HOMOTOPY CLASSIFICATION OF NANOPHRASES IN TURAEV’S THEORY OF WORDS

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ABSTRACT

The purpose of this paper is to give the homotopy classification of nanophrases of length 2 with 4 letters. To do it we construct some new invariants of nanophrases $\gamma$, $T$. The invariant $\gamma$ defined in this paper is an extension of the invariant $\gamma$ for nanowords introduced in [5]. The invariant $T$ is a new invariant of nanophrases. As a corollary of these results, we give the classification of two-components pointed, ordered, oriented curves on surfaces with minimum crossing number $\leq 2$.

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1. Introduction.

Words are finite sequences of letters in a given alphabet. In [2] C. F. Gauss introduced a method to investigate closed planar curves by words of a certain type now called Gauss words. We can apply this method to encode surface curves. (See [10].)

V. Turaev introduced word theory in [5], [6]. The key of new concepts introduced in those papers are those of étale words and nanowords. An étale word over an alphabet $\alpha$ endowed with an involution $\tau : \alpha \rightarrow \alpha$ is a word in an alphabet $\mathcal{A}$ endowed with a projection $\mathcal{A} \ni A \mapsto |A| \in \alpha$. Every word in the alphabet $\alpha$ becomes an étale word over $\alpha$ by using the identity mapping $\text{id} : \alpha \rightarrow \alpha$ as the projection. An étale word over $\alpha$ is called nanoword if every letter appears twice or not at all. In the case where the alphabet $\alpha$ consists of two elements permuted by $\tau$, the notion of a nanoword over $\alpha$ is equivalent to the notion of an open virtual string introduced in [9].

Turaev introduced an equivalence relation of homotopy on the set of étale words over $\alpha$. The relation of homotopy is generated by three transformations or moves on nanowords. The first move consists in deleting two consecutive entries of the same letter. The second move has the form $xAByBAz \mapsto xyz$ where $x, y, z$ are words and $A, B$ are letters such that $|A| = \tau(|B|)$. The third move has the form $xAByACzBCt \mapsto xBAyCAzCBt$ where $x, y, z, t$ are words and $A, B, C$ are letters such that $|A| = |B| = |C|$. These moves are suggested by the Reidemeister moves.
in knot theory. In fact the first (resp. second, third) homotopy move is similar to
the first (resp. second, third) Reidemeister move. (See [6] for more details.) Turaev
applied topological methods to a semigroup consisting letters to study properties
and characteristics of nanowords preserved under homotopy. For instance, these are
applications of colorings of knot diagrams, the theory of knot quandles, etc. (See
[5], [6], [7], [4] for more details.) As an application of those methods, Turaev gave
the homotopy classification of nanowords of length \( \leq 6 \) in [5].

On the other hand, in [6] Turaev showed that a stable equivalence class of an
oriented pointed curve on a surface is identified with a homotopy class of nanoword
in a 2-letter alphabet. Moreover Turaev extended this result to multi-component
curves. In fact a stable equivalence class of an oriented, ordered, pointed multi-
component curve on a surface is identified with a homotopy classes of a nanophrase
in a 2-letter alphabet. Roughly speaking, a nanophrase is a sequence of étale words
which concatenation of those words is a nanoword. (See [6], [8].) We can define
homotopy moves similarly as in the case of nanowords.

Now the purpose of this paper is to give the homotopy classification of
nanophrases of length 2 with 4 letters. (Theorem 4.6.) To do it we construct some
new invariants of nanophrases. As a corollary of these results, we give the classification
of two-components pointed, ordered, oriented curves on surfaces with minimum
crossing number \( \leq 2 \). (See also [1].)

Another application of the theory of words was introduced by N. Ito in [3].
By using the theory of words, Ito reconstructed the Arnold basic invariants and
constructed some other invariants for plane closed curves, long curves, and fronts.

In section 2 we review the theory of words and phrases which are introduced
by Turaev in [5], [6]. In section 3 we construct some new homotopy invariants of
nanophrases \( \gamma, T \). The invariant \( \gamma \) defined in this paper is an extension of the
invariant \( \gamma \) for nanowords introduced in [5]. The invariant \( T \) is a new invariant of
nanophrases. In section 4 we generalize Turaev’s result to the case of nanophrases.
In fact we give the homotopy classification of nanophrases of length 2 with 4 letters
using homotopy invariants constructed in section 3.

2. Nanowords and Nanophrases.
In this section we review the theory of words and phrases (cf. [5], [6]).

2.1. Nanowords and their homotopy.
An alphabet is a set and letters are its elements. A word of length \( n \geq 1 \) on an
alphabet \( \mathcal{A} \) is a mapping \( w : \hat{n} \to \mathcal{A} \) where \( \hat{n} = \{1, 2, \cdots, n\} \). A word usually
encoded by the sequence of letters \( w(1)w(2) \cdots w(n) \). A word \( w : \hat{n} \to \mathcal{A} \) is a Gauss
word if each element of \( \mathcal{A} \) is the image of precisely two elements of \( \hat{n} \).

For a set \( \alpha \), an \( \alpha \)-alphabet is a set \( \mathcal{A} \) endowed with a mapping \( \mathcal{A} \to \alpha \) called
projection. the image of \( A \in \mathcal{A} \) under this mapping is denoted \( |A| \). A étale word
over \( \alpha \) is a pair (an \( \alpha \)-alphabet, a word on \( \mathcal{A} \)). A nanoword over \( \alpha \) is a pair (an \( \alpha \)-alphabet, a Gauss word on \( \alpha \)). An empty étale word in an empty \( \alpha \)-alphabet is a nanoword called the empty nanoword \( \emptyset \) of length 0.

A morphism of \( \alpha \)-alphabets \( \mathcal{A}_1, \mathcal{A}_2 \) is a set-theoretic mapping \( f : \mathcal{A}_1 \rightarrow \mathcal{A}_2 \) such that \( |A| = |f(A)| \) for all \( A \in \mathcal{A}_1 \). If \( f \) is bijective, then this morphism is an isomorphism. Two étale words \((\mathcal{A}_1, w_1)\) and \((\mathcal{A}_2, w_2)\) over \( \alpha \) are isomorphic if there is an isomorphism \( f : \mathcal{A}_1 \rightarrow \mathcal{A}_2 \) such that \( w_2 = f \circ w_1 \).

