A Remark on the Discrete Deterministic Game Approach for Curvature Flow Equations

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Abstract. This paper constructs a family of discrete two-person games, whose values converge to the unique viscosity solution of a general curvature flow equation in dimension two. We summarize all of the techniques needed for such second-order games. We introduce barrier games, which can be considered as a combination of the classical barrier argument and game perspectives.

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1 Introduction

As for applications of viscosity solution theory, it is known that deterministic differential games are always connected with first-order Hamilton-Jacobi-Isaacs equations (see [1]), while geometric flows are usually formulated through the level set method by second-order degenerate parabolic equations (see [6, 8] or [10]). However, Kohn and Serfaty [12] initialized a deterministic game-theoretic approach to explore second-order geometric flows, giving a lot of new ideas in understanding motion by curvature. They constructed a family of discrete games whose values converge to the unique solution of the mean curvature flow equation.

Such an approximation seems unexpected at first glance, since only stochastic games are supposed to give rise to a second-order equation. The novelty could actually be attributed to the singularity in the limit taking. When tracing it back, we find that Brezis and Pazy [3] first proved the nonlinear Chernoff formula for nonlinear semigroups and Evans [7] recast and developed the convergence theorem in the viscosity sense for the BMO algorithm [1] of motion by curvature; see also a paper [11] for recent development of the BMO scheme. A discovery was then made by Catté, Dibos and Koepfler [5] that a certain proper "inf sup" operator satisfies all of the requirements in [7] and can be applied in image processing. Consult also [14] for generalization in this direction. Hence the scheme [12] may be considered as a successful and reasonable application of the "inf sup" operators or, more generally, morphological operators (see [4]), since such kinds of structures often appear in dynamic programming for classical differential games.

We are concerned with the question posed in [12] how widely this game-theoretic approach can be applied. There might be several ways, among which we here choose to give an answer based on a general but standard game setting. More precisely, we establish rules for a family of discrete games of finite horizon and set the objective functional to be of the Bolza type. Our game values are proved to converge to the solution of the following two-dimensional Cauchy problem:

$$-u_t + H(x, \nabla u, \nabla^2 u) = 0 \quad \text{in } \mathbb{R}^2 \times (0, T), \tag{1.1a}$$

$$u(x, T) = u_0(x) \qquad \text{in } \mathbb{R}^2, \tag{1.1b}$$

where H is of some particular form and meets a list of assumptions. We discuss more details in Section 2.

Roughly speaking, two aspects lead to second-order deterministic games. One is to use a scaled clock. Namely, we let the time step size of each game be the square of its space step size, which causes the term u_t to coexist with second-order space derivatives of u. The other is a null condition which plays a role of singularizing the first-order terms of game values. It allows, from the viewpoint of games, both parties to sacrifice their directional controls so as to avoid more severe loss. There is certainly no way to realize the condition for the one-dimensional case.

Expressed explicitly as a second-order Hamilton-Jacobi equation with singularity, (1.1a) is not so typical, so its comparison principle is stated in Section 3. At last we briefly prove our main theorem in Section 4. Our main tool is still the dynamic programming principle as usual but we shall place more emphasis on a barrier argument, for which a generalized "continuity in discrete time" of value functions is shown. The argument essentially reflects the properties of morphological operators. It is worth mentioning that one may modify our game setting into a time-optimal problem, trying to obtain a game interpretation for an elliptic equation. However, the indispensable comparison theorem is not always clear, especially when the domain is not star-shaped.

The main result of this paper is included in [13].

2 Discrete Deterministic Games

Consider two compact *control sets* A, B and a discrete system in $\mathbb{R}^2$,

$$y(k+1) = y(k) + \sqrt{2}\varepsilon f(y(k), a_k, b_k) + \varepsilon^2 g(y(k), a_k, b_k) \tag{2.1}$$

with an initial datum

$$y(0) = x \in \mathbb{R}^2.$$

Here $a_k \in A$, $b_k \in B$ and f, g: $\mathbb{R}^2 \times A \times B \to \mathbb{R}^2$ are involved in the different order terms of step size ε. Accordingly, we obtain the N*-step state*, denoted by $y(N; \vec{a}, \vec{b})$, governed by the system (2.1) with inputs $\vec{a} \in A^N$ and $\vec{b} \in B^N$.

