HOMOTOPY CLASSIFICATION OF NANOPHRASES IN TURAEV'S THEORY OF WORDS

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ABSTRACT

The purpose of this paper is to give the homotopy classification of nanophrases of length 2 with 4 letters. To do it we construct some new invariants of nanophrases γ, T. The invariant γ defined in this paper is an extension of the invariant γ for nanowords introduced in [5]. The invariant T is a new invariant of nanophrases. As a corollary of these results, we give the classification of two-components pointed, ordered, oriented curves on surfaces with minimum crossing number ≤ 2.

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1. Introduction.

Words are finite sequences of letters in a given alphabet. In [2] C. F. Gauss introduced a method to investigate closed planar curves by words of a certain type now called Gauss words. We can apply this method to encode surface curves. (See [10].)

V.Turaev introduced word theory in [5], [6]. The key of new concepts introduced in those papers are those of étale words and nanowords. An étale word over an alphabet α endowed with an involution $\tau : \alpha \longrightarrow \alpha$ is a word in an alphabet $\mathcal{A}$ endowed with a projection $\mathcal{A} \ni A \mapsto |A| \in \alpha$. Every word in the alphabet α becomes an étale word over α by using the identity mapping $id : \alpha \longrightarrow \alpha$ as the projection. An étale word over α is called nanoword if every letter appears twice or not at all. In the case where the alphabet α consists of two elements permuted by τ, the notion of a nanoword over α is equivalent to the notion of an open virtual string introduced in [9].

Turaev introduced an equivalence relation of homotopy on the set of étale words over α. The relation of homotopy is generated by three transformations or moves on nanowords. The first move consists in deleting two consecutive entries of the same letter. The second move has the form $xAByBAz \mapsto xyz$ where x, y, z are words and A, B are letters such that $|A| = \tau(|B|)$. The third move has the form $xAByACzBCt \mapsto xBAyCAzCBt$ where x, y, z, t are words and A, B, C are letters such that $|A| = |B| = |C|$. These moves are suggested by the Reidemeister moves in knot theory. In fact the first (resp. second, third) homotopy move is similar to the first (resp. second, third) Reidemeister move.(See [6] for more details). Turaev applied topological methods to a semigroup consisting letters to study properties and characteristics of nanowords preserved under homotopy. For instance,these are applications of colorings of knot diagrams, the theory of knot quandles, etc. (See [5], [6], [7], [4] for more details.) As an application of those methods, Turaev gave the homotopy classification of nanowords of length ≤ 6 in [5].

On the other hand, in [6] Turaev showed that a stable equivalence class of an oriented pointed curve on a surface is identified with a homotopy class of nanoword in a 2-letter alphabet. Moreover Turaev extended this result to multi-component curves. In fact a stable equivalence class of an oriented, ordered, pointed multi-component curve on a surface is identified with a homotopy classes of a nanophrase in a 2-letter alphabet. Roughly speaking, a nanophrase is a sequence of étale words which concatenation of those words is a nanoword. (See [6], [8].) We can define homotopy moves similarly as in the case of nanowords.

Now the purpose of this paper is to give the homotopy classification of nanophrases of length 2 with 4 letters. (Theorem 4.6.) To do it we construct some new invariants of nanophrases. As a corollary of these results, we give the classification of two-components pointed, ordered, oriented curves on surfaces with minimum crossing number ≤ 2. (See also [1].)

Another application of the theory of words was introduced by N.Ito in [3]. By using the theory of words, Ito reconstructed the Arnold basic invariants and constructed some other invariants for plane closed curves, long curves, and fronts.

In section 2 we review the theory of words and phrases which are introduced by Turaev in [5], [6]. In section 3 we construct some new homotopy invariants of nanophrases γ, T. The invariant γ defined in this paper is an extension of the invariant γ for nanowords introduced in [5]. The invariant T is a new invariant of nanophrases. In section 4 we generalize Turaev's result to the case of nanophrases. In fact we give the homotopy classification of nanophrases of length 2 with 4 letters using homotopy invariants constructed in section 3.

2. Nanowords and Nanophrases.

In this section we review the theory of words and phrases (cf.[5], [6]).

2.1. *Nanowords and their homotopy.*

An *alphabet* is a set and *letters* are its elements. A *word of length* $n \geq 1$ *on an alphabet* $\mathcal{A}$ is a mapping $w : \hat{n} \rightarrow \mathcal{A}$ where $\hat{n} = \{1, 2, \cdots, n\}$. A word usually encoded by the sequence of letters $w(1)w(2)\cdots w(n)$. A word $w : \hat{n} \rightarrow \mathcal{A}$ is a *Gauss word* if each element of $\mathcal{A}$ is the image of precisely two elements of $\hat{n}$.

For a set α, an α-alphabet is a set $\mathcal{A}$ endowed with a mapping $\mathcal{A} \rightarrow \alpha$ called *projection.* the image of $A \in \mathcal{A}$ under this mapping is denoted $|A|$. A *étale word* over α is a pair (an α-alphabet, a word on $\mathcal{A}$). A *nanoword over* α is a pair (an α-alphabet, a Gauss word on α). An empty étale word in an empty α-alphabet is a nanoword called the *empty nanoword* $\emptyset$ of length 0.

A *morphism* of α-alphabets $\mathcal{A}_1$, $\mathcal{A}_2$ is a set-theoric mapping $f : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ such that $|A| = |f(A)|$ for all $A \in \mathcal{A}_1$. If f is bijective, then this morphism is an *isomorphism.* Two étale words $(\mathcal{A}_1, w_1)$ and $(\mathcal{A}_2, w_2)$ over α are *isomorphic* if there is an isomorphism $f : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ such that $w_2 = f \circ w_1$.

To define homotopy of nanowords we fix a set α with an involution $\tau : \alpha \longrightarrow \alpha$ and a subset $S \subset \alpha \times \alpha \times \alpha$. We call the pair (α, S) *homotopy data.*

Definition 2.1. Let (α, S) be homotopy data. We define a *homotopy moves* (1) - (3) as follows:

(1) $(\mathcal{A}, xAAy) \longrightarrow (\mathcal{A} \setminus \{A\}, xy)$
for all $A \in \mathcal{A}$ and x, y are words in $\mathcal{A} \setminus \{A\}$.

