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# Inverse scattering problem for the Klein Gordon equation in quantum field theory

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## Abstract

An inverse scattering problem for a quantized scalar field  $\phi$  interacting with a classical source  $J$  is considered. Assuming that  $J$  is of the form  $J(t, x) = J_T(t)J_X(x)$ ,  $(t, x) \in \mathbb{R} \times \mathbb{R}^3$ , we represent  $J_X$  (resp.  $J_T$ ) in terms of  $J_T$  (resp.  $J_X$ ) and the asymptotic fields of  $\phi$  and its conjugate field.

## 1 Introduction

We consider an inverse scattering problem for a quantized scalar field  $\phi$  interacting with a classical source  $J$  [8] which obeys the Klein-Gordon equation

$$(\square + m^2)\phi(t, x) = J(t, x) \tag{1.1}$$

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in  $\mathbb{R} \times \mathbb{R}^3$ . Here  $\square = \frac{\partial^2}{\partial t^2} - \Delta$ ,  $\Delta$  is the Laplacian in  $\mathbb{R}^3$ ,  $m > 0$  and  $J$  is a real function.  $\phi(t, x)$  and its conjugate field  $\pi(t, x) = \frac{\partial}{\partial t}\phi(t, x)$  are operator valued distributions [14] and the time-zero fields  $\phi_0(x) = \phi(0, x)$  and  $\pi_0(x) = \pi(0, x)$  satisfy the canonical commutation relations

$$[\phi_0(x), \pi_0(y)] = i\delta(x - y), \quad (1.2)$$

$$[\phi_0(x), \phi_0(y)] = [\pi_0(x), \pi_0(y)] = 0. \quad (1.3)$$

A typical example of this abstract system is the nucleon-pion interaction, that is,  $\phi$  describes the pion field and  $J$  the distribution function of the nucleons [3]. In this case, the pions mediate the nuclear force between nucleons.

The inverse scattering problem for the equation (1.1) is to recover  $J$  from the knowledge of the asymptotic fields  $\phi_{\pm}$  of  $\phi$  and  $\pi_{\pm}$  of  $\pi$ . Under suitable conditions for  $J$  the asymptotic field  $\phi_{\pm}$  and  $\pi_{\pm}$  are formally given by

$$\phi_{\pm}(t, x) = \phi_0(t, x) + \int_0^{\pm\infty} ds \frac{\sin(t-s)\omega(i\nabla)J(s, x)}{\omega(i\nabla)} \quad (1.4)$$

$$\pi_{\pm}(t, x) = \pi_0(t, x) + \int_0^{\pm\infty} ds \cos(t-s)\omega(i\nabla)J(s, x), \quad (1.5)$$

where  $\omega(k) = \sqrt{m^2 + k^2}$ ,  $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$  and  $\phi_0(t, x)$  is the free solution of the Klein Gordon equation satisfying  $\phi_0(0, x) = \phi_0(x)$  and  $\pi(0, x) = \pi_0(x)$ . It is well known that the operator-valued distribution  $\phi_0(t, x)$  and  $\pi_0(t, x)$  are ill-defined as operators and hence  $\phi_{\pm}(t, x)$  and  $\pi_{\pm}(t, x)$  also make no sense. But the smeared fields  $\phi_{\pm}(t, f) = \int dx \phi_{\pm}(t, x)f(x)$  and  $\pi_{\pm}(t, f) = \int dx \pi_{\pm}(t, x)f(x)$  ( $f \in \mathcal{S}(\mathbb{R}^3)$ ) may be well-defined as operators on the Boson Fock space  $\mathcal{F}(L^2(\mathbb{R}^3))$  over  $L^2(\mathbb{R}^3)$ , where  $\mathcal{S}(\mathbb{R}^d)$  denotes the rapidly decreasing  $C^{\infty}$ -function space on  $\mathbb{R}^d$  ( $d \geq 1$ ). To state our results, we define functional  $Z[f]$  by

$$Z[f] = X[f] + iY[f],$$

where

$$X[f] := \langle \Phi, [\pi_+(f) - \pi_-(f)]\Phi \rangle$$

$$Y[f] := \langle \Psi, [\phi_+(\omega(i\nabla)f) - \phi_-(\omega(i\nabla)f)]\Psi \rangle.$$

Here  $\phi_{\pm}(f) = \phi_{\pm}(0, f)$ ,  $\pi_{\pm}(f) = \pi_{\pm}(0, f)$  and  $\Psi, \Phi \in \mathcal{F}(L^2(\mathbb{R}^3))$  are normalized vectors. Let us make some comments.

