Periodicity of non-central integral arrangements modulo positive integers

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March 2008

Abstract

An integral coefficient matrix determines an integral arrangement of hyperplanes in $\mathbb{R}^m$. After modulo q reduction ($q \in \mathbb{Z}_{>0}$), the same matrix determines an arrangement $\mathcal{A}_q$ of "hyperplanes" in $\mathbb{Z}_q^m$. In the special case of central arrangements, Kamiya, Takemura and Terao [*J. Algebraic Combin.*, to appear] showed that the cardinality of the complement of $\mathcal{A}_q$ in $\mathbb{Z}_q^m$ is a quasi-polynomial in $q \in \mathbb{Z}_{>0}$. Moreover, they proved in the central case that the intersection lattice of $\mathcal{A}_q$ is periodic from some q on. The present paper generalizes these results to the case of non-central arrangements. The paper also studies the arrangement $\hat{\mathcal{B}}_m^{[0,a]}$ of Athanasiadis [*J. Algebraic Combin.* **10** (1999), 207–225] to illustrate our results.

Key words: characteristic quasi-polynomial, elementary divisor, hyperplane arrangement, intersection poset.

1 Introduction

An $m \times n$ integral coefficient matrix $C \in \mathrm{Mat}_{m\times n}(\mathbb{Z})$ and a vector $b \in \mathbb{Z}^n$ of integral constant terms determine an arrangement $\mathcal{A}$ of n hyperplanes H_j, $1 \le j \le n$, in $\mathbb{R}^m$. In the same way, for a positive integer q, the modulo q reductions of C and b determine an arrangement $\mathcal{A}_q = \mathcal{A}_q(C, b)$ of n "hyperplanes" $H_{j,q}$, $1 \le j \le n$, in $\mathbb{Z}_q^m$, where $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$. In the special case $b = 0$, which we call the central case, we showed in [3] that the cardinality of the complement $M(\mathcal{A}_q)$ of this arrangement in $\mathbb{Z}_q^m$ is a quasi-polynomial in $q \in \mathbb{Z}_{>0}$. In [3] we called this quasi-polynomial the *characteristic quasi-polynomial*, and gave an explicit period ρ_0 (called the lcm period in [4]) of this quasi-polynomial,

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although a “period collapse” might occur and ρ_0 may not be the minimum period in general. The characteristic quasi-polynomials of the arrangements of root systems and the mid-hyperplane arrangements ([2]) are studied in [4]. In the present paper, we consider the general case where b is not necessarily zero, which we call the *non-central case*.

In the non-central case, we show that the cardinality of the complement of the arrangement after modulo q reduction remains to be a quasi-polynomial in q with ρ_0 as a period. However, unlike the central case $b = 0$, this periodicity holds true not for all $q > 0$ but for all $q > q_0$ for some $q_0 \in \mathbb{Z}_{\geq 0}$. We give a bound q_0 explicitly, though it may not be the strict one in general.

Besides, for each $q \in \mathbb{Z}_{>0}$, we can define a poset $L_q = L_q(C, b)$, called the *intersection poset*, consisting of nonempty intersections of some of $H_{j,q},\ 1 \leq j \leq n$, with partial order defined by reverse inclusion. Then we can consider a sequence $L_1, L_2, \ldots$ and study its periodicity. With an appropriate definition of an isomorphism of the intersection posets, the sequence of isomorphism classes of $L_q,\ q = 1, 2, \ldots$, is shown to be periodic from some q on (Section 4). We have to recognize the distinction between the periodicity of $|M(\mathcal{A}_q)|$ and that of L_q. This distinction is illustrated with a simple example in Section 4.

For the concepts in the theory of arrangements of hyperplanes, the reader is referred to [5]. Concerning general properties of quasi-polynomials, [7] is a basic reference.

The organization of this paper is as follows. In Section 2 we present a basic theory of elementary divisors modulo q. In Section 3, we prove that the cardinality of the complement of the arrangement after modulo q reduction is a quasi-polynomial for all sufficiently large q. In Section 4, we investigate the periodicity of the intersection posets. In the final section, Section 5, we study the arrangement $\hat{\mathcal{B}}_m^{[0,a]}$ of Athanasiadis [1] to illustrate our general results in the non-central case.

2 Elementary divisors modulo a positive integer

In this section we present a basic theory of canonical forms and elementary divisors of matrices with entries in $\mathbb{Z}_q$. This is needed in our developments in Sections 3 and 4.

Suppose an $m \times n$ matrix $A \in \mathrm{Mat}_{m\times n}(\mathbb{Z}_q)$ is given. The dimensions m, n in this section are general and are not necessarily equal to those in other sections. Let

$$GL_k(R) := \{M \in \mathrm{Mat}_{k\times k}(R) : \det M \text{ is a unit in } R\}$$

for an arbitrary commutative ring R. We say that A is equivalent to $B \in \mathrm{Mat}_{m\times n}(\mathbb{Z}_q)$, denoted by $A \sim B$, if $B = PAQ$ for some $P \in GL_m(\mathbb{Z}_q)$ and $Q \in GL_n(\mathbb{Z}_q)$. Our purpose is to find a canonical form of A using elementary divisors. For $a \in \mathbb{Z}$, we denote its q reduction by $[a]_q := a + q\mathbb{Z} \in \mathbb{Z}_q$. For an integral matrix or vector A', let $[A']_q$ stand for the element-wise q reduction of A'.

Proposition 2.1. *Let $A \in \mathrm{Mat}_{m\times n}(\mathbb{Z}_q)$ be an $m \times n$ matrix with entries in $\mathbb{Z}_q$. Then,*

(i) *The matrix A is equivalent to*

$$\operatorname{diag}([d_1]_q, [d_2]_q, \dots, [d_s]_q, 0, 0, \dots, 0)$$

for some integers $d_1, d_2, \dots, d_s$ such that $0 < d_1 \le d_2 \le \cdots \le d_s < q$ and $d_1 \mid d_2 \mid \cdots \mid d_s \mid q$.

(ii) *The integers $d_1, d_2, \dots, d_s$ are uniquely determined by A.*

We call $d_1, \dots, d_s \in \mathbb{Z}$, or $[d_1]_q, \dots, [d_s]_q \in \mathbb{Z}_q$, the *elementary divisors* of $A \in \mathrm{Mat}_{m\times n}(\mathbb{Z}_q)$.