(2) $(\mathcal{A}, xAByBAz) \longrightarrow (\mathcal{A} \setminus \{A, B\}, xyz)$
if $A, B \in \mathcal{A}$ with $|B| = \tau(|A|)$. x, y, z are words in $\mathcal{A} \setminus \{A, B\}$.

(3) $(\mathcal{A}, xAByACzBCt) \longrightarrow (\mathcal{A}, xBAyCAzCBt)$
if $A, B, C \in \mathcal{A}$ satisfy $(|A|, |B|, |C|) \in S$. x, y, z, t are words in $\mathcal{A}$.

Definition 2.2. Let (α, S) be a homotopy data. Then nanowords $(\mathcal{A}_1, w_1)$ and $(\mathcal{A}_2, w_2)$ over α are *S-homotopic* (denote $(\mathcal{A}_1, w_1) \simeq_S (\mathcal{A}_2, w_2)$) if $(\mathcal{A}_2, w_2)$ can be obtained from $(\mathcal{A}_1, w_1)$ by a finite sequence of isomorphism, S-homotopy moves (1) - (3) and the inverse moves.

The set of S-homotopy classes of nanowords over α is denoted as $\mathcal{N}(\alpha, S)$.

To define S-homotopy of étale words. We define *desingularization* of étale words $(\mathcal{A}, w)$ over α as follows: $\mathcal{A}^d := \{A_{i,j} := (A, i, j) | A \in \mathcal{A}, 1 \leq i < j \leq m_w(A)\}$ with projection $|A_{i,j}| := |A| \in \alpha$ for all A, i, j (where $m_w(A) := Card(w^{-1}(A))$). The word w^d is obtained from w by first deleting all $A \in \mathcal{A}$ with $m_w(A) = 1$. Then for each $A \in \mathcal{A}$ with $m_w(A) \geq 2$ and each $i = 1, 2, \ldots m_w(A)$, we replace the i-th entry of A in w by

$$A_{1,i} A_{2,i} \ldots A_{i-1,i} A_{i,i+1} A_{i,i+2} \ldots A_{i,m_w(A)}.$$

The resulting $(\mathcal{A}^d, w^d)$ is a nanoword of length $\Sigma m_w(A)(m_w(A) - 1)$ and called a *desingularization of* $(\mathcal{A}, w)$. Then we define S-homotopy of étale words as following:

Definition 2.3. Let w_1 and w_2 be étale words over α. Then w_1 and w_2 are S-homotopic if w_1^d and w_2^d are S-homotopic.

Recall the following three lemmas from [5].

Lemma 2.4. *Let (α, S) be a homotopy data and $\mathcal{A}$ be an α-alphabet. A, B, C are distinct letters in $\mathcal{A}$. x, y, z, w are words in $\mathcal{A} \setminus \{A, B, C\}$ with $xyzt$ is Gauss word. Then following (i)-(iii) are hold :*

(i) $(\mathcal{A}, xAByCAzBCt) \simeq_S (\mathcal{A}, xAByACzCBt)$

if $(|A|, \tau(|B|), |C|) \in S$,
(ii) $(\mathcal{A}, xAByCAzCBt) \simeq_S (\mathcal{A}, xBAyACzBCt)$
if $(\tau(|A|), \tau(|B|), |C|) \in S$,
(iii)$(\mathcal{A}, xAByACzCBt) \simeq_S (\mathcal{A}, xBAyCAzBCt)$
if $(|A|, \tau(|B|), \tau(|C|)) \in S$.

Lemma 2.5. *Suppose that* $S \cap (\alpha \times b \times b) \neq \emptyset$ *for all* $b \in \alpha$. *Let* $(\mathcal{A}, xAByABz)$ *be nanoword over* α *with* $|B| = \tau(|A|).x, y, z$ *are words in* $\mathcal{A} \setminus \{A, B\}$*.Then*

$$(\mathcal{A}, xAByABz) \simeq_S (\mathcal{A} \setminus \{A, B\}, xyz).$$

In the remaining part of the paper we assume that S is the diagonal of α^3 that is $\{(a, a, a)\}_{a \in \alpha}$. Under this convention, we shall omit the prefix S- and speak simply of homotopy rather than S-homotopy. We shall also omit index S and write $\simeq$, $||\cdot||$, $\mathcal{N}_\bullet(\alpha)$ for $\simeq_S$, $||\cdot||_S$, $\mathcal{N}_\bullet^S(\alpha)$.

Lemma 2.6. *Let* β *be* τ*-invariant subset of* α. *If two étale words over* β *are homotopic in the class of étale words over* α*, then they are homotopic in the class of étale words over* β.

V.Turaev gives a homotopy classification of nanowords of length 4 in [5].

Theorem 2.7. *Let* w *be a nanoword of length* 4 *over* α. *Then* w *is either* $w \simeq \emptyset$ *or isomorphic to the nanoword* $w_{a,b} := (\mathcal{A} = \{A, B\}, ABAB)$ *where* $|A| = a, |B| = b \in \alpha$ *with* $a \neq \tau(b)$. *Moreover for* $a \neq \tau(b)$*, the nanoword* $w_{a,b}$ *is non-contractible and two nanowords* $w_{a,b}$ *and* $w_{a',b'}$ *are homotopic if and only if* $(a, b) = (a', b')$.

In this paper we generalize Turaev's result to the case of "nanophrases".

2.2. *Nanophrases and their homotopy.*

Definition 2.8. A *nanophrase* $(\mathcal{A}, (w_1|w_2|\cdots|w_k))$ of length $k \geq 0$ over a set α is a pair consisting of an α-alphabet $\mathcal{A}$ and a sequence of k words $w_1, \cdots, w_k$ on $\mathcal{A}$ such that $w_1 w_2 \cdots w_k$ is a Gauss word on $\mathcal{A}$. We denote it shortly by $(w_1|w_2|\cdots|w_k)$. We denote a set of nanophrases of length k over α by $\mathcal{P}_k(\alpha)$.