To define homotopy of nanowords we fix a set \( \alpha \) with an involution \( \tau : \alpha \rightarrow \alpha \) and a subset \( S \subset \alpha \times \alpha \times \alpha \). We call the pair \((\alpha, S)\) homotopy data.

**Definition 2.1.** Let \((\alpha, S)\) be homotopy data. We define a homotopy moves (1) - (3) as follows:

1. \((\mathcal{A}, xAYy) \rightarrow (\mathcal{A} \setminus \{A\}, xy)\)
   
   for all \( A \in \mathcal{A} \) and \( x, y \) are words in \( \mathcal{A} \setminus \{A\} \).

2. \((\mathcal{A}, xAByBAz) \rightarrow (\mathcal{A} \setminus \{A, B\}, xyz)\)
   
   if \( A, B \in \mathcal{A} \) with \( |B| = \tau(|A|) \). \( x, y, z \) are words in \( \mathcal{A} \setminus \{A, B\} \).

3. \((\mathcal{A}, xAByACzBCt) \rightarrow (\mathcal{A}, xBAyCAzCBl)\)
   
   if \( A, B, C \in \mathcal{A} \) satisfy \((|A|, |B|, |C|) \in S, x, y, z, t \) are words in \( \mathcal{A} \).

**Definition 2.2.** Let \((\alpha, S)\) be a homotopy data. Then nanowords \((\mathcal{A}_1, w_1)\) and \((\mathcal{A}_2, w_2)\) over \( \alpha \) are \( S \)-homotopic (denote \((\mathcal{A}_1, w_1) \simeq_S (\mathcal{A}_2, w_2)\)) if \((\mathcal{A}_2, w_2)\) can be obtained from \((\mathcal{A}_1, w_1)\) by a finite sequence of isomorphism, \( S \)-homotopy moves (1) - (3) and the inverse moves.

The set of \( S \)-homotopy classes of nanowords over \( \alpha \) is denoted as \( \mathcal{N}(\alpha, S) \).

To define \( S \)-homotopy of étale words. We define desingularization of étale words \((\mathcal{A}, w)\) over \( \alpha \) as follows: \( \mathcal{A}^d := \{A_{i,j} := (A, i, j)|A \in \mathcal{A}, 1 \leq i < j \leq m_w(A)\} \) with projection \( |A_{i,j}| := |A| \in \alpha \) for all \( A, i, j \) (where \( m_w(A) := \text{Card}(w^{-1}(A)) \)). The word \( w^d \) is obtained from \( w \) by first deleting all \( A \in \mathcal{A} \) with \( m_w(A) = 1 \). Then for each \( A \in \mathcal{A} \) with \( m_w(A) \geq 2 \) and each \( i = 1, 2, \ldots m_w(A) \), we replace the \( i \)-th entry of \( A \) in \( w \) by

\[ A_{1,i}A_{2,i} \ldots A_{i-1,i}A_{i,i+1}A_{i,i+2} \ldots A_{i,m_w(A)}. \]

The resulting \((\mathcal{A}^d, w^d)\) is a nanoword of length \( \Sigma m_w(A)(m_w(A) - 1) \) and called a desingularization of \((\mathcal{A}, w)\). Then we define \( S \)-homotopy of étale words as following:

**Definition 2.3.** Let \( w_1 \) and \( w_2 \) be étale words over \( \alpha \). Then \( w_1 \) and \( w_2 \) are \( S \)-homotopic if \( w_1^d \) and \( w_2^d \) are \( S \)-homotopic.

Recall the following three lemmas from [5].

**Lemma 2.4.** Let \((\alpha, S)\) be a homotopy data and \( \mathcal{A} \) be an \( \alpha \)-alphabet. \( A, B, C \) are distinct letters in \( \mathcal{A} \). \( x, y, z, w \) are words in \( \mathcal{A} \setminus \{A, B, C\} \) with \( xyz \) is Gauss word. Then following (i)-(iii) hold:

1. \((\mathcal{A}, xAByACzBCt) \simeq_S (\mathcal{A}, xAByACzCBl)\)
Lemma 2.5. Suppose that $S \cap \alpha \times b \times b \neq \emptyset$ for all $b \in \alpha$. Let $(A, xAByABz)$ be nanoword over $\alpha$ with $|B| = \tau(|A|), x, y, z$ are words in $A \setminus \{A, B\}$. Then

$$(A, xAByABz) \simeq_S (A \setminus \{A, B\}, xyz).$$

In the remaining part of the paper we assume that $S$ is the diagonal of $\alpha^3$ that is $\{(a, a, a)\}_{a \in \alpha}$. Under this convention, we shall omit the prefix $S$- and speak simply of homotopy rather than $S$-homotopy. We shall also omit index $S$ and write $\simeq, ||\cdot||, N_\bullet(\alpha)$ for $\simeq_S, ||\cdot||_S, N_\bullet^S(\alpha)$.

Lemma 2.6. Let $\beta$ be $\tau$-invariant subset of $\alpha$. If two étale words over $\beta$ are homotopic in the class of étale words over $\alpha$, then they are homotopic in the class of étale words over $\beta$.

V.Turaev gives a homotopy classification of nanowords of length 4 in [5].

Theorem 2.7. Let $w$ be a nanoword of length 4 over $\alpha$. Then $w$ is either $w \simeq \emptyset$ or isomorphic to the nanoword $w_{a,b} := (A = \{A, B\}, ABAB)$ where $|A| = a, |B| = b \in \alpha$ with $a \neq \tau(b)$. Moreover for $a \neq \tau(b)$, the nanoword $w_{a,b}$ is non-contractible and two nanowords $w_{a,b}$ and $w_{a',b'}$ are homotopic if and only if $(a, b) = (a', b')$.

