Singularities of Anti de Sitter Torus Gauss maps

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Abstract

We study timelike surfaces in Anti de Sitter 3-space as an application of singularity theory. We define two mappings associated to a timelike surface which are called a *Anti de Sitter nullcone Gauss image* and a *Anti de Sitter torus Gauss map.* We also define a family of functions named the *Anti de Sitter null height function* of the timelike surface. We use this family of functions as a basic tool to investigate the geometric meanings of singularities of the Anti de Sitter nullcone Gauss image and the Anti de Sitter torus Gauss map.

Keywords: Anti de Sitter 3-space; timelike surface; AdS-nullcone Gauss image; AdS-torus Gauss map; Legendrian singularities.

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1 Introduction

This paper is written as one of the research projects on differential geometry of submanifolds in Anti de Sitter 3-space from the viewpoint of singularity theory. It is well known that Minkowski space is a flat Lorentzian space form and de Sitter space is the Lorentzian space form with positive curvature. There are several articles for the study of submanifolds in these two Lorentzian space forms [10, 12, 13, 14, 15]. The Lorentzian space form with the negative curvature is called Anti de Sitter space which is one of the vacuum solutions of the Einstein equation in the theory of relativity. Singularity theory tools, as illustrated by several papers which appeared so far ([2, 4, 5, 6, 7, 9, 10, 11, 16, 20, 21, 22, 23, 25, 28, 29, 30]), have proven to be useful in the description, of geometrical properties of submanifolds immersed in different ambient spaces, from both the local and global viewpoint. The natural connection between

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Geometry and Singularities relies on the basic fact that the contacts of a submanifold with the models (invariant under the action of a suitable transformation group) of the ambient space can be described by means of the analysis of the singularities of appropriate families of contact functions, or equivalently, of their associated Lagrangian and/or Legendrian maps ([1, 24, 26]). However, there are not much results on submanifolds immersed in Anti de Sitter space, in particular from the view point of singularity theory. We remark that although Anti de Sitter space is diffeomorphic to de Sitter space, their causalities are quite different. In [8] we have studied the spacelike surfaces in Anti de Sitter 3-space as an application of Legendrian singularity theory. We construct a basic framework for the study of timelike surfaces in Anti de Sitter 3-space here. As it was to be expected, the situation presents certain peculiarities when compared with the Minkowski case and the de Sitter case. For instance, in our case it is always possible to choose two lightlike normal directions along the timelike surface in the frame of its normal bundle. This is similar to the de Sitter case, but the normalized image is located in the Lorentzian torus T_1^2. For the de Sitter case, the normalized image of the lightlike normal is located in the spacelike sphere S_+^2. Moreover, there are no closed timelike surfaces in de Sitter space but there are such surfaces in Anti de Sitter space.

In §2 we prepare the basic notions on timelike surfaces in Anti de Sitter 3-space. We define the Anti de Sitter nullcone Gauss image (briefly, AdS-nullcone Gauss image) and Anti de Sitter torus Gauss map (briefly, AdS-torus Gauss map). We will find the AdS-nullcone Gauss image is more computable than the AdS-torus Gauss map. We also define the Anti de Sitter null Gauss-Kronecker curvature and Anti de Sitter torus Gauss-Kronecker curvature. We investigate their relations. We can prove that Anti de Sitter torus Gauss-Kronecker curvature is not a Lorentz invariant but it is an $SO(2)\times SO(2)$-invariant. Moreover, these two Gauss-Kronecker curvatures have the same zero points set. In §3 We introduce the notion of height functions on timelike surfaces, named AdS-null height function, which is useful to show that the AdS-nullcone Gauss image has a singular point if and only if the Anti de Sitter null Gauss-Kronecker curvature vanished at such point. In §4 We apply mainly the Legendrian singularity theory to interpret the AdS-nullcone Gauss image as a Legendrian map. In §5 we define a surface, named Anti de Sitter torus cylindrical pedal, as a tool to study the relationship between the AdS-nullcone Gauss image and the AdS-torus Gauss map. We also study the contact of timelike surfaces with some model surfaces (i.e., AdS-horospheres) in §6. In §7 we give a generic classification of singularities of AdS-nullcone Gauss image and AdS-torus Gauss map. In the last part, §8, we introduce the notion of the AdS-null Monge form of a timelike surface in Anti de Sitter 3-space and as an application of this notion we give two examples.

We shall assume throughout the whole paper that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

2 The local differential geometry of timelike surfaces

In this section we introduce the local differential geometry of timelike surfaces in Anti de Sitter 3-space. For details of Lorentzian geometry, see [27].

Let $\mathbb{R}^4 = \{(x_1, \cdots, x_4) | x_i \in \mathbb{R}\ (i = 1, \cdots, 4)\ \}$ be a 4-dimensional vector space. For any vectors $\boldsymbol{x} = (x_1, \cdots, x_4)$ and $\boldsymbol{y} = (y_1, \cdots, y_4)$ in $\mathbb{R}^4$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined to be $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4$. We call $(\mathbb{R}^4, \langle , \rangle)$ a *smei-Euclidean 4-space with index 2* and write $\mathbb{R}^4_2$ instead of $(\mathbb{R}^4, \langle , \rangle)$.

We say that a non-zero vector $\boldsymbol{x}$ in $\mathbb{R}^4_2$ is *spacelike*, *null* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^4_2$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. We denote the signature of a vector $\boldsymbol{x}$ by

$$\mathrm{sign}(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \textit{ is spacelike} \\ 0 & \boldsymbol{x} \textit{ is null} \\ -1 & \boldsymbol{x} \textit{ is timelike} \end{cases}$$

For a vector $\boldsymbol{n} \in \mathbb{R}^4_2$ and a real number c, we define the *hyperplane with pseudo-normal* $\boldsymbol{n}$ by

$$HP(\boldsymbol{n}, c) = \{\boldsymbol{x} \in \mathbb{R}^4_2 | \langle \boldsymbol{x}, \boldsymbol{n} \rangle = c\}.$$

We call $HP(\boldsymbol{n}, c)$ a *Lorentz hyperplane*, a *semi-Euclidean hyperplane with index 2* or a *null hyperplane* if $\boldsymbol{n}$ is *timelike, spacelike or null* respectively.

We now define *Anti de Sitter 3-space* (briefly, *AdS 3-space*) by

$$H^3_1 = \{\boldsymbol{x} \in \mathbb{R}^4_2 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\},$$

a *unit pseudo 3-sphere with index 2* by

$$S^3_2 = \{\boldsymbol{x} \in \mathbb{R}^4_2 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\},$$

and a *closed nullcone* with vertex $\boldsymbol{a}$ by

$$\Lambda_a = \{\boldsymbol{x} \in \mathbb{R}^4_2 | \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a} \rangle = 0\}.$$

In particular we call Λ_0 the *nullcone* at the origin. We also define the *Lorentz torus* by

$$T^2_1 = \{\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \Lambda_0 | x_1^2 + x_2^2 = x_3^2 + x_4^2 = 1\}.$$

If non-zero vector $\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \Lambda_0$, we have

$$\tilde{\boldsymbol{x}} = \pm \frac{1}{\sqrt{x_1^2 + x_2^2}} (x_1, x_2, x_3, x_4) = \pm \frac{1}{\sqrt{x_1^2 + x_2^2}} \boldsymbol{x} \in T^2_1.$$

For any $\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3 \in \mathbb{R}^4_2$, we define a vector $\boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3$ by

$$\boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3 = \begin{vmatrix} -\boldsymbol{e}_1 & -\boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \\ x_1^1 & x_1^2 & x_1^3 & x_1^4 \\ x_2^1 & x_2^2 & x_2^3 & x_2^4 \\ x_3^1 & x_3^2 & x_3^3 & x_3^4 \end{vmatrix},$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ is the canonical basis of $\mathbb{R}^4_2$ and $\boldsymbol{X}_i = (x_i^1, x_i^2, x_i^3, x_i^4)$. We can easily check that $\langle \boldsymbol{X}, \boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3 \rangle = \det(\boldsymbol{X}, \boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3)$, so that $\boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3$ is pseudo-orthogonal to any $\boldsymbol{X}_i$ (for $i = 1, 2, 3$).

We now study the extrinsic differential geometry of timelike surfaces in Anti de Sitter 3-space. Let $\boldsymbol{X} : U \longrightarrow H^3_1$ be a regular surface (i.e., an embedding), where $U \subset \mathbb{R}^2$ is an open subset. We denote $M = \boldsymbol{X}(U)$ and identify M with U through the embedding $\boldsymbol{X}$. The embedding $\boldsymbol{X}$ is said to be timelike if the induced metric $\mathbf{I}$ of M is Lorentzian. Throughout the remainder in this paper we assume that M is an timelike surface in H^3_1. We define a vector $\boldsymbol{N}(u)$ by

$$\boldsymbol{N}(u) = \frac{\boldsymbol{X}(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \boldsymbol{X}_{u_2}(u)}{\|\boldsymbol{X}(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \boldsymbol{X}_{u_2}(u)\|}.$$

By definition, we have

$$\langle \boldsymbol{N}(u), \boldsymbol{X}(u) \rangle \equiv \langle \boldsymbol{N}(u), \boldsymbol{X}_{u_i}(u) \rangle \equiv 0 \text{ and } \langle \boldsymbol{X}(u), \boldsymbol{X}_{u_i}(u) \rangle \equiv 0 \text{ (for } i = 1, 2).$$

This means that $\boldsymbol{X}(u), \boldsymbol{N}(u) \in N_pM$, where $u = (u_1, u_2) \in U$ and $p = \boldsymbol{X}(u) \in M$. Since the embedding is timelike and $\boldsymbol{X}(u) \in H^3_1$, $\boldsymbol{N}$ is spacelike. Therefore $\langle \boldsymbol{N}(u), \boldsymbol{N}(u)\rangle \equiv 1$. It follows that $\boldsymbol{X}(u) \pm \boldsymbol{N}(u) \in \Lambda_0 \cap N_pM$ and $\widetilde{\boldsymbol{X}(u) \pm \boldsymbol{N}}(u) \in T^2_1 \cap N_pM$. Thus we can define a map $\mathbb{G}^{\pm}_n : U \longrightarrow \Lambda_0$ by $\mathbb{G}^{\pm}_n(u) = \boldsymbol{X}(u) \pm \boldsymbol{N}(u)$. This map is analogous to the hyperbolic Gauss indicatrix of hypersurfaces in $H^n_+(-1)$ which was defined in [11]. Here, we call it the *Anti de Sitter nullcone Gauss image* (briefly, *AdS-nullcone Gauss image*) of $\boldsymbol{X}$(or M). We also define a map $\widetilde{\mathbb{G}^{\pm}_n} : U \longrightarrow T^2_1$ by $\widetilde{\mathbb{G}^{\pm}_n}(u) = \widetilde{\boldsymbol{X}(u) \pm \boldsymbol{N}}(u) = \frac{1}{\xi(u)}\mathbb{G}^{\pm}_n(u)$, where $\xi(u) = \pm\sqrt{(x_1(u) \pm n_1(u))^2 + (x_2(u) \pm n_2(u))^2}$. We call it the *Anti de Sitter torus Gauss map* (or, *AdS-torus Gauss map*) of $\boldsymbol{X}$.