Proof. Let $A' \in \mathrm{Mat}_{m\times n}(\mathbb{Z})$ with $A = [A']_q$. By the theory of elementary divisors over $\mathbb{Z}$, there exist $P' \in GL_m(\mathbb{Z})$ and $Q' \in GL_n(\mathbb{Z})$ such that $P'A'Q' = \operatorname{diag}(e_1, e_2, \dots, e_\ell, 0, 0, \dots, 0)$ with $e_1 \mid e_2 \mid \cdots \mid e_\ell,\ 1 \le e_1 \le e_2 \le \cdots \le e_\ell$. Note that

$$\gcd\{a, q\} = \gcd\{b, q\} \iff [a]_q \doteq [b]_q \iff [a]_q\mathbb{Z}_q = [b]_q\mathbb{Z}_q$$

for $a, b \in \mathbb{Z}_{>0}$. Here $\doteq$ stands for the equality up to a unit multiplication in $\mathbb{Z}_q$. Define

$$s = \max\{j : q \text{ does not divide } e_j\}, \qquad d_i = \gcd\{e_i, q\}\ \ (1 \le i \le s).$$

Then $[d_i]_q \doteq [e_i]_q,\ 1 \le i \le s$, which proves (i). The uniqueness (ii) follows from Lemma 2.2 below. □

Lemma 2.2. *Let R be an arbitrary commutative ring. Let A and B be both $m\times n$ matrices with entries in R. Suppose that*

$$A = \operatorname{diag}(\alpha_1, \alpha_2, \dots, \alpha_s, 0, \dots, 0) \ \text{ and } \ B = \operatorname{diag}(\beta_1, \beta_2, \dots, \beta_t, 0, \dots, 0)$$

satisfy

$$\alpha_1 R \supseteq \alpha_2 R \supseteq \cdots \supseteq \alpha_s R \ne (0), \quad \beta_1 R \supseteq \beta_2 R \supseteq \cdots \supseteq \beta_t R \ne (0)$$

and $B = PAQ^{-1}$ with $P \in GL_m(R),\ Q \in GL_n(R)$. Then

$$s = t \quad \text{and} \quad \alpha_i R = \beta_i R \ \ (1 \le i \le s).$$

Proof. We may assume $m \le n$ without loss of generality. Define

$$\alpha_{s+1} = \cdots = \alpha_m = \beta_{t+1} = \cdots = \beta_m = 0.$$

Let $1 \le k \le m$. Define I_k to be the ideal of R generated by $\{p_{ij} : 1 \le j \le k \le i \le m\}$, where p_{ij} is the (i,j)-entry of P. (For example, I_1 is the ideal generated by all entries in the first column of P.) Then $\det P \in I_k$ for each k because the product $p_{1\sigma(1)}p_{2\sigma(2)}\cdots p_{m\sigma(m)}$ for every permutation σ belongs to the ideal I_k. Since $\alpha_j p_{ij}$ is the (i,j)-entry of $PA = BQ$, one has $\alpha_j p_{ij} \in \beta_i R$. Thus

$$\alpha_k p_{ij} R \subseteq \alpha_j p_{ij} R \subseteq \beta_i R \subseteq \beta_k R$$

for $1 \le j \le k \le i \le m$. This shows $\alpha_k(\det P) \in \beta_k R$ and $\alpha_k R \subseteq \beta_k R$. The converse is symmetric. □

For $A \in \mathrm{Mat}_{m\times n}(\mathbb{Z}_q)$, let $\mathrm{im}\, A$ denote the submodule of the $\mathbb{Z}_q$-module $\mathbb{Z}_q^m$ generated by the columns of A. For $b \in \mathbb{Z}_q^m$, we are interested in determining whether $b \in \mathrm{im}\, A$ or not. Let $(A, b) \in \mathrm{Mat}_{m\times(n+1)}(\mathbb{Z}_q)$ be an $m \times (n+1)$ matrix obtained by appending b to A as the last column.

Lemma 2.3. *For $A \in \mathrm{Mat}_{m\times n}(\mathbb{Z}_q)$ and $b \in \mathbb{Z}_q^m$, we have $b \in \mathrm{im}\, A$ if and only if the elementary divisors of (A, b) are the same as those of $(A, 0)$.*

Proof. Suppose that $b \in \mathrm{im}\, A$. Then by elementary column operations, we can transform (A, b) to $(A, 0)$. Therefore, the elementary divisors of (A, b) are the same as those of $(A, 0)$.

Suppose that both (A, b) and $(A, 0)$ have the same elementary divisors $d_1, \dots, d_s$. Then $\mathrm{im}(A, b)$ and $\mathrm{im}(A, 0)$ have the same cardinality $q^s/(d_1 \cdots d_s)$. Since $\mathrm{im}(A, b) \supseteq \mathrm{im}(A, 0)$ always holds, we have $\mathrm{im}(A, b) = \mathrm{im}(A, 0)$. □

3 Characteristic quasi-polynomial in non-central case

Let $m, n \in \mathbb{Z}_{>0}$ be positive integers. Suppose an $(m + 1) \times n$ integral matrix

$$\begin{pmatrix} C \\ b \end{pmatrix} \in \mathrm{Mat}_{(m+1)\times n}(\mathbb{Z})$$

is given, where $C = (c_1, \dots, c_n) \in \mathrm{Mat}_{m\times n}(\mathbb{Z})$ with $c_j = (c_{1j}, \dots, c_{mj})^T \neq (0, \dots, 0)^T,\ 1 \leq j \leq n$, and $b = (b_1, \dots, b_n) \in \mathbb{Z}^n$. Define

$$H_{j,q} = H_q(c_j, b_j) := \{z = (z_1, \dots, z_m) \in \mathbb{Z}_q^m : z[c_j]_q = [b_j]_q\}$$

and

$$\mathcal{A}_q = \mathcal{A}_q(C, b) := \{H_{j,q} : 1 \leq j \leq n\}.$$

When q is not a prime, it is not appropriate to call $H_{j,q}$ a hyperplane, but by abuse of terminology we call $H_{j,q}$ a *hyperplane* also in such cases. Denote the complement of $\mathcal{A}_q$ by

$$M(\mathcal{A}_q) := \mathbb{Z}_q^m \setminus \bigcup_{H\in\mathcal{A}_q} H = \{z \in \mathbb{Z}_q^m : z[C]_q - [b]_q \in (\mathbb{Z}_q^\times)^n\},$$

where $\mathbb{Z}_q^\times := \mathbb{Z}_q \setminus \{0\}$. Then we have

$$|M(\mathcal{A}_q)| = q^m + \sum_{\emptyset\neq J\subseteq[n]} (-1)^{|J|} |H_{J,q}| \tag{1}$$

with

$$H_{J,q} := \bigcap_{j\in J} H_{j,q} \quad \text{for } \ J \subseteq [n] := \{1, 2, \dots, n\}$$

(In Section 4, we have to consider $H_{J,q}$ for $J = \emptyset$, in which case we understand that $H_{\emptyset,q} = \mathbb{Z}_q^m$).