In this paper we generalize Turaev’s result to the case of “nanophrases”.

2.2. Nanophrases and their homotopy.

Definition 2.8. A nanophrase $(A, (w_1|w_2|\cdots|w_k))$ of length $k \geq 0$ over a set $\alpha$ is a pair consisting of an $\alpha$-alphabet $A$ and a sequence of $k$ words $w_1, \cdots, w_k$ on $A$ such that $w_1w_2\cdots w_k$ is a Gauss word on $A$. We denote it shortly by $(w_1|w_2|\cdots|w_k)$. We denote a set of nanophrases of length $k$ over $\alpha$ by $P_k(\alpha)$.

By definition, there is a unique empty nanophrase of length 0 (the corresponding $\alpha$-alphabet $\emptyset$ is void).

Remark 2.9. Any nanoword $w$ over $\alpha$ yields a nanophrase $(w)$ of length 1.

A mapping $f : A_1 \longrightarrow A_2$ is isomorphism of two nanophrases if $f$ is an isomorphism of $\alpha$-alphabets transforming the first nanophrase into second one.

Given a homotopy data $(\alpha, \tau, S)$, we define homotopy move on nanophrases as in section 2.1 with the only difference that the 2-letter subwords $AA, AB, BA, AC, BC$ modified by these moves may occur in different words of phrase. Isomorphism
and homotopy moves generate an equivalence relation $\simeq_S$ of $S$-homotopy on the class of nanophrases over $\alpha$. We denote a set of $S$-homotopy class of nanophrases of length $k$ by $\mathcal{P}_k(\alpha, S)$.

**Example 2.10.** Nanophrases $(AB|ADDCBC)$ and $(BA|CACB)$ with $|A| = |B| = |C| \in \alpha$ over $\alpha$ are homotopic. Indeed

$$(AB|ADDCBC) \simeq (AB|ACBC) \simeq (BA|CACB).$$

Lemmas 2.4 and 2.5 extend to nanophrases with the only change that the 2-letter subwords $AB, BA, CA$, and so forth may occur in different word of the phrase.

3. Some Homotopy Invariants of Nanophrases.

In this section, we define three new homotopy invariants of nanophrases. They will be used in the next section.

3.1. Invariant $\gamma$.

Recall that an orbit of the involution $\tau: \alpha \to \alpha$ is a subset of $\alpha$ consisting either of one element or of two elements; in latter case the orbit is called free. Let $\Pi$ be the group which defined as follows:

$$\Pi := \left\lbrace \left( f_{z}a, g_{a} \right) \mid \tau(a) = 1 \text{ for all } a \in \alpha \right\rbrace.$$

Let $\mathbb{Z}\Pi$ be the integral group-ring of $\Pi$.

**Definition 3.1.** Let $P = (A, (w_1|w_2| \cdots |w_k))$ be a nanophrase of length $k$ over $\alpha$ and $n_i$ the length of nanoword $w_i$. Set $n = \sum_{1 \leq i \leq k} n_i$. Then we define $n$ elements $\gamma^1_i, \gamma^2_i, \cdots, \gamma^n_i, (i \in \{1, 2, \cdots, k\})$ of $\Pi$ by $\gamma^1_i := z_{w_j(i)}$ if $w_j(i) \neq w_j(m)$ for all $l < j$ and for all $m < i$ when $l = j$. Otherwise $\gamma^1_i := z_{\tau(w_j(i))}$. Then we define $\gamma(P) \in \otimes^k \mathbb{Z}\Pi$ by

$$\gamma(P) := \gamma^1_1 \gamma^2_1 \cdots \gamma^n_1 \otimes \gamma^1_2 \gamma^2_2 \cdots \gamma^n_2 \otimes \cdots \otimes \gamma^1_k \gamma^2_k \cdots \gamma^n_k.$$

Then we obtain following theorem.

**Theorem 3.2.** The $\gamma$ is a homotopy invariant of nanophrases.

**Remark 3.3.** By definition, for nanophrases of length 1 the invariant $\gamma$ for nanophrases is equal to Turaev’s invariant $\gamma$ defined in [5].

**Example 3.4.** Let $A := \{A, B, C\}$ be an $\alpha$-alphabet. Set $|A| = a, |B| = b, |C| = c \in \alpha$. Consider a nanophrase $P = (ABC|CB|A)$, then

$$\gamma(P) = z_aza_bz_c \otimes z_{\tau(c)}z_{\tau(b)} \otimes z_{\tau(a)}.$$
3.2. Invariant $T$.

In this subsection we define homotopy invariants of nanophrases over $\alpha_0 := \{a, b\}$ with involution $\tau_0$ permuting $a, b$ and nanophrases over one-point set. At first, we define a homotopy invariant of nanophrases $T$ over $\alpha_0$. To define this invariant, we define some notation as follows.

**Definition 3.5.** Let $P = (A, (w_1|\cdots|w_k))$ be a nanophrase over $\alpha_0$ and $A, B \in \mathcal{A}$. Then we define $\sigma_P(A, B)$ as follows: If $A$ and $B$ form $\cdots A \cdots B \cdots A \cdots B \cdots$ in $P$ and $|B| = a$, or $\cdots B \cdots A \cdots B \cdots A \cdots$ in $P$ and $|B| = b$, then $\sigma_P(A, B) := 1$. If $\cdots A \cdots B \cdots A \cdots B \cdots$ in $P$ and $|B| = b$, or $\cdots B \cdots A \cdots B \cdots A \cdots$ in $P$ and $|B| = a$, then $\sigma_P(A, B) := -1$. Otherwise $\sigma_P(A, B) := 0$.