We remak that the map $\mathbb{G}^{\pm}_n(u)$ was used by S. Lee [17] to study the timelike sufaces of constant mean curvature ± 1 in Anti de Sitter 3-space. He called $\mathbb{G}^{\pm}_n(u)$ the *hyperbolic Gauss map*. By a direct calculation we know that $\mathbb{G}^{\pm}_n$ is constant if and only if $\widetilde{\mathbb{G}^{\pm}_n}$ is constant.

It is easy to show that $\boldsymbol{N}_{u_i}$ $(i = 1, 2)$ are tangent vector of M. Therefore we have a linear transformation $S^{\pm}_p = -d\mathbb{G}^{\pm}_n(u) = -(d\boldsymbol{X}(u) \pm d\boldsymbol{N}(u)) : T_pM \longrightarrow T_pM$ which is called the *Anti de Sitter null shape operator* (briefly, *AdS-null shape operator*) of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. Under the identification of U and M, the derivation $d\boldsymbol{X}(u)$ can be identified with the identity mapping id_{T_pM}, this means that $S^{\pm}_p = -d\mathbb{G}^{\pm}_n(u) = -(\mathrm{id}_{T_pM} \pm d\boldsymbol{N}(u))$. We have another linear mapping

$$d\widetilde{\mathbb{G}^{\pm}_n}(u) : T_pM \longrightarrow T_p\mathbb{R}^4_2 = T_pM \oplus N_pM.$$

If we consider the orthogonal projection $\pi^T : T_pM \oplus N_pM \longrightarrow T_pM$, then we have

$$\widetilde{S^{\pm}_p} = -(d\widetilde{\mathbb{G}^{\pm}_n}(u))^T = -\pi^T \circ d\widetilde{\mathbb{G}^{\pm}_n}(u) : T_pM \longrightarrow T_pM,$$

and call it the *Anti de Sitter torus shape operator* (briefly, *AdS-torus shape operator*)of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. We remark that $S^{\pm}_p$ (resp., $\widetilde{S^{\pm}_p}$) does not always have real eigenvalues. If the eigenvalues are real numbers, we denote it by $k^{\pm}_i$ (resp., $\widetilde{k^{\pm}_i}$).

We define $K^{\pm}_{AdSn}(u) = \det S^{\pm}_p = k^{\pm}_1 \cdot k^{\pm}_2$ and $\widetilde{K^{\pm}_{AdSt}}(u) = \det\widetilde{S^{\pm}_p} = \widetilde{k^{\pm}_1} \cdot \widetilde{k^{\pm}_2}$. We respectively call $K^{\pm}_{AdSn}(u)$ the *Anti de Sitter null Gauss-Kronecker curvature* (briefly, *AdS-null G-K curvature*) and $\widetilde{K^{\pm}_{AdSt}}(u)$ the *Anti de Sitter torus Gauss-Kronecker curvature* (briefly, *AdS-torus G-K curvature*) of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. We say that a point $p = \boldsymbol{X}(u)$ is a (*positive or negative*) *Anti de Sitter horospherical parabolic point* (briefly, *AdSh$^{\pm}$-parabolic point*) (resp. *positive or negative Anti de Sitter torus parabolic point, briefly, AdSt$^{\pm}$-parabolic point*) of $M = \boldsymbol{X}(U)$ if $K^{\pm}_{AdSn}(u) = 0$(resp. $\widetilde{K^{\pm}_{AdSt}}(u) = 0$). By a straightforward calculation we have the relation $S^{\pm}_p = \xi(u)\widetilde{S^{\pm}_p}$, so that we have $k^{\pm}_i(p) = \xi(u)\widetilde{k^{\pm}_i}(p)$ and $K^{\pm}_{AdSn}(u) = \xi^2(u)\widetilde{K^{\pm}_{AdSt}}(u)$. Then we have the following relations.

$$\begin{cases} k^{\pm}_i(p) = 0 \Longleftrightarrow \widetilde{k^{\pm}_i}(u) = 0 \\ K^{\pm}_{AdSn}(u) = 0 \Longleftrightarrow \widetilde{K^{\pm}_{AdSt}}(u) = 0 \end{cases}$$

We say that a point $u \in U$ or $p = \boldsymbol{X}(u)$ is an *umbilic point* if $S^{\pm}_p = k^{\pm}(p)id_{T_pM}$. We also say that $M = \boldsymbol{X}(U)$ is *totally umbilic* if all points on M are umbilic.

We now consider the geometric meaning of the AdS-nullcone Gauss image of a timelike surface. First, we consider a surface given by the intersection of H^3_1 with hyperplane $HP(\boldsymbol{n}, c)$. We denote it by $AH(\boldsymbol{n}, c) = H^3_1 \cap HP(\boldsymbol{n}, c)$ and call it a *Anti de Sitter pseudohyperbolid with index* 1 (briefly, *AdS-pseudohyperboloid*), a *Anti de Sitter pseudosphere with index* 1 (briefly, *AdS-pseudosphere*) or a *Anti de Sitter horosphere* (briefly, *AdS-horosphere*) if $\boldsymbol{n}$ is spacelike, timelike and $\|\boldsymbol{n}\| < |c|$ or null respectively. Especially, we call $AH(\boldsymbol{n}, 0)$ the *Anti de Sitter small pseudohyperbolid with index* 1 (briefly, *AdS-small pseudohyperboloid*) if $\boldsymbol{n}$ is spacelike and $c = 0$. Then we have the following proposition.

Proposition 2.1 *Let $\boldsymbol{X} : U \longrightarrow H^3_1$ be a timelike surface in Anti de Sitter 3-space. If the AdS-nullcone Gauss image $\mathbb{G}^{\pm}_n$ is constant, then the timelike surface $\boldsymbol{X}(U) = M$ is a part of a AdS-horosphere.*

Proof. We consider the set $V = \{\boldsymbol{y} \in \mathbb{R}^4_2 | \langle \boldsymbol{y}, \boldsymbol{X} \pm \boldsymbol{N} \rangle = -1\}$. Since $\mathbb{G}^{\pm}_n = \boldsymbol{X} \pm \boldsymbol{N}$ is constant, the set $V = HP(\mathbb{G}^{\pm}_n, -1)$ is a null hyperplane. We also have $\langle \boldsymbol{X}, \mathbb{G}^{\pm}_n \rangle \equiv -1$, so $\boldsymbol{X}(U) = M \subset V \cap H^3_1$. □

We also have the following classification theorem on umbilic points.

Proposition 2.2 *Suppose that $M = \boldsymbol{X}(U)$ is totally umbilic. Then $k^{\pm}(p)$ is constant $k^{\pm}$. Under this condition, we have the following classification.*

(1) *Suppose $k^{\pm} \neq 0$.*

(a) *If $0 < |k^{\pm} + 1| < 1$, then M is a part of a AdS-pseudohyperboloid;*

(b) *If $|k^{\pm} + 1| > 1$, then M is a part of a AdS-pseudosphere;*

(c) *If $k^{\pm} = -1$, then M is a part of a AdS-small pseudohyperboloid.*

(2) *Suppose $k^{\pm} = 0$ then M is a part of a AdS-horosphere.*

The proof is almost the same as that of Proposition 2.3 in [11], so that we omit it. We also call a point $p \in M$ the *Anti de Sitter horospherical point* (briefly, *AdS-horospherical point*) if $k^{\pm}_i(p) = 0$ $(i = 1, 2)$.

We now introduce the pseudo-Riemannian metric $ds^2 = \sum_{i,j=1}^{2} g_{ij} du_i du_j$ on $M = \boldsymbol{X}(U)$, where $g_{ij}(u) = \langle \boldsymbol{X}_{u_i}(u), \boldsymbol{X}_{u_j}(u) \rangle$ for any $u \in U$. We also define the *Anti de Sitter null second fundamental invariant* by $h^{\pm}_{ij}(u) = \langle -\mathbb{G}^{\pm}_{n\,u_i}(u), \boldsymbol{X}_{u_j}(u) \rangle$, *Anti de Sitter torus second fundamental invariant* by $\widetilde{h^{\pm}_{ij}}(u) = \langle -\widetilde{\mathbb{G}^{\pm}_{n\,u_i}}(u), \boldsymbol{X}_{u_j}(u) \rangle = \frac{1}{\xi(u)} h^{\pm}_{ij}(u)$ for any $u \in U$. We can also show the following Weingarten formulas by exactly the same arguments as those of [8, 11, 15].

Proposition 2.3 *With the above notation, we have the followings*

(1) *Anti de Sitter null Weingarten formula*:

$$\mathbb{G}^{\pm}_{n\,u_i} = -\sum_{j=1}^{2} (h^{\pm})^j_i \boldsymbol{X}_{u_j},$$

where $((h^{\pm})^j_i) = (h^{\pm}_{ik})(g^{kj})$ and $(g^{kj}) = (g_{kj})^{-1}$.

(2) *Anti de Sitter torus Weingarten formula*:

$$(\widetilde{\mathbb{G}^{\pm}_{n\,u_i}})^T = \pi^T \circ \widetilde{\mathbb{G}^{\pm}_{n\,u_i}} = -\textstyle\sum_{j=1}^{2} (\widetilde{h^{\pm}})^j_i \boldsymbol{X}_{u_j} = -\dfrac{1}{\xi(u)} \displaystyle\sum_{j=1}^{2} (h^{\pm})^j_i \boldsymbol{X}_{u_j},$$

where $((\widetilde{h^{\pm}})^j_i) = (\widetilde{h^{\pm}_{ik}})(g^{kj})$ and $(g^{kj}) = (g_{kj})^{-1}$. □

As a corollary of the above proposition, we have the following expression of the AdS-null G-K curvature and AdS-torus G-K curvature .

Corollary 2.4 *With the same notations as in the above Proposition, we have*:

$$K^{\pm}_{AdSn} = \frac{det(h^{\pm}_{ij})}{det(g_{ij})} = \xi^2 \frac{det(\widetilde{h^{\pm}_{ij}})}{det(g_{ij})} = \xi^2 \widetilde{K^{\pm}_{AdSt}}.$$ □

3 Height functions on timelike surfaces

In this section we define two families of functions on a timelike surface in Anti de Sitter 3-space which are useful for the study of singularities of AdS-nullcone Gauss image and AdS-torus Gauss map.

Let $\boldsymbol{X} : U \longrightarrow H^3_1$ be a timelike surface. We define a family of functions $H : U \times \Lambda_0 \longrightarrow \mathbb{R}$ by $H(u, \boldsymbol{v}) = \langle \boldsymbol{X}(u), \boldsymbol{v}\rangle + 1$. We call H an *Anti de Sitter null height function* (or, an *AdS-null height function*) on $M = \boldsymbol{X}(U)$. We denote the *Hessian matrix* of the AdS-null height function $h_{v_0}(u) = H(u, \boldsymbol{v}_0)$ at u_0 by $\mathrm{Hess}(h_{v_0})(u_0)$. Then we have the following proposition.