- Since, by (1.4) and (1.5), we find that  $X[f]$  (resp.  $Y[f]$ ) is independent of  $\Phi \in \mathcal{F}(L^2(\mathbb{R}^3))$  (resp.  $\Psi \in \mathcal{F}(L^2(\mathbb{R}^3))$ ),  $X[f]$  and  $Y[f]$  can be independently determined.

- Since, by (1.3), (1.4) and (1.5), we find that the incoming field  $\phi_-$  (resp.  $\pi_-$ ) and the outgoing field  $\phi_+$  (resp.  $\pi_+$ ) are mutually commuting and hence compatible observables, the functional  $Y[f]$  (resp.  $X[f]$ ) can be determined.
- The asymptotic completeness for this system, which is proved later, and the equations (1.4) and (1.5) ensure the existence and variety of vectors  $\Phi, \Psi \in \mathcal{F}(L^2(\mathbb{R}^3))$  to define  $X[f], Y[f]$ .
- If  $\phi_0$  and  $\pi_0$  are given by a classical function, then (1.1) is a classical Klein-Gordon equation. In this case,  $Z[f]$  is equivalent to

$$\left\langle (S_{\text{cl}} - id) \begin{pmatrix} \phi_- \\ \pi_- \end{pmatrix}, \begin{pmatrix} i\omega f \\ f \end{pmatrix} \right\rangle_{L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)},$$

where  $S_{\text{cl}}$  is the scattering operator for the classical Klein-Gordon equation and  $id$  is the identity operator. From the equivalence, we can also apply our method below to the classical theory. For other results of inverse scattering problem for classical Klein-Gordon equations, see, e.g., [2, 7, 4, 5, 6, 1, 12, 13, 11, 10]. As far as the authors know, there is no result of inverse scattering problem for the equation (1.1).

In this paper we suppose that the source  $J$  is of the form  $J(t, x) = J_T(t)J_X(x)$ , where  $J_T \in \mathcal{S}(\mathbb{R})$  and  $J_X \in \mathcal{S}(\mathbb{R}^3)$  are real functions. We show the following results: (1) The first result is concerned with a reconstruction formula of  $J_X$ . Assuming that  $J_T$  is a given function and that the Fourier transform  $\widehat{J}_T$  of  $J_T$  is analytic, we represent  $J_X$  in terms of  $J_T$  and the functional  $Z[f]$ . (2) The second result is concerned with a reconstruction formula of  $J_T$ . We assume that  $J_T$  satisfies the following condition:

$$\text{For some } \delta > 0, \quad (1 + |t|)|J_T(t)|e^{\delta|t|} \in L^1(\mathbb{R}). \quad (1.6)$$

Then we express  $J_T$  by means of  $J_X$  and the functional  $Z[f]$ .

This paper is organized as follows. The section 2 is devoted to the direct scattering problem for the equation (1.1) in quantum field theory. In the subsection 2.1, we review the Boson Fock space and operators therein. In the subsection 2.2, we first solve the equation (1.1), construct the asymptotic fields  $\phi_{\pm}$  and  $\pi_{\pm}$  by the LSZ method [9], and prove the asymptotic completeness for this system. We also construct the scattering operator  $S$ . The precise definitions of the functional  $X[f], Y[f]$  are given in the end of this subsection. In the section 3, we give the reconstruction formulas of  $J_X$  and  $J_T$ , respectively. To this end, we first state the basic property of the functional  $Z[f]$  and formulate our assumptions for  $J$ . The subsection 3.1 and 3.2 are devoted to reconstructing  $J_X$  and  $J_T$ , respectively.

## 2 Scattering theory in quantum field theory

In general we denote the inner product and the associated norm of a Hilbert space  $\mathcal{H}$  by  $\langle *, \cdot \rangle_{\mathcal{H}}$  and  $\| \cdot \|_{\mathcal{H}}$ , respectively. The inner product is linear in  $\cdot$  and antilinear in  $*$ . If there is no danger of confusion, we omit the subscript  $\mathcal{H}$  in  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  and  $\| \cdot \|_{\mathcal{H}}$ . For a linear operator  $T$  on  $\mathcal{H}$ , we denote the domain of a linear operator  $T$  by  $D(T)$  and, if  $D(T)$  is dense in  $\mathcal{H}$ , the adjoint of  $T$  by  $T^*$ .