We also assume there is a *maturity time*, denoted by T, so that an *objective functional* can be introduced for any $(x, t) \in \mathbb{R}^2 \times [0, T]$:

$$J^\varepsilon(x, t, \vec{a}, \vec{b}) := u_0(y(N^\varepsilon)) + \varepsilon^2 \sum_{k=1}^{N^\varepsilon} l(y(k), a_k, b_k).$$

Here the functions $u_0 : \mathbb{R}^2 \to \mathbb{R}$ and $l : \mathbb{R}^2 \times A \times B \to \mathbb{R}$ denote respectively the *terminal* and *running costs*. The integer N^ε stands for the total of game steps such that $t+N^\varepsilon\varepsilon^2$ converges to T as $\varepsilon \to 0$. We hereafter rewrite N^ε as N and let it just be the integer part of $\frac{T-t}{\varepsilon^2}$. We then also call such a quantity $T^\varepsilon := t+N^\varepsilon\varepsilon^2$ the *actual maturity time* for t. As a result, $J^\varepsilon(x, t, \vec{a}, \vec{b}) = u_0(x)$ provided $T - \varepsilon^2 \leq t \leq T$. We summarize this part of explanation by saying that each step costs time ε^2.

About the functions f, g and l here, we shall impose several assumptions on them:

$$
\text{(A1)} \qquad \begin{cases} f(x,a,b) \text{ and } g(x,a,b) \text{ are uniformly bounded on } \mathbb{R}^2 \times A \times B, \\ f(x,a,b),\ g(x,a,b) \text{ and } l(x,a,b) \text{ are Lipschitz continuous in } x, \\ \text{i.e., } \exists L > 0 \text{ such that for all } a \in A,\ b \in B \text{ and } x_1, x_2 \in \mathbb{R}^n, \\ \quad |f(x_1,a,b) - f(x_2,a,b)| + |g(x_1,a,b) - g(x_2,a,b)| \\ \quad + |l(x_1,a,b) - l(x_2,a,b)| \leq L|x_1 - x_2|. \end{cases}
$$

Moreover, we assume

$$
\text{(A2)} \qquad A = A_1 \times A_2, \quad B = B_1 \times B_2,
$$

where A_1, A_2, B_1 and B_2 are all compact sets. In addition, f, g and l are assumed to possess special structures:
(A3)

$$
\begin{cases} \text{There exists a function } v_f : A_2 \times B_2 \to S^1 = \{v \in \mathbb{R}^2 : |v| = 1\} \text{ such that for} \\ \text{each } a = (a^1, a^2) \in A \text{ and } b = (b^1, b^2) \in B,\ f(x,a,b) = |f(x,a^1,b^1)| v_f(x,a^2,b^2) \end{cases}
$$

and

$$
\text{(A4)} \qquad \begin{cases} g(x,a,b) = g(x,a^1,b^1) \text{ and } l(x,a,b) = l(x,a^1,b^1) \\ \text{for all } a = (a^1,a^2) \in A \text{ and } b = (b^1,b^2) \in B. \end{cases}
$$

That is to say, each controller of f have in fact two independent sub-controllers, one managing its magnitude and the other steering its direction. In contrast, g and l are immune to the impact of those sub-controllers related to f's vectorial part.

Two persons play a game for the duration from t to T with opposite aims about the objective functional. One is a minimizing player, called Paul, who has a command of the control A, while Carol, the other one, targets to maximize $J^\varepsilon(x,t)$ by exerting influence on the system through B. Yet another problem of importance is about the information pattern. Naturally, both players have perfect information concerning their past, but neither of them knows the future behavior of his opponent. In addition, we assume Carol is advantageous in information, that is, for each round she can observe Paul's choice before she makes her own decision.

The preceding description being followed, an *upper value function* is well defined by

$$u^\varepsilon(x,t) = \sup_{\beta\in\Theta^N} \inf_{\vec{a}\in A^N} J^\varepsilon(x,t), \tag{2.2}$$

where Θ^N stands for the set of *non-anticipating strategies* of Carol. More specifically, an element β of Θ^N is a map carrying every $\vec{a} = (a_1,\dots,a_N) \in A^N$ to $\beta[\vec{a}] = \vec{b} = (b_1,\dots,b_N) \in B^N$ such that for arbitrary $1 \le k \le N$, $\beta, \underline{\beta} \in \Theta^N$ and $\vec{a}, \underline{\vec{a}} \in A^N$, $a_j = \underline{a}_j$ for all $j = 1,2,\dots,k$ implies $b_k = \underline{b}_k$. We remark that the notation in (2.2), followed from [1], should be understood as

$$u^\varepsilon(x,t) = \inf_{a_1\in A} \sup_{b_1\in B} \dots \inf_{a_N\in A} \sup_{b_N\in B} J^\varepsilon(x,t), \tag{2.3}$$

which is used by Friedman [9] to define discrete game values. Equation (2.3) specifies the notion more accurately but less concisely.