Proposition 3.1 *Let $M = \boldsymbol{X}(U)$ be a timelike surface in H^3_1 and $H : U \times \Lambda_0 \longrightarrow \mathbb{R}$ be an AdS-null height function. Then we have the following*:

(1) $H(u_0, \boldsymbol{v}) = \frac{\partial H}{\partial u_i}(u_0, \boldsymbol{v}) = 0$ *(for $i = 1, 2$) if and only if $\boldsymbol{v} = \boldsymbol{X}(u_0) \pm \boldsymbol{N}(u_0) = \mathbb{G}^{\pm}_n(u_0)$;*

(2) *Let $\boldsymbol{v}^{\pm}_0 = \boldsymbol{X}(u_0) \pm \boldsymbol{N}(u_0)$, then $p = \boldsymbol{X}(u_0)$ is a AdSh$^{\pm}-$parabolic point if and only if* $\mathrm{detHess}\ (h_{v^{\pm}_0})(u_0) = 0$;

(3) *Let $\boldsymbol{v}^{\pm}_0 = \boldsymbol{X}(u_0) \pm \boldsymbol{N}(u_0)$, then $p = \boldsymbol{X}(u_0)$ is a AdS-horospherical point if and only if* $\mathrm{rankHess}(h_{v^{\pm}_0})(u_0) = 0$.

Proof. (1) Since $\{\boldsymbol{X}, \boldsymbol{N}, \boldsymbol{X}_{u_1}, \boldsymbol{X}_{u_2}\}$ is a basis of the vector space $T_p\mathbb{R}^4_2$ where $p = \boldsymbol{X}(u)$, there exist real numbers $\lambda, \eta, \alpha_1, \alpha_2$ such that $\boldsymbol{v} = \lambda\boldsymbol{X} + \eta\boldsymbol{N} + \alpha_1\boldsymbol{X}_{u_1} + \alpha_2\boldsymbol{X}_{u_2}$. Therefore $H(u, \boldsymbol{v}) = 0$ if and only if $\lambda = -\langle \boldsymbol{X}(u), \boldsymbol{v}\rangle = 1$. Since $0 = \frac{\partial H}{\partial u_i}(u, \boldsymbol{v}) = \langle \boldsymbol{X}_{u_i}, \boldsymbol{v}\rangle = \sum_{j=1}^{2} g_{ij}\alpha_i$ and (g_{ij}) is non-degenerate, we have $\alpha_i = 0$ (for $i = 1, 2$). Therefore we have $\boldsymbol{v} = \boldsymbol{X} + \eta\boldsymbol{N}$. From the fact that $\langle \boldsymbol{v}, \boldsymbol{v}\rangle = 0$, we have $\eta = \pm 1$.

(2) By definition, we have

$$\mathrm{Hess}(h_{v^{\pm}_0})(u_0) = (\langle \boldsymbol{X}_{u_iu_j}(u_0), \mathbb{G}^{\pm}_n(u_0)\rangle) = (-\langle \boldsymbol{X}_{u_i}(u_0), \mathbb{G}^{\pm}_{n\,u_j}(u_0)\rangle).$$

From the AdS-null Weingarten formula, we have

$$-\langle \boldsymbol{X}_{u_i}, \mathbb{G}^{\pm}_{n\,u_j}\rangle = \textstyle\sum_{\alpha=1}^{2}(h^{\pm})^{\alpha}_i\langle \boldsymbol{X}_{u_\alpha}, \boldsymbol{X}_{u_j}\rangle = \sum_{\alpha=1}^{2}(h^{\pm})^{\alpha}_i g_{\alpha j} = h^{\pm}_{ij}.$$

Therefore we have

$$K^{\pm}_{AdSn}(u_0) = \frac{\det(h^{\pm}_{i,j}(u_0))}{\det(g_{ij}(u_0))} = \frac{\mathrm{detHess}(h_{v^{\pm}_0})(u_0)}{\det(g_{ij}(u_0))}.$$

Then assertion (2) is satisfied.

(3) By the AdS-null Weingarten formula, p is an umbilic point if and only if there exists an orthogonal matrix A such that $A^t((h^{\pm})^l_i)A = k^{\pm}I$. Therefore, we have $((h^{\pm})^l_i) = Ak^{\pm}IA^t = k^{\pm}I$. Then we have

$$\mathrm{Hess}(h_{v^{\pm}_0})(u_0) = (h^{\pm}_{ij}(u_0)) = ((h^{\pm})^l_i(u_0))(g_{lj}(u_0)) = k^{\pm}(g_{ij}(u_0)).$$

Thus, $p = \boldsymbol{X}(u_0)$ is a AdS-horospherical point if and only if $\mathrm{rankHess}(h_{v^{\pm}_0})(u_0) = 0$. □

As an application of the above proposition, we have the following direct corollary.

Corollary 3.2 *Let $H : U \times \Lambda_0 \longrightarrow \mathbb{R}$, with $H(u, \boldsymbol{v}) = h_v(u)$ be an AdS-null height function on timelike surface $M = \boldsymbol{X}(U)$ and $\mathbb{G}^{\pm}_n$ be the AdS-nullcone Gauss image, $p = \boldsymbol{X}(u)$. Suppose $\boldsymbol{v}^{\pm} = G^{\pm}_n(u)$, then the following conditions are equivalent*:

(1) *$p \in M$ is a degenerate singular point of AdS-null height function $h_{v^{\pm}}$*

(2) *$p \in M$ is a singular point of AdS-nullcone Gauss image $\mathbb{G}^{\pm}_n$;*

(3) *$K^{\pm}_{AdSn}(u) = 0$.* □

We can also define another family of functions $\widetilde{H} : U \times T_1^2 \longrightarrow \mathbb{R}$ by $\widetilde{H}(u, \boldsymbol{v}) = \langle \boldsymbol{X}(u), \boldsymbol{v} \rangle$. We call $\widetilde{H}$ an *Anti de Sitter torus height function* (briefly, an *AdS-torus height function*) on $\boldsymbol{X} : U \longrightarrow H_1^3$. We denote the *Hessian matrix* of the AdS-torus height function $\widetilde{h_{v_0}}(u) = \widetilde{H}(u, \boldsymbol{v}_0)$ at u_0 by $\mathrm{Hess}(\widetilde{h_{v_0}})(u_0)$. By exactly the same arguments as those of Proposition 3.1, we have the following proposition.

Proposition 3.3 *Let $M = \boldsymbol{X}(U)$ be a timelike surface in H_1^3 and $\widetilde{H} : U \times T_1^2 \longrightarrow \mathbb{R}$ be an AdS-torus height function. Then we have the following*:

(1) $\frac{\partial \widetilde{H}}{\partial u_i}(u_0, \boldsymbol{v}) = 0$ *(for $i = 1, 2$) if and only if $\boldsymbol{v} = \widetilde{\boldsymbol{X}(u_0) \pm \boldsymbol{N}}(u_0) = \widetilde{\mathbb{G}_n^\pm}(u_0)$;*

Suppose that $\boldsymbol{v}_0^\pm = \widetilde{\mathbb{G}_n^\pm}(u_0)$. Then

(2) $p = \boldsymbol{X}(u_0)$ *is a AdSt$^\pm$−parabolic point if and only if* $\det\mathrm{Hess}(\widetilde{h_{v_0^\pm}})(u_0) = 0$;

(3) $p = \boldsymbol{X}(u_0)$ *is a AdS-horospherical point if and only if* $\mathrm{rank}\mathrm{Hess}(\widetilde{h_{v_0^\pm}})(u_0) = 0$. □

Corollary 3.4 *Let $\widetilde{H} : U \times T_1^2 \longrightarrow \mathbb{R}$, with $\widetilde{H}(u, \boldsymbol{v}) = \widetilde{h_v}(u)$ be an AdS-torus height function on timelike surface $M = \boldsymbol{X}(U)$ and $\widetilde{\mathbb{G}_n^\pm}$ be the AdS-torus Gauss map, $p = \boldsymbol{X}(u)$, $\boldsymbol{v}^\pm = \widetilde{G_n^\pm}(u)$. Then the following conditions are equivalent*:

(1) *$p \in M$ is a degenerate singular point of AdS-torus height function $\widetilde{h_{v^\pm}}$,*

(2) *$p \in M$ is a singular point of AdS-torus Gauss map $\widetilde{\mathbb{G}_n^\pm}$,*

(3) $\widetilde{K_{AdSt}^\pm}(u) = 0$. □

4 AdS-nullcone Gauss images as Legendrian maps

In this section we naturally interpret the AdS-nullcone Gauss image of M as a Legendrian map in the framework of Legendrian singularity theory. We give a brief review on Legendrian singularity theory mainly due to Arnold and Zakalyukin[1, 32]. The main tool of Legendrian singularities theory is the notion of generating families. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be a function germ. We say that F is a *Morse family* of hypersurfaces if the mapping

$$\Delta^* F = (F, \frac{\partial F}{\partial q_1}, \cdots, \frac{\partial F}{\partial q_k}) : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, \mathbf{0})$$

is non-singular, where $(q, x) = (q_1, \cdots, q_k, x_1, \cdots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$. In this case we have a smooth $(n-1)$−dimensional submanifold,

$$\Sigma_*(F) = \{(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) | F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0\}$$

and the map germ $\mathcal{L}_F : (\Sigma_*(F), \mathbf{0}) \longrightarrow PT^*\mathbb{R}^n$ defined by

$$\mathcal{L}_F(q, x) = (x, [\frac{\partial F}{\partial x_1}(q, x) : \cdots : \frac{\partial F}{\partial x_n}(q, x)])$$

is a Legendrian immersion germ. Then we have the following fundamental theorem of Arnold and Zakalyukin[1, 32].

Proposition 4.1 *All Legendrian submanifold germs in $PT^*\mathbb{R}^n$ are constructed by the above method.*

We call F a *generating family* of $\mathcal{L}_F(\Sigma_*(F))$. Therefore the corresponding wave front is

$$W(\mathcal{L}_F) = \{x \in \mathbb{R}^n | \exists\ q \in \mathbb{R}^k \text{ such that } F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0\}.$$

We sometimes denote $\mathcal{D}_F = W(\mathcal{L}_F)$ and call it the *discriminant set* of F.

Now we can apply the above arguments to our situation. First, we have the following principle property with respect to the AdS-null height function H.