### 2.1 Boson Fock space

We first recall the abstract Boson Fock space and operators therein. The Boson Fock space over  $L^2(\mathbb{R}^3)$  is defined by

$$\begin{aligned} \mathcal{F}(L^2(\mathbb{R}^3)) &:= \bigoplus_{n=0}^{\infty} \bigotimes_s^n L^2(\mathbb{R}^3) \\ &= \left\{ \Psi = \{ \Psi^{(n)} \}_{n=0}^{\infty} \left| \Psi^{(n)} \in \bigotimes_s^n L^2(\mathbb{R}^3), \quad \sum_{n=0}^{\infty} \left\| \Psi^{(n)} \right\|_{\bigotimes_s^n L^2(\mathbb{R}^3)}^2 < \infty \right. \right\}, \end{aligned}$$

where  $\bigotimes_s^n L^2(\mathbb{R}^3)$  denotes the symmetric tensor product of  $L^2(\mathbb{R}^3)$  with the convention  $\bigotimes_s^0 L^2(\mathbb{R}^3) = \mathbb{C}$ .

The creation operator  $a^*(f)$  ( $f \in L^2(\mathbb{R}^3)$ ) on  $\mathcal{F}(L^2(\mathbb{R}^3))$  is defined by

$$(a^*(f)\Psi)^{(n)} := \sqrt{n} S_n \left( f \otimes \Psi^{(n-1)} \right)$$

with the domain

$$D(a^*(f)) := \left\{ \Psi = \{ \Psi^{(n)} \}_{n=0}^{\infty} \left| \sum_{n=0}^{\infty} n \left\| S_n \left( f \otimes \Psi^{(n-1)} \right) \right\|_{\bigotimes_s^n L^2(\mathbb{R}^3)}^2 < \infty \right. \right\},$$

where  $S_n$  denotes the symmetrization operator on  $\bigotimes^n L^2(\mathbb{R}^3)$  satisfying  $S_n = S_n^* = S_n^2$  and  $S_n(\bigotimes^n L^2(\mathbb{R}^3)) = \bigotimes_s^n L^2(\mathbb{R}^3)$ . The annihilation operator  $a(f)$  ( $f \in L^2(\mathbb{R}^3)$ ) is defined by the adjoint of  $a^*(\bar{f})$ , i.e.,  $a(f) := a^*(\bar{f})^*$ . By definition,  $a^*(f)$  (resp.  $a(f)$ ) is linear (resp. antilinear) in  $f \in L^2(\mathbb{R}^3)$ . As is well known, the creation and annihilation operators leave the finite particle subspace

$$\mathcal{F}_0 = \bigcup_{m=1}^{\infty} \left\{ \Psi = \{ \Psi^{(n)} \}_{n=0}^{\infty} \mid \Psi^{(n)} = 0, \quad n \geq m \right\}$$

and satisfy the canonical commutation relations

$$[a(f), a^*(g)] = \langle \bar{f}, g \rangle_{L^2(\mathbb{R}^3)}, \quad [a(f), a(g)] = [a^*(f), a^*(g)] = 0. \quad (2.1)$$

The Fock vacuum  $\Omega = \{\Omega^{(n)}\}_{n=0}^\infty \in \mathcal{F}(L^2(\mathbb{R}^3))$  is defined by  $\Omega^{(0)} = 1$ ,  $\Omega^{(n)} = 0$  ( $n \geq 1$ ) and satisfies

$$a(f)\Omega = 0, \quad f \in L^2(\mathbb{R}^3). \quad (2.2)$$

It is well known that  $\Omega$  is a unique vector satisfying (2.2) up to a constant factor.

Let  $C$  be a contraction operator on  $L^2(\mathbb{R}^3)$ , i.e.,  $\|C\| \leq 1$ . We define a contraction operator  $\Gamma(C)$  on  $\mathcal{F}(L^2(\mathbb{R}^3))$  by

$$(\Gamma(C)\Psi)^{(n)} = (\otimes^n C)\Psi^{(n)}, \quad \Psi = \{\Psi^{(n)}\}_{n=0}^\infty$$

with the convention  $\otimes^0 C = 1$ . If  $u$  is unitary, i.e.  $U^{-1} = U^*$ , then  $\Gamma(U)$  is also unitary and satisfies  $\Gamma(U)^* = \Gamma(U^*)$  and

$$\Gamma(U)a(f)\Gamma(U)^* = a(Uf), \quad \Gamma(U)a^*(f)\Gamma(U)^* = a^*(Uf).$$

For a self-adjoint operator  $T$  on  $L^2(\mathbb{R}^3)$ , i.e.,  $T = T^*$ ,  $\{\Gamma(e^{itT})\}_{t \in \mathbb{R}}$  is a strongly continuous one-parameter unitary group on  $\mathcal{F}(L^2(\mathbb{R}^3))$ . Then, by the Stone theorem, there exists a unique self-adjoint operator  $d\Gamma(T)$  such that

$$\Gamma(e^{itT}) = e^{itd\Gamma(T)}.$$

## 2.2 Scattering theory

We denote by  $\hat{u}$  the Fourier transform of a function  $u$  on  $\mathbb{R}^d$ :  $\hat{u}(k) = \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d} e^{-ikx} u(x) dx$  ( $d \geq 1$ ). Henceforth, we assume that  $J \in \mathcal{S}(\mathbb{R}^4)$  for simplicity.