For $J = \{j_1, \dots, j_{|J|}\} \subseteq [n],\ 1 \le |J| \le n,\ j_1 < \cdots < j_{|J|}$, denote

$$\begin{aligned} C_J &:= (c_{j_1}, \dots, c_{j_{|J|}}) \in \mathrm{Mat}_{m\times|J|}(\mathbb{Z}), \\ b_J &:= (b_{j_1}, \dots, b_{j_{|J|}}) \in \mathbb{Z}^{|J|}. \end{aligned}$$

Using C_J and b_J, we can write $H_{J,q}$ as

$$H_{J,q} = H_q(C_J, b_J) := \{z \in \mathbb{Z}_q^m : z[C_J]_q = [b_J]_q\}.$$

Now, let $f_J : \mathbb{Z}^m \to \mathbb{Z}^{|J|}$ be a $\mathbb{Z}$-homomorphism defined by $z \mapsto zC_J$, and

$$f_{J,q} : \mathbb{Z}_q^m \to \mathbb{Z}_q^{|J|} \tag{2}$$

the induced morphism $z \mapsto z[C_J]_q$. When $[b_J]_q \in \mathrm{im} f_{J,q}$, i.e., $z_0[C_J]_q = [b_J]_q$ for some $z_0 \in \mathbb{Z}_q^m$, we have $H_q(C_J, b_J) = z_0 + H_q(C_J, 0) := \{z_0 + z : z \in H_q(C_J, 0)\}$; when $[b_J]_q \notin \mathrm{im} f_{J,q}$, on the other hand, we have $H_q(C_J, b_J) = \emptyset$. Hence

$$|H_q(C_J, b_J)| = \begin{cases} |H_q(C_J, 0)| & \text{if } [b_J]_q \in \mathrm{im} f_{J,q}, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Let

$$A_J := \begin{pmatrix} C_J \\ b_J \end{pmatrix} \in \mathrm{Mat}_{(m+1)\times|J|}(\mathbb{Z}).$$

By Lemma 2.3, we know that $[b_J]_q \in \mathrm{im} f_{J,q}$ if and only if the elementary divisors of $[C_J]_q$ are the same as those of $[A_J]_q$. As seen in the proof of Proposition 2.1, the elementary divisors of $[A_J]_q$ and $[C_J]_q$ are obtained by q reduction of the elementary divisors of A_J and C_J over $\mathbb{Z}$, respectively. Let $e_{J,1}|e_{J,2}|\cdots|e_{J,\ell(J)}$ denote the elementary divisors of C_J and let $e'_{J,1}|e'_{J,2}|\cdots|e'_{J,\ell'(J)}$ denote the elementary divisors of A_J. Denote $d_{J,j}(q) := \gcd\{e_{J,j}, q\},\ 1 \le j \le \ell(J)$, and $d'_{J,j}(q) := \gcd\{e'_{J,j}, q\},\ 1 \le j \le \ell'(J)$. The elementary divisors of $[C_J]_q$ are $d_{J,j},\ 1 \le j \le s(J)$, where

$$s(J) = s(J, q) := \max\{j : 1 \le j \le \ell(J),\ q \not| \, e_{J,j}\} = \max\{j : 1 \le j \le \ell(J),\ d_{J,j}(q) < q\}.$$

Similarly, the elementary divisors of $[A_J]_q$ are $d'_{J,j}(q),\ 1 \le j \le s'(J)$, with $s'(J) = s'(J, q) := \max\{j : 1 \le j \le \ell'(J),\ q \not| \, e'_{J,j}\}$. Note that $\ell(J) = \mathrm{rank}\, C_J$, $\ell'(J) = \mathrm{rank}\, A_J$ and that $\ell'(J) - \ell(J) = 0$ or 1.

First, consider the case of J for which $\mathrm{rank}\, A_J = \mathrm{rank}\, C_J$. For those J, we have the following equivalence:

$$\begin{aligned} & s(J) = s'(J) \text{ and } d_{J,j}(q) = d'_{J,j}(q),\ 1 \le j \le s(J) \\ & \qquad \iff d_{J,j}(q) = d'_{J,j}(q),\ 1 \le j \le \ell(J). \end{aligned} \tag{4}$$

(The implication $\Longrightarrow$ is seen by noting that for any $j > s(J) = s'(J)$ we have $d_{J,j}(q) = q = d'_{J,j}(q)$.) Then, (3) and (4) imply

$$|H_q(C_J, b_J)| = \begin{cases} |H_q(C_J, 0)| & \text{if } d_{J,j}(q) = d'_{J,j}(q) \text{ for all } j = 1, \dots, \ell(J), \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

In Lemma 2.1 of [3], we showed that

$$|H_q(C_J,0)| = d_J(q)q^{m-\ell(J)}, \tag{6}$$

where $d_J(q) := \prod_{j=1}^{\ell(J)} d_{J,j}(q)$. By (5) and (6), we get

$$|H_{J,q}| = |H_q(C_J,b_J)| = \tilde{d}_J(q)q^{m-\ell(J)}, \tag{7}$$

where

$$\tilde{d}_J(q) = \begin{cases} d_J(q) & \text{if } d_{J,j}(q) = d'_{J,j}(q) \text{ for all } j = 1,\dots,\ell(J), \\ 0 & \text{otherwise.} \end{cases}$$

Now, write $e(J) := e_{J,\ell(J)}$ and define

$$\rho_0 := \operatorname{lcm}\{e(J) : J \subseteq [n],\ J \neq \emptyset\}.$$

Then $d_{J,j}(q+\rho_0) = \gcd\{e_{J,j}, q+\rho_0\} = d_{J,j}(q)$ because $e_{J,j}|\rho_0,\ 1 \le j \le \ell(J)$. Moreover, $e'(J) := e'_{J,\ell'(J)}|e(J)|\rho_0$ by Lemma 2.3 of [3], and hence $d'_{J,j}(q+\rho_0) = d'_{J,j}(q),\ 1 \le j \le \ell(J)$, in a similar manner. Therefore,

$$\tilde{d}_J(q+\rho_0) = \tilde{d}_J(q) \tag{8}$$

for all nonempty $J \subseteq [n]$.

Next, consider the case of J for which $\operatorname{rank} A_J = \operatorname{rank} C_J + 1$. Then for any $q > e'(J)$, there are $s'(J) = \ell'(J) = \ell(J)+1$ elementary divisors of $[A_J]_q$, whereas there are only $s(J) \le \ell(J)$ elementary divisors of $[C_J]_q$. Thus we can conclude

$$|H_{J,q}| = |H_q(C_J,b_J)| = 0 \quad \text{for all } q > e'(J). \tag{9}$$

Equations (1), (7) and (9) yield

$$|M(\mathcal{A}_q)| = q^m + \sum_{J:\, \operatorname{rank} A_J = \operatorname{rank} C_J} (-1)^{|J|}\tilde{d}_J(q)q^{m-\ell(J)} \tag{10}$$

for all $q \in \mathbb{Z}_{>0}$ with

$$q > \max\{e'(J) : \operatorname{rank} A_J = \operatorname{rank} C_J + 1,\ J \neq \emptyset\} =: q_0 \tag{11}$$

(When $\{J \subseteq [n] : \operatorname{rank} A_J = \operatorname{rank} C_J + 1,\ J \neq \emptyset\} = \emptyset$, we understand that $q_0 = 0$). Note that whether $\operatorname{rank} A_J = \operatorname{rank} C_J$ or $\operatorname{rank} A_J = \operatorname{rank} C_J + 1$ does not depend on q.