Proposition 4.2 *The AdS-null height function $H : U \times \Lambda_0 \longrightarrow \mathbb{R}$ is a Morse family of hypersurfaces.*

Proof. For any $\boldsymbol{v} = (v_1, v_2, v_3, v_4) \in \Lambda_0$, we have $\boldsymbol{v} \neq \boldsymbol{0}$. Without loss of the generality, we might assume that $v_1 > 0$, then $v_1 = \sqrt{v_3^2 + v_4^2 - v_2^2}$. So that

$$H(u, \boldsymbol{v}) = -x_1(u)\sqrt{1 + v_3^2 + v_4^2 - v_2^2} - x_2(u)v_2 + x_3(u)v_3 + x_4(u)v_4 + 1$$

where $\boldsymbol{X}(u) = (x_1(u), x_2(u), x_3(u), x_4(u))$. We have to prove the mapping

$$\Delta^* H = (H, \frac{\partial H}{\partial u_1}, \frac{\partial H}{\partial u_2})$$

is non-singular at any point. The Jacobian matrix of $\Delta^* H$ is given as follows:

$$\begin{pmatrix} \langle \boldsymbol{X}_{u_1}, \boldsymbol{v}\rangle & \langle \boldsymbol{X}_{u_2}, \boldsymbol{v}\rangle & x_1\frac{v_2}{v_1} - x_2 & -x_1\frac{v_3}{v_1} + x_3 & -x_1\frac{v_4}{v_1} + x_4 \\ \langle \boldsymbol{X}_{u_1u_1}, \boldsymbol{v}\rangle & \langle \boldsymbol{X}_{u_1u_2}, \boldsymbol{v}\rangle & x_{1u_1}\frac{v_2}{v_1} - x_{2u_1} & -x_{1u_1}\frac{v_3}{v_1} + x_{3u_1} & -x_{1u_1}\frac{v_4}{v_1} + x_{4u_1} \\ \langle \boldsymbol{X}_{u_2u_1}, \boldsymbol{v}\rangle & \langle \boldsymbol{X}_{u_2u_2}, \boldsymbol{v}\rangle & x_{1u_2}\frac{v_2}{v_1} - x_{2u_2} & -x_{1u_2}\frac{v_3}{v_1} + x_{3u_2} & -x_{1u_2}\frac{v_4}{v_1} + x_{4u_2} \end{pmatrix}.$$

We claim that it will suffice to show that the determinant of the matrix

$$A = \begin{pmatrix} x_1\frac{v_2}{v_1} - x_2 & -x_1\frac{v_3}{v_1} + x_3 & -x_1\frac{v_4}{v_1} + x_4 \\ x_{1u_1}\frac{v_2}{v_1} - x_{2u_1} & -x_{1u_1}\frac{v_3}{v_1} + x_{3u_1} & -x_{1u_1}\frac{v_4}{v_1} + x_{4u_1} \\ x_{1u_2}\frac{v_2}{v_1} - x_{2u_2} & -x_{1u_2}\frac{v_3}{v_1} + x_{3u_2} & -x_{1u_2}\frac{v_4}{v_1} + x_{4u_2} \end{pmatrix},$$

does not vanish at $(u, \boldsymbol{v}) \in \Delta^* H^{-1}(\boldsymbol{0})$. In this case, $\boldsymbol{v} = \mathbb{G}_n^{\pm}(u)$ and we denote

$$\boldsymbol{b}_1 = \begin{pmatrix} x_1 \\ x_{1u_1} \\ x_{1u_2} \end{pmatrix}, \ \boldsymbol{b}_2 = \begin{pmatrix} x_2 \\ x_{2u_1} \\ x_{2u_2} \end{pmatrix}, \ \boldsymbol{b}_3 = \begin{pmatrix} x_3 \\ x_{3u_1} \\ x_{3u_2} \end{pmatrix}, \ \boldsymbol{b}_4 = \begin{pmatrix} x_4 \\ x_{4u_1} \\ x_{4u_2} \end{pmatrix}.$$

Then we have

$$\det A = -\frac{v_1}{v_1}\det(\boldsymbol{b}_2,\ \boldsymbol{b}_3,\ \boldsymbol{b}_4) + \frac{v_2}{v_1}\det(\boldsymbol{b}_1,\ \boldsymbol{b}_3,\ \boldsymbol{b}_4) - \frac{v_3}{v_1}\det(\boldsymbol{b}_1,\ \boldsymbol{b}_2,\ \boldsymbol{b}_4) + \frac{v_4}{v_1}\det(\boldsymbol{b}_1,\ \boldsymbol{b}_2,\ \boldsymbol{b}_3).$$

On the other hand, we have

$$\boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2} = (-\det(\boldsymbol{b}_2,\ \boldsymbol{b}_3,\ \boldsymbol{b}_4),\ \det(\boldsymbol{b}_1,\ \boldsymbol{b}_3,\ \boldsymbol{b}_4),\ \det(\boldsymbol{b}_1,\ \boldsymbol{b}_2,\ \boldsymbol{b}_4),\ -\det(\boldsymbol{b}_1,\ \boldsymbol{b}_2,\ \boldsymbol{b}_3))$$

Therefore we have

$$\begin{aligned} \det A &= \langle (-\frac{v_1}{v_1}, -\frac{v_2}{v_1}, -\frac{v_3}{v_1}, -\frac{v_4}{v_1}), \boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2}\rangle \\ &= -\frac{1}{v_1}\langle \mathbb{G}_n^{\pm}, \| \boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2} \| \boldsymbol{N}\rangle \\ &= \mp\frac{\| \boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2} \|}{v_1} \neq 0. \end{aligned}$$

□

Let $\boldsymbol{X} : U \longrightarrow H_1^3$ be a timelike surface in H_1^3 and $\mathbb{G}_n^{\pm}$ be the AdS-nullcone Gauss image on $M = \boldsymbol{X}(U)$. We denote $\boldsymbol{X}(u) = (x_1(u), x_2(u), x_3(u), x_4(u))$ and $\mathbb{G}_n^{\pm}(u) = (v_1(u), v_2(u), v_3(u), v_4(u))$ as coordinate representations. We define a smooth mapping

$$\mathcal{G}^{\pm} : U \longrightarrow PT^*(\Lambda_0)$$

by $\mathcal{G}^{\pm}(u) = (\mathbb{G}_n^{\pm}(u), [(x_1v_2 - x_2v_1) : (-x_1v_3 + x_3v_1) : (-x_1v_4 + x_4v_1)])$. Then by the above proposition we have the following corollary.

Corollary 4.3 *For any timelike surfaces $\boldsymbol{X} : U \longrightarrow H_1^3$, the AdS-null height function $H : U \times \Lambda_0 \longrightarrow \mathbb{R}$ of $\boldsymbol{X}$ is a generating family of the Legendrian embedding $\mathcal{G}^{\pm}$.* □

Therefore we conclude that the AdS-nullcone Gauss image $\mathbb{G}_n^{\pm}$ can be regarded as a Legendrian map and $\mathbb{G}_n^{\pm}(U)$ can be regarded as a wave front set of $\mathcal{G}^{\pm}$.

5 The AdS-torus cylindrical pedals of timelike surfaces

In this section we consider a surface associate to $M = \boldsymbol{X}(U)$, whose singular points set is diffeomophism to those of AdS-nullcone Gauss image. We can use this surface to investigate the relationship between the AdS-nullcone Gauss image $\mathbb{G}_n^{\pm}$ and the AdS-Gauss map $\widetilde{\mathbb{G}_n^{\pm}}$ of a timelike surface in the Anti de Sitter 3−space. For any timelike surface $\boldsymbol{X} : U \longrightarrow H_1^3$, we define a smooth mapping $ACP_M : U \longrightarrow T_1^2 \times \mathbb{R}^*$ by

$$ACP_M(u) = (\widetilde{\mathbb{G}_n^{\pm}}(u), -\langle \boldsymbol{X}(u), \widetilde{\mathbb{G}_n^{\pm}}(u)\rangle) = (\widetilde{\mathbb{G}_n^{\pm}}(u), \tfrac{1}{\xi(u)}).$$

We call it the *AdS-torus cylindrical pedal* of $M = \boldsymbol{X}(U)$, where $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$. We define a diffeomorphism $\phi : T_1^2 \times \mathbb{R}^* \longrightarrow \Lambda_0$ by $\phi(\boldsymbol{v}, \lambda) = \lambda^{-1}\boldsymbol{v}$. It is easy to check that $\phi(ACP_M(u)) = \mathbb{G}_n^{\pm}(u)$, this means that the singular points sets of $\mathbb{G}_n^{\pm}$ and ACP_M are diffeomorphism.

We now consider a family functions $\overline{H} : U \times T_1^2 \times \mathbb{R}^* \longrightarrow \mathbb{R}$ defined by

$$\overline{H}(u, \boldsymbol{v}, \lambda) = \langle \boldsymbol{X}(u), \boldsymbol{v}\rangle + \lambda = \widetilde{H}(u, \boldsymbol{v}) + \lambda,$$

we call it the *extended AdS-torus height function* on $M = \boldsymbol{X}(U)$. By the similar calculation to the proof of Proposition 3.1(1), we have

$$\mathcal{D}_{\overline{H}} = \{(\widetilde{\mathbb{G}_n^{\pm}}(u), \tfrac{1}{\xi(u)})|u \in U\} = \{ACP_M(u)|u \in U\}.$$

On the other hand, we consider the canonical projection $\pi_1 : T_1^2 \times \mathbb{R}^* \longrightarrow T_1^2$. Then we have $\pi_1|\mathcal{D}_{\overline{H}}$ can be identified with the AdS-torus Gauss map $\widetilde{\mathbb{G}_n^{\pm}}$ of X. Since

$$\mathbb{G}_n^{\pm}(u) = -\frac{1}{\langle \boldsymbol{X}(u), \widetilde{\mathbb{G}_n^{\pm}}(u)\rangle}\widetilde{\mathbb{G}_n^{\pm}}(u) = \xi(u)\widetilde{\mathbb{G}_n^{\pm}}(u),$$

we have $\phi(\mathcal{D}_{\overline{H}}) = \{\mathbb{G}_n^{\pm}(u)|u \in U\} = \mathcal{D}_H$. Therefore, we may say that the AdS-nullcone Gauss image $\mathbb{G}_n^{\pm}$ is a *lift* of the AdS-Gauss map $\widetilde{\mathbb{G}_n^{\pm}}$. In fact, we also have

$$\Sigma_*(\overline{H}) = \{(u,\ \widetilde{\mathbb{G}_n^{\pm}}(u),\ -\langle \boldsymbol{X}(u), \widetilde{\mathbb{G}_n^{\pm}}(u)\rangle)|u \in U\}.$$

We remark that similar discussions apply to the extended AdS-torus height function $\overline{H}$ and AdS-torus height function $\widetilde{H}$, we have $\overline{H}$ and $\widetilde{H}$ are Morse family.