For each  $f \in \mathcal{S}(\mathbb{R}^3)$  we set

$$\begin{aligned} \phi_0(f) &= \frac{1}{\sqrt{2}} \left[ a^* \left( \frac{\hat{f}}{\sqrt{\omega}} \right) + a \left( \frac{\hat{f}(-\cdot)}{\sqrt{\omega}} \right) \right], \\ \pi_0(f) &= \frac{i}{\sqrt{2}} \left[ a^* \left( \sqrt{\omega} \hat{f} \right) - a \left( \sqrt{\omega} \hat{f}(-\cdot) \right) \right]. \end{aligned}$$

Let us first solve the following operator-valued Cauchy problem:

$$\begin{aligned} \frac{d^2}{dt^2} \phi(t, f) &= \phi(t, (\Delta - m^2)f) + \langle J(t, \cdot), f \rangle \\ \phi(0, f) &= \phi_0(f) \\ \pi(0, f) &= \pi_0(f), \end{aligned}$$

where  $\pi(t, f) = \frac{d}{dt} \phi(t, f)$  and we denote by  $\frac{d}{dt}$  the time derivative in the strong topology. Here we assume that, for each  $t \in \mathbb{R}$ ,  $\phi(t, f)$  and  $\pi(t, f)$  are operator-valued distributions on  $\mathcal{F}_0$ , i.e.,  $\phi(t, f)$  and  $\pi(t, f)$  possess the following properties:

- $\mathcal{F}_0 \subset D(\phi(t, f)) \cap D(\phi(t, f))^* \cap D(\pi(t, f)) \cap D(\pi(t, f))^*$ ,
- the maps  $\mathcal{S}(\mathbb{R}^3) \ni f \mapsto \langle \Psi, \phi(t, f) \Phi \rangle \in \mathbb{C}$  and  $\mathcal{S}(\mathbb{R}^3) \ni f \mapsto \langle \Psi, \pi(t, f) \Phi \rangle \in \mathbb{C}$  are the tempered distributions.

Then the unique solution of the above Cauchy problem is given by

$$\begin{aligned}\phi(t, f) &= \phi_0(t, f) + \phi_{\text{cl}}(t, f), \\ \pi(t, f) &= \pi_0(t, f) + \pi_{\text{cl}}(t, f),\end{aligned}$$

where

$$\phi_0(t, f) = \Gamma(e^{it\omega})\phi_0(f)\Gamma(e^{-it\omega}), \quad \pi_0(t, f) = \Gamma(e^{it\omega})\pi_0(f)\Gamma(e^{-it\omega})$$

and

$$\begin{aligned}\phi_{\text{cl}}(t, f) &= \frac{1}{2i} \int_0^t ds \left[ \left\langle \frac{e^{is\omega} \widehat{J}(s, \cdot)}{\omega}, e^{it\omega} \widehat{f} \right\rangle - \left\langle \frac{e^{-is\omega} \widehat{J}(s, \cdot)}{\omega}, e^{-it\omega} \widehat{f} \right\rangle \right] \\ \pi_{\text{cl}}(t, f) &= \frac{1}{2} \int_0^t ds \left[ \left\langle e^{is\omega} \widehat{J}(s, \cdot), e^{it\omega} \widehat{f} \right\rangle + \left\langle e^{-is\omega} \widehat{J}(s, \cdot), e^{-it\omega} \widehat{f} \right\rangle \right]\end{aligned}$$

In what follows, we construct the asymptotic fields  $\phi_{\pm}$  and  $\pi_{\pm}$  by the LSZ method [9]. For each  $h \in \mathcal{S}(\mathbb{R}^3)$ , we set

$$a_t(h) \equiv i [\pi(t, g(t, \cdot)) - \phi(t, \dot{g}(t, \cdot))]$$

and

$$a_t^*(h) = a_t(\bar{h})^*,$$

where

$$\begin{aligned}g(t, x) &= \frac{1}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} \frac{dk}{\sqrt{2\omega(k)}} e^{i\omega(k)t - ikx} h(k), \\ \dot{g}(t, x) &= \frac{i}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} dk \sqrt{\frac{\omega(k)}{2}} e^{i\omega(k)t - ikx} h(k).\end{aligned}$$