By definition, there is a unique *empty nanophrase* of length 0 (the corresponding α-alphabet $\mathcal{A}$ is void).

Remark 2.9. Any nanoword w over α yields a nanophrase (w) of length 1.

A mapping $f : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ is *isomorphism of two nanophrases* if f is an isomorphism of α-alphabets transforming the first nanophrase into second one.

Given a homotopy data (α, τ, S), we define homotopy move on nanophrases as in section 2.1 with the only difference that the 2-letter subwords AA, AB, BA, AC, BC modified by these moves may occur in different words of phrase. Isomorphism and homotopy moves generate an equivalence relation $\simeq_S$ of S-homotopy on the class of nanophrases over α. We denote a set of S-homotopy class of nanophrases of length k by $\mathcal{P}_k(\alpha, S)$.

Example 2.10. Nanophrases $(AB|ADDCBC)$ and $(BA|CACB)$ with $|A| = |B| = |C| \in \alpha$ over α are homotopic. Indeed

$$(AB|A\underline{DD}CBC) \simeq (\underline{AB}|\underline{ACBC}) \simeq (BA|CACB).$$

Lemmas2.4 and 2.5 extend to nanophrases with the only change that the 2-letter subwords AB, BA, CA, and so forth may occur in different word of the phrase.

3. Some Homotopy Invariants of Nanophrases.

In this section, we define three new homotopy invariants of nanophrases. They will be used in the next section.

3.1. *Invariant γ.*

Recall that an orbit of the involution $\tau : \alpha \longrightarrow \alpha$ is a subset of α consisting either of one element or of two elements; in latter case the orbit is called *free*. Let Π be the group which defined as follows:

$$\Pi := (\{z_a\}_{a\in\alpha} | z_a z_{\tau(a)} = 1 \ for\ all\ a \in \alpha).$$

Let $\mathbb{Z}\Pi$ be the integral group-ring of Π.

Definition 3.1. Let $P = (\mathcal{A}, (w_1|w_2|\cdots|w_k))$ be a nanophrase of length k over α and n_i the length of nanoword w_i. Set $n = \sum_{1\leq i\leq k} n_i$. Then we define n elements $\gamma_1^i, \gamma_2^i, \cdots, \gamma_{n_i}^i$ $(i \in \{1, 2, \cdots, k\})$ of Π by $\gamma_i^j := z_{|w_j(i))|}$ if $w_j(i) \neq w_l(m)$ for all $l < j$ and for all $m < i$ when $l = j$. Otherwise $\gamma_i^j := z_{\tau(|w_j(i)|)}$. Then we define $\gamma(P) \in \otimes^k \mathbb{Z}\Pi$ by

$$\gamma(P) := \gamma_1^1\gamma_2^1 \cdots \gamma_{n_1}^1 \otimes \gamma_1^2\gamma_2^2 \cdots \gamma_{n_2}^2 \otimes \cdots \otimes \gamma_1^k\gamma_2^k \cdots \gamma_{n_k}^k.$$

Then we obtain following theorem.

Theorem 3.2. *The γ is a homotopy invariant of nanophrases.*

Remark 3.3. By definition, for nanophrases of length 1 the invariant γ for nanophrases is equal to Turaev's invariant γ defined in [5].

Example 3.4. Let $\mathcal{A} := \{A, B, C\}$ be an α-alphabet. Set $|A| = a, |B| = b, |C| = c \in \alpha$. Consider a nanophrase $P = (ABC|CB|A)$, then

$$\gamma(P) = z_a z_b z_c \otimes z_{\tau(c)} z_{\tau(b)} \otimes z_{\tau(a)}.$$

3.2. *Invariant T.*

In this subsection we define homotopy invariants of nanophrases over $\alpha_0 := \{a, b\}$ with involution τ_0 permuting a, b and nanophrases over one-point set. At first, we define a homotopy invariant of nanophrases T over α_0. To define this invariant, we define some notation as follows.

Definition 3.5. Let $P = (\mathcal{A}, (w_1|\cdots|w_k))$ be a nanophrase over α_0 and $A, B \in \mathcal{A}$. Then we define $\sigma_P(A, B)$ as follows: If A and B form $\cdots A \cdots B \cdots A \cdots B \cdots$ in P and $|B| = a$, or $\cdots B \cdots A \cdots B \cdots A \cdots$ in P and $|B| = b$, then $\sigma_P(A, B) := 1$. If $\cdots A \cdots B \cdots A \cdots B \cdots$ in P and $|B| = b$, or $\cdots B \cdots A \cdots B \cdots A \cdots$ in P and $|B| = a$, then $\sigma_P(A, B) := -1$. Otherwise $\sigma_P(A, B) := 0$.

Definition 3.6. For $A \in \mathcal{A}$ we define $\varepsilon(A) \in \{\pm 1\}$ by

$$\varepsilon(A) := \begin{cases} 1 \ (\ if\ |A| = a\ ), \\ -1 \ (\ if\ |A| = b\ ). \end{cases}$$

Definition 3.7. Let $P = (\mathcal{A}, (w_1|w_2|\cdots|w_k))$ be a nanophrase of length k over α_0. For $A \in \mathcal{A}$ such that there exist $i \in \{1, 2, \cdots, k\}$ such that $Card(w_i^{-1}(A)) = 2$, we define $T_P(A) \in \mathbb{Z}$ by

$$T_P(A) := \varepsilon(A) \sum_{B \in \mathcal{A}} \sigma_P(A, B),$$

and we define $T_P(w_i) \in \mathbb{Z}$ by

$$T_P(w_i) := \sum_{A \in \mathcal{A},\ Card(w_i^{-1}(A))=2} T_P(A).$$

Then we define $T(P) \in \mathbb{Z}^k$ by

$$T(P) := (T_P(w_1), T_P(w_2), \cdots, T_P(w_k)).$$

Theorem 3.8. *The T is a homotopy invariant of nanophrases over α_0.*

Proof. Consider the 1-st homotopy move
$P_1 := (w_1|\cdots|w_{l-1}|xAAy|w_{l+1}|\cdots|w_k) \longrightarrow$
$$P_2 := (w_1|\cdots|w_{l-1}|xy|w_{l+1}|\cdots|w_k).$$
It is clear that $T_{P_1}(w_i) = T_{P_2}(w_i)$ for all $i \neq l$. We show that $T_{P_1}(xAAy) = T_{P_2}(xy)$. Note that $\sigma_{P_1}(A, B) = 0$ for all $B \in \mathcal{A}$ by definition. Therefore $T_{P_1}(A) = 0$. Moreover $\sigma_{P_1}(E, A) = 0$ for all $E \in \mathcal{A}$. So A does not contribute to $T_{P_1}(E)$ for all $E \in \mathcal{A}$. Therefore $T_{P_1}(xAAy) = T_{P_2}(xy)$.