**Definition 3.6.** For $A \in \mathcal{A}$ we define $\epsilon(A) \in \{\pm 1\}$ by
\[
\epsilon(A) := \begin{cases} 
1 & \text{if } |A| = a, \\
-1 & \text{if } |A| = b.
\end{cases}
\]

**Definition 3.7.** Let $P = (A, (w_1|w_2|\cdots|w_k))$ be a nanophrase of length $k$ over $\alpha_0$. For $A \in \mathcal{A}$ such that there exist $i \in \{1, 2, \cdots, k\}$ such that $\text{Card}(w_i^{-1}(A)) = 2$, we define $T_P(A) \in \mathbb{Z}$ by
\[
T_P(A) := \epsilon(A) \sum_{B \in \mathcal{A}} \sigma_P(A, B),
\]
and we define $T_P(w_i) \in \mathbb{Z}$ by
\[
T_P(w_i) := \sum_{A \in \mathcal{A}, \text{Card}(w_i^{-1}(A))=2} T_P(A).
\]

Then we define $T(P) \in \mathbb{Z}^k$ by
\[
T(P) := (T_P(w_1), T_P(w_2), \cdots, T_P(w_k)).
\]

**Theorem 3.8.** The $T$ is a homotopy invariant of nanophrases over $\alpha_0$.

**Proof.** Consider the 1-st homotopy move
\[
P_1 := (w_1|\cdots|w_{l-1}|xAAy|w_{l+1}|\cdots|w_k) \longrightarrow P_2 := (w_1|\cdots|w_{l-1}|xy|w_{l+1}|\cdots|w_k).
\]
It is clear that $T_{P_i}(w_i) = T_{P_2}(w_i)$ for all $i \neq l$. We show that $T_{P_1}(xAYy) = T_{P_2}(xy)$.

Note that $\sigma_{P_1}(A, B) = 0$ for all $B \in \mathcal{A}$ by definition. Therefore $T_{P_1}(A) = 0$.
Moreover $\sigma_{P_1}(E, A) = 0$ for all $E \in \mathcal{A}.$ So $A$ does not contribute to $T_{P_1}(E)$ for all $E \in \mathcal{A}$. Therefore $T_{P_1}(xAYy) = T_{P_2}(xy)$.

Consider the 2-nd homotopy move such that $A$ and $B$ occur in some words once
\[
P_1 := (w_1|\cdots|x_1ABy_1|\cdots|x_2BAy_2|\cdots|w_k) \longrightarrow P_2 := (w_1|\cdots|x_1y_1|\cdots|x_2y_2|\cdots|w_k).
\]
with $|A| = \tau(|B|)$. 

It is sufficient to show that \( T_{P_1}(x_1ABy_1) = T_{P_2}(x_1y_1) \) and \( T_{P_1}(x_2BAy_2) = T_{P_2}(x_2y_2) \). Note that \( A \) and \( B \) occur in \( P \) once. Moreover for all \( E \) such that \( \cdots E \cdots AB \cdots E \cdots \) in \( P_1 \)

\[
T_{P_1}(E) = \varepsilon(E)(n_1 + \sigma_{P_1}(E, A) + \sigma_{P_1}(E, B) + n_2) \\
= \varepsilon(E)(n_1 + n_2) \\
= T_{P_2}(E)
\]

where \( n_1, n_2 \) are integers. Therefore \( T_{P_1}(x_1ABy_1) = T_{P_2}(x_1y_1) \). \( T_{P_1}(x_2BAy_2) = T_{P_2}(x_2y_2) \) is proved similarly.

Consider the 2-nd homotopy move such that \( A \) and \( B \) occur in some word twice \( P_1 := (w_1 \cdots |w_{t-1}|x_1ABy_1BAz|w_{t+1} \cdots |w_k) \) \( \rightarrow \) \( P_2 := (w_1 \cdots |xyz \cdots |w_k) \) with \( |A| = \tau(|B|) \). It is sufficient to show that \( T_{P_1}(w_1) = T_{P_2}(w_1) \). At first we show \( T_{P_1}(A) + T_{P_2}(B) = 0 \). Indeed

\[
T_{P_1}(A) = \varepsilon(A)(\sigma_{P_1}(A, B) + n + \sigma_{P_1}(A, B)) \\
= \varepsilon(A)n \\
= -\varepsilon(B)n \\
= -T_{P_2}(B)
\]

where \( n \) is an integer. Now we show \( T_{P_2}(E) = T_{P_2}(E) \) for all \( E \neq A, B \). If \( \cdots E \cdots AB \cdots E \cdots BA \cdots \) or \( \cdots AB \cdots E \cdots BA \cdots E \), then

\[
T_{P_1}(E) = \varepsilon(E)(n_1 + \sigma_{P_1}(E, A) + \sigma_{P_1}(E, B) + n_2) \\
= \varepsilon(E)(n_1 + n_2) \\
= T_{P_2}(E)
\]

where \( n_1, n_2, n_3 \) are integers If \( \cdots E \cdots AB \cdots BA \cdots \), then

\[
T_{P_1}(E) = \varepsilon(E)(n_1 + \sigma_{P_1}(E, A) + \sigma_{P_1}(E, B) + n_2) \\
= \varepsilon(E)(n_1 + n_2 + n_3) \\
= T_{P_2}(E)
\]

where \( n_1, n_2, n_3 \) are integers. Therefore \( T_{P_1}(E) = T_{P_2}(E) \) for all \( E \neq A, B \).

Consider the 3-rd homotopy move

\[
P_1 := (w_1 \cdots |x_1ABy_1| \cdots |x_2ACy_2| \cdots |x_3BCy_3| \cdots |w_k) \rightarrow \quad P_2 := (w_1 \cdots |x_1BAy_1| \cdots |x_2CAy_2| \cdots |x_3CBy_3| \cdots |w_k)
\]

with \( |A| = |B| = |C| \). In this case it is clear that \( T(P_1) = T(P_2) \).