As usual, we can obtain the discrete dynamic programming equation,

$$u^\varepsilon(x,t) = \inf_{a\in A} \sup_{b\in B} \{u^\varepsilon(x + \sqrt{2}\varepsilon f(x,a,b) + \varepsilon^2 g(x,a,b), t+\varepsilon^2) + \varepsilon^2 l(x,a,b)\}, \tag{2.4}$$

for which Taylor expansion up to the second order formally yields

$$\begin{aligned} 0 \approx \inf_{a\in A} \sup_{b\in B} \big\{ &\sqrt{2}\varepsilon\langle \nabla u^\varepsilon, f(x,a,b)\rangle + \varepsilon^2 \left\langle D^2u^\varepsilon f(x,a,b), f(x,a,b)\right\rangle \\ &+ \varepsilon^2\partial_t u^\varepsilon + \varepsilon^2\langle \nabla u^\varepsilon, g(x,a,b)\rangle + \varepsilon^2 l(x,a,b)\big\}. \end{aligned} \tag{2.5}$$

Recalling the setting (A2) and inserting the next null condition,

$$\inf_{a_2\in A_2} \sup_{b_2\in B_2} \langle p, v_f(x,a_2,b_2)\rangle = 0, \quad \text{for } \forall p \in S^1 \text{ and } x \in \mathbb{R}^2, \tag{NC}$$

we guarantee when ε is sufficiently small,

$$\inf_{a\in A} \sup_{b\in B} \langle \nabla u^\varepsilon, f(x,a,b)\rangle = 0, \tag{2.6}$$

and meanwhile

$$\left\langle v_f(x,a_2,b_2), \frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|} \right\rangle = 0. \tag{2.7}$$

We now plug (2.7) into (2.5), take into account that $J^\varepsilon(x,T) = u_0(x)$, and let ε go to zero, then the limit function u of u^ε, if exists, should satisfy (1.1a) and (1.1b) where $H : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbf{S}^2 \to \mathbb{R}$ is a function such that

(H)

$$\begin{aligned} H(x,p,X) = -\inf_{a_1\in A_1}\sup_{b_1\in B_1} \Bigg\{ &\langle p, g(x,a_1,b_1)\rangle \\ &+ |f(x,a_1,b_1)|^2 \mathrm{tr}\left((I - \frac{p\otimes p}{|p|^2})X \right) + l(x,a_1,b_1) \Bigg\}. \end{aligned}$$

Here and in the sequel $\mathbf{S}^n$ denotes the set of all real symmetric $n \times n$ matrices.

All of the above can be realized in a rigorous manner.

Theorem 2.1. *Assume* (A1)-(A4). *Let u^ε be the game value introduced as above and the condition* (NC) *hold, and if u_0 is continuous and constant outside some compactum in $\mathbb{R}^2$, then u^ε converges uniformly on every compact set of $\mathbb{R}^2 \times (0,T]$ to the unique solution of* (1.1a), (1.1b) *with H given in* (H).

Remark 2.1. In particular, when we explicitly take

$$\begin{aligned} &A_1 = B_1 = \emptyset, A_2 = S^1,\ B_2 = \{\pm 1\}, \\ &f(x,a,b) = b_2 a_2,\ g(x,a,b) \equiv 0 \text{ and } l(x,a,b) \equiv 0, \end{aligned}$$

(1.1a) is reduced to backward mean curvature flow, and the condition (NC) turns out to be self-evident:

$$\inf_{v\in S^1}\sup_{b=\pm 1} b\langle p, v\rangle = 0, \quad \text{for } \forall p \in S^1.$$

It indicates that the minimizing player Paul has better controllability, though Carol can switch the sign freely. This superiority well compensates Paul's deficiency in information and consequently, with rescaled time, the antagonism between them escalates into a higher level, justified mathematically as a second-order PDE.

Remark 2.2. Our equation (1.1a) together with (H) is general. Besides the mean curvature flow mentioned above, one might let either A_1 or B_1 be empty and employ the Fenchel transform to derive "motion by a convex or concave function of curvature" including the "positive curvature flow" or "negative curvature flow" backward in time.