Consider the 2-nd homotopy move such that A and B occur in some words once
$P_1 := (w_1|\cdots|x_1ABy_1|\cdots|x_2BAy_2|\cdots|w_k) \longrightarrow$
$$P_2 := (w_1|\cdots|x_1y_1|\cdots|x_2y_2|\cdots|w_k).$$
with $|A| = \tau(|B|)$.

It is sufficient to show that $T_{P_1}(x_1ABy_1) = T_{P_2}(x_1y_1)$ and $T_{P_1}(x_2BAy_2) = T_{P_2}(x_2y_2)$. Note that A and B occur in P once. Moreover for all E such that $\cdots E \cdots AB \cdots E \cdots$ in P_1

$$\begin{aligned} T_{P_1}(E) &= \varepsilon(E)(n_1 + \sigma_{P_1}(E,A) + \sigma_{P_1}(E,B) + n_2) \\ &= \varepsilon(E)(n_1 + n_2) \\ &= T_{P_2}(E) \end{aligned}$$

where n_1, n_2 are integers. Therefore $T_{P_1}(x_1ABy_1) = T_{P_2}(x_1y_1)$. $T_{P_1}(x_2BAy_2) = T_{P_2}(x_2y_2)$ is proved similarly.

Consider the 2-nd homotopy move such that A and B occur in some word twice $P_1 := (w_1|\cdots|w_{l-1}|xAByBAz|w_{l+1}|\cdots|w_k) \longrightarrow P_2 := (w_1|\cdots|xyz|\cdots|w_k)$ with $|A| = \tau(|B|)$. It is sufficient to show that $T_{P_1}(w_l) = T_{P_2}(w_l)$. At first we show $T_{P_1}(A) + T_{P_1}(B) = 0$. Indeed

$$\begin{aligned} T_{P_1}(A) &= \varepsilon(A)(\sigma_{P_1}(A,B) + n + \sigma_{P_1}(A,B)) \\ &= \varepsilon(A)n \\ &= -\varepsilon(B)n \\ &= -T_{P_1}(B) \end{aligned}$$

where n is an integer. Now we show $T_{P_1}(E) = T_{P_2}(E)$ for all $E \neq A, B$. If $\cdots E \cdots AB \cdots E \cdots BA \cdots$ or $\cdots AB \cdots E \cdots BA \cdots E$, then

$$\begin{aligned} T_{P_1}(E) &= \varepsilon(E)(n_1 + \sigma_{P_1}(E,A) + \sigma_{P_1}(E,B) + n_2) \\ &= \varepsilon(E)(n_1 + n_2) \\ &= T_{P_2}(E) \end{aligned}$$

where n_1, n_2, n_3 are integers If $\cdots E \cdots AB \cdots BA \cdots E \cdots$, then

$$\begin{aligned} T_{P_1}(E) &= \varepsilon(E)(n_1 + \sigma_{P_1}(E,A) + \sigma_{P_1}(E,B) + n_2 \\ &\qquad\qquad + \sigma_{P_1}(E,B) + \sigma_{P_1}(E,A) + n_3) \\ &= \varepsilon(E)(n_1 + n_2 + n_3) \\ &= T_{P_2}(E) \end{aligned}$$

where n_1, n_2, n_3 are integers. Therefore $T_{P_1}(E) = T_{P_2}(E)$ for all $E \neq A, B$.

Consider the 3-rd homotopy move

$$P_1 := (w_1|\cdots|x_1ABy_1|\cdots|x_2ACy_2|\cdots|x_3BCy_3|\cdots|w_k) \longrightarrow$$
$$P_2 := (w_1|\cdots|x_1BAy_1|\cdots|x_2CAy_2|\cdots|x_3CBy_3|\cdots|w_k)$$

with $|A| = |B| = |C|$. In this case it is clear that $T(P_1) = T(P_2)$.

Consider the 3-rd homotopy move

$$P_1 := (w_1|\cdots|x_1ABy_1ACz_1|\cdots|x_2BCy_2|\cdots|w_k) \longrightarrow$$
$$P_2 := (w_1|\cdots|x_1BAy_1CAz_1|\cdots|x_2CBy_2|\cdots|w_k)$$

with $|A| = |B| = |C|$.

It is sufficient to show $T_{P_1}(x_1ABy_1ACz_1) = T_{P_2}(x_1BAy_1CAz_1)$ and $T_{P_1}(x_2BCy_2) = T_{P_2}(x_2CBy_2)$.

$$\begin{aligned} T_{P_1}(A) &= \varepsilon(A)(\sigma_{P_1}(A,B) + n_1) \\ &= \varepsilon(A)(n_1 + \sigma_{P_2}(A,C)) \\ &= T_{P_2}(A), \end{aligned}$$

where n_1 an integer. So $T_{P_1}(x_1ABy_1ACz_1) = T_{P_2}(x_1BAy_1CAz_1)$ holds. $T_{P_1}(x_2BCy_2) = T_{P_2}(x_2CBy_2)$ is clear.

Consider the 3-rd homotopy move
$P_1 := (w_1|\cdots|x_1ABy_1|\cdots|x_2ACy_2BCz_2|\cdots|w_k) \longrightarrow$
$P_2 := (w_1|\cdots|x_1BAy_1|\cdots|x_2CAy_2CBz_2|\cdots|w_k)$
with $|A| = |B| = |C|$. In this case $T(P_1) = T(P_2)$ is proved similarly to above case.