On the other hand, for any $\boldsymbol{v} = (v_1, v_2, v_3, v_4) \in T_1^2$, we consider a coordinate neighborhood $U_{24}^+ = \{\boldsymbol{v} = (v_1, v_2, v_3, v_4) \in T_1^2 | v_2 > 0 \text{ and } v_4 > 0\}$, then

$$\overline{H}(u, \boldsymbol{v}, \lambda) = \widetilde{H}(u, \boldsymbol{v}) + \lambda = -x_1v_1 - x_2\sqrt{1 - v_1^2} + x_3v_3 + x_4\sqrt{1 - v_3^2} + \lambda.$$

We now consider smooth mappings $\mathcal{L}_{\overline{H}} : \widetilde{\mathbb{G}_n^{\pm}}^{-1}(U_{24}^+) \longrightarrow T^*(T_1^2) \times \mathbb{R}^*$ defined by

$$\mathcal{L}_{\overline{H}}(u) = (\widetilde{\mathbb{G}_n^{\pm}}(u),\ [\frac{\partial \overline{H}}{\partial v_1} : \frac{\partial \overline{H}}{\partial v_3} : \frac{\partial \overline{H}}{\partial \lambda}],\ \frac{1}{\xi(u)}) = (\widetilde{\mathbb{G}_n^{\pm}}(u),\ \frac{\partial \overline{H}}{\partial v_1},\ \frac{\partial \overline{H}}{\partial v_3},\ \frac{1}{\xi(u)})$$

and $L_{\widetilde{H}} : \widetilde{\mathbb{G}_n^{\pm}}^{-1}(U_{24}^+) \longrightarrow T^*(T_1^2)$ defined by

$$L_{\widetilde{H}}(u) = (\widetilde{\mathbb{G}_n^{\pm}}(u),\ \frac{\partial \widetilde{H}}{\partial v_1},\ \frac{\partial \widetilde{H}}{\partial v_3}) = (\widetilde{\mathbb{G}_n^{\pm}}(u),\ \frac{\partial \overline{H}}{\partial v_1},\ \frac{\partial \overline{H}}{\partial v_3}).$$

According by these definitions we have $\mathcal{L}_{\overline{H}}$ is a Legendrian embedding whose generating family is the extended AdS-torus height function $\overline{H}$ and $L_{\widetilde{H}}$ is a Lagrangian embedding whose generating family is the AdS-torus height function $\widetilde{H}$. The details on Lagrangian singularities can be found in [1, 32]. We now consider the canonical projection

$$\pi : T^*(T_1^2) \times \mathbb{R}^* \longrightarrow T^*(T_1^2),\ \pi(\boldsymbol{v}, \lambda) = \boldsymbol{v},$$

then $\pi(\mathcal{L}_{\overline{H}}) = L_{\widetilde{H}}$. We remark that if we adopt other local coordinates on T_1^2, exactly the same results hold. Therefore we have the following proposition.

Proposition 5.1 *Under the same assumptions as in the above arguments, we have the following:*

(1) *The AdS-Gauss map $\widetilde{\mathbb{G}_n^\pm}$ is a Lagrangian map. The corresponding Lagrangian embedding $L_{\widetilde{H}}$ is called the Lagrangian lift of the AdS-Gauss map $\widetilde{\mathbb{G}_n^\pm}$;*

(2) *The Legendrian lift $\mathcal{G}^\pm$ of the AdS-nullcone Gauss image $\mathbb{G}_n^\pm$ is a covering of the Lagrangian lift $L_{\widetilde{H}}$ of the AdS-Gauss map $\widetilde{\mathbb{G}_n^\pm}$.*

Proof. The assertion (1) follows from the above arguments.

On the other hand, for any $\boldsymbol{v} \in T_1^2$, without loss of the generality, we can assume that $v_2 > 0$ and $v_4 > 0$. Then we have $v_2 = \sqrt{1 - v_1^2}$, $v_4 = \sqrt{1 - v_3^2}$, so we can regard (v_1, v_3) as the coordinate system of T_1^2. Therefore, the homogeneous coordinates of $PT^*(T_1^2 \times \mathbb{R}^*)$ can be expressed as $(v_1, v_3, \lambda, [\varsigma_1 : \varsigma_2 : \varsigma])$. Moreover, if $\varsigma \neq 0$, we have

$$(v_1, v_3, \lambda, [\varsigma_1 : \varsigma_2 : \varsigma]) = (v_1, v_3, \lambda, [\frac{\varsigma_1}{\varsigma} : \frac{\varsigma_2}{\varsigma} : 1]),$$

so that we can adopt the corresponding affine coordinates $(v_1, v_3, \lambda, \rho_1, \rho_2)$, where $\rho_i = \varsigma_1/\varsigma$. By the above argument we can naturally regard $T^*(T_1^2) \times \mathbb{R}^*$ as the affine part of $PT^*(T_1^2 \times \mathbb{R}^*)$. We also have the following relation:

$$H \circ (id_U \times \phi)(u, \boldsymbol{v}, \lambda) = H(u, \lambda^{-1}\boldsymbol{v}) = \lambda^{-1}\overline{H}(u, \boldsymbol{v}, \lambda).$$

This means that $H \circ (id_U \times \phi)$ and $\overline{H}$ are $\mathcal{C}$-equivalent in the sense of Mather[18]. So that these generating families correspond to the same Legendrian submanifold (cf., [1, 32]. Then we have a unique contact diffeomorphism $\Phi : PT^*(T_1^2 \times \mathbb{R}^*) \longrightarrow PT^*\Lambda_0$ covering $\phi : T_1^2 \times \mathbb{R}^* \longrightarrow \Lambda_0$ such that $\Phi \circ \mathcal{L}_{\overline{H}} = \mathcal{G}^\pm$. Therefore, $\mathcal{G}^\pm$ is a covering of $L_{\widetilde{H}}$. □

6 Contact with AdS-horospheres

In this section we consider the geometric meaning of the singularities of the AdS-nullcone Gauss image of a timelike surface $M = \boldsymbol{X}(U)$ in H_1^3. We consider the contact of timelike surfaces with AdS-horosphere type surfaces. We now briefly review the theory of contact due to Montaldi [24]. Let $X_i, Y_i (i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that the *contact* of X_1 and Y_1 at y_1 is the same type as the *contact* of X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. It is clear that in the definition $\mathbb{R}^n$ could be replaced by any manifold. In his paper [24], Montaldi gives a characterization of the notion of contact by using the terminology of singularity theory.

Theorem 6.1 *Let $X_i, Y_i (i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $dim X_1 = dim X_2$ and $dim Y_1 = dim Y_2$. Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, \mathbf{0})$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(\mathbf{0}), y_i)$. Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ if and only* *if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.*

For the definition of the $\mathcal{K}$-equivalent, See Martinet [19]. We now consider a function $\mathcal{H}$: $H^3_1 \times \Lambda_0 \longrightarrow \mathbb{R}$ defined by $\mathcal{H}(\boldsymbol{u},\boldsymbol{v}) = \langle \boldsymbol{u},\boldsymbol{v}\rangle + 1$. For any $\boldsymbol{v}_0 \in \Lambda_0$, we denote $\mathfrak{h}_{v_0}(\boldsymbol{u}) = \mathcal{H}(\boldsymbol{u},\boldsymbol{v}_0)$ and we define the AdS-horosphere by $\mathfrak{h}^{-1}_{v_0}(0) = H^3_1 \cap HP(\boldsymbol{v}_0,-1)$. We write $AH(\boldsymbol{v}_0,-1) = H^3_1 \cap HP(\boldsymbol{v}_0,-1)$. For any $u_0 \in U$, we consider the null vector $\boldsymbol{v}^{\pm}_0 = \mathbb{G}^{\pm}_n(u_0)$. Then we have

$$\mathfrak{h}_{v^{\pm}_0} \circ \boldsymbol{X}(u_0) = \mathcal{H} \circ (\boldsymbol{X} \times id_{\Lambda_0})(u_0,\boldsymbol{v}_0) = H(u_0,\mathbb{G}^{\pm}_n(u_0)) = 0.$$

We also have relations
$$\frac{\partial \mathfrak{h}_{v^{\pm}_0} \circ \boldsymbol{X}}{\partial u_i}(u_0) = \frac{\partial H}{\partial u_i}(u_0,\mathbb{G}^{\pm}_n(u_0)) = 0,$$

for $i = 1,2$. This means that the AdS-horosphere $AH(\boldsymbol{v}^{\pm}_0,-1)$ is tangent to $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u_0)$. In this case, we call $AH(\boldsymbol{v}^{\pm}_0,-1)$ the *tangent AdS-horosphere* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u_0)$ (or, u_0), which we write $AH^{\pm}(\boldsymbol{X},u_0)$. Let $\boldsymbol{v}_1,\boldsymbol{v}_2$ be null vectors. If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly dependent, then $HP(\boldsymbol{v}_1,-1)$ and $HP(\boldsymbol{v}_2,-1)$ are parallel. Therefore, we say that AdS-horosphere $AH(\boldsymbol{v}_1,-1)$ and $AH(\boldsymbol{v}_2,-1)$ are *parallel*, if $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly dependent. Then we have the following lemma.

Lemma 6.2 *Let $\boldsymbol{X} : U \longrightarrow H^3_1$ be a timelike surface. Consider two points $u_1,u_2 \in U$. Then we have the following assertions*:

(1) $\mathbb{G}^{\pm}_n(u_1) = \mathbb{G}^{\pm}_n(u_2)$ *if and only if* $\mathrm{AH}^{\pm}(\boldsymbol{X},u_1) = \mathrm{AH}^{\pm}(\boldsymbol{X},u_2)$.

(2) $\widetilde{\mathbb{G}^{\pm}_n}(u_1) = \widetilde{\mathbb{G}^{\pm}_n}(u_2)$ *if and only if* $\mathrm{AH}^{\pm}(\boldsymbol{X},u_1)$ *and* $\mathrm{AH}^{\pm}(\boldsymbol{X},u_2)$ *are parallel.* □

We now consider the contact of M with tangent AdS-horosphere at $p \in M$ as an application of Legendrian singularity theory. We introduce an equivalence relation among Legendrian immersion germs. Let $i : (L,p) \subset (PT^*\mathbb{R}^n,p)$ and $i' : (L',p') \subset (PT^*\mathbb{R}^n,p')$ be Legendrian immersion germs. Then we say that i and i' are *Legendrian equivalent* if there exists a contact diffeomorphism germ $\psi : (PT^*\mathbb{R}^n,p) \longrightarrow (PT^*\mathbb{R}^n,p')$ such that H preserves fibres of π and $\psi(L) = L'$. A Legendrian immersion germ into $PT^*\mathbb{R}^n$ at a point is said to be *Legendrian stable* if for every map with the given germ there are a neighborhood in the space of Legendrian immersion (in the Whitney $C^\infty-$topology) and a neighborhood of the original point such that each Legendrian immersion belonging to the first neighborhood has, in the second neighborhood, a point at which its germ is Legendrian equivalent to the original germ.

Since the Legendrian lift $i : (L,p) \subset (PT^*\mathbb{R}^n,p)$ is uniquely determined on the regular part of the wave front $W(i)$, we have the following proposition due to Zakalyukin[33].

Proposition 6.3 *Let $i : (L,p) \subset (PT^*\mathbb{R}^n,p)$ and $i' : (L',p') \subset (PT^*\mathbb{R}^n,p')$ be Legendrian immersion germs such that regular sets of $\pi \circ i$ and $\pi \circ i'$ respectively are dense. Then i and i' are Legemndrian equivalent if and only if wave front sets $W(i)$ and $W(i')$ are diffeomorphic as set germs.*

We remark that the assumption in the above proposition is a generic condition for i and i'. In particular, if i and i' are Legendrian stable, then these satisfy the assumption.