Remark 2.3. The assumption that u_0 is constant outside some compactum can be weakened. It is sufficient to assume that u_0 is bounded and uniformly continuous in $\mathbb{R}^2$, i.e., $u_0 \in BUC(\mathbb{R}^2)$.

3 The Limit Equation and Comparison Principle

The equation (1.1a) actually gathers several common difficulties in the proofs of comparison theorems for nonlinear second-order parabolic PDEs:

(1) The region is unbounded;

(2) The equation is singular;

(3) The equation is spatially inhomogeneous up to the highest order. Namely, the first and second order terms of H depend on the spatial variable x.

In spite of them, we could still follow [10] and define viscosity solutions for (1.1a). Set $Q := \mathbb{R}^n \times (0, T)$.

Definition 3.1. Assume that (A1)-(A4) hold for H.

1. An upper semicontinuous function u is called a *subsolution* of (1.1a) if whenever there are $(\hat{x}, \hat{t}) \in Q$, a neighborhood $\mathcal{O}$ of $(\hat{x}, \hat{t})$ and a smooth function ϕ on $\mathcal{O}$ such that

$$\max_{\mathcal{O}}(u - \phi) = (u - \phi)(\hat{x}, \hat{t}),$$

we have

$$-\phi_t(\hat{x}, \hat{t}) + H(\hat{x}, \nabla\phi(\hat{x}, \hat{t}), \nabla^2\phi(\hat{x}, \hat{t})) \leq 0$$

if $\nabla\phi(\hat{x}, \hat{t}) \neq 0$ and

$$-\phi_t(\hat{x}, \hat{t}) - \inf_{a_1 \in A_1} \sup_{b_1 \in B_1} l(\hat{x}, a_1, b_1) \leq 0$$

if $\nabla\phi(\hat{x}, \hat{t}) = 0$ and $\nabla^2\phi(\hat{x}, \hat{t}) = 0$.

2. A lower semicontinuous function u is called a *supersolution* of (1.1a) if whenever there are $(\hat{x}, \hat{t}) \in Q$, a neighborhood $\mathcal{O}$ of $(\hat{x}, \hat{t})$ and a smooth function ϕ on $\mathcal{O}$ such that

$$\min_{\mathcal{O}}(u - \phi) = (u - \phi)(\hat{x}, \hat{t}),$$

we have

$$-\phi_t(\hat{x}, \hat{t}) + H(\hat{x}, \nabla\phi(\hat{x}, \hat{t}), \nabla^2\phi(\hat{x}, \hat{t})) \geq 0$$

if $\nabla\phi(\hat{x}, \hat{t}) \neq 0$ and

$$-\phi_t(\hat{x}, \hat{t}) - \inf_{a_1 \in A_1} \sup_{b_1 \in B_1} l(\hat{x}, a_1, b_1) \geq 0$$

if $\nabla\phi(\hat{x}, \hat{t}) = 0$ and $\nabla^2\phi(\hat{x}, \hat{t}) = 0$.

3. A function u is a *solution* of (1.1a) if it is both a subsolution and a supersolution.

We can prove a comparison theorem for (1.1a), (1.1b) to show the uniqueness of solutions. Let us treat the n-dimensional case and a more general Hamiltonian defined on $\mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}) \times \mathbf{S}^n$ fulfilling a list of hypotheses. One may verify that (H) under the assumptions (A1)-(A4) is its special case. By transforming with respect to time variable t in (1.1a), we virtually need to study

$$u_t + F(x, \nabla u, \nabla^2 u) = 0 \quad \text{in } \mathbb{R}^n \times (0, T), \tag{3.1a}$$

$$u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}^n. \tag{3.1b}$$

We assume F satisfies

$$F : \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}) \times \mathbf{S}^n \to \mathbb{R} \text{ is continuous;} \tag{F1}$$

$$\begin{cases} F(x, p, X) \leq F(x, p, Y), \\ \text{for all } X \geq Y,\ X, Y \in \mathbf{S}^n,\ x \in \mathbb{R}^n \text{ and } p \in \mathbb{R}^n \backslash \{0\}; \end{cases} \tag{F2}$$

$$\begin{cases} \displaystyle\lim_{\substack{|p_1 - p_2| \to 0 \\ |X_1 - X_2| \to 0}} \sup_{x \in \mathbb{R}^n} |F(x, p_1, X_1) - F(x, p_2, X_2)| = 0, \\ \text{and there exists a function } G(x) \text{ such that} \\ \displaystyle\lim_{\substack{p \to 0 \\ X \to 0}} \sup_{x \in \mathbb{R}^n} |F(x, p, X) - G(x)| = 0, \end{cases} \tag{F3}$$