Consider the 3-rd homotopy move
$P_1 := (w_1|\cdots|xAByACzBCt|\cdots|w_k) \longrightarrow$
$P_2 := (w_1|\cdots|xBAyCAzCBt|\cdots|w_k)$
with $|A| = |B| = |C|$. In this case it is sufficient to show that $T_{P_1}(xAByACzBCt) = T_{P_2}(xBAyCAzCBt)$. $T_{P_1}(A) = T_{P_2}(A)$ and $T_{P_1}(C) = T_{P_2}(C)$ is clear. Note that $\sigma_{P_1}(B,A) = -\sigma_{P_1}(B,C)$ and $\sigma_{P_2}(B,A) = \sigma_{P_2}(B,C) = 0$. We obtain $T_{P_1}(B) = T_{P_2}(B)$. $T_{P_1}(E) = T_{P_2}(E)$ for all $E \neq A,B,C$ is checked easily. So we obtain $T(P_1) = T(P_2)$. □

Next we define invariant T for nanophrases over one-point set. To define this invariant, we define some notation as followings.

Definition 3.9. Let $P := (\mathcal{A}, (w_1|\cdots|w_k))$ be a nanophrase over one-point set $\alpha := \{a\}$. Let $A, B \in \mathcal{A}$ be letters. Then we define $\tilde{\sigma}_P(A,B) \in \mathbb{Z}/2\mathbb{Z}$ as followings: If A and B forms $\cdots A \cdots B \cdots A \cdots B \cdots$ or $\cdots B \cdots A \cdots B \cdots A \cdots$ in P, then $\tilde{\sigma}_P(A,B) := 1$. Otherwise $\tilde{\sigma}_P(A,B) := 0$.

Definition 3.10. Let $P := (\mathcal{A}, (w_1|\cdots|w_k))$ be a nanophrase over $\alpha := \{a\}$. For $A \in \mathcal{A}$ such that there exist $i \in \{1, 2, \cdots, k\}$ such that $Card(w_i^{-1}(A)) = 2$, we define $T_P(A) \in \mathbb{Z}/2\mathbb{Z}$ by

$$T_P(A) := \sum_{B \in \mathcal{A}} \tilde{\sigma}_P(A,B) \in \mathbb{Z}/2\mathbb{Z},$$

and $T_P(w_i) \in \mathbb{Z}/2\mathbb{Z}$ by

$$T_P(w_i) := \sum_{A \in \mathcal{A},\ Card(w_i^{-1}(A))=2} T_P(A).$$

Then we define $T(P) \in (\mathbb{Z}/2\mathbb{Z})^k$ by

$$T(P) := (T_P(w_1), T_P(w_2), \cdots, T_P(w_k)).$$

Then next theorem follows.

Theorem 3.11. *The T is a homotopy invariant of nanophrases over one-point set.*

Proof. Consider the 1-st homotopy move
$P_1 := (w_1|\cdots|w_{l-1}|xAAy|w_{l+1}|\cdots|w_k) \longrightarrow$
$$P_2 := (w_1|\cdots|w_{l-1}|xy|w_{l+1}|\cdots|w_k).$$
It is clear that $T_{P_1}(w_i) = T_{P_2}(w_i)$ for all $i \neq l$. We show that $T_{P_1}(xAAy) = T_{P_2}(xy)$. Note that $\tilde{\sigma}_{P_1}(A, B) = 0$ for all $B \in \mathcal{A}$ by definition. Therefore $T_{P_1}(A) = 0$. Moreover $\tilde{\sigma}_{P_1}(E, A) = 0$ for all $E \in \mathcal{A}$. So A does not contribute to $T_{P_1}(E)$ for all $E \in \mathcal{A}$. Therefore $T_{P_1}(xAAy) = T_{P_2}(xy)$.

Consider the 2-nd homotopy move such that A and B occur in some words once
$P_1 := (w_1|\cdots|x_1ABy_1|\cdots|x_2BAy_2|\cdots|w_k) \longrightarrow$
$$P_2 := (w_1|\cdots|x_1y_1|\cdots|x_2y_2|\cdots|w_k)$$
with $|A| = \tau(|B|)$. It is sufficient to show that $T_{P_1}(x_1ABy_1) = T_{P_2}(x_1y_1)$ and $T_{P_1}(x_2BAy_2) = T_{P_2}(x_2y_2)$. Note that A and B occur in P once. Moreover for all E such that $\cdots E \cdots AB \cdots E \cdots$ in P_1

$$\begin{aligned} T_{P_1}(E) &= \varepsilon(E)(n_1 + \tilde{\sigma}_{P_1}(E, A) + \tilde{\sigma}_{P_1}(E, B) + n_2) \\ &= \varepsilon(E)(n_1 + 2 + n_2) \\ &= \varepsilon(E)(n_1 + n_2) \\ &= T_{P_2}(E) \end{aligned}$$

where n_1, n_2 are integers. Therefore $T_{P_1}(x_1ABy_1) = T_{P_2}(x_1y_1)$. $T_{P_1}(x_2BAy_2) = T_{P_2}(x_2y_2)$ is proved similarly. The case of other type homotopy moves is proved similarly to above. □

Remark 3.12. Any nanoword w over α yields a nanophrase (w) of length 1. So we can consider the invariant of nanophrases over α_0 (resp. one-point set) T as a invariant of nanowords over α_0 (resp. one-point set). But these invariants are useless. In fact it is easily checked that $T((w)) = 0$ for all nanowords over α_0 and nanowords over one-point set.

4. Classification of Nanophrases of Length 2 with 4 Letters.

In this section we give the homotopy classification of nanophrases of length 2 less than 4 letters.

4.1. *Classification of nanophrases of length 2 with 2 letters.*

In this subsection we give the homotopy classification of nanowords of length 2 with 2 letter.

Consider a nanophrase of length 2 with 2 letter $P_a := (A|A)$ with $|A| = a$.