Consider the 3-rd homotopy move

\[
P_1 := (w_1 \cdots |x_1ABy_1ACz_1| \cdots |x_2BCy_2| \cdots |w_k) \rightarrow \quad P_2 := (w_1 \cdots |x_1BAy_1CAz_1| \cdots |x_2CBy_2| \cdots |w_k)
\]

with \( |A| = |B| = |C| \).
It is sufficient to show $T_{P_1}(x_1 ABy_1 ACz_1) = T_{P_2}(x_1 BAY_1 CAz_1)$ and $T_{P_1}(x_2 BCA_y_2) = T_{P_2}(x_2 CBY_2)$.

$$T_{P_1}(A) = \varepsilon(A)(\sigma_{P_1}(A, B) + n_1)$$
$$= \varepsilon(A)(n_1 + \sigma_{P_1}(A, C))$$
$$= T_{P_2}(A),$$

where $n_1$ an integer. So $T_{P_1}(x_1 ABy_1 ACz_1) = T_{P_2}(x_1 BAY_1 CAz_1)$ holds.

$T_{P_1}(x_2 BCA_y_2) = T_{P_2}(x_2 CBY_2)$ is clear.

Consider the 3-nd homotopy move

$$P_1 := (w_1 \cdots |x_1 ABy_1| \cdots |x_2 ACz_2| \cdots |w_k) \longrightarrow$$
$$P_2 := (w_1 \cdots |x_1 BAY_1| \cdots |x_2 CAz_2| \cdots |w_k)$$

with $|A| = |B| = |C|$. In this case $T(P_1) = T(P_2)$ is proved similarly to above case.

Consider the 3-nd homotopy move

$$P_1 := (w_1 \cdots |xAByACzBC| \cdots |w_k) \longrightarrow$$
$$P_2 := (w_1 \cdots |xBAYCAzCB| \cdots |w_k)$$

with $|A| = |B| = |C|$. In this case it is sufficient to show that $T_{P_1}(xAByACzBCt) = T_{P_2}(xAByACzBCt)$. $T_{P_1}(A) = T_{P_2}(A)$ and $T_{P_1}(C) = T_{P_2}(C)$ is clear. Note that

$$\sigma_{P_1}(B, A) = -\sigma_{P_1}(B, C)$$
$$\sigma_{P_2}(B, A) = \sigma_{P_2}(B, C) = 0.$$

We obtain $T_{P_1}(B) = T_{P_2}(B)$. $T_{P_1}(E) = T_{P_2}(E)$ for all $E \neq A, B, C$ is checked easily. So we obtain $T(P_1) = T(P_2)$.

Next we define invariant $T$ for nanophrases over one-point set. To define this invariant, we define some notation as followings.

**Definition 3.9.** Let $P := (A, (w_1 \cdots |w_k))$ be a nanophrase over one-point set $A := \{a\}$. Let $A, B \in A$ be letters. Then we define $\tilde{\sigma}_P(A, B) \in \mathbb{Z}/2\mathbb{Z}$ as followings:

If $A$ and $B$ forms $\cdots A \cdots B \cdots A \cdots B \cdots$ or $\cdots B \cdots A \cdots B \cdots A \cdots$ in $P$, then $\tilde{\sigma}_P(A, B) := 1$. Otherwise $\tilde{\sigma}_P(A, B) := 0$.

**Definition 3.10.** Let $P := (A, (w_1 \cdots |w_k))$ be a nanophrase over $\alpha := \{a\}$. For $A \in A$ such that there exist $i \in \{1, 2, \cdots, k\}$ such that $Card((w_i^{-1}(A)) = 2$, we define $T_P(A) \in \mathbb{Z}/2\mathbb{Z}$ by

$$T_P(A) := \sum_{B \in A} \tilde{\sigma}_P(A, B) \in \mathbb{Z}/2\mathbb{Z},$$

and $T_P(w_i) \in \mathbb{Z}/2\mathbb{Z}$ by

$$T_P(w_i) := \sum_{A \in A, Card((w_i^{-1}(A)) = 2} T_P(A).$$

Then we define $T(P) \in (\mathbb{Z}/2\mathbb{Z})^k$ by

$$T(P) := (T_P(w_1), T_P(w_2), \cdots, T_P(w_k)).$$
Then next theorem follows.

**Theorem 3.11.** The $T$ is a homotopy invariant of nanophrases over one-point set.

**Proof.** Consider the 1-st homotopy move

\[ P_1 := (w_1 \cdots w_{l-1} | xA | y | w_{l+1} \cdots w_k) \mapsto \]

\[ P_2 := (w_1 \cdots w_{l-1} | x | y | w_{l+1} \cdots w_k). \]

It is clear that $T_{P_1}(w_i) = T_{P_2}(w_i)$ for all $i \neq l$. We show that $T_{P_1}(xAY) = T_{P_2}(xy)$. Note that $\tilde{T}_{P_1}(A, B) = 0$ for all $B \in \mathcal{A}$ by definition. Therefore $T_{P_1}(A) = 0$. Moreover $\tilde{T}_{P_1}(E, A) = 0$ for all $E \in \mathcal{A}$. So $A$ does not contribute to $T_{P_1}(E)$ for all $E \in \mathcal{A}$. Therefore $T_{P_1}(xAY) = T_{P_2}(xy)$.

Consider the 2-nd homotopy move such that $A$ and $B$ occur in some words once

\[ P_1 := (w_1 \cdots | x_1 AB | y_1 | x_2 B | A | y_2 | \cdots | w_k) \mapsto \]

\[ P_2 := (w_1 \cdots | x_1 y_1 | \cdots | x_2 y_2 | \cdots | w_k) \]

with $|A| = \tau(|B|)$. It is sufficient to show that $T_{P_1}(x_1 AB y_1) = T_{P_2}(x_1 y_1)$ and $T_{P_1}(x_2 B A y_2) = T_{P_2}(x_2 y_2)$. Note that $A$ and $B$ occur in $P$ once. Moreover for all $E$ such that $\cdots E \cdots AB \cdots E \cdots$ in $P_1$

\[ T_{P_1}(E) = \varepsilon(E)(n_1 + \tilde{T}_{P_1}(E, A) + \tilde{T}_{P_1}(E, B) + n_2) \]

\[ = \varepsilon(E)(n_1 + 2 + n_2) \]

\[ = \varepsilon(E)(n_1 + n_2) \]

\[ = T_{P_2}(E) \]

where $n_1, n_2$ are integers. Therefore $T_{P_1}(x_1 AB y_1) = T_{P_2}(x_1 y_1)$. $T_{P_1}(x_2 B A y_2) = T_{P_2}(x_2 y_2)$ is proved similarly. The case of other type homotopy moves is proved similarly to above.