Remark 3.1. *Consider the case of a central arrangement: $b = 0$. In that case, we have (i) $\{J : \operatorname{rank} A_J = \operatorname{rank} C_J\}$ equals the set of all nonempty subsets $J \subseteq [n]$, and (ii) $\tilde{d}_J(q) = d_J(q)$ because $d_{J,j}(q) = d'_{J,j}(q)$ for all $j = 1,\dots,\ell(J)$. Therefore, in the case of a central arrangement,* (10) *with* (11) *reduces to the result obtained in [3].*

When actually calculating q_0, we can do without J's with $|J| > m+1$:

$$q_0 = \max\{e'(J) : \operatorname{rank} A_J = \operatorname{rank} C_J + 1,\ 1 \le |J| \le m+1\}. \tag{12}$$

Equation (12) follows from the following argument: For any J with $\operatorname{rank} A_J = \operatorname{rank} C_J + 1$ and $|J| > m+1$, we can find a subset $J' \subset J$ such that $|J'| = \operatorname{rank} A_{J'} = \operatorname{rank} A_J$ $(\le m+1)$. This J' satisfies $\operatorname{rank} A_{J'} = \operatorname{rank} C_{J'} + 1$ because $\operatorname{rank} A_{J'} = \operatorname{rank} A_J = \operatorname{rank} C_J + 1 \ge \operatorname{rank} C_{J'} + 1$. Now, since $J' \subset J$ and $\operatorname{rank} A_{J'} = \operatorname{rank} A_J$, Lemma 2.3 of [3] implies that $e'(J)|e'(J') \neq 0$. Therefore, we have (12).

Now, (10) together with (8) implies that $|M(\mathcal{A}_q)|$ is a quasi-polynomial in $q > q_0$ with a period ρ_0. In fact, it is a monic integral quasi-polynomial of degree m. Furthermore, since $d_{J,j}(\gcd\{\rho_0, q\}) = \gcd\{e_{J,j}, \gcd\{\rho_0, q\}\} = \gcd\{e_{J,j}, \rho_0, q\} = d_{J,j}(q),\ 1 \le j \le \ell(J)$, and $d'_{J,j}(\gcd\{\rho_0, q\}) = d'_{J,j}(q),\ 1 \le j \le \ell'(J)$, we have

$$\tilde{d}_J(q) = \tilde{d}_J(\gcd\{\rho_0, q\})$$

when $\ell(J) = \ell'(J)$. So we can see from (10) that the constituents of the quasi-polynomial $|M(\mathcal{A}_q)|,\ q > q_0$, coincide for all q with the same value $\gcd\{\rho_0, q\}$.

By the discussions so far, we obtain the following theorem:

Theorem 3.2. *There exist monic polynomials* $P_1(t), \dots, P_{\rho_0}(t) \in \mathbb{Z}[t]$ *of degree* m *such that* $|M(\mathcal{A}_q)| = P_r(q)$ $(q \in r + \rho_0\mathbb{Z}_{\ge 0},\ 1 \le r \le \rho_0)$ *for all integers* $q > q_0$. *Moreover, polynomials* $P_r(t)$ $(1 \le r \le \rho_0)$ *depend on* r *only through* $\gcd\{\rho_0, r\}$.

A period ρ_0 is the same period that was used in the central case in [3] and [4]. In [4] this period was called the lcm period. Theorem 3.2 implies that this ρ_0 continues to be a period of $|M(\mathcal{A}_q)|,\ q \in \mathbb{Z}_{>0}$, for the general case of non-centrality. However, unlike the central case $b = 0$, we have to ignore $q \le q_0$ when $b \neq 0$. This exclusion of small q's in the case $b \neq 0$ is actually needed in general. This can be seen from the following simple example.

Let $m = 1,\ n = 2,\ C = (1, 1)$ and $b = (1, -1)$. Then $H_{1,q} = \{[1]_q\}$ and $H_{2,q} = \{[-1]_q\}$. Thus $H_{1,q} = H_{2,q}$ if and only if $q = 1, 2$. Therefore,

$$|M(\mathcal{A}_q)| = \begin{cases} q - 1 & \text{for } q = 1, 2, \\ q - 2 & \text{for } q \ge 3. \end{cases}$$

This expression implies that $|M(\mathcal{A}_q)|$ is a polynomial (a quasi-polynomial with the minimum period one) in $q \in \mathbb{Z}_{>0}$ for $q > 2$ but not for all $q \ge 1$. We can calculate $q_0 = 2$, from which we see that q_0 is the strict bound for q in this example. In addition, $\rho_0 = 1$, so ρ_0 equals the minimum period in this case.

4 Periodicity of intersection posets

In this section, we study the periodicity of the intersection posets.

For each $q \in \mathbb{Z}_{>0}$, the intersection poset is defined to be the set

$$L_q = L_q(C, b) := \{H_{J,q} \neq \emptyset : J \subseteq [n]\}$$

equipped with the partial order by reverse inclusion. When considering the periodicity of the sequence $L_1, L_2, \ldots$, we have to be careful about the definition of an isomorphism of $L_q,\ q \in \mathbb{Z}_{>0}$. By way of example, let us consider the following simple case.

Let $m = 1,\ n = 2$, and consider the central case

$$\begin{pmatrix} C \\ b \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix}. \tag{13}$$

For this pair of coefficient matrix and constant vector (13), we have $\rho_0 = \mathrm{lcm}\{3, 4\} = 12$, and $H_{\{1,2\},q} = \{0\}$ for all $q \in \mathbb{Z}_{>0}$ (Note that $3z = 4z = 0$ implies $z = 4z - 3z = 0$). The intersection posets $L_q = L_q(C, b),\ q \in \mathbb{Z}_{>0}$, are given as follows:

(14)

$$\begin{cases} V < H_{1,q} = H_{2,q} = H_{\{1,2\},q} & \gcd\{12, q\} = 1, \\ V < H_{2,q} = \{0, \frac{q}{2}\} < H_{1,q} = H_{\{1,2\},q} & \gcd\{12, q\} = 2, \\ V < H_{1,q} = \{0, \frac{q}{3}, \frac{2q}{3}\} < H_{2,q} = H_{\{1,2\},q} & \gcd\{12, q\} = 3, \\ V < H_{2,q} = \{0, \frac{q}{4}, \frac{2q}{4}, \frac{3q}{4}\} < H_{1,q} = H_{\{1,2\},q} & \gcd\{12, q\} = 4, \\ V < H_{1,q} = \{0, \frac{q}{3}, \frac{2q}{3}\} < H_{\{1,2\},q},\ V < H_{2,q} = \{0, \frac{q}{2}\} < H_{\{1,2\},q} & \gcd\{12, q\} = 6, \\ V < H_{1,q} = \{0, \frac{q}{3}, \frac{2q}{3}\} < H_{\{1,2\},q},\ V < H_{2,q} = \{0, \frac{q}{4}, \frac{2q}{4}, \frac{3q}{4}\} < H_{\{1,2\},q} & \gcd\{12, q\} = 12, \end{cases}$$

where $V := \mathbb{Z}_q^m$. Here, we are writing, e.g., $\{0, \frac{q}{3}, \frac{2q}{3}\}$ instead of $\{[0]_q, [\frac{q}{3}]_q, [\frac{2q}{3}]_q\}$ for simplicity.