We can interpret the Legendrian equivalence by using the notion of generating families. We denote $\mathcal{E}_n$ the local ring of function germs $(\mathbb{R}^n,\mathbf{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal $\mathcal{M}_n = \{h \in \mathcal{E}_n | h(\mathbf{0}) = 0\}$. Let $F,G : (\mathbb{R}^k \times \mathbb{R}^n,\mathbf{0}) \longrightarrow (\mathbb{R},0)$ be function germs. We say that F and G are *$P-\mathcal{K}$ equivalent* if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n,\mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n,\mathbf{0})$ of the form $\Psi(q,x) = (\psi_1(q,x),\psi_2(x))$ for $(q,x) \in (\mathbb{R}^k \times \mathbb{R}^n,\mathbf{0})$ such that $\Psi^*(\langle F\rangle_{\mathcal{E}_{k+n}}) = \langle G\rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \longrightarrow \mathcal{E}_{k+n}$ is the pull back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$.

Let $F:(\mathbb{R}^k\times\mathbb{R}^n,\mathbf{0})\longrightarrow(\mathbb{R},0)$ be a function germ. We say that F is a *$\mathcal{K}$-versal deformation* of $f=F|\mathbb{R}^k\times\{\mathbf{0}\}$ if

$$\mathcal{E}_k=T_e(\mathcal{K})(f)+\langle\frac{\partial F}{\partial x_1}|\mathbb{R}^k\times\{\mathbf{0}\},\cdots,\frac{\partial F}{\partial x_n}|\mathbb{R}^k\times\{\mathbf{0}\}\rangle_{\mathbb{R}},$$

where $T_e(\mathcal{K})(f)=\langle\frac{\partial f}{\partial q_1},\cdots,\frac{\partial f}{\partial q_k},f\rangle_{\mathcal{E}_k}$. The main result in the theory of Legendrian singularities [1, 32] is the following:

Theorem 6.4 *Let $F,G:(\mathbb{R}^k\times\mathbb{R}^n,\mathbf{0})\longrightarrow(\mathbb{R},0)$ be Morse families. Then*
(1) *$\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian equivalent if and only if F and G are P–$\mathcal{K}$ equivalent;*
(2) *$\mathcal{L}_F$ is Legendrian stable if and only if F is a $\mathcal{K}$-versal deformation of $f=F|\mathbb{R}^k\times\{\mathbf{0}\}$.*

By the uniqueness result of the $\mathcal{K}$-versal deformation of a function germ, Proposition 6.3 and Theorem 6.4, we have the following classification result of Legendrian stable germs in the appendix of [11]. For any map germ $f:(\mathbb{R}^n,\mathbf{0})\longrightarrow(\mathbb{R}^p,\mathbf{0})$, we define the *local ring* of f by $Q(f)=\mathcal{E}_n/f^*(\mathcal{M}_p)\mathcal{E}_n$. Then we have the following proposition.

Proposition 6.5 *Let $F,G:(\mathbb{R}^k\times\mathbb{R}^n,\mathbf{0})\longrightarrow(\mathbb{R},0)$ be Morse families. Suppose that $\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian stable. Then the following conditions are equivalent:*
(1) *$(W(\mathcal{L}_F),\mathbf{0})$ and $(W(\mathcal{L}_G),\mathbf{0})$ are diffeomorphic as germs;*
(2) *$\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian equivalent;*
(3) *$Q(f)$ and $Q(g)$ are isomorphic as $\mathbb{R}$-algebras, where $f=F|\mathbb{R}^k\times\{\mathbf{0}\}$ and $g=G|\mathbb{R}^k\times\{\mathbf{0}\}$.*

We have the tools for study of the contact of timelike surfaces with AdS-horospheres. Let $\mathbb{G}^{\pm}_{n\,i}:(U,u_i)\longrightarrow(\Lambda_0,\boldsymbol{v}^{\pm}_i)$ (for $i=1,2$) be AdS-nullcone Gauss image germs of timelike surface germs $\boldsymbol{X}_i:(U,u_i)\longrightarrow(H^3_1,\boldsymbol{X}_i(u_i))$. We say that $\mathbb{G}^{\pm}_{n\,1}$ and $\mathbb{G}^{\pm}_{n\,2}$ are *$\mathcal{A}$-equivalent* if there exist diffeomorphism germs $\phi:(U,u_1)\longrightarrow(U,u_2)$ and $\Phi:(H^3_1,\boldsymbol{v}^{\pm}_1)\longrightarrow(H^3_1,\boldsymbol{v}^{\pm}_2)$ such that $\Phi\circ\mathbb{G}^{\pm}_{n\,1}=\mathbb{G}^{\pm}_{n\,2}\circ\phi$. Suppose the regular set of $\mathbb{G}^{\pm}_{n\,i}$ is dense in (U,u_i) for each $i=1,2$. It follows from Proposition 6.3 that $\mathbb{G}^{\pm}_{n\,1}$ and $\mathbb{G}^{\pm}_{n\,2}$ are $\mathcal{A}$-equivalent if and only if the corresponding Legendrian embedding germs $\mathcal{G}^{\pm}_1:(U,u_1)\longrightarrow(\Delta_1,\mathbf{z}_1)$ and $\mathcal{G}^{\pm}_2:(U,u_2)\longrightarrow(\Delta_1,\mathbf{z}_2)$ are Legendrian equivalent. This condition is also equivalent to the condition that two generating families H_1 and H_2 are P–$\mathcal{K}$ equivalent by Theorem 6.4. Here, $H_i:(U\times\Lambda_0,(u_i,\boldsymbol{v}^{\pm}_i))\longrightarrow\mathbb{R}$ is the corresponding AdS-null height function germ of $\boldsymbol{X}_i$.

On the other hand, we denote $h_{i,v^{\pm}_i}=H_i(u,\boldsymbol{v}^{\pm}_i)$; then we have $h_{i,v^{\pm}_i}(u)=\mathfrak{h}_{v^{\pm}_i}\circ\boldsymbol{X}_i(u)$. By Theorem 6.1,

$$K(\boldsymbol{X}_1(U),\mathrm{AH}^{\pm}(\boldsymbol{X}_1,u_1),\boldsymbol{v}^{\pm}_1)=K(\boldsymbol{X}_2(U),\mathrm{AH}^{\pm}(\boldsymbol{X}_2,u_2),\boldsymbol{v}^{\pm}_2)$$

if and only if $h_{1,v^{\pm}_1}$ and $h_{2,v^{\pm}_2}$ are $\mathcal{K}$-equivalent. Therefore, we can apply the above arguments to our situation. We denote by $Q^{\pm}(x,u_0)$ the local ring of the function germ $h_{v^{\pm}_0}:(U,u_0)\longrightarrow\mathbb{R}$, where $\boldsymbol{v}^{\pm}_0=\mathbb{G}^{\pm}_n(u_0)$. We remark that we can write the local ring explicitly as follows:

$$Q^{\pm}(x,u_0)=\frac{C^{\infty}_{u_0}(U)}{\langle\langle\boldsymbol{X}(u),\mathbb{G}^{\pm}_n(u_0)\rangle+1\rangle_{C^{\infty}_{u_0}(U)}}$$

where $C^{\infty}_{u_0}(U)$ is the local ring of function germs at u_0 with the unique maximal ideal $\mathcal{M}_{u_0}(U)$.

Theorem 6.6 *Let $\boldsymbol{X}_i:(U,u_i)\longrightarrow(H^3_1,\boldsymbol{X}_i(u_i))$ (for $i=1,2$) be timelike surface germs such that the corresponding Legendrian embedding germs $\mathcal{G}^{\pm}_i:(U,u_i)\longrightarrow(\Delta_1,\boldsymbol{z}_i)$ are Legendrian stable. Then the following conditions are equivalent:*
(1) *AdS-nullcone Gauss image germs $\mathbb{G}^{\pm}_{n\,1}$ and $\mathbb{G}^{\pm}_{n\,2}$ are $\mathcal{A}$-equivalent;*
(2) *H_1 and H_2 are P–$\mathcal{K}$-equivalent;*

(3) $h_{1,v_1^\pm}$ *and* $h_{2,v_2^\pm}$ *are* $\mathcal{K}$*-equivalent;*

(4) $K(\boldsymbol{X}_1(U), AH^\pm(\boldsymbol{X}_1, u_1), \boldsymbol{v}_1^\pm) = K(\boldsymbol{X}_2(U), AH^\pm(\boldsymbol{X}_2, u_2), \boldsymbol{v}_2^\pm)$

(5) $Q^\pm(\boldsymbol{X}_1, u_1)$ *and* $Q^\pm(\boldsymbol{X}_2, u_2)$ *are isomorphic as* $\mathbb{R}$*-algebras.* □

For a timelike surface germ $\boldsymbol{X} : (U, u_0) \longrightarrow (H_1^3, \boldsymbol{X}(u_0))$, we call $\boldsymbol{X}^{-1}(AH(\mathbb{G}_n^\pm(u_0), -1), u_0)$ the *tangent Anti de Sitter horospherical indicatrix germ* (briefly, *tangent AdS-horospherical indicatrix germ*) of $\boldsymbol{X}$. In general we have the following proposition:

Proposition 6.7 *Let* $\boldsymbol{X}_i : (U, u_i) \longrightarrow (H_1^3, \boldsymbol{X}_i(u_i))$ *(for* $i = 1, 2$*) be timelike surface germs such that their AdSh*$^\pm$*-parabolic sets have no interior points as subspaces of* U*. If AdS-nullcone Gauss image germs* $\mathbb{G}_{n\,1}^\pm$ *and* $\mathbb{G}_{n\,2}^\pm$ *are* $\mathcal{A}$*-equivalent, then*

$$K(\boldsymbol{X}_1(U), AH^\pm(\boldsymbol{X}_1, u_1), \boldsymbol{v}_1^\pm) = K(\boldsymbol{X}_2(U), AH^\pm(\boldsymbol{X}_2, u_2), \boldsymbol{v}_2^\pm).$$

In this case, $\boldsymbol{X}_1^{-1}(AH(\mathbb{G}_{n\,1}^\pm(u_1), -1), u_1)$ *and* $\boldsymbol{X}_2^{-1}(AH(\mathbb{G}_{n\,2}^\pm(u_2), -1), u_2)$ *are diffeomorphic as set germs.* □

From the above proposition, the diffeomorphism type of the tangent AdS-horospherical indicatrix germ is an invariant of $\mathcal{A}$-classification of the AdS-nullcone Gauss image germ of $\boldsymbol{X}$. Moreover, we can borrow some basic invariants from the singularity theory on function germs. We need $\mathcal{K}$-invariants for a function germ. The local ring of a function is a complete $\mathcal{K}$-invariant for generic function germs. It is, however, not a numerical invariant. The $\mathcal{K}$-codimension of a function germ is a numerical $\mathcal{K}$-invariant of function germs. We denote

$$\text{Ah}^\pm\text{-ord}(\boldsymbol{X}, u_0) = \dim \frac{C_{u_0}^\infty(U)}{\langle h_{v_0^\pm}(u_0), \partial h_{v_0^\pm}(u_0)/\partial u_i \rangle_{C_{u_0}^\infty(U)}},$$

where $\boldsymbol{v}_0^\pm = \mathbb{G}_n^\pm(u_0)$. Usually $\text{Ah}^\pm\text{-ord}(\boldsymbol{X}, u_0)$ is called the $\mathcal{K}$-codimension of $h_{v_0^\pm}$. However, We call it the *order of contact with tangent AdS-horosphere type surface* at $\boldsymbol{X}(u_0)$. We also have the notion of *corank* of function germs:

$$\text{Ah}^\pm\text{-corank}(\boldsymbol{X}, u_0) = 2 - \text{rankHess}(h_{v_0^\pm})(u_0),$$