Theorem 4.1. *Let P be a nanophrase of length 2 with 2 letters. Then $P \not\simeq (\emptyset|\emptyset)$ if and only if $P \approx P_a$. Moreover $P_a \simeq P_{a'}$ if and only if $a = a'$.*

Proof. The first part of this theorem is clear. We show the second part of this theorem. Suppose $P_a \simeq P_{a'}$. Then $\gamma(P_a) = \gamma(P_{a'})$. This implies $z_a \otimes z_{\tau(a)} = z_{a'} \otimes z_{\tau(a')}$. Therefore $z_a = z_{a'}$ in Π. It is possible only if $a = a'$. So the theorem is proved. □

4.2. *Classification of nanophrases of length 2 with 4 letters.*

First, we show following lemmas.

Lemma 4.2. *Let β be τ-invariant subset of α. If two nanophrases over β are homotopic in the class of nanophrases over α, then they are homotopic in the class of nanophrases over β.*

Proof. This lemma is proved similarly to Lemma 2.6. □

Lemma 4.3. *Let $P_1 = (w_1|w_2|\cdots|w_k)$ and $P_2 = (v_1|v_2|\cdots|v_k)$ be nanophrases of length k over α. If P_1 and P_2 are homotopic as nanophrases, then $w_1w_2\cdots w_k$ and $v_1v_2\cdots v_k$ are homotopic as nanowords over α.*

Proof. It follows from definitions of homotopy of nanowords and homotopy of nanophrases. □

Lemma 4.4. *Let $P_1 = (w_1|w_2|\cdots|w_k)$ and $P_2 = (v_1|v_2|\cdots|v_k)$ be nanophrases of length k over α. If P_1 and P_2 are homotopic as nanophrases, then w_i and v_i are homotopic as étale words for all $i \in \{1, 2, , \cdots, k\}$.*

Proof. This follows from the definition of homotopy moves and the desingularization of étale words. □

The following lemma follows from the definition of homotopy moves of nanophrases.

Lemma 4.5. *Let $P_1 = (w_1|\cdots|w_k)$ and $P_2 = (v_1|\cdots|v_k)$ are nanophrases of length k. If P_1 and P_2 are homotopic, then length of w_i is equal to length of v_i modulo 2 for all $i \in \{1, 2, \cdots, k\}$.*

Take two letters $a, b \in \alpha$ (possibly $a = b$). Let $\mathcal{A}$ be the α-alphabet consisting the three letters A, B with $|A| = a$, $|B| = b \in \alpha$. Consider the following nanophrases: $P^{4,0}_{a,b} := (ABAB|\emptyset)$, $P^{3,1}_{a,b} := (ABA|B)$, $P^{2,2I}_{a,b} := (AB|AB)$, $P^{2,2II}_{a,b} := (AB|BA)$, $P^{1,3}_{a,b} := (A|BAB)$, $P^{0,4}_{a,b} := (\emptyset|ABAB)$. If $a = \tau(b)$, then $P^{4,0}_{a,b} \simeq P^{2,2I}_{a,b} \simeq P^{2,2II}_{a,b} \simeq P^{0,4}_{a,b} \simeq (\emptyset|\emptyset)$. So in this paper, if we write $P^{4,0}_{a,b}$, $P^{2,2I}_{a,b}$, $P^{2,2II}_{a,b}$, $P^{0,4}_{a,b}$, then we always assume that $a \neq \tau(b)$. The following theorem gives the classification of nanophrases of length 2 with 4 letters.

Theorem 4.6. *Let P be a nanophrase of length 2 with 4 letters, then P is either homotopic to nanophrases of length 2 with 2 letters or isomorphic to a nanophrase one of followings: $P^{4,0}_{a,b}$, $P^{3,1}_{a,b}$, $P^{2,2I}_{a,b}$, $P^{2,2II}_{a,b}$, $P^{1,3}_{a,b}$, $P^{0,4}_{a,b}$. For $(i,j) \in \{(4,0),(3,1),(2,2I),(2,2II),(1,3),(0,4)\}$ and any $a,b \in \alpha$. The nanophrase $P^{i,j}_{a,b}$ is neither homotopic to $(\emptyset|\emptyset)$ nor homotopic to nanophrases of length 2 with 2 letters. The nanophrases $P^{i,j}_{a,b}$ and $P^{i,j}_{a',b'}$ are homotopic if and only if $(a,b)=(a',b')$. For $(i,j) \neq (i',j')$, the nanophrases $P^{i,j}_{a,b}$ and $P^{i',j'}_{a',b'}$ are not homotopic for any $a,b,a',b' \in \alpha$.*

In [6], Turaev showed a stable equivalence class of an oriented, ordered, pointed multi-component curve on a surface is identified with a homotopy classes of a nanophrase in a 2-letter alphabet. So we obtain a following corollary.

Corollary 4.7. ([1]).
There are exactly 19 *stable equivalence classes of two components pointed ordered, oriented, curves on surfaces with minimum crossing number ≤ 2.*

Proof of Theorem 4.6. The first claim of this theorem is clear. We prove latter part of this theorem.