According to the usual definition of a poset isomorphism, L_q's are isomorphic to one another for all q's with $\gcd\{12, q\} = 2, 3$ or 4, and so are L_q's for all q's with $\gcd\{12, q\} = 6$ or 12. However, here we do not want to consider L_q for $\gcd\{12, q\} = 2, 4$:

$$V < H_{2,q} < H_{1,q} = H_{\{1,2\},q}$$

and L_q for $\gcd\{12, q\} = 3$:

$$V < H_{1,q} < H_{2,q} = H_{\{1,2\},q}$$

to be isomorphic to each other. This is because of the following.

We are dealing not with one intersection poset but with a sequence of intersection posets $L_1, L_2, \ldots$ obtained for a fixed numbering of hyperplanes $H_{j,q},\ j \in [n]$. Thus, it is not appropriate to allow a permutation of indices $j \in [n]$ tailored for each $L_q,\ q \in \mathbb{Z}_{>0}$, separately.

On the other hand, because our concern is the periodicity of the sequence $L_1, L_2, \ldots$, we may take the numbering of hyperplanes as a given one, and do not have to care about what the fixed numbering on which the sequence $L_1, L_2, \ldots$ is based is.

Based on these considerations, we adopt, in this paper, the following definition of the isomorphism of the intersection posets $L_q = L_q(C, b),\ q \in \mathbb{Z}_{>0}$.

Definition 4.1. *Intersection posets* $L_q = \{H_{J,q} \neq \emptyset : J \subseteq [n]\}$ *and* $L_{q'} = \{H_{J,q'} \neq \emptyset : J \subseteq [n]\}$, $q, q' \in \mathbb{Z}_{>0}$, *are defined to be isomorphic to each other iff the following conditions hold true:*

$$H_{J,q} \in L_q \iff H_{J,q'} \in L_{q'}$$

for all $J \subseteq [n]$, *and*

$$H_{J_1,q} \leq H_{J_2,q} \iff H_{J_1,q'} \leq H_{J_2,q'}$$

for all $J_1, J_2 \subseteq [n]$ *such that* $H_{J_1,q}, H_{J_2,q} \in L_q$ *and* $H_{J_1,q'}, H_{J_2,q'} \in L_{q'}$.

For our example (13), we see from (14) that L_q with $\gcd\{12, q\} = 2$ or 4 is not isomorphic to L_q with $\gcd\{12, q\} = 3$.

We are now in a position to investigate the periodicity of the sequence of isomorphism classes of L_q, $q \geq 1$. We continue to assume $c_j = (c_{1j}, \ldots, c_{mj})^T \neq (0, \ldots, 0)^T$ for all $j = 1, \ldots, n$. Let

$$\begin{aligned} q_1 &:= \max\{e(J \cup \{j\}) : \operatorname{rank} C_{J\cup\{j\}} = \operatorname{rank} C_J + 1,\ j \in [n],\ J \neq \emptyset\} \\ &= \max\{e(J \cup \{j\}) : \operatorname{rank} C_{J\cup\{j\}} = \operatorname{rank} C_J + 1,\ j \in [n],\ 1 \leq |J| \leq m-1\} \end{aligned}$$

(When there is no pair (J, j) satisfying $\operatorname{rank} C_{J\cup\{j\}} = \operatorname{rank} C_J + 1$, we understand that $q_1 = 0$). Define

$$q^* := \max\left\{q_0, q_1, \max_{1\leq j\leq n} \gcd\{c_{1j}, \ldots, c_{mj}\}\right\} \geq 1,$$

where $\gcd\{c_{1j}, \ldots, c_{mj}\}$, $1 \leq j \leq n$, are taken to be positive. Then we can prove the following theorem.

Theorem 4.2. *Suppose* $q, q' \in \mathbb{Z}_{>0}$ *satisfy* $q, q' > q^*$ *and* $\gcd\{\rho_0, q\} = \gcd\{\rho_0, q'\}$. *Then we have the following:*

(i) For any $J \subseteq [n]$, *we have* $H_{J,q} = \emptyset$ *if and only if* $H_{J,q'} = \emptyset$.

(ii) For any $j \in [n]$ *and any* $J \subseteq [n]$ *such that* $H_{J,q} \neq \emptyset$ *and* $H_{J,q'} \neq \emptyset$, *we have* $H_{j,q} \supseteq H_{J,q}$ *if and only if* $H_{j,q'} \supseteq H_{J,q'}$.

In order to verify this theorem, we first present the following lemma.

Lemma 4.3. *Fix an arbitrary* $q \in \mathbb{Z}_{>0}$. *Then, for any* $j \in [n]$ *and any* $J \subseteq [n]$ *such that* $H_{J,q} \neq \emptyset$, *we have* $H_{j,q} \supseteq H_{J,q}$ *if and only if the following two conditions hold true:*

(a) $([b_j]_q, [b_J]_q) \in \mathbb{Z}_q^{|J|+1}$ *lies in the submodule of the* $\mathbb{Z}_q$*-module* $\mathbb{Z}_q^{|J|+1}$ *generated by the rows of* $([c_j]_q, [C_J]_q) \in \mathrm{Mat}_{m\times(|J|+1)}(\mathbb{Z}_q)$*;*

(b) $[c_j]_q \in \mathbb{Z}_q^m$ *lies in the submodule of the* $\mathbb{Z}_q$*-module* $\mathbb{Z}_q^m$ *generated by the columns of* $[C_J]_q \in \mathrm{Mat}_{m\times|J|}(\mathbb{Z}_q)$.

Proof. First suppose $H_{j,q} \supseteq H_{J,q}$. Then there exists $z_0 \in H_{j,q} \cap H_{J,q} = H_{J,q} \neq \emptyset$. Since $z_0([c_j]_q, [C_J]_q) = ([b_j]_q, [b_J]_q)$, we see that condition (a) holds. Further, we have $H_{j,q} = H_q(c_j, b_j) = z_0 + H_q(c_j, 0)$ and $H_{J,q} = H_q(C_J, b_J) = z_0 + H_q(C_J, 0)$. Hence $H_{j,q} \supseteq H_{J,q}$ is equivalent to $H_q(c_j, 0) \supseteq H_q(C_J, 0)$, which in turn is equivalent to $[c_j]_q$ being in the submodule of $\mathbb{Z}_q^m$ generated by the columns of $[C_J]_q$ (Proposition 3.2 of [3]). Thus condition (b) holds.