(F4) $$\begin{cases} F(x,p,X+\sigma p\otimes p)=F(x,p,X), \\ \text{for all } \sigma\in\mathbb{R},\ x\in\mathbb{R}^n,\ p\in\mathbb{R}^n\backslash\{0\} \text{ and } X\in\mathbf{S}^n. \end{cases}$$

(F5) $$\begin{cases} \exists C>0 \text{ such that} \\ |G(x_1)-G(x_2)|\le C|x_1-x_2| \quad \text{for all } x_1,x_2\in\mathbb{R}^n. \end{cases}$$

(F6)
$$\begin{cases} \text{For any } R>0, \text{ there is a modulus } \omega_R \text{ such that} \\ \quad F(x_2,p,X_2)-F(x_1,p,X_1)\le\omega_R(|x_1-x_2|(|p|+1)+\eta|x_1-x_2|^2) \\ \text{for } x_1,x_2\in\mathbb{R}^n,\ p\in\mathbb{R}^n\backslash\{0\},\ X_1,X_2\in\mathbf{S}^n \text{ and } \eta>0 \text{ whenever} \\ \quad -3\eta I\le\begin{pmatrix} X_1 & O \\ O & -X_2\end{pmatrix}\le 3\eta J,\ J=\begin{pmatrix} I & -I \\ -I & I\end{pmatrix}. \end{cases}$$

We remark that (F4) is implied by the well-known geometricity but more general in that we do not require positive homogeneity in (p,X) here.

Theorem 3.1 (Comparison Principle)**.** *Assume* (F1)-(F6) *hold. Let u and v be sub- and supersolutions of* (3.1a)*. If*

$$\lim_{\delta\to 0}\sup\{u(x,t)-v(y,s):|x-y|+|t-s|+t+s\le\delta, \\ (x,t),(y,s)\in\mathbb{R}^n\times[0,T']\}\le 0 \tag{3.2}$$

for every $T'\in(0,T)$, then $u(x,t)\le v(x,t)$ in Q.

For its proof, one may refer to Theorem 3.6.4 in [10] for the case $G\equiv 0$, or to [13] for the general case.

4 Convergence of the Game Values

We now turn to the proof of Theorem 2.1. Let us first introduce the *upper* and *lower relaxed limits* of u^ε as

$$\overline{u}(x,t):=\limsup_{\varepsilon\to 0}{}^{*}u^\varepsilon(x,t)=\lim_{\delta\to 0}\sup\{u^\varepsilon(y,s):\varepsilon<\delta,\ |x-y|+|t-s|<\delta\}$$

and

$$\underline{u}(x,t):=\liminf_{\varepsilon\to 0}{}_{*}u^\varepsilon(x,t)=\lim_{\delta\to 0}\inf\{u^\varepsilon(y,s):\varepsilon<\delta,\ |x-y|+|t-s|<\delta\},$$

The proof then comprises three parts:

Step 1. Verify that $\underline{u}$ and $\overline{u}$ have a terminal comparison result of the type (3.2);

Step 2. Show the lower relaxed limit $\underline{u}$ of game values u^ε is a supersolution of (1.1a);

Step 3. Show the upper relaxed limit $\overline{u}$ of game values u^ε is a subsolution of (1.1a).

It is quite clear that in light of the three steps above, Theorem 2.1 is a consequence of Theorem 3.1. Before carrying out our strategy, we first prepare some lemmas for later use.

Lemma 4.1. *Suppose ξ is a unit vector in $\mathbb{R}^2$ and X is a real symmetric 2×2 matrix, then there exists a constant $M > 0$ that depends only on the norm of X, such that for any unit vector $v \in \mathbb{R}^2$,*

$$|\langle X\xi^\perp, \xi^\perp\rangle - \langle Xv, v\rangle| \leq M|\langle \xi, v\rangle|, \tag{4.1}$$

where $\xi^\perp$ denotes a unit orthonormal vector of ξ.

We skip the simple proof of Lemma 4.1 and concentrate on a more important property about the value functions. It is extended from the so-called "equicontinuous in discrete time" proposed in [12].