By Proposition 4.1, $\boldsymbol{X}(u_0)$ is an AdSh$^\pm$-parabolic point if and only if $\text{Ah}^\pm\text{-corank}\ (\boldsymbol{X}, u_0) \geq 1$ and $\boldsymbol{X}(u_0)$ is an AdS-horospherical point if and only if $\text{Ah}^\pm\text{-corank}\ (\boldsymbol{X}, u_0) = 2$. On the other hand, a function germ $f : (\mathbb{R}^{n-1}, \boldsymbol{a}) \longrightarrow \mathbb{R}$ has the A_k-singularity if f is $\mathcal{K}$-equivalent to the germ $\pm u_1^2 \pm \cdots \pm u_{n-2}^2 + u_{n-1}^{k+1}$. If $\text{Ah}^\pm\text{-corank}(\boldsymbol{X}, u_0) = 1$, the AdS-null height function $h_{v_0^\pm}$ has the A_k-singularity at u_0 and is generic. In this case we have $\text{Ah}^\pm\text{-ord}(\boldsymbol{X}, u_0) = k$. This number is equal to the order of contact in the classical sense (cf., [3]). This is the reason why we call $\text{Ah}^\pm\text{-ord}(\boldsymbol{X}, u_0)$ the order of contact with the AdS-horosphere type surface at $\boldsymbol{X}(u_0)$.

7 Classification of singularities of AdS-nullcone Gauss images

In this section we give the generic classification of singularities of AdS-nullcone Gauss images. We have almost the same arguments as those of [11], so that we omit the details. We consider the space of timelike embeddings $\text{Emb}_T(U, H_1^3)$ with the Whitney C^∞-topology. By the classification of stable Legendrian singularities of $n = 3$ and the transversality theorem of [11] (Proposition 7.1), we have the following theorem.

Theorem 7.1 *There exists an open dense subset* $\mathcal{O} \subset \text{Emb}_T(U, H_1^3)$ *such that for any* $\boldsymbol{X} \in \mathcal{O}$ *the following conditions hold.*

(1) *The AdSh$^{\pm}$-parabolic set* ${K^{\pm}_{AdSn}}^{-1}(0)$ *is a regular curve. We call such a curve the AdSh$^{\pm}$-parabolic curve.*

(2) *The AdS-nullcone Gauss image* $\mathbb{G}^{\pm}_n$ *along the AdSh$^{\pm}$-parabolic curve is a cuspidal edge except at isolated points. At such the point* $\mathbb{G}^{\pm}_n$ *is the swallowtail.*

(3) *The cuspidal edge points (swallowtail points) of AdS-nullcone Gauss image* $\mathbb{G}^{\pm}_n$ *correspond to fold points (cusp points) of AdS-torus Gauss map.*

Here, a map germ $f : (\mathbb{R}^2, \boldsymbol{a}) \longrightarrow (\mathbb{R}^3, \boldsymbol{b})$ *is called a cuspidal edge if it is* $\mathcal{A}$*-equivalent to the germ* (u_1, u_2^2, u_2^3) *and a swallowtail if it is* $\mathcal{A}$*-equivalent to the germ* $(3u_1^4 + u_1^2u_2, 4u_1^3 + 2u_1u_2, u_2)$*. A map germ* $f : (\mathbb{R}^2, \boldsymbol{a}) \longrightarrow (\mathbb{R}^2, \boldsymbol{b})$ *is called a fold if it is* $\mathcal{A}$*-equivalent to the germ* (u_1, u_2^2) *and a cusp if it is* $\mathcal{A}$*-equivalent to the germ* $(u_1, u_2^3 + u_1u_2)$ (cf., Figure 1).

Cuspital edge Swallowtail

Figure 1

Following the terminology of Whitney[31], we say that a timelike surface $\boldsymbol{X} : U \longrightarrow H^3_1$ has the *excellent AdS-nullcone Gauss image* $\mathbb{G}^{\pm}_n$, the AdS-nullcone Gauss image $\mathbb{G}^{\pm}_n$ has only cuspidal edges and swallowtails as singularities.

We now consider the geometric meanings of cuspidal edges and swallowtails of the AdS-nullcone Gauss image. We have the following results analogous to the results of [11].

Theorem 7.2 *Let* $\mathbb{G}^{\pm}_n : (U, u_0) \longrightarrow (\Lambda_0, \boldsymbol{v}^{\pm}_0)$ *be the excellent AdS-nullcone Gauss image germ of a timelike surface* $\boldsymbol{X}$ *and* $h_{v_0} : (U, u_0) \longrightarrow \mathbb{R}$ *be the AdS-null height function germ at* u_0*, where* $\boldsymbol{v}^{\pm}_0 = \mathbb{G}^{\pm}_n(u_0)$*. Then we have the following.*

(1) *The point* u_0 *is an AdSh$^{\pm}$-parabolic point of* $\boldsymbol{X}$ *if and only if* Ah$^{\pm}$-corank$(\boldsymbol{X}, u_0)= 1$.

(2) *If* u_0 *is an AdSh$^{\pm}$-parabolic point of* $\boldsymbol{X}$*, then* $h_{v^{\pm}_0}$ *has the* A_k*−singularity for* $k = 2, 3$.

(3) *Suppose that* u_0 *is an AdSh$^{\pm}$-parabolic point of* $\boldsymbol{X}$*. Then the following conditions are equivalent:*

(a) $\mathbb{G}^{\pm}_n$ *has the cuspidal edge at* u_0*;*

(b) $h_{v^{\pm}_0}$ *has the* A_2*−singularity;*

(c) Ah$^{\pm}$-order$(\boldsymbol{X}, u_0) = 2$*;*

(d) *the tangent AdS-horospherical indicatrix is an ordinary cusp, where a curve* $C \subset \mathbb{R}^2$ *is called an ordinary cusp if it is diffeomorphic to the curve given by* $\{(u_1, u_2) | u_1^2 - u_2^3 = 0\}$*.*

(e) *for each* $\varepsilon > 0$*, there exit two points* $u_1, u_2 \in U$ *such that* $|u_0 - u_i| < \varepsilon$ *for* $i = 1, 2$*, neither of* u_1 *nor* u_2 *is an AdSh$^{\pm}$-parabolic point and the tangent AdS-horospheres to* $M = \boldsymbol{X}(U)$ *at* u_1 *and* u_2 *are parallel.*

(4) *Suppose that* u_0 *is an AdSh$^{\pm}$-parabolic point of* $\boldsymbol{X}$*. Then the following conditions are equivalent:*

(a) $\mathbb{G}^{\pm}_n$ *has the swallowtail at* u_0*;*

(b) $h_{v^{\pm}_0}$ *has the* A_3*−singularity;*

(c) Ah$^{\pm}$-order$(\boldsymbol{X}, u_0) = 3$*;*

(d) *the tangent AdS-horospherical indicatrix is an point or a tachnodal, where a curve* $C \subset \mathbb{R}^2$ *is called a tachnodal if it is diffeomorphic to the curve given by* $\{(u_1, u_2) | u_1^2 - u_2^4 = 0\}$.

(e) *for each* $\varepsilon > 0$, *there exit three points* $u_1, u_2, u_3 \in U$ *such that* $|u_0 - u_i| < \varepsilon$ *for* $i = 1, 2, 3$, *none of* u_1, u_2, u_3 *are an AdSh*$^{\pm}$*-parabolic points and the tangent AdS-horospheres to* $M = \boldsymbol{X}(U)$ *at* u_1, u_2 *and* u_3 *are parallel.*

(f) *for each* $\varepsilon > 0$, *there exit two points* $u_1, u_2 \in U$ *such that* $|u_0 - u_i| < \varepsilon$ *for* $i = 1, 2$, *neither of* u_1 *nor* u_2 *is an AdS-parabolic point and the tangent AdS-horospheres to* $M = \boldsymbol{X}(U)$ *at* u_1 *and* u_2 *are equal.*

Proof. By the Proposition 3.1, we have shown that u_0 is an AdSh$^{\pm}$-parabolic point if and only if Ah$^{\pm}$-corank$(\boldsymbol{X}, u_0) \geq 1$. Since $n = 3$, we have Ah$^{\pm}$-corank$(\boldsymbol{X}, u_0) \leq 2$. Since AdS-null height function germ $H : (U \times \Lambda_0, (u_0, \boldsymbol{v}_0^{\pm})) \longrightarrow \mathbb{R}$ can be considered as a generating family of the Legendrian embedding germ $\mathcal{G}^{\pm}$, $h_{v_0^{\pm}}$ has only the A_k-singularities ($k = 1, 2, 3$). This means that the corank of the Hessian matrix of the $h_{v_0^{\pm}}$ at an AdSh$^{\pm}$-parabolic point is 1. The assertion (2) also follows. For the same reason, the conditions (3)$\{(a), (b), (c)\}$(respectively, (4)$\{(a), (b), (c)\}$) are equivalent.

On the other hand, if the AdS-null height function germ $h_{v_0^{\pm}}$ has the A_2-singularity, it is $\mathcal{K}$-equivalent to the germ $\pm u_1^2 + u_2^3$. Since the $\mathcal{K}$-equivalence preserves the zero level sets, the tangent AdS-horospherical indicatrix is diffeomorphic to the curve given by $\pm u_1^2 + u_2^3 = 0$. This is the ordinary cusp. The normal form for the A_3-singularity is given by $\pm u_1^2 + u_2^4$, so the tangent AdS-horospherical indicatrix is diffeomorphic to the curve given by $\pm u_1^2 + u_2^4 = 0$. This means that the condition (3)$\{(d)\}$(respectively, (4)$\{(d)\}$) is also equivalent to the other conditions.

Suppose that u_0 is an AdSh$^{\pm}$-parabolic point, by Proposition 5.1, the AdS-torus Gauss map has only folds or cusps. If the point u_0 is a fold point, there is a neighborhood of u_0 on which the AdS-torus Gauss map is 2 to 1 except the AdSh$^{\pm}$-parabolic line (i.e, fold curve). By Lemma 6.2, the condition (3)(e) holds. If the point u_0 is a cusp, the critical value set is an ordinary cusp. By the normal form, we can understand that the AdS-Gauss map is 3 to 1 inside region of the critical value. Moreover, the point u_0 is in the closure of the region. This means that the condition (4)(e) is satisfied. We can also observe that near by the cusp point, there are 2 to 1 points which approach to u_0. However, one of those points are always AdSh$^{\pm}$-parabolic points. Since other singularities do not appear in this case, so that the condition (3)(e) (respectively, (4)(e)) characterizes a fold (respectively, a cusp).