Consider a nanophrase $P^{4,0}_{a,b}$. $P^{4,0}_{a,b} \not\simeq (\emptyset|\emptyset)$ and $P^{4,0}_{a,b} \not\simeq P_{a'}$ for any $a' \in \alpha$ are follows from Lemma 4.5. $P^{4,0}_{a,b} \not\simeq P^{3,1}_{a',b'}$ and $P^{4,0}_{a,b} \not\simeq P^{1,3}_{a',b'}$ are follows from Lemma 4.5. $P^{4,0}_{a,b} \not\simeq P^{0,4}_{a',b'}$ is follows from Lemma 4.4. Indeed the first étale word of $P^{4,0}_{a,b}$ is $ABAB$ and the first étale word of $P^{0,4}_{a',b'}$ is $\emptyset$. $ABAB$ is not homotopic to $\emptyset$ by Theorem 2.7. (Note that we assume $a \neq \tau(b)$ and $a' \neq \tau(b')$ in this case). $P^{4,0}_{a,b} \not\simeq P^{2,2II}_{a',b'}$ follows from Lemma 4.3. Indeed a nanoword $ABBA$ with $|A|=a', |B|=b'$ is homotopic to $\emptyset$. On the other hand, a nanoword $ABAB$ with $|A|=a, |B|=b$ with $a \neq \tau(b)$ is not homotopic to $\emptyset$. Suppose that $P^{4,0}_{a,b} \simeq P^{2,2I}_{a',b'}$. Then $\gamma(P^{4,0}_{a,b}) = \gamma(P^{2,2I}_{a',b'})$. $\gamma(P^{4,0}_{a,b}) = z_a z_b z_{\tau(a)} z_{\tau(b)} \otimes 1$ and $\gamma(P^{2,2I}_{a',b'}) = z_{a'} z_{b'} \otimes z_{\tau(a')} z_{\tau(b')}$. So $z_{\tau(a')} z_{\tau(b')} = 1$. This implies $a' = \tau(b')$. But this contradicts to $a' \neq \tau(b')$. Therefore $P^{4,0}_{a,b} \not\simeq P^{2,2I}_{a',b'}$. $P^{4,0}_{a,b} \simeq P^{4,0}_{a',b'}$ only if $(a,b)=(a',b')$ follows from Lemma 4.3 and Theorem 2.7.

Consider the nanophrase $P^{3,1}_{a,b}$. $P^{3,1}_{a,b} \not\simeq (\emptyset|\emptyset)$ follows from Lemma 4.5. $P^{3,1}_{a,b} \not\simeq P_{a'}$ is proved later. $P^{3,1}_{a,b} \not\simeq P^{2,2I}_{a',b'}$ and $P^{3,1}_{a,b} \not\simeq P^{2,2II}_{a',b'}$ follows by Lemma 4.5. $P^{3,1}_{a,b} \not\simeq P^{1,3}_{a',b'}$ is proved later. $P^{3,1}_{a,b} \not\simeq P^{0,4}_{a',b'}$ follows from Lemma 4.5. Suppose $P^{a,b}_{3,1} \simeq P^{a',b'}_{3,1}$. If $a \neq \tau(b)$, then $(a,b)=(a',b')$ by Theorem 2.7. If $a = \tau(b)$, then $a' = \tau(b')$ by Theorem2.7 and Lemma 4.3. So $\gamma(P^{a,b}_{3,1}) = z_a z_b z_{\tau(a)} \otimes z_{\tau(b)} = z_{\tau(a)} \otimes z_{\tau(b)}$ and $\gamma(P^{a',b'}_{3,1}) = z'_a z'_b z_{\tau(a')} \otimes z_{\tau(b')} = z_{\tau(a')} \otimes z_{\tau(b')}$. This implies $z_{\tau(a)} = z_{\tau(a')}$ and $z_{\tau(b)} = z_{\tau(b')}$. Therefore $(a,b)=(a',b')$.

Consider the nanophrase $P^{2,2I}_{a,b}$. $P^{2,2I}_{a,b} \not\simeq (\emptyset|\emptyset)$ and $P^{2,2I}_{a,b} \not\simeq P_{a'}$ follows from Lemma 4.3. $P^{2,2I}_{a,b} \not\simeq P^{2,2II}_{a',b'}$ follows from Lemma 4.3. $P^{2,2I}_{a,b} \not\simeq P^{1,3}_{a',b'}$ follows from

Lemma4.5. Suppose $P^{2,2I}_{a,b} \simeq P^{0,4}_{a',b'}$. Then $\gamma(P^{2,2I}_{a,b}) = \gamma(P^{0,4}_{a',b'})$. This implies $z_a z_b = 1$. This is possible only if $a = \tau(b)$. But this contradicts to assumption. So $P^{2,2I}_{a,b} \not\simeq P^{0,4}_{a',b'}$. $P^{2,2I}_{a,b} \simeq P^{2,2I}_{a',b'}$ if and only if $(a,b) = (a',b')$ follows by Lemma 4.3.

Consider the nanophrase $P^{2,2II}_{a,b}$. Suppose $P^{2,2II}_{a,b} \simeq (\emptyset|\emptyset)$. Then $\gamma(P^{2,2II}_{a,b}) = \gamma((\emptyset|\emptyset)) = 1 \otimes 1$. This implies $z_a z_b = 1$. So $a = \tau(b)$. But this contradicts to $a \neq \tau(b)$. Therefore $P^{2,2II}_{a,b} \not\simeq (\emptyset|\emptyset)$. $P^{2,2II}_{a,b} \not\simeq P_{a'}$ follows from Lemma 4.5. $P^{2,2II}_{a,b} \not\simeq P^{1,3}_{a',b'}$ follows from Lemma4.5. Suppose $P^{2,2II}_{a,b} \simeq P^{0,4}_{a',b'}$. Then $\gamma(P^{2,2II}_{a,b}) = \gamma(P^{0,4}_{a',b'})$. This implies $z_a z_b = 1$. This is possible only if $a = \tau(b)$. But this contradicts to assumption. So $P^{2,2II}_{a,b} \not\simeq P^{0,4}_{a',b'}$. Suppose $P^{2,2II}_{a,b} \simeq P^{2,2II}_{a',b'}$. Then $\gamma(P^{2,2II}_{a,b}) = \gamma(P^{2,2II}_{a',b'})$. This implies to $z_a z_b = z_{a'} z_{b'}$. This is possible only if either "$a = a'$ and $b = b'$" or "$a = \tau(b)$ and $a' = \tau(b')$". The latter case contradicts to $a \neq \tau(b)$. So $(a,b) = (a',b')$. Therefore $P^{2,2II}_{a,b} \simeq P^{2,2II}_{a',b'}$ if and only if $(a,b) = (a',b')$.