Lemma 4.2. *Assume* (A1)-(A4) *hold. Assume u_0 is continuous and constant outside a compact set in $\mathbb{R}^2$. Let $\{u^\varepsilon\}_{\varepsilon>0}$ be the associated game values. Then there exists a constant $L_\lambda > 0$ for every $\lambda > 0$ and a modulus ω such that*

$$|u^\varepsilon(x,t) - u^\varepsilon(y,s)| \leq \lambda + \omega(|x-y|) + L_\lambda|t-s|. \tag{4.2}$$

for all $\varepsilon > 0$, $x, y \in \mathbb{R}^2$ and $t, s \in [0,T]$ satisfying $|t-s| = k\varepsilon^2$ for some $k \in \mathbb{Z}$.

Let us see how to prove this relation. Since f, g, l and u_0 are all bounded and uniformly continuous in x, it is not hard to prove based on the definition of u^ε that there is a modulus ω such that for any $t \in [0,T]$,

$$|u^\varepsilon(x,t) - u^\varepsilon(y,t)| \leq \omega(|x-y|). \tag{4.3}$$

In consequence, it is sufficient to look into the continuity in time t. We first observe for any $x \in \mathbb{R}^2$ and $0 \leq s \leq t \leq T$ such that $t - s = k\varepsilon^2$, the following inequality holds:

$$|u^\varepsilon(x,t) - u^\varepsilon(x,s)| \leq \sup_{z\in\mathbb{R}^2}\{|u_0(z) - u^\varepsilon(z, T^\varepsilon - k\varepsilon^2)|\}, \tag{4.4}$$

where T^ε is the actual maturity time for t. It says that comparison between two game values could be postponed whenever they share the same starting point. A special situation of $l \equiv 0$ and $k = 1$ is presented in the next lemma. Our argument could certainly be generalized so as to obtain (4.4) rigorously.

Lemma 4.3. *Assume* $(x, t) \in \mathbb{R}^2 \times [\varepsilon^2, T]$, *then*

(i) $u^\varepsilon(x, t - \varepsilon^2) - u^\varepsilon(x, t) \leq \sup_{z \in \mathbb{R}^2} \{u^\varepsilon(z, T^\varepsilon - \varepsilon^2) - u_0(z)\}$;

(ii) $u^\varepsilon(x, t - \varepsilon^2) - u^\varepsilon(x, t) \geq \inf_{z \in \mathbb{R}^2} \{u^\varepsilon(z, T^\varepsilon - \varepsilon^2) - u_0(z)\}$.

Proof. Let us only prove the inequality (ii) for example. A symmetric argument will lead to (i). What we use here are the definition of u^ε in (2.2) and a multi-step version of dynamic programming principle:

$$u^\varepsilon(x, t - \varepsilon^2) = \sup_{\beta \in \Theta^N} \inf_{\vec{a} \in A^N} u^\varepsilon(y(N; \vec{a}, \beta[\vec{a}]), T^\varepsilon - \varepsilon^2). \tag{4.5}$$

From (2.2), for any $h > 0$, there is $\beta_0 \in \Theta^N$ such that

$$u^\varepsilon(x, t) \leq \inf_{\vec{a} \in A^N} u^\varepsilon(y(N; \vec{a}, \beta_0[\vec{a}]), T^\varepsilon) + h. \tag{4.6}$$

On the other hand, we can take a certain $\vec{a}_0$ satisfying

$$u^\varepsilon(x, t - \varepsilon^2) \geq u^\varepsilon(y(N; \vec{a}_0, \beta_0[\vec{a}_0]), T^\varepsilon - \varepsilon^2) - h. \tag{4.7}$$

Combining (4.6) and (4.7), we get

$$\begin{aligned} &u^\varepsilon(x, t) - u^\varepsilon(x, t - \varepsilon^2) \\ &\qquad \leq u_0(y(N; \vec{a}_0, \beta_0[\vec{a}_0])) - u^\varepsilon(y(N; \vec{a}_0, \beta_0[\vec{a}_0]), T^\varepsilon - \varepsilon^2) + 2h. \end{aligned}$$

We therefore have, by taking the supremum and sending $h \to 0$,

$$u^\varepsilon(x, t) - u^\varepsilon(x, t - \varepsilon^2) \leq \sup_{z \in \mathbb{R}^2} \{u_0(z) - u^\varepsilon(z, T^\varepsilon - \varepsilon^2)\}.$$

□

The proof of Lemma 4.2 will be completed via a barrier argument at its final stage. One could use the mollifier to construct with ease an uppper barrier as follows.