For the swallowtail, point u_0, there is a self-intersection curve approaching u_0. On this curve, there are two distinct points u_1 and u_2 such that $\mathbb{G}_n^{\pm}(u_1) = \mathbb{G}_n^{\pm}(u_2)$. By Lemma 7.2, this means that the tangent AdS-horospheres to $M = \boldsymbol{X}(U)$ at u_1 and u_2 are equal. Since there are no other singularities in this case, the condition (4)$\{(f)\}$ characterizes a swallowtail point of $\mathbb{G}_n^{\pm}$. This completes the proof. □

8 AdS-null Monge form

The notion of the Monge form of a surface in Euclidean 3-space is one of the powerful tools for the study of local properties of the surface from the view point of differential geometry.In this section we consider the analogous notion for a timelike surface in H_1^3.

We now consider a function $f(u_1, u_2)$ with $f(0) = f_{u_i}(0) = 0$. Then we have a timelike surface in H^3_1 defined by

$$\boldsymbol{X}_f(u_1,u_2)=(\sqrt{1+\varepsilon_1u_1^2+\varepsilon_2u_2^2+f^2(u_1,u_2)},\ \frac{(1-\varepsilon_1)u_1+(1-\varepsilon_2)u_2}{2},$$
$$f(u_1,u_2),\ \frac{(1+\varepsilon_1)u_1+(1+\varepsilon_2)u_2}{2}),$$

where $\varepsilon_i=\mathrm{sign}(\boldsymbol{X}_i)$ $(i=1,2)$. We can easily calculate $\boldsymbol{N}(0)=(0,0,\frac{\varepsilon_2-\varepsilon_1}{2},0)$; therefore $\mathbb{G}^{\pm}_n(0)=(1,0,\pm1,0)$. We call $\boldsymbol{X}_f$ a *Anti de Sitter null Monge form* (briefly, *AdS-null Monge form*). Then we have the following proposition.

Proposition 8.1 *Any timelike surface in H^3_1 is locally given by the AdS-null Monge form.*

Proof. Let $\boldsymbol{X}:U\longrightarrow H^3_1$ be a timelike surface. We consider Lorentzian motion of H^3_1 which is a transitive action. Therefore, without loss of the generality, we assume that $p=\boldsymbol{X}(0)=(1,0,0,0)$. We denote $M=\boldsymbol{X}(U)$, we have a basis $\{\boldsymbol{X}(0),\boldsymbol{N}(0),\boldsymbol{X}_{u_1}(0),\boldsymbol{X}_{u_2}(0)\}$ of $T_p\mathbb{R}^4_2$ such that $T_pM=\langle\boldsymbol{X}_{u_1}(0),\boldsymbol{X}_{u_2}(0)\rangle_{\mathbb{R}}$. Applying the Gram-Schmidt procedure we have a pseudo-orthonormal basis $\{\boldsymbol{X}(0),\boldsymbol{N}(0),\ \boldsymbol{e}_1,\boldsymbol{e}_2\}$ of $T_p\mathbb{R}^4_2$ such that $T_pM=\langle\boldsymbol{e}_1,\boldsymbol{e}_2\rangle_{\mathbb{R}}$. In particular, $\{\boldsymbol{e}_1,\boldsymbol{e}_2\}$ is an orthonormal basis of T_pM. Since $p=(1,0,0,0)$, T_pM is considered to be a subspace of ${}_0\mathbb{R}^3_1=\{(0,x_1,x_2,x_3)|x_i\in\mathbb{R}\}$. By a rotation of the space ${}_0\mathbb{R}^3_1$, we might assume that $T_pM=\{(0,\frac{(1-\varepsilon_1)u_1+(1-\varepsilon_2)u_2}{2},0,\frac{(1+\varepsilon_1)u_1+(1+\varepsilon_2)u_2}{2})|u_i\in\mathbb{R}\}\subset\mathbb{R}^4_2$. Then the germ (M,p) might be written in the form

$$(f_0(u_1,u_2),\ \frac{(1-\varepsilon_1)u_1+(1-\varepsilon_2)u_2}{2},f(u_1,u_2),\ \frac{(1+\varepsilon_1)u_1+(1+\varepsilon_2)u_2}{2})$$

with function germs $f_0(u_1,u_2),f(u_1,u_2)$. Since $M\subset H^3_1$, we have the relation

$$f_0(u_1,u_2)=\sqrt{1+\varepsilon_1u_1^2+\varepsilon_2u_2^2+f^2(u_1,u_2)}.$$

Since we have

$$T_pM=\{(0,\frac{(1-\varepsilon_1)u_1+(1-\varepsilon_2)u_2}{2},0,\frac{(1+\varepsilon_1)u_1+(1+\varepsilon_2)u_2}{2})|u_i\in\mathbb{R}\},$$

the condition $f(0)=0,\ f_{u_i}(0)=0$ are automatically satisfied. □

For the null vector $\boldsymbol{v}_0^{\pm}=(1,0,\pm1,0)$, we consider the AdS-horosphere type surface $\mathrm{AH}(\boldsymbol{v}_0^{\pm},-1)$. Then we have the AdS-null Monge form of $\mathrm{AH}(\boldsymbol{v}_0^{\pm},-1)$:

$$\boldsymbol{a}^{\pm}(u_1,u_2)=(\frac{\varepsilon_1u_1^2+\varepsilon_2u_2^2}{2}+1,\ \frac{(1-\varepsilon_1)u_1+(1-\varepsilon_2)u_2}{2},\pm\frac{\varepsilon_1u_1^2+\varepsilon_2u_2^2}{2},\ \frac{(1+\varepsilon_1)u_1+(1+\varepsilon_2)u_2}{2}).$$

Here, we can easily check the relation $\langle\boldsymbol{a}(u),\boldsymbol{v}_0^{\pm}\rangle=-1$.

On the other hand, $\boldsymbol{a}^{\pm}(0)=(1,0,0,0)=p$ and $\boldsymbol{a}^{\pm}_{u_i}(0)$ is equal to the $x_{3+\varepsilon_i}$-axis for $i=1,2$. This means that $T_pM=T_p(\boldsymbol{a}^{\pm}(U))$. Therefore $\boldsymbol{a}^{\pm}(U)=\mathrm{AH}(\boldsymbol{v}_0^{\pm},-1)$ is the tangent AdS-horosphere of $M=\boldsymbol{X}_f(U)$ at $p=\boldsymbol{X}_f(0)$. It follows from this fact that the tangent AdS-horospherical indicatrix of the AdS-null Monge form germ $(\boldsymbol{X}_f,0)$ is given as follows:

$$\boldsymbol{X}_f^{-1}(AH(\boldsymbol{v}_0^{\pm},0))=\{(u_1,u_2)|\pm2f(u_1,u_2)=\varepsilon_1u_1^2+\varepsilon_2u_2^2\}.$$

On the other hand, since $f(0)=f_{u_i}(0)=0$, we may write

$$f(u_1,u_2)=\frac{1}{2}\bar{k}_1u_1^2+\frac{1}{2}\bar{k}_2u_2^2+g(u_1,u_2)$$

where $g \in \mathcal{M}_2^3$ and $\bar{k_1}, \bar{k_2}$ are eigenvalues of $(f_{u_1 u_2}(0))$. Under this representation, we can easily calculate $\boldsymbol{X}_{f,u_1u_2}(0) = (\varepsilon_i \delta_{ij}, 0, \bar{k_i}, \delta_{ij}, 0)$. It follows from this fact that

$$h_{ij}^{\pm}(0) = \langle \mathbb{G}_n^{\pm}(0), \boldsymbol{X}_{f,u_iu_j}(0) \rangle = \varepsilon_i \delta_{ij}(-1 \pm \bar{k_i}),$$

and

$$g_{ij}(0) = \langle \boldsymbol{X}_{f,u_i}(0), \boldsymbol{X}_{f,u_j}(0) \rangle = \varepsilon_i \delta_{ij}.$$

Therefore, we have $k_i^{\pm}(0) = -1 \pm \varepsilon_i \bar{k_i}$ and

$$K_{AdSn}^{\pm}(0) = k_1^{\pm}(0) k_2^{\pm}(0) = (-1 \pm \varepsilon_1 \bar{k_1})(-1 \pm \varepsilon_2 \bar{k_2}).$$

The tangent AdS-horospherical indicatrix is given by

$$\begin{aligned} \boldsymbol{X}_f^{-1}(AH(\boldsymbol{v}_0^{\pm}, -1)) &= \{(u_1, u_2) | \pm \bar{k_1} u_1^2 \pm \bar{k_2} u_2^2 \pm 2g(u_1, u_2) - \varepsilon_1 u_1^2 - \varepsilon_2 u_2^2 = 0\} \\ &= \{(u_1, u_2) | \varepsilon_1 k_1^{\pm}(0) u_1^2 + \varepsilon_2 k_2^{\pm}(0) u_2^2 \pm 2g(u_1, u_2) = 0\}. \end{aligned}$$

If we try to draw picture of the AdS-nullcone Gauss image, it might be very hard to give a parameterization. However, by the AdS-null Monge form of the tangent AdS-horospherical indicatrix germ, we can easy to detect the type of singularities of the AdS-nullcone Gauss image $\mathbb{G}_n^{\pm}$ (or, AdS-torus Gauss map $\widetilde{\mathbb{G}_n^{\pm}}$).

Example 8.1 *Consider the function given by*

$$f(u_1, u_2) = \frac{1}{2} u_1^2 + u_2^2 + \frac{1}{2} u_1^3.$$

Suppose that $\varepsilon_1 = -1, \varepsilon_2 = 1$ *Then* $\bar{k_1} = 1$, $\bar{k_2} = 2$. *We have* $k_1^+ = -2$, $k_2^+ = 1$, $k_1^- = 0, k_2^- = -3$. *So that the origin is an AdSh*$^-$*- parabolic point. The tangent AdS-horospherical indicatrix germ at the origin is the ordinary cusp* $u_2^2 = -\frac{1}{3} u_1^3$. *By Theorem 7.2,* $\mathbb{G}_n^-$ $(\widetilde{\mathbb{G}_n^-})$ *is the cuspidal edge* (*fold*) *at the origin.*

Example 8.2 *Consider the function given by*

$$f(u_1, u_2) = \frac{1}{2} u_1^2 + \frac{1}{4} u_2^2 + \frac{3}{2} u_1^4.$$

Suppose that $\varepsilon_1 = 1, \varepsilon_2 = -1$ *Then* $\bar{k_1} = 1$, $\bar{k_2} = 1/2$. *We have* $k_1^+ = 0$, $k_2^+ = -3/2$, $k_1^- = -2, k_2^- = -1/2$. *So that the origin is an AdSh*$^+$*- parabolic point. The tangent AdS-horospherical indicatrix germ at the origin is the tachnodal* $u_2^2 = u_1^4$. *By Theorem 7.2,* $\mathbb{G}_n^-$ $(\widetilde{\mathbb{G}_n^-})$ *is the swallowtail* (*cusp*) *at the origin.*

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