By direct calculation, we have the following lemma:

**Lemma 2.1.** *Let  $\Psi \in \mathcal{F}_0$ . Then the strong limits*

$$a_{\pm}(h)\Psi = s - \lim_{t \rightarrow \pm\infty} a_t(h)\Psi, \quad a_{\pm}^*(h)\Psi = s - \lim_{t \rightarrow \pm\infty} a_t^*(h)\Psi$$

*exist and are given explicitly by*

$$a_{\pm}(h) = a(h) + \langle \bar{g}_{\pm}, h \rangle, \quad a_{\pm}^*(h) = a(h) + \langle g_{\pm}, h \rangle, \quad (2.3)$$

where

$$g_{\pm} \equiv i \int_0^{\pm\infty} ds e^{is\omega} \widehat{J}(s, \cdot) \in L^2(\mathbb{R}^3).$$

Henceforth, for  $h \in L^2(\mathbb{R}^3)$ , we define the asymptotic annihilation  $a_{\pm}(h)$  and creation  $a_{\pm}^*(h)$  by the right hand sides in equations (2.3).

For each  $h \in L^2(\mathbb{R}^3)$  such that  $h/\sqrt{\omega} \in L^2(\mathbb{R}^3)$ , the operator

$$\chi(h) = \frac{i}{\sqrt{2}} \left[ a^* \left( \frac{h}{\sqrt{\omega}} \right) - a \left( \frac{\bar{h}}{\sqrt{\omega}} \right) \right]$$

is an essentially self-adjoint on  $\mathcal{F}_0$ . We denote the closure of  $\chi(h)$  by the same symbol  $\chi(h)$  and define a unitary operator  $\mathcal{U}(h)$  by

$$\mathcal{U}(h) = e^{i\chi(h)}.$$

By (2.1) and (2.2) one can prove the following lemma:

**Lemma 2.2.** *It follows that*

$$\mathcal{U}(g_{\pm})a(h)\mathcal{U}(g_{\pm})^{-1} = a_{\pm}(h), \quad \mathcal{U}(g_{\pm})a^*(h)\mathcal{U}(g_{\pm})^{-1} = a_{\pm}^*(h)$$

and that  $\Omega_{\pm} = \mathcal{U}(g_{\pm})\Omega$  is a unique vector satisfying

$$a_{\pm}(h)\Omega_{\pm} = 0, \quad h \in L^2(\mathbb{R}^3).$$

Let

$$\mathcal{F}_{\text{fin}}^{\pm} = \text{L.H.}\{\Omega_{\pm}, a_{\pm}^*(h_1) \cdots a_{\pm}^*(h_n)\Omega_{\pm} \mid h_1, \dots, h_n \in \mathcal{S}(\mathbb{R}^3), n \geq 1\},$$

where  $\text{L.H.}\{\dots\}$  stands for the linear hull of  $\{\dots\}$ . The asymptotic outgoing Fock space  $\mathcal{F}^+$  (resp. the asymptotic incoming Fock space  $\mathcal{F}^-$ ) is defined the closure of the subspace  $\mathcal{F}_{\text{fin}}^+$  (resp.  $\mathcal{F}_{\text{fin}}^-$ ).

**Proposition 2.3.** *The asymptotic completeness holds, i.e.,*

$$\mathcal{F}(L^2(\mathbb{R}^3)) = \mathcal{F}^+ = \mathcal{F}^-.$$

*Proof.* This lemma follows from the fact that the subspace  $\mathcal{F}_{\text{fin}} = \text{L.H.}\{a^*(h_1) \cdots a^*(h_n)\Omega \mid h_1, \dots, h_n \in \mathcal{S}(\mathbb{R}^3), n \geq 0\}$  is dense in  $\mathcal{F}(L^2(\mathbb{R}^3))$ .  $\square$

By the above lemma, the scattering operator  $S$  is uniquely determined up to a phase by the following relations  $S\Omega_+ = \Omega_-$  and

$$Sa_+^*(h_1) \cdots a_+^*(h_n)\Omega_+ = a_-^*(h_1) \cdots a_-^*(h_n)\Omega_-$$

for  $h_1, \dots, h_n \in L^2(\mathbb{R}^3)$  ( $n \geq 1$ ), and is given explicitly by

$$S = \mathcal{U}(g_{\infty}),$$

where

$$g_{\infty} = -i \int_{\mathbb{R}} ds e^{is\omega} \widehat{J}(s, \cdot).$$

The asymptotic fields  $\phi_{\pm}(t, f)$  and  $\pi_{\pm}(t, f)$  are defined by the solution of the following Cauchy problem:

$$\begin{aligned}\frac{d^2}{dt^2}\phi_{\pm}(t, f) &= \phi_{\pm}(t, (\Delta - m^2)f) \\ \phi_{\pm}(0, f) &= \phi_{\pm}(f) \\ \pi_{\pm}(0, f) &= \pi_{\pm}(f),\end{aligned}$$

where  $\pi_{\pm}(t, f) = \frac{d}{dt}\phi_{\pm}(t, f)$  and

$$\begin{aligned}\phi_{\pm}(f) &:= \frac{1}{\sqrt{2}} \left[ a_{\pm}^* \left( \frac{\widehat{f}}{\sqrt{\omega}} \right) + a_{\pm} \left( \frac{\widehat{f}(-\cdot)}{\sqrt{\omega}} \right) \right], \\ \pi_{\pm}(f) &:= \frac{i}{\sqrt{2}} \left[ a_{\pm}^* \left( \sqrt{\omega}\widehat{f} \right) - a_{\pm} \left( \sqrt{\omega}\widehat{f}(-\cdot) \right) \right].\end{aligned}$$

For all  $\Psi \in \mathcal{F}_0 \setminus \{0\}$  with  $\|\Psi\| = 1$  and  $f \in \mathcal{S}(\mathbb{R}^3)$ , we set

$$\begin{aligned}X[f] &:= \langle \Psi, [\pi_+(f) - \pi_-(f)]\Psi \rangle \\ Y[f] &:= \langle \Psi, [\phi_+(\omega(i\nabla)f) - \phi_-(\omega(i\nabla)f)]\Psi \rangle.\end{aligned}$$

Since, by direct calculation, we find that the functionals  $X[f]$  and  $Y[f]$  are bounded on  $\mathcal{S}(\mathbb{R}^3)$ , one can uniquely extend  $X[f]$  and  $Y[f]$  to bounded functionals on  $L^2(\mathbb{R}^3)$ . We denote by the same symbol  $X[f]$  and  $Y[f]$  the extended functionals of  $X[f]$  and  $Y[f]$ , respectively.

### 3 Inverse scattering

Our aim of this section is to represent the classical source  $J$  in terms of the outgoing field  $\phi_+(f)$  and the incoming field  $\phi_-(f)$ . We define functional  $Z : L^2(\mathbb{R}^3) \rightarrow \mathbb{C}$  by

$$Z[f] = X[f] + iY[f].$$

For  $f \in L^2(\mathbb{R}^3)$ ,  $\lambda > 0$  and  $k_0, x \in \mathbb{R}^3$ , we denote  $e^{-ik_0 \cdot x} f(\lambda x)$  by  $f_{k_0}^{\lambda}(x)$ . Before we give main results, we first state the following lemma:

**Lemma 3.1.** *Assume that  $J \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^3)$ . If  $f \in L^2(\mathbb{R}^3)$  satisfies  $\widehat{f} \in L^1(\mathbb{R}^3)$ , then we have, for any  $k \in \mathbb{R}^3$ ,*

$$\lim_{\lambda \rightarrow 0} Z[f_k^{\lambda}] = \left( \int_{\mathbb{R}^3} \widehat{f}(k') dk' \right) \int_{\mathbb{R}} dt \int_{\mathbb{R}^3} dx e^{-i\omega(k)t - ik \cdot x} J(t, x). \quad (3.1)$$

*Proof.* Since, by , we have

$$\begin{aligned} X[f] &= \frac{1}{2} \left[ \left\langle \int_{\mathbb{R}} dt e^{it\omega} J(t, \cdot), f \right\rangle + \left\langle \int_{\mathbb{R}} dt e^{-it\omega} J(t, \cdot), f \right\rangle \right] \\ Y[f] &= \frac{1}{2i} \left[ \left\langle \int_{\mathbb{R}} dt e^{it\omega} J(t, \cdot), f \right\rangle - \left\langle \int_{\mathbb{R}} dt e^{-it\omega} J(t, \cdot), f \right\rangle \right], \end{aligned}$$

it follows that

$$\begin{aligned} Z[f] &= \left\langle \int_{\mathbb{R}} e^{it\omega(i\nabla)} J(t, \cdot) dt, f \right\rangle \\ &= \int_{\mathbb{R}} dt \left\langle e^{it\omega(i\nabla)} J(t, \cdot), f \right\rangle. \end{aligned}$$

We immediately see that

$$\widehat{f_{k_0}^\lambda}(k) = \lambda^{-3} \widehat{f}(\lambda^{-1}(k + k_0)).$$