**Remark 3.12.** Any nanoword $w$ over $\alpha$ yields a nanophrase $(w)$ of length 1. So we can consider the invariant of nanophrases over $\alpha_0$ (resp. one-point set) $T$ as a invariant of nanowords over $\alpha_0$ (resp. one-point set). But these invariants are useless. In fact it is easily checked that $T((w)) = 0$ for all nanowords over $\alpha_0$ and nanowords over one-point set.

4. **Classification of Nanophrases of Length 2 with 4 Letters.**

In this section we give the homotopy classification of nanophrases of length 2 less than 4 letters.

4.1. **Classification of nanophrases of length 2 with 2 letters.**

In this subsection we give the homotopy classification of nanowords of length 2 with 2 letter.

Consider a nanophrase of length 2 with 2 letter $P_a := (A|A)$ with $|A| = a$. 
Theorem 4.1. Let $P$ be a nanophrase of length 2 with 2 letters. Then $P \not\in (\emptyset|\emptyset)$ if and only if $P \simeq P_0$. Moreover $P_a \simeq P_{a'}$ if and only if $a = a'$.

Proof. The first part of this theorem is clear. We show the second part of this theorem. Suppose $P_a \simeq P_{a'}$. Then $\gamma(P_a) = \gamma(P_{a'})$. This implies $z_a \otimes z_{\tau(a)} = z_{a'} \otimes z_{\tau(a')}$. Therefore $z_a = z_{a'}$ in $\Pi$. It is possible only if $a = a'$. So the theorem is proved. \hfill \Box

4.2. Classification of nanophrases of length 2 with 4 letters.

First, we show following lemmas.

Lemma 4.2. Let $\beta$ be $\tau$-invariant subset of $\alpha$. If two nanophrases over $\beta$ are homotopic in the class of nanophrases over $\alpha$, then they are homotopic in the class of nanophrases over $\beta$.

Proof. This lemma is proved similarly to Lemma 2.6. \hfill \Box

Lemma 4.3. Let $P_1 = (w_1|w_2|\cdots|w_k)$ and $P_2 = (v_1|v_2|\cdots|v_k)$ be nanophrases of length $k$ over $\alpha$. If $P_1$ and $P_2$ are homotopic as nanophrases, then $w_1w_2\cdots w_k$ and $v_1v_2\cdots v_k$ are homotopic as nanowords over $\alpha$.

Proof. It follows from definitions of homotopy of nanowords and homotopy of nanophrases. \hfill \Box

Lemma 4.4. Let $P_1 = (w_1|w_2|\cdots|w_k)$ and $P_2 = (v_1|v_2|\cdots|v_k)$ be nanophrases of length $k$ over $\alpha$. If $P_1$ and $P_2$ are homotopic as nanophrases, then $w_i$ and $v_i$ are homotopic as étale words for all $i \in \{1,2,\cdots,k\}$.

Proof. This follows from the definition of homotopy moves and the desingularization of étale words. \hfill \Box

The following lemma follows from the definition of homotopy moves of nanophrases.

Lemma 4.5. Let $P_1 = (w_1|\cdots|w_k)$ and $P_2 = (v_1|\cdots|v_k)$ are nanophrases of length $k$. If $P_1$ and $P_2$ are homotopic, then length of $w_i$ is equal to length of $v_i$ modulo 2 for all $i \in \{1,2,\cdots,k\}$.

Take two letters $a, b \in \alpha$ (possibly $a = b$). Let $\mathcal{A}$ be the $\alpha$-alphabet consisting the three letters $A, B$ with $|A| = a, |B| = b \in \alpha$. Consider the following nanophrases: $P_{a,b}^{4,0} := (AB|AB|\emptyset), P_{a,b}^{4,1} := (AB|A|B), P_{a,b}^{2,2I} := (AB|AB), P_{a,b}^{2,2II} := (AB|BA), P_{a,b}^{1,3} := (A|BAB), P_{a,b}^{0,4} := (\emptyset|ABAB)$. If $a = \tau(b)$, then $P_{a,b}^{4,0} \simeq P_{a,b}^{2,2I} \simeq P_{a,b}^{2,2II} \simeq P_{a,b}^{0,4} \simeq (\emptyset|\emptyset)$. So in this paper, if we write $P_{a,b}^{4,0}, P_{a,b}^{2,2I}, P_{a,b}^{2,2II}, P_{a,b}^{0,4}, P_{a,b}^{1,3}$, then we always
assume that \( a \neq \tau(b) \). The following theorem gives the classification of nanophrases of length 2 with 4 letters.

**Theorem 4.6.** Let \( P \) be a nanophrase of length 2 with 4 letters, then \( P \) is either homotopic to nanophrases of length 2 with 2 letters or isomorphic to a nanophrase one of followings: \( P_{a,b}^{2,21}, P_{a,b}^{2,21I}, P_{a,b}^{1,3}, P_{a,b}^{0,4} \). For \( (i,j) \in \{(4,0),(3,1),(2,2I),(2,2II),(1,3),(0,4)\} \) and any \( a,b \in \alpha \). The nanophrase \( P_{a,b}^{i,j} \) is not homotopic to \((\emptyset)\) nor homotopic to nanophrases of length 2 with 2 letters.

The nanophrases \( P_{a,b}^{i,j} \) and \( P_{a',b'}^{i,j} \) are homotopic if and only if \((a,b) = (a',b')\).

In [6], Turaev showed a stable equivalence class of an oriented, ordered, pointed multi-component curve on a surface is identified with a homotopy classes of a nanophrase in a 2-letter alphabet. So we obtain a following corollary.

**Corollary 4.7.** \([1]\).

There are exactly 19 stable equivalence classes of two components pointed ordered, oriented, curves on surfaces with minimum crossing number \( \leq 2 \).