Consider the nanophrase $P^{1,3}_{a,b}$. $P^{1,3}_{a,b} \not\simeq (\emptyset|\emptyset)$ follows from Lemma 4.5. $P^{1,3}_{a,b} \not\simeq P_{a'}$ is proved later. $P^{1,3}_{a,b} \not\simeq P^{0,4}_{a',b'}$ follows from Lemma 4.5. Suppose $P^{1,3}_{a,b} \simeq P^{1,3}_{a',b'}$. If $a \neq \tau(b)$, then $(a,b) = (a',b')$ by Lemma 4.3 and Theorem 2.7. If $a = \tau(b)$, then $a' = \tau(b')$. So if $\gamma(P^{1,3}_{a,b}) = \gamma(P^{1,3}_{a',b'})$, then $z_a = z_{a'}$ and $z_b = z_{b'}$. This implies $(a,b) = (a',b')$.

Consider the nanophrase $P^{0,4}_{a,b}$. $P^{0,4}_{a,b} \not\simeq (\emptyset|\emptyset)$ and $P^{0,4}_{a,b} \not\simeq P_{a'}$ follow from Lemma 4.4. $P^{0,4}_{a,b} \simeq P^{0,4}_{a',b'}$ if and only if $(a,b) = (a',b')$ follows from Lemma 4.3.

Now we proof following three remain parts of proof: $P^{3,1}_{a,b} \not\simeq P_{a'}$, $P^{3,1}_{a,b} \not\simeq P^{1,3}_{a',b'}$, and $P^{1,3}_{a,b} \not\simeq P_{a'}$.

Suppose $P^{3,1}_{a,b} \simeq P_{a'}$. $\gamma(P^{3,1}_{a,b}) = z_a z_b z_{\tau(a)} \otimes z_{\tau(b)}$ and $\gamma(P_{a'}) = z_{a'} \otimes z_{\tau(a')}$. This implies $a' = b$. Moreover $a = \tau(b)$ by Lemma 4.3. So if $a \neq b$, then $P^{3,1}_{a,b} \simeq P_b$ as nanophrases over α_0. However,

$$\begin{aligned} T(P^{3,1}_{a,b}) &= (T_{P^{3,1}_{a,b}}(ABA), T_{P^{3,1}_{a,b}}(B)) \\ &= (\varepsilon(A)\sigma_{P^{3,1}_{a,b}}(A,B), 0) \\ &= (-1, 0), \end{aligned}$$

and

$$T(P_b) = (0,0).$$

This contradicts to homotopy invariance of T. If $a = b$, then $P^{3,1}_{a,a} \simeq P_a$ as nanophrases over $\alpha = \{a\}$. However

$$\begin{aligned} T(P^{3,1}_{a,a}) &= (1,0) \in (\mathbb{Z}/2\mathbb{Z})^2, \\ T(P_a) &= (0,0) \in (\mathbb{Z}/2\mathbb{Z})^2. \end{aligned}$$

This contradicts to homotopy invariance of T. Therefore $P^{3,1}_{a,b} \not\simeq P_{a'}$.

$P^{1,3}_{a,b} \not\simeq P_{a'}$ is proved similarly to above.

Suppose $P^{3,1}_{a,b} \simeq P^{1,3}_{a',b'}$. If $a \neq \tau(b)$, then $a' \neq \tau(b')$ and $(a,b) = (a',b')$ by Lemma 4.3. Moreover $\gamma(P^{1,3}_{a,b}) = \gamma(P^{1,3}_{a',b'})$ implies $z_a z_b z_{\tau(a)} = z_a$ and $z_{\tau(b)} = z_b z_{\tau(a)} z_{\tau(b)}$. So

$z_b z_{\tau(a)} = 1$ and this is possible only if $a = b$. Therefore $P^{3,1}_{a,a} \simeq P^{1,3}_{a,a}$ as nanowords over $\alpha_0 = \{a, \tau(a)\}$ by Lemma 4.2. However,

$$\begin{aligned} T(P^{3,1}_{a,a}) &= (T_{P^{3,1}_{a,a}}(ABA), T_{P^{3,1}_{a,a}}(B)) \\ &= (\varepsilon(A)\sigma_{P^{3,1}_{a,a}}(A,B), 0) \\ &= (1, 0), \end{aligned}$$

and

$$\begin{aligned} T(P^{1,3}_{a,a}) &= (T_{P^{1,3}_{a,a}}(A), T_{P^{1,3}_{a,a}}(BAB)) \\ &= (0, \varepsilon(B)\sigma_{P^{1,3}_{a,a}}(B,A)) \\ &= (0, -1). \end{aligned}$$

This contradicts to homotopy invariance of T. If $a = \tau(b)$, then $a' = \tau(b')$ by Lemma 4.3. Moreover $\gamma(P^{1,3}_{a,b}) = \gamma(P^{1,3}_{a',b'})$. This implies $z_{\tau(a)} = z_{a'}$ and $z_{\tau(b)} = z_{b'}$. So $a = \tau(a')$ and $b = \tau(b')$. If $a = \tau(a)$, then $a = a' = b = b'$ by above equations. Therefore $P^{3,1}_{a,a} \simeq P^{1,3}_{a,a}$ as nanophrases over $\alpha = \{a\}$. However,

$$\begin{aligned} T(P^{3,1}_{a,a}) &= (1, 0) \in (\mathbb{Z}/2\mathbb{Z})^2, \\ T(P^{1,3}_{a,a}) &= (0, 1) \in (\mathbb{Z}/2\mathbb{Z})^2. \end{aligned}$$

This contradicts to homotopy invariance of T. If $a \neq \tau(a)$, then $P^{3,1}_{a,b} \simeq P^{1,3}_{b,a}$ as nanophrases over α_0. However,

$$\begin{aligned} T(P^{3,1}_{a,b}) &= (\varepsilon(A)\sigma_{P^{3,1}_{a,b}}(A,B), 0) = (-1, 0) \\ T(P^{1,3}_{b,a}) &= (0, \varepsilon(B)\sigma_{P^{1,3}_{b,a}}(B,A)) = (0, 1). \end{aligned}$$

This contradicts to homotopy invariance of T. Therefore $P^{3,1}_{a,b} \not\simeq P^{1,3}_{a',b'}$.

Now we have completed the homotopy classification of nanophrases of length 2 with 4 letters. □

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