Therefore, it follows from the Plancherel theorem that

$$\begin{aligned} Z[f_{k_0}^\lambda] &= \lambda^{-3} \int_{\mathbb{R}} dt \left\langle e^{it\omega(k)} \widehat{J}(t, k), \widehat{f}(\lambda^{-1}(k + k_0)) \right\rangle \\ &= \int_{\mathbb{R}} dt \left\langle e^{it\omega(\lambda k - k_0)} \widehat{J}(t, \lambda k - k_0), \widehat{f}(k) \right\rangle. \end{aligned}$$

By using the Lebesgue dominated convergence theorem, we obtain (3.1).  $\square$

Henceforth, we suppose that  $J$  is expressed by

$$J(t, x) = J_T(t) J_X(x), \quad (3.2)$$

where  $J_T \in \mathcal{S}(\mathbb{R})$  and  $J_X \in \mathcal{S}(\mathbb{R}^3)$ . In Subsection 3.1 below, assuming that  $J_T$  is a given function and that  $\widehat{J}_T$  is analytic, we will represent  $J_X$  in terms of  $Z$  and  $J_T$ . In Subsection 3.2 below, we next assume that  $J_X$  is a given function. We will show that  $J_T$  is determined by  $Z$  and  $J_X$  if  $J_T$  satisfies the following condition:

$$\text{For some } \delta > 0, \quad (1 + |t|) |J_T(t)| e^{\delta|t|} \in L^1(\mathbb{R}). \quad (3.3)$$

### 3.1 Reconstruction of $J_X$

Let  $J_T$  be a given function belonging to  $\mathcal{S}(\mathbb{R})$ . Then we immediately obtain the reconstruction formula for determining  $J_X$ :

**Theorem 3.2.** Assume that  $J \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^3)$  satisfies (3.2) and  $J_T \in \mathcal{S}(\mathbb{R})$  is a nonzero, real analytic given function. Suppose that  $f \in L^2(\mathbb{R}^3)$  satisfies  $\widehat{f} \in L^1(\mathbb{R}^3)$  and  $\int_{\mathbb{R}^3} \widehat{f} \neq 0$ . If  $\omega(k) \notin (\widehat{J}_T)^{-1}(0)$ , then we have

$$(\widehat{J}_X)(k) = \frac{\lim_{\lambda \rightarrow 0} Z[f_k^\lambda]}{\sqrt{2\pi} \left( \int_{\mathbb{R}^3} (\widehat{f})(k') dk' \right) (\widehat{J}_T)(\omega(k))}. \quad (3.4)$$

**Remark 3.1.** Since  $J_T \in \mathcal{S}(\mathbb{R})$  is real analytic function,  $(\widehat{J}_T)^{-1}(0)$  is a discrete set and hence countable. Thus, (3.4) holds for almost all  $k \in \mathbb{R}^3$  and we have

$$J_X(x) = \left( 2\pi \left( \int_{\mathbb{R}^3} (\widehat{f})(k') dk' \right) \right)^{-1} \int_{\mathbb{R}^3} \frac{e^{ix \cdot k} \lim_{\lambda \rightarrow 0} Z[f_k^\lambda]}{(\widehat{J}_T)(\omega(k))} dk$$

for any  $x \in \mathbb{R}^3$ .

## 3.2 Reconstruction of $J_T$

Let  $J_X$  be a nonzero, given function belonging to  $\mathcal{S}(\mathbb{R}^3)$ . The following lemma will help us to identify  $J_T$ :

**Lemma 3.3.** Suppose that  $J_T$  satisfies (3.3). Then  $\widehat{J}_T$  is real analytic. Furthermore, the radius of convergence of a Taylor series of  $\widehat{J}_T$  is larger than  $\delta$ .

*Proof.* A function  $\varphi(z) = \int_{\mathbf{R}} e^{-itz} J_T(t) dt$  is well-defined on a strip  $\mathbb{R} + (-\delta, \delta)i$ . Let  $\varphi_R$  and  $\varphi_I$  be real-valued functions given by

$$\begin{aligned} \varphi_R(x, y) &= \int_{\mathbf{R}} \cos(tx) e^{ty} J_T(t) dt, \\ \varphi_I(x, y) &= - \int_{\mathbf{R}} \sin(tx) e^{ty} J_T(t) dt, \end{aligned}$$

respectively. Then we see from (3.3) that

$$\varphi(x + iy) = \varphi_R(x, y) + i\varphi_I(x, y) \in C^1(\mathbb{R} \times (-\delta, \delta)),$$

$\partial_x \varphi_R = \partial_y \varphi_I$  and  $\partial_y \varphi_R = -\partial_x \varphi_I$ . Therefore,  $\varphi$  is analytic and  $\widehat{J}_T$  is real analytic.

