(W)/K) \) and \( M(H^p(W)/K) = M(H^p(W)/K) \).

**Corollary 3.** For \( 1 \leq p \leq \infty \), \( M(H^p(W)/QH^p(W)) = H^\infty/QH^\infty \) for any inner function \( Q \).
References


Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060-0810, Japan
nakazi@math.hokudai.ac.jp