Next, suppose conditions (a) and (b) hold. Then by condition (a), we have $H_{j,q} \cap H_{J,q} \neq \emptyset$. Hence, by the same argument as above, $H_{j,q} \supseteq H_{J,q}$ is equivalent to condition (b). So $H_{j,q} \supseteq H_{J,q}$ holds true. $\square$

Proof of Theorem 4.2. *Part (i):* Since $H_{\emptyset,q} = \mathbb{Z}_q^m$ and $H_{\emptyset,q'} = \mathbb{Z}_{q'}^m$, the equivalence $H_{J,q} = \emptyset \Leftrightarrow H_{J,q'} = \emptyset$ is trivially true for $J = \emptyset$. So let us consider nonempty $J \subseteq [n]$. We have $H_{J,q} = \{z \in \mathbb{Z}_q^m : z[C_J]_q = [b_J]_q\} \neq \emptyset$ if and only if $[b_J]_q \in \operatorname{im} f_{J,q}$, where $f_{J,q}$ is defined in (2). First, consider the case of J for which $\operatorname{rank} A_J = \operatorname{rank} C_J$. Then we know that $[b_J]_q \in \operatorname{im} f_{J,q}$ if and only if $d_{J,i}(q) = d'_{J,i}(q)$ for all $i = 1, \ldots, \ell(J)$. This argument is valid when q is replaced by q'. Now, since $d_{J,i}(q) = d_{J,i}(q')$ and $d'_{J,i}(q) = d'_{J,i}(q')$ $(1 \leq i \leq \ell(J))$ because of the assumption $\gcd\{\rho_0, q\} = \gcd\{\rho_0, q'\}$, we find that $H_{J,q} = \emptyset$ if and only if $H_{J,q'} = \emptyset$. Next, consider the case of J for which $\operatorname{rank} A_J = \operatorname{rank} C_J + 1$. In that case, we have by (9) that $H_{J,q} = H_{J,q'} = \emptyset$ because $q, q' > q^* \geq e'(J)$, so the equivalence $H_{J,q} = \emptyset \Leftrightarrow H_{J,q'} = \emptyset$ is trivially true.

Part (ii): Thanks to $q, q' > q^* \geq \gcd\{c_{1j}, \ldots, c_{mj}\} \geq 1$, we have $H_{j,q} \neq \mathbb{Z}_q^m$ and $H_{j,q'} \neq \mathbb{Z}_{q'}^m$, so neither $H_{j,q} \supseteq H_{J,q}$ nor $H_{j,q'} \supseteq H_{J,q'}$ can happen for $J = \emptyset$; therefore, we may assume $J \neq \emptyset$. Moreover, when $j \in J$, the equivalence $H_{j,q} \supseteq H_{J,q} \Leftrightarrow H_{j,q'} \supseteq H_{J,q'}$ is trivially true, so we may further assume $j \notin J$. Now, by Lemma 4.3 it suffices to show that the two conditions (a), (b) in Lemma 4.3 hold true if and only if the same two conditions hold true with q replaced by q'. By part (i) of the present theorem with $J \cup \{j\}$ regarded as the J in (i), we know that (a) holds true if and only if (a) with q replaced by q' holds true. By essentially the same discussion as above, we find that (b) holds true if and only if (b) with q replaced by q' holds true, because $q, q' > q^* \geq e(J \cup \{j\})$ for J and j such that $\operatorname{rank} C_{J \cup \{j\}} = \operatorname{rank} C_J + 1$. $\square$

Remark 4.4. *Consider the central case $b = 0$. In that case, we have $q_0 = 0$ and $q^* = \max\{q_1, \max_{1 \leq j \leq n} \gcd\{c_{1j}, \ldots, c_{mj}\}\}$. Note that $H_{J,q} \neq \emptyset$ for any $q \in \mathbb{Z}_{>0}$ and $J \subseteq [n]$ when $b = 0$.*

From Theorem 4.2, we obtain the following corollary.

Corollary 4.5. *Suppose $q, q' \in \mathbb{Z}_{>0}$ satisfy $q, q' > q^*$ and $\gcd\{\rho_0, q\} = \gcd\{\rho_0, q'\}$. Then L_q is isomorphic to $L_{q'}$. In particular, the sequence of isomorphism classes of L_q, $q = 1, 2, \ldots$, is periodic in $q > q^*$ with a period ρ_0 : $L_q \simeq L_{q+\rho_0}$ for $q > q^*$.*

We must emphasize that the periodicity of the sequence of isomorphism classes of L_q, $q \in \mathbb{Z}_{>0}$, in the sense of Definition 4.1 does not imply the periodicity of $|M(\mathcal{A}_q)|$, $q \in$

$\mathbb{Z}_{>0}$, with the same minimum period. In our example (13), we can see from (14) that

$$|M(\mathcal{A}_q)| = \begin{cases} q-1 & \text{if } \gcd\{12,q\} = 1, \\ q-2 & \text{if } \gcd\{12,q\} = 2, \\ q-3 & \text{if } \gcd\{12,q\} = 3, \\ q-4 & \text{if } \gcd\{12,q\} = 4, \\ q-4 & \text{if } \gcd\{12,q\} = 6, \\ q-6 & \text{if } \gcd\{12,q\} = 12. \end{cases}$$

So the minimum period of the quasi-polynomial $|M(\mathcal{A}_q)|,\ q \in \mathbb{Z}_{>0}$, is 12. On the other hand, the minimum period of the sequence of isomorphism classes of $L_q,\ q \in \mathbb{Z}_{>0}$, is 6 by (14). This is due to the following fact: L_q with $\gcd\{12,q\} = 2$ is isomorphic to L_q with $\gcd\{12,q\} = 4$, while $|H_{2,q}| = 2$ for $\gcd\{12,q\} = 2$ is not equal to $|H_{2,q}| = 4$ for $\gcd\{12,q\} = 4$; similarly, L_q with $\gcd\{12,q\} = 6$ is isomorphic to L_q with $\gcd\{12,q\} = 12$, while $|H_{2,q}| = 2$ for $\gcd\{12,q\} = 6$ is not equal to $|H_{2,q}| = 4$ for $\gcd\{12,q\} = 12$.

Concerning the coarseness of the intersection posets, we can obtain the following result:

Corollary 4.6. *Suppose $J_1 \subseteq [n]$ and $J_2 \subseteq [n]$ satisfy $H_{J_1,q} = H_{J_2,q}$ for some $q > q^*$. Then $H_{J_1,q'} = H_{J_2,q'}$ for any $q' > q^*$ such that* $\gcd\{\rho_0, q'\} | \gcd\{\rho_0, q\}$.