Let  $m$  be a positive integer. Since

$$\sup_{t>0} t^m e^{-\delta t} = m^m \delta^{-m} e^{-m},$$

we obtain for any  $\tau \in \mathbb{R}$ ,

$$\begin{aligned} \left| (\widehat{J}_T)^{(m)}(\tau) \right| &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |t|^m |J_T(t)| dt \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |J_T(t)| e^{\delta|t|} |t|^m e^{-\delta|t|} dt \\ &\leq \frac{1}{\sqrt{2\pi}} m^m \delta^{-m} e^{-m} \int_{\mathbb{R}} |J_T(t)| e^{\delta|t|} dt \end{aligned}$$

Thus, we have

$$\left| \frac{(\widehat{J}_T)^{(m)}(\tau)}{m!} \right| \leq \frac{1}{\sqrt{2\pi}} \left( \int_{\mathbb{R}} |J_T(t)| e^{\delta|t|} dt \right) \frac{m^m e^{-m}}{m!} \delta^{-m}.$$

Using Stirling's formula

$$m! = \sqrt{2\pi m} m^{m+1/2} e^{-m} e^{\theta(m)/12m}, \quad 0 < \theta(m) < 1,$$

we see that

$$\limsup_{m \rightarrow \infty} \left| \frac{(\widehat{J}_T)^{(m)}(\tau)}{m!} \right|^{1/m} \leq \delta^{-1},$$

which completes the lemma.  $\square$

Applying Lemmas 3.1 and 3.3, we have the following result:

**Theorem 3.4.** *Assume that  $J \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^3)$  satisfies (3.2) and  $J_X \in \mathcal{S}(\mathbb{R}^3)$  is a nonzero, given function. Suppose that  $f \in L^2(\mathbb{R}^3)$  satisfies  $\widehat{f} \in L^1(\mathbb{R}^3)$  and  $\int_{\mathbb{R}^3} f \neq 0$ . If  $J_T$  satisfies (3.3), then  $J_T$  is determined by the following steps:*

(Step I) *Fix a point  $k_0 \in (\widehat{J}_T)^{-1}(0)$ . Let  $U_0$  be a  $k_0$ -neighborhood such that  $0 \notin (\widehat{J}_T)(U_0)$ . Then we have*

$$(\widehat{J}_T)(\omega(k)) = \frac{\lim_{\lambda \rightarrow 0} Z[f_k^\lambda]}{2\pi \left( \int_{\mathbb{R}^3} \widehat{f}(k') dk' \right) (\widehat{J}_X)(k)}, \quad k \in U_0. \quad (3.5)$$

*Therefore, we see exact values of  $(\widehat{J}_T)(\tau_0)$ ,  $m = 0, 1, 2, \dots$ , where  $\tau_0 = \omega(k_0)$ .*

(Step II) *For any  $\tau \in (\tau_0 - \delta, \tau_0 + \delta)$ , we have*

$$(\widehat{J}_T)(\tau) = \sum_{m=0}^{\infty} \frac{(\widehat{J}_T)(\tau_0)}{m!} (\tau - \tau_0)^m.$$

(Step III) Let  $l$  be a positive integer. Suppose that we have already determined  $(\widehat{J}_T)(\tau)$  with  $\tau \in (\tau_0 - \frac{(l+1)\delta}{2}, \tau_0 + \frac{(l+1)\delta}{2})$ . For any  $\tau \in [\tau_0 + \frac{(l+1)\delta}{2}, \tau_0 + \frac{(l+2)\delta}{2})$ , we see the value of  $(\widehat{J}_T)(\tau)$  by

$$(\widehat{J}_T)(\tau) = \sum_{m=0}^{\infty} \frac{(\widehat{J}_T)(\tau_0 + l\delta/2)}{m!} (\tau - \tau_0 - l\delta/2).$$

On the other hand, For any  $\tau \in (\tau_0 - \frac{(l+2)\delta}{2}, \tau_0 - \frac{(l+1)\delta}{2}]$ , we see the value of  $(\widehat{J}_T)(\tau)$  by

$$(\widehat{J}_T)(\tau) = \sum_{m=0}^{\infty} \frac{(\widehat{J}_T)(\tau_0 - l\delta/2)}{m!} (\tau - \tau_0 + l\delta/2).$$

(Step IV) From (Step III),  $\widehat{J}_T$  is reconstructed completely. Hence we can determine  $J_T$  by the inverse Fourier transform.

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