Lemma 4.4. *Let $u_0 \in BUC(\mathbb{R}^2)$. Then for any $\lambda > 0$, there exists a function $V_\lambda^+ \in C^2(\mathbb{R}^2)$ and a constant C_λ such that*

(i) $u_0(x) \le V_\lambda^+(x) \le u_0(x) + \lambda$, *for all* $x \in \mathbb{R}^2$.

(ii) $\sup_{x\in\mathbb{R}^2}\{|\nabla V_\lambda^+(x)| + \|\nabla^2 V_\lambda^+(x)\|\} \le C_\lambda$.

Imagine there is a game with the same rules but a different terminal cost V_λ^+ and a zero running cost $l \equiv 0$. Let U^+ denote its game value. A direct comparison yields

$$u^\varepsilon(x,t) \le U^+(x,t) + \|l\|_\infty |T-t|. \tag{4.8}$$

Since $V^+\lambda$ is of class C^2, we may implement critical strategies through the null condition (NC) to make the first-order term vanish in the Taylor expansion of U^+. Such a singularization gives

$$U^+(x,t) \le V_\lambda^+(x) + C_\lambda(\|f\|_\infty^2 + \|g\|_\infty)|T-t|, \tag{4.9}$$

with which we obtain from (4.8) and Lemma 4.4

$$u^\varepsilon(x,t) \le \lambda + u_0(x) + L_\lambda|T-t|,$$

where $L_\lambda := \|l\|_\infty + C_\lambda(\|f\|_\infty^2 + \|g\|_\infty)$. Also, we may construct a lower barrier game to get a bound from below and thus

$$|u^\varepsilon(x,t) - u_0(x)| \le \lambda + L_\lambda|T-t| \quad \text{for all } x \in \mathbb{R}^2. \tag{4.10}$$

Gathering (4.3), (4.4) and (4.10), we get (4.2) and end our proof of Lemma 4.2.

Remark 4.1. We notice that our barrier argument only requires u_0 to be bounded and uniformly continuous. Consequently we are able to relax the assumption in Lemma 4.2 for a general u_0.

Remark 4.2. As was mentioned, (4.2) is a variant of the continuity of the value functions in classical Optimal Control and Game theory. Indeed, when u_0 is additionally of class C^2, (4.2) reduces to

$$|u^\varepsilon(x,t) - u^\varepsilon(y,s)| \le \omega(|x-y|) + L|t-s|.$$

Then there is no need to put a constant λ any longer because the best barrier for u_0 will be itself.

We will see that Lemma 4.2 actually implies Step 1 for the proof of Theorem 2.1.

Proof of Step 1. Since the right hand side of (4.2) is independent of ε, for any fixed $t \in [0, T]$, set $s = T^\varepsilon$ in (4.2), and then we have

$$\overline{u}(x, t) \le \lambda + u_0(x) + L_\lambda |T - t|.$$

Similarly, one may get

$$\underline{u}(y, s) \ge -\lambda + u_0(y) - L_\lambda |T - s|.$$

These two inequalities are combined to yield

$$\overline{u}(x, t) - \underline{u}(y, s) \le 2\lambda + |u_0(x) - u_0(y)| + L_\lambda(|T - t| + |T - s|).$$

It follows that

$$\lim_{\delta \to 0} \sup\{\overline{u}(x, t) - \underline{u}(y, s) : |x-y| + |t-s| + |T-t| + |T-s| \le \delta, (x, t), (y, s) \in \mathbb{R}^2 \times [0, T]\} \le 2\lambda.$$

Sending $\lambda \downarrow 0$, we are done.

$\square$

We follow [12] for the rest parts of the proof.

Proof of Step 2. We argue by contradiction. Contrary to Definition 3.1, assume there exist $\eta_0 \ge 0$, $(x_0, t_0) \in \mathbb{R}^2$, a δ-neighborhood of (x_0, t_0) and a smooth function ϕ such that for all (x, t) in the δ-neighborhood:

(1) $\underline{u}(x_0, t_0) - \phi(x_0, t_0) < \underline{u}(x, t) - \phi(x, t)$;

(2) $\nabla\phi(x_0, t_0) \ne 0$ and thus $\nabla\phi(x, t) \ne 0$.
(The case $\nabla\phi(x_0, t_0) = 0$ will be explained later.)