Proof. Just notice $d_{J,i}(q') | d_{J,i}(q),\ 1 \le i \le \ell(J)$, for $J = J_1, J_2$ because of $\gcd\{\rho_0, q'\} | \gcd\{\rho_0, q\}$. Then the corollary follows from essentially the same argument as in the proof of Theorem 4.2. $\square$

5 Example

In this section, we study the non-central arrangement $\hat{\mathcal{B}}_m^{[0,a]}$ $(a \in \mathbb{Z}_{>0})$ of Athanasiadis [1] to illustrate our results.

The arrangement $\hat{\mathcal{B}}_m^{[0,a]}$ $(a \in \mathbb{Z}_{>0})$ in $\mathbb{R}^m$ is a deformation of the Coxeter arrangement of type B_m, consisting of the hyperplanes defined by the following equations:

$$\begin{array}{rcll} x_i & = & 0,1,\dots,a, & 1 \le i \le m, \\ x_i - x_j & = & 0,1,\dots,a, & 1 \le i < j \le m, \\ x_i + x_j & = & 0,1,\dots,a, & 1 \le i < j \le m. \end{array}$$

We take the coefficient matrix C and the vector b so that C consists of $1, -1$ or 0.

When q is odd, Athanasiadis [1] states in the proof of his Proposition 4.3 that the characteristic polynomial (e.g., [5]) $\chi(\hat{\mathcal{B}}_m^{[0,a]}, t)$ of the real arrangement $\hat{\mathcal{B}}_m^{[0,a]}$ satisfies (15)

$$\chi(\hat{\mathcal{B}}_m^{[0,a]}, q) = [y^{\frac{q-1}{2}-m}] \left(\{\phi_a(y)\}^{m+1} \sum_{j=0}^{\infty} (2j+1)^m y^{aj} - f_{a-2}(y) \{\phi_a(y)\}^{m-1} \sum_{j=0}^{\infty} a'_j y^{aj} \right)$$

for all sufficiently large odd $q \in \mathbb{Z}_{>0}$, where $a'_0 := 1$,

$$(16)\quad a'_j := \sum_{k=2}^{m} \binom{m}{k} (2^k - 2)(2j-1)^{m-k} = (2j+1)^m - 2(2j)^m + (2j-1)^m, \quad j = 1, 2, \ldots,$$

$$\phi_b(y) := 1 + y + y^2 + \cdots + y^{b-1}, \quad b \in \mathbb{Z}_{\geq 0}$$

(we understand that $\phi_0(y) = 0$) and

$$f_{a-2}(y) := \sum_{s \geq 0,\ t \geq 0,\ s+2t \leq a-2} y^{s+t} = \begin{cases} \left\{\phi_{\frac{a}{2}}(y)\right\}^2 & \text{if } a \text{ is even;} \\ \phi_{\frac{a-1}{2}}(y)\phi_{\frac{a+1}{2}}(y) & \text{if } a \text{ is odd.} \end{cases}$$

(For a formal power series or a polynomial $F(y)$, we denote by $[y^k]F(y),\ k \in \mathbb{Z}_{\geq 0}$, the coefficient of y^k in $F(y)$.) He obtained (15) by representing an m-tuple $z = (z_1, z_2, \ldots, z_m) \in \mathbb{Z}_q^m$ satisfying $z_i \neq 0$ $(1 \leq i \leq m)$ and $z_i \pm z_j \neq 0$ $(1 \leq i < j \leq m)$ as a placement of m integers $\epsilon_1 1, \epsilon_2 2, \ldots, \epsilon_m m$ ($\epsilon_i = 1$ or -1 for each $i = 1, 2, \ldots, m$) and $(q-1)/2 - m$ indistinguishable balls along a line, with an extra zero in the leftmost position.

By inspecting his arguments, we can see that the right-hand side of (15) is equal to $|M((\hat{\mathcal{B}}_m^{[0,a]})_q)|$ for all odd $q \geq 2a+1$. Thus we have

$$(17)\quad |M((\hat{\mathcal{B}}_m^{[0,a]})_q)| = [y^{\frac{q-1}{2}-m}] \left(\{\phi_a(y)\}^{m+1} \sum_{j=0}^{\infty} (2j+1)^m y^{aj} - f_{a-2}(y)\{\phi_a(y)\}^{m-1} \sum_{j=0}^{\infty} a'_j y^{aj} \right)$$

for all odd $q \geq 2a+1$.

Now, let us move on to the case of even q. By modifying the arguments of Athanasiadis [1] for the case of odd q, we can count $|M((\hat{\mathcal{B}}_m^{[0,a]})_q)|$ for even q in the following way. Similarly to the case of odd q, we consider a placement of m integers $\epsilon_1 1, \ldots, \epsilon_m m$ and $q/2 - m$ indistinguishable balls with an extra zero in the left most position. When the rightmost position is occupied by an integer, its sign is always taken to be positive. Then we can obtain

$$\begin{aligned}(18)\qquad |M((\hat{\mathcal{B}}_m^{[0,a]})_q)| &= [y^{\frac{q}{2}-m}] \left(\{\phi_a(y)\}^m \sum_{j=1}^{\infty} \{(2j)^m - (2j-1)^m\} y^{aj} \right) \\ &\quad + [y^{\frac{q}{2}-m-1}] \left(\{\phi_a(y)\}^{m+1} \sum_{j=0}^{\infty} (2j+1)^m y^{aj} \right) \\ &\quad - [y^{\frac{q}{2}-m-1}] \left(f_{a-3}(y)\{\phi_a(y)\}^{m-1} \sum_{j=0}^{\infty} a'_j y^{aj} \right)\end{aligned}$$

for all even $q \geq 2a+2$, where a'_j $(j = 0, 1, 2, \ldots)$ are defined in (16) and

$$f_{a-3}(y) = \sum_{s \geq 0,\ t \geq 0,\ s+2t \leq a-3} y^{s+t} = \begin{cases} \left\{\phi_{\frac{a-1}{2}}(y)\right\}^2 & \text{if } a \text{ is odd;} \\ \phi_{\frac{a}{2}-1}(y)\phi_{\frac{a}{2}}(y) & \text{if } a \text{ is even.} \end{cases}$$

The first term on the right-hand side of (18) corresponds to the first term on the right-hand side of (17) with the restriction that the rightmost position on the line is occupied by an integer (which is necessarily positive). The second term on the right-hand side of (18) corresponds to the first term on the right-hand side of (17) with the restriction that the rightmost position is occupied by a ball. The third term on the right-hand side of (18) corresponds to the second term on the right-hand side of (17).

As in Athanasiadis [1], let S be the shift operator:

$$Sf(y) := f(y-1)$$

for polynomials $f(y)$.