**Proof of Theorem 4.6.** The first claim of this theorem is clear. We prove latter part of this theorem.

Consider a nanophrase \( P_{a,b}^{2,0} \), \( P_{a,b}^{0,0} \neq (\emptyset) \) and \( P_{a,b}^{2,0} \neq P_{a'} \) for any \( a' \in \alpha \) are follows from Lemma 4.5. \( P_{a,b}^{0,0} \neq P_{a',b'}^{2,0} \) and \( P_{a,b}^{0,0} \neq P_{a',b'}^{1,3} \) are follows from Lemma 4.5. \( P_{a,b}^{0,0} \neq P_{a',b'}^{0,4} \) is follows from Lemma 4.4. Indeed the first étaile word of \( P_{a,b}^{0,0} \) is ABAB and the first étaile word of \( P_{a',b'}^{0,4} \) is \( \emptyset \). ABAB is not homotopic to \( \emptyset \) by Theorem 2.7. (Note that we assume \( a \neq \tau(b) \) and \( a' \neq \tau(b') \) in this case). \( P_{a,b}^{0,0} \neq P_{a',b'}^{2,2I} \) follows from Lemma 4.3. Indeed a nanoword ABBA with \( |A| = a' \), \( |B| = b' \) is homotopic to \( \emptyset \). On the other hand, a nanoword ABAB with \( |A| = a, |B| = b \) with \( a \neq \tau(b) \) is not homotopic to \( \emptyset \). Suppose that \( P_{a,b}^{2,0} \ll P_{a',b'}^{2,2I} \). Then \( \gamma(P_{a,b}^{2,0}) = \gamma(P_{a',b'}^{2,2I}) \). \( \gamma(P_{a,b}^{2,0}) = z_{a}z_{b}z_{\tau(a)}z_{\tau(b)} \) and \( \gamma(P_{a',b'}^{2,2I}) = z_{a'}z_{b'}z_{\tau(a')}z_{\tau(b')} \). So \( z_{\tau(a')}z_{\tau(b')} = 1 \). This implies \( a' = \tau(b') \). But this contradicts to \( a' \neq \tau(b') \). Therefore \( P_{a,b}^{2,0} \neq P_{a',b'}^{2,2I} \). \( P_{a,b}^{0,0} \sim P_{a',b'}^{0,4} \) only if \((a,b) = (a',b') \) follows from Lemma 4.3 and Theorem 2.7.

Consider the nanophrase \( P_{a,b}^{3,1}, P_{a,b}^{3,1} \neq (\emptyset) \) follows from Lemma 4.5. \( P_{a,b}^{3,1} \neq P_{a',b'}^{2,2I} \) is proved later. \( P_{a,b}^{3,1} \neq P_{a',b'}^{2,2I} \) and \( P_{a,b}^{3,1} \neq P_{a',b'}^{1,3} \) follows by Lemma 4.5. \( P_{a,b}^{3,1} \neq P_{a',b'}^{1,3} \) is proved later. \( P_{a,b}^{3,1} \neq P_{a',b'}^{0,4} \) follows from Lemma 4.5. Suppose \( P_{a,b}^{3,1} \sim P_{a',b'}^{1,3} \). If \( a \neq \tau(b) \), then \((a,b) = (a',b') \) by Theorem 2.7. If \( a = \tau(b) \), then \( a' = \tau(b) \) by Theorem 2.7 and Lemma 4.3. So \( \gamma(P_{a,b}^{3,1}) = z_{a}z_{b}z_{\tau(a)}z_{\tau(b)} = z_{\tau(a)}z_{\tau(b)} \) and \( \gamma(P_{a,b}^{3,1}) = z_{a}z_{b}z_{\tau(a')}z_{\tau(b')} = z_{\tau(a')}z_{\tau(b')} \). This implies \( z_{\tau(a')}z_{\tau(a)} = z_{\tau(b')}z_{\tau(b)} \). Therefore \((a,b) = (a',b') \).

Consider the nanophrase \( P_{a,b}^{2,2I}, P_{a,b}^{2,2I} \neq (\emptyset) \) and \( P_{a,b}^{2,2I} \neq P_{a',b'}^{3,1} \) follows from Lemma 4.3. \( P_{a,b}^{2,2I} \neq P_{a',b'}^{2,2I} \) follows from Lemma 4.3. \( P_{a,b}^{2,2I} \neq P_{a',b'}^{1,3} \) follows from
Lemma 4.5. Suppose $P_{a,b}^{2,2I} \simeq P_{a',b'}^{0,4}$. Then $\gamma(P_{a,b}^{2,2I}) = \gamma(P_{a',b'}^{0,4})$. This implies $z_a z_b = 1$. This is possible only if $a = \tau(b)$. But this contradicts to assumption. So $P_{a,b}^{2,2I} \neq P_{a',b'}^{0,4}$.

Consider the nanophrase $P_{a,b}^{2,2I}$. Suppose $P_{a,b}^{2,2I} \simeq (\emptyset)$. Then $\gamma(P_{a,b}^{2,2I}) = \gamma((\emptyset \emptyset)) = 1 \otimes 1$. This implies $z_a z_b = 1$. So $a = \tau(b)$. But this contradicts to $a \neq \tau(b)$. Therefore $P_{a,b}^{2,2I} \neq (\emptyset)$.

Proof of Lemma 4.5. Suppose $P_{a,b}^{2,2I}$. Then $\gamma(P_{a,b}^{2,2I}) = \gamma(P_{a',b'}^{0,4})$. This implies $z_a z_b = 1$. This is possible only if $a = \tau(b)$. But this contradicts to assumption. So $P_{a,b}^{2,2I} \neq P_{a',b'}^{0,4}$. But this contradicts to assumption. So $P_{a,b}^{2,2I} \neq P_{a',b'}^{0,4}$.