(3) $-\phi_t(x, t) + H(x, \nabla\phi(x, t), \nabla^2\phi(x, t)) \le -\eta_0$.

We next find $(x_0^\varepsilon, t_0^\varepsilon)$ such that $(x_0^\varepsilon, t_0^\varepsilon) \to (x_0, t_0)$ and $u^\varepsilon(x_0^\varepsilon, t_0^\varepsilon) \to \underline{u}(x_0, t_0)$ as $\varepsilon \to 0$.

Now let us construct, starting from $(x_0^\varepsilon, t_0^\varepsilon)$, a game path, for which both players' choices will be determined from the dynamic programming equation and null condition (NC). In fact, one can define inductively

$X_1 := (x_0^\varepsilon, t_0^\varepsilon)$,
$X_{k+1} = (x_{k+1}, t_{k+1}) := X_k + (\sqrt{2}\varepsilon f(x_k, a_k^1, a_k^2, b_k^1, b_k^2) + \varepsilon^2 g(x_k, a_k^1, b_k^1), \varepsilon^2)$ for $k \ge 1$,

where $a_k^1 = \bar{a}_k^1$ and $a_k^2 = \bar{a}_k^2$ are picked so as to deduce from (2.4) that

$$u^\varepsilon(X_k) \geq u^\varepsilon(X_{k+1}) + \varepsilon^2 l(x_k, \bar{a}_k^1, b_k^1) - \varepsilon^3. \tag{4.11}$$

We then determine $b_k^2 = \bar{b}_k^2$ to satisfy

$$\left\langle \frac{\nabla\phi(X_k)}{|\nabla\phi(X_k)|}, v_f(x_k, \bar{a}_k^2, \bar{b}_k^2) \right\rangle \geq 0. \tag{4.12}$$

We next expand $\phi(X_{k+1}) - \phi(X_k)$, applying (4.12) and Lemma 4.1, and get

$$\begin{aligned} \phi(X_{k+1}) - \phi(X_k) \geq \varepsilon^2 \Big\{ & \phi_t(X_k) + \langle \nabla\phi(X_k), g(x_k, \bar{a}_k^1, b_k^1) \rangle \\ & + |f(x_k, \bar{a}_k, b_k^1)|^2 \left\langle \nabla^2\phi(X_k) \frac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|}, \frac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|} \right\rangle \Big\} \end{aligned} \tag{4.13}$$

Since b_k^1 are arbitrary, combining (4.11) and (4.13) and applying our assumption (3), we are led to

$$(u^\varepsilon(X_{k+1}) - \phi(X_{k+1})) - (u^\varepsilon(X_k) - \phi(X_k)) \leq -2\eta_0\varepsilon^2 \tag{4.14}$$

whenever ε is small enough and X_k belongs to the δ-neighborhood of (x_0, t_0). Summing (4.14) up, we get

$$(u^\varepsilon(X_{k+1}) - \phi(X_{k+1})) - (u^\varepsilon(X_1) - \phi(X_1)) \leq -2k\eta_0\varepsilon^2. \tag{4.15}$$

Again, this inequality holds only if none of $X_i (1 \leq i \leq k)$ leaves the δ-neighborhood. On the other hand, since ϕ is smooth and u^ε satisfies (4.2), inequality (4.15) also indicates that X_k may get out of the neighborhood when k is faster than the order $\frac{1}{\varepsilon^2}$. We can take an intermediate subsequence to balance the relationship. In other words, it is possible to let k be at some order of $\frac{1}{\varepsilon}$ such that X_k stays in the neighborhood but $X_k \to (x', t') \neq (x_0, t_0)$ as $\varepsilon \to 0$ and $k \to \infty$. Passing the limit in (4.15), we are led to

$$\underline{u}(x', t') - \phi(x', t') \leq \underline{u}(x_0, t_0) - \phi(x_0, t_0),$$

which is apparently a contradiction to our assumption (1).

If $\nabla\phi(x_0) = 0$, we also assume $\nabla^2\phi(x_0) = 0$. We then could replace all $\frac{\nabla\phi}{|\nabla\phi|}$ above by arbitrary unit vectors and our argument is still valid and even easier. $\square$

The proof of Step 3, quite analogous to that of Step 2, is omitted here.

Remark 4.3. Relaxing the assumption on u_0 to $u_0 \in BUC(\mathbb{R}^2)$ makes no difference to our proof of Step 2 and 3. Hence, in terms of Remark 4.1, we may replace the assumption in Theorem 2.1 too.

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