Lemma 5.1. *Let $p, a, m \in \mathbb{Z}_{>0}$, $l \in \mathbb{Z}_{\geq 0}$ and $h \in \mathbb{Z}_{\geq 0}$ with $h \leq m-1$. Suppose $b \in \mathbb{Z}$ and $c \in \mathbb{Z} \setminus \{0\}$ satisfy $c|ab$. Furthermore, assume $p \geq l + m(a-1)$. Then we have*

$$[y^{p-l}]\psi(y)\{\phi_a(y)\}^m \sum_{j=0}^{\infty}(cj+b)^h y^{aj} = \frac{1}{a^{h+1}}\psi(S)\{\phi_a(S)\}^m S^{l-\frac{ab}{c}}(cp)^h \tag{19}$$

for any polynomial $\psi(y)$.

Proof. It suffices to prove the lemma when $\psi(y) = 1$. Define $c_{k;m,a}$ by $\{\phi_a(y)\}^m = \sum_{k=0}^{m(a-1)} c_{k;m,a} y^k$. Then the left-hand side of (19) is

$$[y^{p-l}]\left(\sum_{k=0}^{m(a-1)}\sum_{j=0}^{\infty} c_{k;m,a}(cj+b)^h y^{aj+k}\right) = \frac{1}{a^h}\sum_{k=p-l-aj,\ 0\leq k\leq m(a-1),\ j\in\mathbb{Z}_{\geq 0}} c_{k;m,a}\{c(p-l-k)+ab\}^h,$$

which can be written as

$$\frac{1}{a^h}\sum_{k\equiv p-l \pmod a,\ 0\leq k\leq m(a-1)} c_{k;m,a}\left\{c\left(p-k-l+\frac{ab}{c}\right)\right\}^h \tag{20}$$

because $p - l \geq m(a-1)$. Since $h \leq m-1$, we have by Lemma 2.2 of Athanasiadis [1] that (20) is equal to

$$\frac{1}{a^h} \times \frac{1}{a}\{\phi_a(S)\}^m\left\{c\left(p-l+\frac{ab}{c}\right)\right\}^h = \frac{1}{a^{h+1}}\{\phi_a(S)\}^m S^{l-\frac{ab}{c}}(cp)^h.$$

□

If we use Lemma 5.1, it is not hard to see that the constituent $P_2(q)$ of the characteristic quasi-polynomial $|M((\hat{\mathcal{B}}_m^{[0,a]})_q)|$ for even q is

$$P_2(q) = \frac{\{\phi_a(S^2)\}^{m-1}S^{2m-a}(1-S^a)^2}{a^{m+1}}\left(S^a\cdot\frac{1+S^a}{1-S^2} + S^2\cdot\frac{(1+S^a)^2}{(1-S^2)^2} - f_{a-3}(S^2)S^2\right)q^m. \tag{21}$$

Moreover, $|M((\hat{\mathcal{B}}_m^{[0,a]})_q)| = P_2(q)$ for all even $q \geq 2a(m+1)$.

Remark 5.2. *When a is odd, we cannot directly apply Lemma 5.1 to the terms on the right-hand side of* (18) *because the condition $c|ab$ of the lemma is not met. But the result* (21) *itself obtained by formally applying the lemma is correct. Similarly, the exponent of each term of $(2j)^m - (2j-1)^m$ is m and this fact violates the condition $h \le m-1$ of the lemma. But the degree of the polynomial $(2j)^m - (2j-1)^m$ in j is $m-1$, and we can formally apply the lemma to each of $(2j)^m$ and $(2j-1)^m$ and then take the difference. The same remark applies to $a'_j = (2j+1)^m - 2(2j)^m + (2j-1)^m,\ j \ge 1$, as well.*

When a is odd, we can calculate

$$f_{a-3}(S^2) = \left(\frac{1-S^{a-1}}{1-S^2}\right)^2$$

and

$$P_2(q) = \frac{\{\phi_a(S^2)\}^{m-1}S^{2m-a}(1-S^a)^2}{a^{m+1}} \cdot \frac{S^a(1+S)^2}{(1-S^2)^2}q^m = \frac{1}{a^{m+1}}S^{2m}\{\phi_a(S^2)\}^{m-1}\{\phi_a(S)\}^2q^m.$$

By Propositions 4.2 and 4.3 of Athanasiadis [1], this is equal to $P_1(q)$ for odd a, so the period collapse occurs for odd a.

When a is even, on the other hand, we can obtain

$$f_{a-3}(S^2) = \frac{(1-S^{a-2})(1-S^a)}{(1-S^2)^2}$$

and

$$P_2(q) = \frac{2}{a^{m+1}}S^{2m}\{\phi_a(S^2)\}^{m-1}\{\phi_{\frac{a}{2}}(S^2)\}^2(1+S^2)q^m.$$

By Propositions 4.2 and 4.3 of Athanasiadis [1], we know

$$P_1(q) = \frac{4}{a^{m+1}}S^{2m+1}\{\phi_a(S^2)\}^{m-1}\{\phi_{\frac{a}{2}}(S^2)\}^2q^m$$

for even a. Hence, unlike the case of odd a, the period collapse does not occur for even a.

Thus we have obtained the following theorem.

Theorem 5.3. *For odd a, we have*

$$|M((\hat{\mathcal{B}}_m^{[0,a]})_q)| = \frac{1}{a^{m+1}}S^{2m}\{\phi_a(S^2)\}^{m-1}\{\phi_a(S)\}^2q^m \tag{22}$$

for all $q \ge 2a(m+1)-1$. For even a, we have

$$|M((\hat{\mathcal{B}}_m^{[0,a]})_q)| = \begin{cases} \frac{4}{a^{m+1}}S^{2m+1}\{\phi_a(S^2)\}^{m-1}\{\phi_{\frac{a}{2}}(S^2)\}^2q^m & \text{if } q \text{ is odd;} \\ \frac{2}{a^{m+1}}S^{2m}\{\phi_a(S^2)\}^{m-1}\{\phi_{\frac{a}{2}}(S^2)\}^2(1+S^2)q^m & \text{if } q \text{ is even} \end{cases} \tag{23}$$

for all $q \ge 2a(m+1)-1$.

Using PARI/GP [6], we have calculated q_0 for some values of m and a. Denote by $\bar{q} = \bar{q}(m, a)$ the greatest integer $q \in \mathbb{Z}_{>0}$ for which both sides do not agree in (22) or in (23). We list $q_0, 2a(m+1)-2$ and $\bar{q}$ for some combinations of values of m and a:

$m = 4,\ a = 3$:

q_0	$2a(m+1)-2$	$\bar{q}$
30	28	21

$m = 4,\ a = 4$:

q_0	$2a(m+1)-2$	$\bar{q}$
38	38	28

$m = 5,\ a = 3$:

q_0	$2a(m+1)-2$	$\bar{q}$
42	34	27

$m = 5,\ a = 4$:

q_0	$2a(m+1)-2$	$\bar{q}$
54	46	36

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