Consider the nanophrase $P_{a,b}^{0,4}$. Suppose $P_{a,b}^{0,4} \neq (\emptyset)$. Then $\gamma(P_{a,b}^{0,4}) = \gamma(P_{a',b'}^{0,4})$. This implies $a = a'$ and $b = b'$. This is possible only if either $a = a'$ and $b = b'$ or $a = \tau(b)$ and $a' = \tau(b')$. The latter case contradicts to assumption. So $a \neq \tau(b)$. So $(a, b) = (a', b')$.

Consider the nanophrase $P_{a,b}^{0,4}$. Suppose $P_{a,b}^{0,4} \neq (\emptyset)$. Then $\gamma(P_{a,b}^{0,4}) = \gamma(P_{a',b'}^{0,4})$. This implies $a = a'$ and $b = b'$. This is possible only if either $a = a'$ and $b = b'$ or $a = \tau(b)$ and $a' = \tau(b')$. The latter case contradicts to assumption. So $a \neq \tau(b)$. So $(a, b) = (a', b')$.

Consider the nanophrase $P_{a,b}^{0,4}$. Suppose $P_{a,b}^{0,4} \neq (\emptyset)$. Then $\gamma(P_{a,b}^{0,4}) = \gamma(P_{a',b'}^{0,4})$. This implies $a = a'$ and $b = b'$. This is possible only if either $a = a'$ and $b = b'$ or $a = \tau(b)$ and $a' = \tau(b')$. The latter case contradicts to assumption. So $a \neq \tau(b)$. So $(a, b) = (a', b')$. However, $P_{a,b}^{0,4} \neq P_{a',b'}^{0,4}$.

Suppose $P_{a,b}^{0,4} \neq P_{a',b'}^{0,4}$. $\gamma(P_{a,b}^{0,4}) = z_a z_b \tau(a) \otimes \tau(b)$ and $\gamma(P_{a',b'}^{0,4}) = z_{a'} \otimes z_{\tau(a')}$. This implies $a = b$. Moreover $a = \tau(b)$ by Lemma 4.3. So if $a \neq b$, then $P_{a,b}^{0,4} \simeq P_{a,b}^{0,4}$ as nanophrases over $\alpha$. However,

\[ T(P_{a,b}^{0,4}) = (T_{P_{a,b}^{0,1}}(ABA), T_{P_{a,b}^{0,1}}(B)) = (\varepsilon(A), 0) = (-1, 0), \]

and

\[ T(P_a) = (0, 0). \]

This contradicts to homotopy invariance of $T$. If $a = b$, then $P_{a,a}^{0,1} \simeq P_{a}^{0,1}$ as nanophrases over $\alpha = \{a\}$. However

\[ T(P_{a,a}^{0,1}) = (1, 0) \in (\mathbb{Z}/2\mathbb{Z})^2, \]

\[ T(P_a) = (0, 0) \in (\mathbb{Z}/2\mathbb{Z})^2. \]

This contradicts to homotopy invariance of $T$. Therefore $P_{a,b}^{0,4} \neq P_{a'}^{0,4}$.

$P_{a,b}^{0,4} \neq P_{a'}^{0,4}$ is proved similarly to above.

Suppose $P_{a,b}^{0,4} \simeq P_{a',b'}^{0,4}$. If $a \neq \tau(b)$, then $a' \neq \tau(b')$ and $(a, b) = (a', b')$ by Lemma 4.3. Moreover $\gamma(P_{a,b}^{0,4}) = \gamma(P_{a',b'}^{0,4})$ implies $z_a z_b \tau(a) = z_{a'} z_{\tau(a')}. This
$z_b z_{\tau(a)} = 1$ and this is possible only if $a = b$. Therefore $P_{a,a}^{3,1} \simeq P_{a,a}^{1,3}$ as nanowords over $\alpha_0 = \{a, \tau(a)\}$ by Lemma 4.2. However,

$$T(P_{a,a}^{3,1}) = (T_{P_{a,a}^{3,1}}(ABA), T_{P_{a,a}^{3,1}}(B))$$

$$= (\varepsilon(A)\sigma_{P_{a,a}^{3,1}}(A, B), 0)$$

$$= (1, 0),$$

and

$$T(P_{a,a}^{1,3}) = (T_{P_{a,a}^{1,3}}(A), T_{P_{a,a}^{1,3}}(BAB))$$

$$= (0, \varepsilon(B)\sigma_{P_{a,a}^{1,3}}(B, A))$$

$$= (0, -1).$$

This contradicts to homotopy invariance of $T$. If $a = \tau(b)$, then $a' = \tau(b')$ by Lemma 4.3. Moreover $\gamma(P_{a,b}^{1,3}) = \gamma(P_{a',b'}^{1,3})$. This implies $z_{\tau(a)} = z_{a'}$ and $z_{\tau(b)} = z_{b'}$. So $a = \tau(a')$ and $b = \tau(b')$. If $a = \tau(a)$, then $a = a' = b = b'$ by above equations. Therefore $P_{a,a}^{3,1} \simeq P_{a,a}^{1,3}$ as nanowords over $\alpha_0 = \{a\}$. However,

$$T(P_{a,a}^{3,1}) = (1, 0) \in (\mathbb{Z}/2\mathbb{Z})^2,$$

$$T(P_{a,a}^{1,3}) = (0, 1) \in (\mathbb{Z}/2\mathbb{Z})^2.$$

This contradicts to homotopy invariance of $T$. If $a \neq \tau(a)$, then $P_{a,b}^{3,1} \simeq P_{b,a}^{1,3}$ as nanowords over $\alpha_0$. However,

$$T(P_{a,b}^{3,1}) = (\varepsilon(A)\sigma_{P_{a,b}^{3,1}}(A, B), 0) = (-1, 0)$$

$$T(P_{b,a}^{1,3}) = (0, \varepsilon(B)\sigma_{P_{b,a}^{1,3}}(B, A)) = (0, 1).$$

This contradicts to homotopy invariance of $T$. Therefore $P_{a,b}^{3,1} \not\simeq P_{a',b'}^{1,3}$.

Now we have completed the homotopy classification of nanowords of length 2 with 4 letters. \qed

References